Classification of Vertex-Transitive Digraphs of Order a Product of Two Distinct Primes via Automorphism Group

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Abstract

In the mid-1990s, two groups of authors independently obtained classifications of vertex-transitive graphs whose order is a product of two distinct primes. In the intervening years it has become clear that there is additional information concerning these graphs that would be useful, as well as making explicit the extensions of these results to digraphs. Additionally, there are several small errors in some of the papers that were involved in this classification. The purpose of this paper is to fill in the missing information as well as correct all known errors.

Mathematics Subject Classifications: 05C25, 20B99

1 Introduction

The initial motivation for this paper came from some work [8] done by the authors that used a well-known classification of vertex-transitive graphs of order pq, where p and q are distinct primes.

The original classification had been obtained by two different groups of authors, each with their own perspective on what properties of these graphs were important. One group (consisting of Marušič and Scapellato) [17] was primarily concerned with determining a minimal transitive subgroup of the automorphism group, while the other (consisting of Praeger, Xu, and several others) [23, 24] was primarily concerned with determining the full automorphism groups of these graphs, and in particular in the cases when the automorphism group is primitive or the graph is arc-transitive. Although the results in the classifications are stated for graphs, the proofs as written apply equally to digraphs.

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Over the years, it has become apparent that there are "gaps" in the information about vertex-transitive digraphs of order pq that are not addressed by either approach but would be useful to fill. Specifically, the classification of vertex-transitive digraphs of order pq that have imprimitive almost simple automorphism groups was incomplete, and the full automorphism groups of the Marušič-Scapellato (di)graphs were unknown. Additionally, there are several errors in this classification, and these have propagated themselves in the literature. The most significant of these errors, at least from the point of view of the difficulty in correcting the error, is with Praeger, Wang, and Xu's classification of arc-transitive Marušič-Scapellato graphs [23].

It is the purpose of this paper to fill the "gaps" described in the preceding paragraph, and to correct the known errors. Finally, widespread reliance on results that contained errors has left a body of results that may or may not be correct; at best, the proofs need to be revised. We have not attempted to address all of these, but we provide a list of those that we are aware of.

2 Preliminaries

Throughout, p and q are distinct primes with $q < p$. We begin with basic definitions. In particular, we define the classes of graphs and digraphs that will appear in what follows (with the exception of the Marušič-Scapellato digraphs, whose definition is best presented in a group-theoretic context and is therefore postponed to Definition 16). We denote the arc-set of a digraph Γ by $A(\Gamma)$. The most commonly studied class of vertex-transitive digraphs are Cayley digraphs.

Definition 1. Let G be a group and $S \subseteq G$. Define the **Cayley digraph of $\boldsymbol{G}$ with connection set $\boldsymbol{S}$**, denoted by $\mathrm{Cay}(G,S)$, to be the digraph with $V(\mathrm{Cay}(G,S)) = G$ and $A(\mathrm{Cay}(G,S)) = \{(g,gs) : g \in G, s \in S\}$.

Note that we use the term digraphs to include graphs. If Γ is a digraph satisfying $(x,y) \in A(\Gamma)$ if and only if $(y,x) \in A(\Gamma)$, then we will say that Γ is a **graph**, and replace each pair (x,y) and (y,x) of symmetric ordered pairs in $A(\Gamma)$ by the unordered pair $\{x,y\}$ in the **edge set** $E(\Gamma)$, which takes the place of the arc set. The next-most-commonly-encountered class of vertex-transitive digraphs are metacirculant digraphs, first defined by Alspach and Parsons [1] (although they only defined metacirculant graphs).

Definition 2. Let $\alpha \in \mathbb{Z}_n^*$, where $\mathbb{Z}_n^*$ is the multiplicative group of units in $\mathbb{Z}_n$, and $S_0,\ldots,S_{m-1} \subseteq \mathbb{Z}_n$ such that $\alpha^m S_i = S_i$, $i \in \mathbb{Z}_m$. Define an **$(\boldsymbol{m,n,\alpha,S_0,\ldots,S_{m-1}})$-metacirculant digraph** $\Gamma = \Gamma(m,n,\alpha,S_0,\ldots,S_{m-1})$ by $V(\Gamma) = \mathbb{Z}_m \times \mathbb{Z}_n$ and $A(\Gamma) = \{(\ell,j),(\ell+i,k)) : k-j \in \alpha^\ell S_i\}$. We also define an **$(\boldsymbol{m,n})$-metacirculant digraph** to be a digraph that is an $(m,n,\alpha,S_0,\ldots,S_{m-1})$-metacirculant digraph for some α and some $S_0,\ldots,S_{m-1}$ as above.

This is not the definition of metacirculant graphs used by Marušič and Scapellato in [17]. They define an (m,n)-metacirculant graph as one whose automorphism group contains a semiregular element of order n whose orbits are cyclically permuted as an n-cycle by some other element. Unfortunately, their definition is not equivalent to the one here (which is basically Alspach and Parsons' original definition [1]). There are at least two infinite families of graphs which satisfy the definition from [17] but that are not isomorphic to metacirculant graphs - see [27, Section 2] and [7, Theorem 4.12]. However, the definition in [17] is equivalent to the one here for metacirculant digraphs of order qp, with q and p distinct primes. See [11, Theorem 9.2.2] for a characterization of metacirculant digraphs of order qp. Finally, there is a mistake in the statement of [17, Proposition 1.1], namely, the condition that $r^q \equiv 1 \pmod{p}$ (the Petersen graph has $r = 2$ which has order 4 modulo 5 - see [1, Example 1]) should be that $r^{q^k} \equiv 1 \pmod{p}$ for some integer k - see [11, Lemma 1.5.12].

Many Cayley digraphs and metacirculant digraphs have the important property of imprimitivity that assists in any effort to understand their automorphisms.

Definition 3. Let $G \leqslant S_X$ be transitive. A subset $B \subseteq X$ is a **block** of G if whenever $g \in G$, then $g(B) \cap B = \emptyset$ or B. For a block B of G, the set $\mathcal{B} = \{g(B) \mid g \in G\}$ is called a ***G*-invariant partition**. If $B = \{x\}$ for some $x \in X$ or $B = X$, then B is a **trivial block**. Any other block is nontrivial. If G has a nontrivial block, then G is **imprimitive**. If G is not imprimitive, we say G is **primitive**.

If Γ is a digraph, then we say that Γ **admits an imprimitive action** if there is some transitive group $G \leqslant \mathrm{Aut}(\Gamma)$ that is imprimitive. We say that Γ **admits no imprimitive action** if every transitive group $G \leqslant \mathrm{Aut}(\Gamma)$ is primitive. We say that Γ is **primitive** if $\mathrm{Aut}(\Gamma)$ is primitive, and Γ is **imprimitive** if $\mathrm{Aut}(\Gamma)$ is imprimitive. We refer to any block of $\mathrm{Aut}(\Gamma)$ as a block of Γ also.

It is important for us to make these definitions about Γ. One of the sources of confusion in the literature is that Marušič and Scapellato referred to a digraph as m-imprimitive whenever it admits an imprimitive action with blocks of size m, even if the full automorphism group is primitive.

We observe that a digraph Γ of order pq must lie in one of three families: Γ is primitive; Γ is imprimitive with blocks of size p; or Γ is imprimitive with blocks of size q. Note that the second and third families are not mutually exclusive.

Marušič provided some of the early analysis of vertex-transitive graphs of order pq.

Proposition 4 (Proposition 3.3, [18]). *The graphs of order pq that admit an imprimitive action with blocks of size p are precisely the (q,p)-metacirculant graphs.*

Theorem 5 (Theorem 3.4, [18]). *Let Γ be a graph of order pq that admits an imprimitive action of the group G with a G-invariant partition $\mathcal{B}$ with blocks of size q. Suppose that Γ is not a metacirculant graph. Then the kernel of the action of G on $\mathcal{B}$ is trivial, and G is nonsolvable.*

The proofs of both of these results as written apply equally to digraphs.

In later work with Scapellato, these results were extended to show the following.

Theorem 6 (Theorem, [19]). *Let* Γ *be a vertex-transitive graph of order* pq *that admits an imprimitive action but is not metacirculant. Then every (transitive) imprimitive subgroup of* $\mathrm{Aut}(\Gamma)$ *admits blocks of size* q*;* $p = 2^{2^s} + 1$ *is a Fermat prime,* q *divides* $p - 2$*, and* Γ *is a Marušič-Scapellato graph (see Definition 16).*

The classification of groups of automorphisms as "primitive" or "imprimitive" is a natural one. Observe that a primitive group G cannot contain an intransitive normal subgroup, because the orbits of such a group would give rise to a G-inviariant partition [31, Proposition 7.1]. However, G-invariant partitions can also arise even if G has no intransitive normal subgroup.

Definition 7. A transitive group is called **quasiprimitive** if every nontrivial normal subgroup is transitive.

As we have just observed, every primitive group is quasiprimitive, and quasiprimitive groups are a generalization of primitive groups.

Vertex-transitive digraphs with quasimprimitive automorphism groups are usually studied via their orbital digraphs, which we now define.

Definition 8. Let G act on $X \times X$ in the canonical way, that is $g(x, y) = (g(x), g(y))$. The orbits of this action are called **orbitals**. One orbital is the diagonal, or $\{(x, x) : x \in X\}$, and is called the **trivial orbital**. We assume here that $\mathcal{O}_1, \ldots \mathcal{O}_r$ are the nontrivial orbitals. Define digraphs $\Gamma_1, \ldots, \Gamma_r$ by $V(\Gamma_i) = X$ and $A(\Gamma_i) = \mathcal{O}_i$. The set $\{\Gamma_i : 1 \leqslant i \leqslant r\}$ is the set of **orbital digraphs of G**. A **generalized orbital digraph of G** is an arc-disjoint union of some orbital digraphs of G (that is, identify vertices in the natural way amongst a set of orbital digraphs, and take the new arc set to be the union of the arcs that are in any of the orbital digraphs). We say an orbital is **self-paired** if the corresponding orbital digraph is a graph.

Orbital digraphs of a group G are often given in terms of their suborbits.

Definition 9. Let $G \leqslant S_n$ be transitive and x a point. The orbits of $\mathrm{Stab}_G(x)$ are the **suborbits of G with respect to x**.

Notice that in an orbital digraph of G, the outneighbors of x and inneighbors of x are both suborbits of G with respect to x. We finish this section with group- and graph-theoretic terms that relate to graph quotients.

Definition 10. Suppose $G \leqslant S_n$ is a transitive group that has a G-invariant partition $\mathcal{B}$ consisting of m blocks of size k. Then G has an **induced action on $\mathcal{B}$**, denoted $G/\mathcal{B}$. Namely, for $g \in G$, define $g/\mathcal{B} : \mathcal{B} \to \mathcal{B}$ by $g/\mathcal{B}(B) = B'$ if and only if $g(B) = B'$, and set $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$. We also define the **fixer of $\mathcal{B}$ in G**, denoted $\mathrm{fix}_G(\mathcal{B})$, to be $\{g \in G : g/\mathcal{B} = 1\}$. That is, $\mathrm{fix}_G(\mathcal{B})$ is the subgroup of G which fixes each block of $\mathcal{B}$ set-wise.

Observe that $\mathrm{fix}_G(\mathcal{B})$ is the kernel of the induced homomorphism $G \to S_{\mathcal{B}}$ that arose previously in the statement of Theorem 5, and as such is normal in G.

Definition 11. Let Γ be a vertex-transitive digraph that admits an imprimitive action of the group G with a G-invariant partition $\mathcal{B}$. Define the **block quotient digraph of Γ with respect to $\mathcal{B}$**, denoted $\Gamma/\mathcal{B}$, to be the digraph with vertex set $\mathcal{B}$ and arc set

$$\{(B, B') : B \neq B' \in \mathcal{B} \textit{ and } (u, v) \in A(\Gamma) \textit{ for some } u \in B \textit{ and } v \in B'\}.$$

Note that $\mathrm{Aut}(\Gamma)/\mathcal{B} \leqslant \mathrm{Aut}(\Gamma/\mathcal{B})$.

3 Automorphism groups of vertex-transitive digraphs of order a product of two distinct prime

Our original interest in this problem arose when we were studying a particular Cayley digraph of the nonabelian group of order 21 whose automorphism group is a nonabelian simple group but is imprimitive. This digraph is included in the Marušič-Scapellato characterization as a metacirculant digraph as its automorphism group contains the nonabelian group of order 21. It does not appear elsewhere in that characterization as Marušič and Scapellato were interested in finding a minimal transitive subgroup (indeed, they define a primitive graph to be one in which *every* transitive subgroup of the automorphism group is primitive), and so they were not concerned with its full automorphism group. This digraph does not occur in the Praeger-Xu characterization, as they were interested in graphs (and occasionally digraphs) whose full automorphism group is primitive (indeed, they define a primitive graph to be one in which the full automorphism group is primitive). So in neither characterization of vertex-transitive graphs of order pq were such digraphs looked for. Finally, this digraph does not arise in [6, Theorem 3.2(1)] since that result only holds for graphs, not digraphs. Thus there is a small gap in the literature here.

The aim of this section of our paper is to fill in this gap. Fortunately, the work by Marušič and Scapellato [19] can be easily modified to help in this goal. Indeed, Marušič and Scapellato's work is actually stronger than advertised through the statement of their results, and an additional goal of this section is to make this stronger work more apparent, as from our work on this paper we believe that such stronger statements may be useful. We note that when writing a wreath product, we use the convention that the first group written is acting on the partition, and the second is acting within each block. Some authors, including Praeger et al, use the opposite order.

Theorem 12. *Let Γ be a vertex-transitive digraph of order pq, where $q < p$ are distinct primes such that $G \leqslant \mathrm{Aut}(\Gamma)$ is quasiprimitive and has a G-invariant partition $\mathcal{B}$ with blocks of size q. Additionally, suppose that $\mathcal{B}$ is also an $\mathrm{Aut}(\Gamma)$-invariant partition. Then G is an almost simple group and exactly one of the following is true:*

1. *there are vertex-transitive digraphs Γ_q of order q and $\Gamma_p = K_p$ or its complement, such that $\Gamma \cong \Gamma_p \wr \Gamma_q$, and $\mathrm{Aut}(\Gamma)$ contains $G/\mathcal{B} \wr (\mathrm{Stab}_G(B)|_B)$, $B \in \mathcal{B}$. This latter group contains a regular cyclic subgroup R, or*

2. *Γ has order 55, and is isomorphic to a generalized orbital digraph of $\mathrm{PSL}(2, 11)$ that is not a generalized orbital digraph of $\mathrm{PGL}(2, 11)$. Moreover, Γ is a Cayley digraph*

of the nonabelian group of order 55*, and its full automorphism group is* $\mathrm{PSL}(2, 11)$*, or*

3. Γ *has order 21, and is isomorphic to a generalized orbital digraph of* $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ *that is not a generalized orbital digraph of* $\mathrm{PGL}(2,7)$*. Moreover,* Γ *is a Cayley digraph of the nonabelian group of order* 21*, and its full automorphism group is* $\mathrm{PSL}(3,2)$*, or*

4. Γ *is a Marušič-Scapellato digraph that is not isomorphic to a metacirculant digraph;* $p = 2^{2^s}+1$ *is a Fermat prime, and* q *divides* $p-2$*. Further, every minimal transitive subgroup* G *of* $\mathrm{Aut}(\Gamma)$ *that admits only a* G*-invariant system of* p *blocks of size* q *is isomorphic to* $\mathrm{SL}(2,2^s)$*, and* $\mathrm{Aut}(\Gamma)$ *is isomorphic to a subgroup of* $\mathrm{Aut}(\mathrm{SL}(2,2^s))$*.*

Proof. We first observe that the digraphs given in the parts of the result are distinct. This follows for the following reasons. First, all digraphs in the first three parts are isomorphic to metacirculant digraphs and the digraphs in (4) are not. The digraphs in the first three cases are all distinct as none of the graphs in (2) or (3) are isomorphic to circulants, while the digraphs in (1) are. Of course, the digraphs in (2) and (3) are distinct as their orders are different.

Almost all of the rest of the proof is contained in [19]. We analyze the digraph structures essentially as they do in the proof of their main theorem.

Since G is quasiprimitive, it has no nontrivial intransitive normal subgroups. So $\mathrm{fix}_G(\mathcal{B}) = 1$ and $G/\mathcal{B} \cong G$ is of prime degree p. As G does not have a normal Sylow p-subgroup, neither does $G/\mathcal{B}$, and so by Burnside's Theorem [5, Corollary 3.5B] $G/\mathcal{B}$ is doubly-transitive, and by another theorem of Burnside [5, Theorem 4.1B], $G/\mathcal{B}$ has nonabelian simple socle. Consequently, G is nonsolvable and $G/\mathcal{B} \cong G$ is almost simple.

The possibilities for $G/\mathcal{B}$ are given in [19, Proposition 2.5]. The various cases, with the one exception of $\mathrm{PSL}(2,2^k)$, $k > 1$, are then analyzed in [19]. They are almost all either rejected as impossible using group theoretic arguments or [19, Proposition 2.1] (which is purely about the permutation group structure and also applies to our situation), or determined to be metacirculants using [19, Proposition 2.2], which is almost sufficient for our purposes. Marušič and Scapellato in fact showed that whenever G is a group satisfying the hypothesis of [19, Proposition 2.2], and Γ is a digraph (they only considered graphs but their proof works for digraphs) with $G \leqslant \mathrm{Aut}(\Gamma)$, then either Γ or its complement is disconnected. This implies that $\mathrm{Aut}(\Gamma)$ is a wreath product (and so Γ is a wreath product $\Gamma_1 \wr \Gamma_2$ with Γ_1 and Γ_2 vertex-transitive digraphs of order p and q, respectively) and $\mathrm{Aut}(\Gamma)$ contains $G/\mathcal{B} \wr (\mathrm{Stab}_G(B)|_B)$ which contains a regular cyclic subgroup R, where $B \in \mathcal{B}$, which is what we need here. We see that $\Gamma_1 = K_p$ or its complement as $G/\mathcal{B}$ is 2-transitive. That $\mathrm{Aut}(\Gamma)$ contains R follows as the automorphism group of every vertex-transitive digraph of prime order contains a regular cyclic subgroup of prime degree, and $G \wr H$ contains $G \times H$ [11, Lemma 4.2.13]. There are two possible group structures for G that do not succumb to this general approach, and [19] uses direct arguments to show that the corresponding (di)graphs are metacirculant. We need to address these exceptional possibilities separately.

The first exception occurs in the proof of [19, Proposition 2.7] when handling the case $G = \mathrm{PSL}(2, 11)$ of degree 55. In this case it is argued that $\mathrm{PSL}(2, 11)$ contains a regular metacyclic subgroup that has blocks of size 11. This is a contradiction to the hypothesis of [19, Proposition 2.7], so finishes the argument for them; for us, it shows that these digraphs are Cayley digraphs (as claimed), and (q, p)-metacirculants. It can be verified in `magma` [3] that the only regular subgroup of $\mathrm{PSL}(2, 11)$ in its action on 55 points is the nonabelian group of order 55. Since $\mathrm{PGL}(2, 11)$ is primitive, the digraphs that arise in this case are precisely those whose full automorphism group is $\mathrm{PSL}(2, 11)$.

The second exception occurs at the beginning of [19, Proposition 3.5], namely when $G = \mathrm{PSL}(3, 2)$ and Γ is of order 21. Here, Marušič and Scapellato noted that $\mathrm{PSL}(3, 2)$ in its action on 21 points has a (transitive) nonabelian subgroup of order 21, and so Γ is a metacirculant (which is enough for their purposes, and for us again shows that Γ is a Cayley graph on the nonabelian group of order 21). By the Atlas of Finite Simple Groups the group $\mathrm{PSL}(3, 2) \cong \mathrm{PSL}(2, 7)$ in its representation on 21 points has suborbits of length 1, 2^2, 4^2, and 8, with the suborbits of lengths 4 being non self-paired. The action of $\mathrm{PGL}(2, 7)$ is primitive, so again orbital digraphs of that group do not meet our hypotheses and we are interested only in those digraphs whose full automorphism group is $\mathrm{PSL}(3, 2)$.

Finally, the case where G has socle $\mathrm{PSL}(2, 2^k) = \mathrm{SL}(2, 2^k)$ is mainly analyzed in [20], where, for example, the orbital digraphs of the groups are determined. In the proof of [19, Theorem], they show that if an imprimitive representation of $\mathrm{SL}(2, 2^k)$ has order qp and is contained in the automorphism group of a metacirculant digraph Γ of order qp, then it either contains the complete p-partite graph where each partition has size q (and are the blocks of $\mathcal{B}$), or is contained in the complement of this graph. These digraphs are easily seen to be circulant as either Γ or its complement is again disconnected. Then G is metacyclic, a contradiction. By [19, Theorem], Γ is a Fermat digraph, and $k = 2^s$, $s \geqslant 1$. Also, that $G = \mathrm{PSL}(2, 2^{2^s}) = \mathrm{SL}(2, 2^{2^s})$ follows from [19, Proposition 2.7], as does that a minimal transitive subgroup of G that has only blocks of size q is isomorphic to $\mathrm{SL}(2, 2^{2^s})$. The arithmetic conditions are also given in [19, Theorem]. Finally, that $\mathrm{Aut}(\Gamma) \leqslant \mathrm{Aut}(\mathrm{SL}(2, 2^{2^s}))$ follows as $\mathrm{Aut}(\Gamma)$, having simple socle, is an almost simple group. □

From a closer analysis of the suborbits of $\mathrm{PSL}(2, 11)$ and of $\mathrm{PSL}(3, 2)$, we can derive additional information about the digraphs that arise in this analysis. The orbital digraphs of $\mathrm{PSL}(2, 11)$ are examined in [16, Example 2.1]. The suborbits are of length $1, 4, 4, 4, 6, 12, 12$, and 12. Two suborbits of length 12 are the only ones that are not self-paired, and the corresponding orbital digraphs have automorphism group $\mathrm{PSL}(2, 11)$ which is imprimitive (as $\mathrm{PSL}(2, 11)$ has disconnected orbital digraphs). Thus, a generalized orbital digraph that is not a graph must use exactly one of these. Two suborbits of length 4 have disconnected orbital graphs and their union is an orbital graph of $\mathrm{PGL}(2, 11)$, while all of the other suborbits are also suborbits of $\mathrm{PGL}(2, 11)$. Thus, in order to avoid $\mathrm{PGL}(2, 11)$ in the automorphism group of an orbital graph, we must include exactly one of these. For $\mathrm{PSL}(3, 2)$, we use `magma` for much of this analysis. We summarize this extra information about $\mathrm{PSL}(2, 11)$ and what we found with `magma` for

PSL(3, 2) in the following remark.

Remark 13. If Γ arises in Theorem 12(2) and is a graph, then it has a subgraph of valency 4 that is a disconnected orbital graph of PSL(2, 11), and the other disconnected orbital graph of PSL(2, 11) (which is the image of this one under the action of PGL(2, 11)) is not a subgraph of Γ (but Γ itself is connected).

If Γ arises in Theorem 12(2) and is not a graph, then it has a subdigraph of valency 12 that is a non-self-paired orbital digraph of PSL(2, 11), and whose paired orbital digraph of PSL(2, 11) is not a subdigraph of Γ.

If Γ arises in Theorem 12(3) then `magma` [3] has been used to verify that Γ cannot be a graph. By [25], PSL(3, 2) has suborbits of length 1, 2, 2, 4, 4, 8. The two suborbits of PSL(3, 2) of length 2 are self-paired (as one of them consists of 3-cycles in blocks of PSL(3, 2)). The two suborbits of length 4 are non-self-paired. Thus Γ has a subdigraph of valency 4 that is a non-self-paired orbital digraph of PSL(3, 2), and whose paired orbital digraph of PSL(3, 2) is not a subdigraph of Γ.

In our next remark, we list those vertex-transitive graphs with simply primitive automorphism group, necessarily contained in the statement of Theorem 12, whose automorphism group contains a subgroup which is imprimitive and quasiprimitive.

Remark 14. There are several instances, other than the complete graph and its complement, where a quasiprimitive group G with nontrivial G-invariant partition $\mathcal{B}$, is contained in the full automorphism group of a digraph Γ of order qp, but the automorphism group Aut(Γ) is simply primitive. We list the exceptions or not in the same order as in Theorem 12:

1. There are no such cases if Theorem 12 (1) holds as $\mathbb{Z}_{qp}$ is a Burnside group [5, Corollary 3.5A]. This implies Aut(Γ) is doubly-transitive and so $\mathrm{Aut}(\Gamma) = S_{qp}$.

2. If Theorem 12 (2) holds then there are graphs whose automorphism group is primitive and equal to PGL(2, 11) on 55 points that contain the quasiprimitive and imprimitive representation of PSL(2, 11) on 55 points. These graphs are explicitly described in [24, Lemma 4.3].

3. If Theorem 12 (3) holds, then there are graphs whose automorphism group is PΓL(3, 2) in its primitive representation on 21 points that contains the quasiprimitive and imprimitive representation of PSL(3, 2) on 21 points. These graphs are explicitly described in [29, Example 2.3].

4. If Theorem 12 (4) holds, then there are graphs whose automorphism group is primitive but contains the quasiprimitive and imprimitive representation of $\mathrm{SL}(2, 2^{2^s})$ on qp points. These graphs are explicitly described in the proof of [17, Theorem 2.1], starting in the last paragraph on page 192.

4 Automorphism groups of Marušič-Scapellato digraphs

We turn now to the next "gap" in information about vertex-transitive digraphs of order pq, where there is also an error. The gap is that the automorphism groups of every vertex-transitive digraph of order pq are not known.

The automorphism groups of circulant digraphs of order pq are found in [12]. One of the authors of this paper determined the automorphism groups of metacirculant graphs of order pq that are not circulant [6] and that argument works for digraphs as well provided that the full automorphism group is not an almost simple group. We dealt with some part of the case where the automorphism group is almost simple (where the automorphism group is imprimitive and almost simple but not a Marušič-Scapellato digraph) in the previous section. Praeger and Xu [24] determined the full automorphism group of graphs of order pq in every case where that group is acting primitively. We will give the full automorphism groups of digraphs that are not graphs with primitive automorphism group in Theorem 26. In light of Theorem 6, this means that the remaining gap in the problem of determining the full automorphism group of vertex-transitive digraphs of order a product of two distinct primes reduces to determining the automorphism groups of imprimitive Marušič-Scapellato digraphs of order pq that are not metacirculant graphs, where p is a Fermat prime, and q divides $p-2$. In the process of filling this gap we will fix an error in [24]. Unfortunately, there is also an error of omission in [24] that we will need to correct, but we leave this for Section 5.

For the first part of this section, we will work in full generality (i.e. with $\mathrm{SL}(2,2^k)$, $k>1$), and when appropriate with will set $k=2^s$, $p=2^{2^s}+1$ a Fermat prime, and q a divisor of $p-2$ (so $s \geqslant 1$). For now though, $p=2^k+1$ and q divides 2^k-1 (and p and q need not be prime).

The Marušič-Scapellato digraphs are vertex-transitive digraphs that are generalized orbital digraphs of $\mathrm{SL}(2,2^k)$. They were first studied by Marušič and Scapellato in [19, 20]. Praeger, Wang, and Xu [23] determined the automorphism groups of Marušič-Scapellato graphs of order pq that are also arc-transitive, partially filling the gap we are addressing here. One of the authors of this paper studied the full automorphism groups of Marušič-Scapellato graphs in [10] and was able to say a great deal about them, but left their complete determination as an open problem [10, Problem 1].

We now discuss the construction of Marušič-Scapellato graphs, using a combination of the approaches followed in [10] and [20].

Let I_2 be the 2×2 identity matrix, and set $Z=\{aI_2 : a\in \mathbb{F}_{2^k}^*\}$, the set of all **scalar matrices**. The name Z is chosen as $Z=\mathrm{Z}(\mathrm{GL}(2,2^k))$, the center of $\mathrm{GL}(2,2^k)$. Let $\mathbb{F}_{2^k}^2$ denote the set of all 2-dimensional vectors whose entries lie in $\mathbb{F}_{2^k}$. Clearly $\mathrm{SL}(2,2^k)$ is transitive on $\mathbb{F}_{2^k}^2-\{(0,0)\}$. It is also clear that $\mathrm{SL}(2,2^k)$ permutes the **projective points** $\mathrm{PG}(1,2^k)$, where a projective point is the set of all vectors other than $(0,0)$ that lie on a line. Notice that there are 2^k+1 projective points, and $\mathrm{PG}(1,2^k)$ is an invariant partition of $\mathrm{SL}(2,2^k)$ in its action on $\mathbb{F}_{2^k}^2-\{(0,0)\}$ with 2^k+1 blocks of size 2^k-1. This action is faithful. That is, $\mathrm{SL}(2,2^k)/\mathrm{PG}(1,2^k)\cong \mathrm{SL}(2,2^k)$, or equivalently, $\mathrm{fix}_{\mathrm{SL}(2,2^k)}(\mathrm{PG}(1,2^k))=1$.

It is traditional to identify the projective points with elements of $\mathbb{F}_{2^k} \cup \{\infty\}$ in the following way: The nonzero vectors in the one-dimensional subspace generated by $(1, 0)$, will be identified with ∞. Any other one-dimensional subspace is generated by a vector of the form $(c, 1)$, where $c \in \mathbb{F}_{2^k}$. The nonzero vectors in the one-dimensional subspace generated by $(c, 1)$ will be identified with c.

For $a \in \mathbb{F}_{2^k}^*$, let $\sqrt{a}$ be the unique element of $\mathbb{F}_{2^k}^*$ whose square is a, and

$$k_a = \begin{bmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{bmatrix}.$$

Set $K = \{k_a : a \in \mathbb{F}_{2^k}^*\}$. It is clear that k_a stabilizes the projective point ∞ and that for any generator ω of $\mathbb{F}_{2^k}^*$, $\langle k_\omega \rangle = K$ is cyclic (of order $2^k - 1$), since $\sqrt{\omega}$ also generates $\mathbb{F}_{2^k}^*$. Additionally, it is clear that every element of the set-wise stabilizer of ∞ in $\mathrm{SL}(2, 2^k)$ has the same action on ∞ as some element of $K|_\infty$ (the entry in the top-right position is irrelevant to the action on ∞). Let $J \leqslant K$ be the unique subgroup of order ℓ, where ℓ is a fixed divisor of $2^k - 1$ (with our notation $\ell = (2^k - 1)/q$). By [5, Exercise 1.5.10], every orbit of $J|_\infty$ is a block of $\mathrm{SL}(2, 2^k)$, and so $\mathrm{SL}(2, 2^k)$ has an invariant partition $\mathcal{D}_\ell$ with blocks of size ℓ (the blocks whose points lie "within" the projective point ∞ of $\mathrm{PG}(1, 2^k)$ – that is, those blocks consisting of points whose second entry is 0 – are the orbits of $J|_\infty$, and the other blocks are the images of these orbits under $\mathrm{SL}(2, 2^k)$). These blocks of $\mathcal{D}_\ell$ will be the vertices of the generalized orbital digraphs of $\mathrm{SL}(2, 2^k)$, and it is the action of $\mathrm{SL}(2, 2^k)$ on these blocks that produces the Marušič-Scapellato digraphs. With our notation, there are pq blocks in $\mathcal{D}_\ell$.

As mentioned above, the blocks of $\mathcal{D}_\ell$ are the images of the orbits of $J|_\infty$ under the action of $\mathrm{SL}(2, 2^k)$, so each lies within a point of $\mathrm{PG}(1, 2^k)$; that is, $\mathcal{D}_\ell \preceq \mathrm{PG}(1, 2^k)$. Now $\mathrm{SL}(2, 2^k)/\mathcal{D}_\ell$ is a faithful representation of $\mathrm{SL}(2, 2^k)$ (as $\mathrm{fix}_{\mathrm{SL}(2,2^k)}(\mathrm{PG}(1, 2^k)) = 1$ and $\mathcal{D}_\ell \preceq \mathrm{PG}(1, 2^k)$). Additionally, the $\mathrm{SL}(2, 2^k)$-invariant partition $\mathrm{PG}(1, 2^k)$ induces the $\mathrm{SL}(2, 2^k)/\mathcal{D}_\ell$-invariant partition $\mathcal{B} = \mathrm{PG}(1, 2^k)/\mathcal{D}_\ell$, and $\mathcal{B}$ consists of $2^k + 1$ blocks whose size in general is $m = (2^k - 1)/\ell$ (under our assumptions, $m = q$). We will use the notation $\mathcal{B} = \mathrm{PG}(1, 2^k)/\mathcal{D}_\ell$ throughout this section. The elements of $\mathcal{B}$ will be the blocks of our digraphs of order pq, and will have size q (and there are p of them), so for our purposes and henceforth in this section, we have $q = m = (2^k - 1)/\ell$. It is shown in [20] that $\mathcal{B}$ is the unique $\mathrm{SL}(2, 2^k)/\mathcal{D}_\ell$-invariant partition with blocks of size q. The following result is [20, Lemma 2.3].

Lemma 15. $\mathrm{SL}(2, 2^k)/\mathcal{D}_\ell$ *has* q *suborbits of length* 1 *and* q *suborbits of length* 2^k. *Additionally, for a suborbit* S *of length* 2^k, $|S \cap (c/\mathcal{D}_\ell)| = 1$ *for every projective point* $c \in \mathrm{PG}(1, 2^k)$.

Note that this implies that the valency of an orbital digraph of $\mathrm{SL}(2, 2^k)$ is either 1 or 2^k. Additionally, as $\mathrm{SL}(2, 2^k)/\mathrm{PG}(1, 2^k) = \mathrm{PSL}(2, 2^k)$ is doubly-transitive, the previous result also implies that the orbital digraphs of $\mathrm{SL}(2, 2^k)/\mathcal{D}_\ell$ having valency 2^k are graphs. We now define Marušič-Scapellato digraphs, and the fact that some orbital digraphs are graphs and some are not will cause us to naturally define these digraphs in terms of the edges which are not arcs as well as arcs that need not be edges.

Definition 16. Let $k > 1$ an integer, and q a divisor of $2^k - 1$, $S \subset \mathbb{Z}_q^*$, $\emptyset \subseteq T \subseteq \mathbb{Z}_q$, and ω a primitive element of $\mathbb{F}_{2^k}$. The digraph $X(2^k, q, S, T)$ has vertex set $\mathrm{PG}(1, 2^k) \times \mathbb{Z}_q$. The outneighbors of (∞, r) are $\{(\infty, r + a) : a \in S\}$ while the inneighbors of (∞, r) are $\{(y, r + b) : y \in \mathbb{F}_{2^k}, b \in T\}$. The outneighbors of (x, r), $x \in \mathbb{F}_{2^k}$, are given by $\{(x, r + a) : a \in S\}$ while the inneighbors of (x, r) are

$$\{(\infty, r - b) : b \in T\} \cup \{(x + \omega^i, -r + b + 2i) : i \in \mathbb{Z}_{2^k - 1}, b \in T\}.$$

The digraph $X(2^k, q, S, T)$ is a **Marušič-Scapellato digraph**.

In [20] Marušič and Scapellato only defined graphs, but their definition, with the obvious modifications, also define digraphs as above - see [21]. Additionally, they required that $\emptyset \subset T \subset \mathbb{Z}_q$ as they wished their family to be disjoint from other already known families of graphs. If $\emptyset = T$ or $T = \mathbb{Z}_q$ then the resulting digraphs are either disconnected or complements of disconnected digraphs, and so have automorphism group either a nontrivial wreath product or a symmetric group. They also showed that with their definition, Marušič-Scapellato digraphs are isomorphic to some, but not all, generalized orbital digraphs of $\mathrm{SL}(2, 2^k)/\mathcal{D}_\ell$. We prefer the more general definition that includes all generalized orbital digraphs of $\mathrm{SL}(2, 2^k)/\mathcal{D}_\ell$. However, the distinction Marušič and Scapellato made is also important, so if $T = \emptyset$ or $\mathbb{Z}_q$, we will call such a Marušič-Scapellato digraph a **degenerate Marušič-Scapellato digraph**.

We have said that the vertices of the Marušič-Scapellato digraphs are the blocks of $\mathcal{D}_\ell$. The blocks of $\mathcal{D}_\ell$ are two-dimensional vectors that are subsets of projective points; in fact, it may be useful to the reader if we describe the blocks of $\mathcal{D}_\ell$ more precisely here. We assume that a primitive element ω of $\mathbb{F}_{2^k}$ has been chosen and is fixed. Then each block $D \in \mathcal{D}_\ell$ has one of the following forms:

$$\begin{aligned} \{(\sqrt{\omega}^{qj+r}, 0) : 0 \leqslant j \leqslant \ell - 1\} \qquad & \text{(in the projective point } \infty) \\ \{(\sqrt{\omega}^{qj+c+r}, \sqrt{\omega}^{qj+r} : 0 \leqslant j \leqslant \ell - 1\} \qquad & \text{(in the projective point } \sqrt{\omega}^c) \\ \{(0, \sqrt{\omega}^{qj+r}) : 0 \leqslant j \leqslant \ell - 1\} \qquad & \text{(in the projective point } 0), \end{aligned}$$

for some fixed $0 \leqslant r \leqslant q - 1$ and $1 \leqslant c \leqslant 2^k - 1$. The action of any element of $\mathrm{SL}(2, 2^k)$ on any one of these sets is easy to calculate. Clearly, the definition that we have given for the Marušič-Scapellato graphs does not have these sets as vertices; its vertices are the elements of $\mathrm{PG}(1, 2^k) \times \mathbb{Z}_q$.

In [20, Theorem 3.1], Marušič and Scapellato show that the imprimitive orbital digraphs of $\mathrm{SL}(2, 2^k)$ whose invariant partitions come from the projective points, are precisely the Marušič-Scapellato digraphs with the correct correspondence chosen between the blocks of $\mathcal{D}_\ell$ and the elements of $\mathrm{PG}(1, 2^k) \times \mathbb{Z}_q$. They describe explicitly how certain matrices act on elements of $\mathrm{PG}(1, 2^k) \times \mathbb{Z}_q$.

The action on the first coordinate is straightforward; the set of blocks of $\mathcal{D}_\ell$ that lie in a particular projective point will correspond to the set of vertices of the digraph whose label has that first coordinate. Thus, any matrix will map a vertex whose first coordinate is some projective point, to a vertex whose first coordinate is the image of that projective point under that matrix. However, the action on the second coordinate is less clear, and this is what they explain in more detail.

In Equations (10) and (12) of [20], they explain that the labeling of the vertices is chosen so that $k_\omega(\infty, r) = (\infty, r+1)$ (where $r \in \mathbb{Z}_q$), and $k_\omega(c, r) = (c\omega, r+1)$ for any projective point c other than ∞. Thus k_ω always adds one in the second coordinate, but in the first it multiplies by ω except when the first coordinate is ∞. They also introduce a family of matrices

$$h_b = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix},$$

where $b \in \mathbb{F}_{2^k}$. Observe that $H = \{h_b : b \in \mathbb{F}_{2^k}\}$ is a group, and in fact since $\mathbb{F}_{2^k}$ has characteristic 2, H is an elementary abelian 2-group. They note in the paper that the stabilizer of ∞ in $\mathrm{SL}(2, 2^k)$ is the set of upper triangular matrices, and this is generated by k_ω together with H. In Equation (14) of [20], they observe that under their labeling, $h_b(\infty, r) = (\infty, r)$, and $h_b(c, r) = (c + b, r)$ when c is any projective point other than ∞.

From this point on, we assume that $k = 2^s$, $s \geqslant 1$, $p = 2^{2^s} + 1$ is a Fermat prime, and q divides $p - 2$. For brevity, we write $t = 2^s$ in proofs and discussions, but for accuracy write 2^{2^s} in the statements of results. With the information we now have in hand, we are ready to understand how diagonal matrices act on the vertex labels from $\mathrm{PG}(1, 2^t) \times \mathbb{Z}_q$; this will be valuable to us.

Lemma 17. *Let ω be a primitive root of $\mathbb{F}_{2^{2^s}}$, so that $\sqrt{\omega}$ is also a primitive root of $\mathbb{F}_{2^{2^s}}$. Then the permutation $\sqrt{\omega} I_2$ acts on vertices labeled with elements of $\mathrm{PG}(1, 2^{2^s}) \times \mathbb{Z}_q$ by*

$$\sqrt{\omega} I_2(\infty, r) = (\infty, r+1) \qquad \text{and} \qquad \sqrt{\omega} I_2(c, r) = (c, r-1), \text{ when } c \neq \infty.$$

Proof. Observe that a point of $\mathbb{F}_{2^t}^2$ that lies in the projective point ∞ has 0 as its second entry, so the action of k_ω on such a point must be identical to the action of $\sqrt{\omega} I_2$. Thus, any set $D \in \mathcal{D}_\ell$ of points lying in the projective point ∞ must have the same image under $\sqrt{\omega} I_2$ as under k_ω. If D corresponds to the vertex labelled (∞, r), then since Marušič and Scapellato have told us that $k_\omega(\infty, r) = (\infty, r+1)$, it must also be the case that $\sqrt{\omega} I_2(\infty, r) = (\infty, r+1)$.

Similarly, a point of $\mathbb{F}_{2^t}^2$ that lies in the projective point 0 has 0 as its first entry, so the action of $k_{\omega^{-1}} = (k_\omega)^{-1}$ on such a point must be identical to the action of $\sqrt{\omega} I_2$. Again, Marušič and Scapellato have told us that $k_\omega(0, r-1) = (0, r)$, it must be the case that $\sqrt{\omega} I_2(0, r) = (0, r-1)$.

Finally, consider any point of $\mathbb{F}_{2^t}^2$ that lies in the projective point c where $c \neq 0, \infty$, so $c = \sqrt{\omega}^x$ for some x. Then the point of $\mathbb{F}_{2^t}^2$ has the form $(\sqrt{\omega}^{qi+x+m}, \sqrt{\omega}^{qi+m})$ for some $0 \leqslant i \leqslant \ell - 1$ and $0 \leqslant m \leqslant q - 1$. Straightforward calculations using the field's characteristic of 2 show that the action of the matrix $\sqrt{\omega}^{-1} I_2$ has the same effect on such a point as the action of the matrix

$$h_c k_\omega h_c = \begin{bmatrix} \sqrt{\omega} & \sqrt{\omega}^x(\sqrt{\omega} + \sqrt{\omega}^{-1}) \\ 0 & \sqrt{\omega}^{-1} \end{bmatrix}.$$

Using the information from Marušič and Scapellato, we know that $h_c(c,r) = (c+c,r) = (0,r)$, $k_\omega(0,r) = (0,r+1)$, and $h_c(0,r+1) = (c,r+1)$. Thus, we must also have $\sqrt{\omega}^{-1} I_2(c,r) = (c,r+1)$, and hence $\sqrt{\omega} I_2(c,r) = (c,r-1)$. $\square$

Let $F : \mathbb{F}_{2^t} \to \mathbb{F}_{2^t}$ be the Frobenius automorphism, and so be given by $F(x) = x^2$. The Frobenius automorphism induces an automorphism f of $\mathrm{GL}(2,2^t)$ in the natural way - by applying F to the entries of the standard matrix of an element of $\mathrm{GL}(2,2^t)$. Observe that since the Frobenius automorphism is an automorphism, we have $Z \cap \mathrm{SL}(2,2^t) = \{I_2\}$. Furthermore, every element of $\mathbb{F}_{2^t}$ is a square, and so every element of $\mathbb{F}_{2^t}$ arises as the determinant of some matrix in Z. Therefore $\langle \mathrm{SL}(2,2^t), Z\rangle = \mathrm{GL}(2,2^t)$. Since we know that $\mathrm{SL}(2,2^t), Z \lhd \mathrm{GL}(2,2^t)$, this implies that $\mathrm{GL}(2,2^t) = \mathrm{SL}(2,2^t) \times Z$.

We need to introduce some additional notation that will be used throughout the remainder of this section. We use $\Gamma\mathrm{L}(2,2^t)$ to denote the group $\mathrm{GL}(2,2^t) \rtimes \langle f\rangle$. We also use $\Sigma\mathrm{L}(2,2^t)$ to denote $\mathrm{SL}(2,2^t) \rtimes \langle f\rangle$. We know that $\mathrm{GL}(2,2^t) = \mathrm{SL}(2,2^t) \times Z$, so $\Gamma\mathrm{L}(2,2^t) = (\mathrm{SL}(2,2^t) \times Z) \rtimes \langle f\rangle$ where the action of f leaves $\mathrm{SL}(2,2^t)$ and Z invariant.

Lemma 18. *Let $p = 2^{2^s}+1$ be a Fermat prime and $q|(2^{2^s}-1)$ a prime, with $q\ell = (2^{2^s}-1)$. Let a be the order of 2 modulo q, let b be a divisor of $\gcd(a,2^s)$ with $b \neq a$, and let $1 \neq L = \langle f^b\rangle$ (where f is the automorphism of $\mathrm{GL}(2,2^{2^s})$ induced by the Frobenius automorphism, as described above). If $1 \neq z/\mathcal{D}_\ell \in Z/\mathcal{D}_\ell$, then $z^{-1}\langle \mathrm{SL}(2,2^{2^s}), L\rangle z/\mathcal{D}_\ell \neq \langle \mathrm{SL}(2,2^{2^s}), L\rangle/\mathcal{D}_\ell$.*

Proof. Let $1 \neq L = \langle f^b\rangle \leqslant \langle f\rangle$ and let $G = \langle \mathrm{SL}(2,2^t), L\rangle$. Towards a contradiction, suppose that $1 \neq z/\mathcal{D}_\ell \in Z/\mathcal{D}_\ell$, and $z^{-1}Gz/\mathcal{D}_\ell = G/\mathcal{D}_\ell$. Let $Y = \langle z/\mathcal{D}_\ell\rangle$. As $Z \lhd \Gamma\mathrm{L}(2,2^t)$ is cyclic and Y is the unique subgroup of $Z/\mathcal{D}_\ell$ of order $|Y|$, $Y \lhd \Gamma\mathrm{L}(2,2^t)/\mathcal{D}_\ell$. Since $z^{-1}Gz/\mathcal{D}_\ell = G/\mathcal{D}_\ell$, it follows that $G/\mathcal{D}_\ell \lhd \langle Y, G/\mathcal{D}_\ell\rangle$. Moreover, since $Y \cap G/\mathcal{D}_\ell = 1$ (this follows from $\mathrm{GL}(2,2^t) = \mathrm{SL}(2,2^t) \times Z$), we see $\langle Y, G/\mathcal{D}_\ell\rangle \cong Y \times G/\mathcal{D}_\ell$. In particular, $z/\mathcal{D}_\ell$ commutes with $f^b/\mathcal{D}_\ell$.

Choose i such that $z = \sqrt{\omega}^i I_2 \in Z$, for some fixed generator ω of $\mathbb{F}_{2^t}^*$ ($\sqrt{\omega}$ also generates $\mathbb{F}_{2^t}^*$). It is straightforward to verify that $z^{-1}f^bzf^{-b} = z^{2^b-1} = \sqrt{\omega}^{i(2^b-1)}I_2$. On the other hand, since $z/\mathcal{D}_\ell$ commutes with $f/\mathcal{D}_\ell$, it follows that $(z^{-1}f^bzf^{-b})/\mathcal{D}_\ell = 1$, implying $\sqrt{\omega}^{i(2^b-1)}I_2/\mathcal{D}_\ell = 1$.

Observe that each block of $\mathcal{D}_\ell$ has the form $\{(x\sqrt{\omega}^{qj}, y\sqrt{\omega}^{qj}) : 0 \leqslant j < \ell\}$, for some $x, y \in \mathbb{F}_{2^t}$. This implies that $\mathrm{fix}_Z(\mathcal{D}_\ell) = \langle \sqrt{\omega}^q I_2\rangle$. Therefore $\sqrt{\omega}^{i(2^b-1)}I_2/\mathcal{D}_\ell = 1$ if and only if $\sqrt{\omega}^{i(2^b-1)} \in \langle \sqrt{\omega}^q\rangle$. We conclude that $i(2^b-1) \equiv uq \pmod{2^t-1}$ for some integer u. Since q divides 2^t-1, it follows that q divides $i(2^b-1)$. Recall that a is the order of 2 modulo q and $b < a$. This implies that $2^b \not\equiv 1 \pmod q$ and therefore q does not divide 2^b-1. Since q is a prime, and q divides $i(2^b-1)$, this implies that q divides i. However, this means that $z \in \langle \sqrt{\omega}^q I_2\rangle = \mathrm{fix}_Z(\mathcal{D}_\ell)$ contradicting the assumption that $z/\mathcal{D}_\ell \neq 1$. This contradiction establishes that $z^{-1}Gz/\mathcal{D}_\ell \neq G/\mathcal{D}_\ell$, as claimed. $\square$

The first error that we will correct concerns the classification of arc-transitive Marušič-Scapellato graphs given in [23, Theorem, as it relates to (3.8)]. In that paper, Lemma 4.9(a) states that the q connected orbital graphs $X(2^t, q, \emptyset, \{x\})$ (where $x \in \mathbb{Z}_q$) of $\mathrm{SL}(2,2^t)$

all have automorphism group $\Sigma L(2, 2^t)$ (this group is written in [23] as $\Gamma L(2, 2^t)$, but it is clear from the proof of [23, Theorem 3.7] that they mean the group we are denoting by $\Sigma L(2, 2^t)$). Using Lemma 17 to understand the action of Z on these graphs, we see that for any $z \in Z$ with $z \neq 1$, there exists some $x' \in \mathbb{Z}_q^*$ such that $X(2^t, q, \emptyset, x\})^z = X(2^t, q, \emptyset, \{x - x'\})$ (more precisely, if $z = \sqrt{\omega}^i I_2$, then $x' = 2i$). Thus, every such z acts as a cyclic permutation on this set of q graphs. Suppose that Γ and Γ^z are two of these orbital digraphs with $z/\mathcal{D}_\ell \neq 1$ (so that the graphs are distinct), and $\mathrm{Aut}(\Gamma) = \Sigma L(2, 2^t)$ as claimed in [23]. Then $\mathrm{Aut}(z(\Gamma)) = z^{-1}(\mathrm{Aut}(\Gamma))z$, and by taking $b = 1$ in Lemma 18 we see that $\Sigma L(2, 2^t)/\mathcal{D}_\ell \neq z^{-1}\Sigma L(2, 2^t)z/\mathcal{D}_\ell$, contradicting their claim that $\mathrm{Aut}(z(\Gamma)) = \mathrm{Aut}(\Gamma)$.

The mathematical error leading to the incorrect statement of [23, Lemma 4.9] actually arises in [23, Lemma 4.8] where it is concluded that the automorphism group G of any Marušič-Scapellato graph satisfies $\mathrm{SL}(2, 2^t)/\mathcal{D}_\ell \leqslant G \leqslant \Sigma L(2, 2^t)/\mathcal{D}_\ell$ (using our notation). The proof of [23, Lemma 4.8] only gives that $G/\mathcal{B} = \Sigma L(2, 2^t)/\mathrm{PG}(1, 2^t) = \mathrm{P}\Sigma L(2, 2^t)$. If we consider any of the groups that are conjugate to $\Sigma L(2, 2^t)$ by a scalar matrix, which we have shown in Lemma 18 are distinct modulo $\mathcal{D}_\ell$, the fact that scalar matrices fix every point of $\mathrm{PG}(1, 2^t)$ shows that every such group satisfies this equation. With that said, the proof of [23, Lemma 4.9 (b)] is correct if we strengthen the hypothesis to assume that $\mathrm{SL}(2, 2^t)/\mathcal{D}_\ell \leqslant G \leqslant \Sigma L(2, 2^t)/\mathcal{D}_\ell$. So we can restate their result correctly as follows, to identify the arc-transitive Marušič-Scapellato digraphs whose automorphism group is contained in $\Sigma L(2, 2^t)/\mathcal{D}_\ell$.

Note that when $G = \mathrm{Aut}(\Gamma)$ where Γ is one of these Marušič-Scapellato graphs, and $\mathrm{SL}(2, 2^t)/\mathcal{D}_\ell \leqslant G \leqslant \Sigma L(2, 2^t)/\mathcal{D}_\ell$, all of these actions on $\mathcal{D}_\ell$ are faithful, so that $\mathrm{SL}(2, 2^t)/\mathcal{D}_\ell \cong \mathrm{SL}(2, 2^t)$, $G/\mathcal{D}_\ell \cong G$, and $\Sigma L(2, 2^t)/\mathcal{D}_\ell \cong \Sigma L(2, 2^t)$. In [23], they were to some extent studying the abstract structure of these groups, and did not make this distinction, which may have contributed to the confusion and does lead to our statement looking somewhat different from theirs.

Theorem 19 (see [23], Lemma 4.9)**.** *Let $p = 2^{2^s} + 1$ be a Fermat prime and $q|(2^{2^s} - 1)$ be prime. Let $\Gamma = X(2^{2^s}, q, S, T)$ be an arc-transitive Marušič-Scapellato digraph and assume that $\mathrm{SL}(2, 2^{2^s})/\mathcal{D}_\ell \leqslant \mathrm{Aut}(\Gamma) \leqslant \Sigma L(2, 2^s)/\mathcal{D}_\ell$. Let a be the order of 2 modulo q. Then $S = \emptyset$ and one of the following is true:*

1. *$T = \{0\}$, Γ has valency q, and automorphism group $\Sigma L(2, 2^{2^s})/\mathcal{D}_\ell$.*
2. *There is a divisor b of $\gcd(a, 2^{2^s})$ and $1 < a/b < q - 1$ such that $T = U_{b,i} = \{i2^{bj} : 0 \leqslant j < a/b\}$. There are exactly $(q - 1)b/a$ distinct graphs of this type for a given b, each of valency qa/b, and the automorphism group of each is $\langle \mathrm{SL}(2, 2^{2^s}), L\rangle/\mathcal{D}_\ell$ where $L \leqslant \langle f\rangle$ is of order $2^{2^s}/b$. Up to isomorphism, there are exactly $(q - 1)/b$ such graphs.*

Before turning to the characterization of arc-transitive Marušič-Scapellato digraphs of order qp, we will need a solution to the isomorphism problem for these graphs. This problem has been solved in [10], but the solution there is not suited to our needs. The solution given in [10] is also perhaps not optimal in the sense that it requires one check

$|\Sigma\mathrm{L}(2,2^t)| = |\mathrm{Aut}(\mathrm{SL}(2,2^t))|$ maps to determine isomorphism, while we show in the next result that one only needs to check qs maps.

Theorem 20. *Let $p = 2^{2^s}+1$ be a Fermat prime, $q|(2^{2^s}-1)$ a prime, and Γ,Γ' be non-degenerate Marušič-Scapellato digraphs. Then Γ and Γ' are isomorphic if and only if $\delta(\Gamma)=\Gamma'$, where $\delta\in\langle Z,f\rangle/\mathcal{D}_\ell$.*

Proof. It is shown in [10, Theorem 1] that $\Gamma'=\delta(\Gamma)$ for some δ if and only if this occurs for a δ that normalizes $\mathrm{SL}(2,2^t)$. This normalizer is $\Gamma\mathrm{L}(2,2^t)/\mathcal{D}_\ell$ as every element of S_{qp} that normalizes $\mathrm{SL}(2,2^t)/\mathcal{D}_\ell$ can be written in the form ab, where $b\in\mathrm{SL}(2,2^t)$, and $a\in\mathrm{Aut}(\mathrm{SL}(2,2^t))$, by [11, Lemma 4.4.9]. It should be clear that the image of Γ under an inner automorphism of $\mathrm{SL}(2,2^2)/\mathcal{D}_\ell$ is just Γ. By [32, Subsection 3.3.4], the outer automorphisms are f and its powers as well as conjugation by an element of Z. The result follows. □

We are now ready to determine the arc-transitive Marušič-Scapellato digraphs of order a product of two distinct primes with imprimitive automorphism group.

Theorem 21. *Let $s>1$, $p=2^{2^s}+1$ be a Fermat prime, and $q|(2^{2^s}-1)$ be prime. Let $\Gamma = X(2^{2^s},q,S,T)$ be a nondegenerate arc-transitive Marušič-Scapellato digraph constructed with the primitive root ω of $\mathbb{F}_{2^{2^s}}$ with an imprimitive automorphism group. Let a be the order of 2 modulo q, and $d=\sqrt{\omega}I_2$. Then $S=\emptyset$ and one of the following is true:*

1. *$T=\{-2k\}$, Γ has valency q, and automorphism group $d^{-k}\Sigma\mathrm{L}(2,2^{2^s})d^k/\mathcal{D}_\ell$, $k\in\mathbb{Z}_q$.*
2. *There is a divisor b of $\gcd(a,2^s)$, $1<a/b<q-1$, and $k\in\mathbb{Z}_q$ such that $T=U_{b,i,k}=\{i2^{bj}-2k : 0\leqslant j<a/b\}$. There are exactly $(q-1)b/a$ distinct graphs of this type for a given b and k, each of valency qa/b, and the automorphism group of each is $d^{-k}\langle\mathrm{SL}(2,2^{2^s}),L\rangle d^k/\mathcal{D}_\ell$ where $L\leqslant\langle f\rangle$ is of order $2^{2^s}/b$. Up to isomorphism, there are exactly $(q-1)/b$ such graphs.*

Proof. For the proof of this result, we will abuse notation by writing H instead of $H/\mathcal{D}_\ell$ where $H/\mathcal{D}_\ell\leqslant d^{-k}\Sigma\mathrm{L}(2,2^t)d^k/\mathcal{D}_\ell$, and will similarly abuse notation for elements of $d^{-k}\Sigma\mathrm{L}(2,2^t)d^k/\mathcal{D}_\ell$. This should cause no confusion. The result follows by Theorem 19 if $\mathrm{Aut}(\Gamma)\leqslant\Sigma\mathrm{L}(2,2^t)$, in which case $k=0$. Suppose that $\mathrm{Aut}(\Gamma)$ is not contained in $\Sigma\mathrm{L}(2,2^t)$. As $\mathrm{Aut}(\Gamma)$ is imprimitive and contains $\mathrm{SL}(2,2^t)$, by [19, Theorem] either Γ is metacirculant or the only invariant partition of $\mathrm{Aut}(\Gamma)$ is $\mathcal{B}=\mathrm{PG}(1,2^t)$ which is also the only invariant partition of $\mathrm{SL}(2,2^t)$. If Γ is metacirculant, then it is degenerate by [17, Theorem 2.1]. Hence Γ is not metacirculant and so $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})=1$ by [18, Theorem 3.4]. Then $\mathrm{Aut}(\Gamma)/\mathcal{B}\cong\mathrm{Aut}(\Gamma)$ is a group of prime degree p. By [5, Corollary 3.5B] we have $\mathrm{Aut}(\Gamma)/\mathcal{B}\leqslant\mathrm{AGL}(1,p)$ or is a doubly-transitive group. By [5, Theorem 4.1B] we see either $\mathrm{Aut}(\Gamma)/\mathcal{B}\leqslant\mathrm{AGL}(1,p)$ or is a doubly-transitive group with nonabelian simple socle. If $\mathrm{Aut}(\Gamma)/\mathcal{B}\leqslant\mathrm{AGL}(1,p)$ then $\mathrm{Aut}(\Gamma)$ contains a normal subgroup of order p, and so has blocks of size p, a contradiction. Thus $\mathrm{Aut}(\Gamma)/\mathcal{B}$ is a doubly-transitive group with nonabelian simple socle. By [23, Lemmas 4.5, 4.6, and 4.7] we have $\mathrm{SL}(2,2^t)\triangleleft\mathrm{Aut}(\Gamma)$.

Now, as above, $N_{S_V}(\mathrm{SL}(2,2^t)) = \langle \mathrm{SL}(2,2^t), f, Z\rangle = \Gamma\mathrm{L}(2,2^t) = (\mathrm{SL}(2,2^t)\times Z)\rtimes\langle f\rangle$ by [32, Subsection 3.3.4] and [11, Lemma 4.4.9]. Thus every element γ of $N_{S_V}(\mathrm{SL}(2,2^t))$, and hence $\gamma\in\mathrm{Aut}(\Gamma)$, can be written as $\gamma = f^iz\omega$, where $\omega\in\mathrm{SL}(2,2^t)$, $z\in Z$, and i is a positive integer. Of course, as $\mathrm{SL}(2,2^t)\leqslant\mathrm{Aut}(\Gamma)$, $f^iz\in\mathrm{Aut}(\Gamma)$ if and only if $f^iz\omega\in\mathrm{Aut}(\Gamma)$ for some $\omega\in\mathrm{SL}(2,2^t)$. Then $H = \mathrm{Aut}(\Gamma)\cap\{f^iz : i\in\mathbb{Z}, z\in Z\}$ is a subgroup of $\mathrm{Aut}(\Gamma)$, and $\mathrm{Aut}(\Gamma)/\mathrm{SL}(2,2^t)\cong H$ by the First Isomorphism Theorem. As $\mathrm{Aut}(\Gamma)\cap Z = 1$, H is isomorphic to a subgroup of $\langle f\rangle$, and $\mathrm{Aut}(\Gamma)/\mathrm{SL}(2,2^t)$ is isomorphic to a cyclic 2-subgroup. Then $\mathrm{Aut}(\Gamma)/\mathrm{SL}(2,2^t)$ is conjugate by an element $z/\mathrm{SL}(2,2^t)\in Z/\mathrm{SL}(2,2^t)$ to a subgroup of $\langle f\rangle/\mathrm{SL}(2,2^t)$, and so $z^{-1}\mathrm{Aut}(\Gamma)z\leqslant\Sigma\mathrm{L}(2,2^t)$. Then $\mathrm{Aut}(\Gamma)\leqslant z\Sigma\mathrm{L}(2,2^t)z^{-1}$ and $z^{-1}(\Gamma)$ is a nondegenerate arc-transitive Marušič-Scapellato digraph with $\mathrm{Aut}(z^{-1}(\Gamma))\leqslant\Sigma\mathrm{L}(2,2^t)$, and so is given by Theorem 19.

In order to verify the numbers of arc-transitive Marušič-Scapellato digraphs are as in the result, we need only to see different scalar matrices do indeed give different arc-transitive Marušič-Scapellato digraphs. Suppose that there exist two different arc-transitive Marušič-Scapellato graphs Γ_1 and Γ_2 with automorphism groups contained in $\Sigma\mathrm{L}(2,2^t)$, such that $z_1(\Gamma_1) = z_2(\Gamma_2)$, for $z_1,z_2\in Z$. Then $\Gamma_2 = z_2^{-1}z_1(\Gamma_1)$. This implies that Γ_1 and Γ_2 are of the same valency and since by Theorem 19 all Marušič-Scapellato graphs with the same valency have the same automorphism groups, it follows that $\mathrm{Aut}(\Gamma_1) = \mathrm{Aut}(\Gamma_2) = \langle\mathrm{SL}(2,2^t), L\rangle$, where $L = \langle f^b\rangle$. By Lemma 18 it follows that $z^{-1}\langle\mathrm{SL}(2,2^t), L\rangle z = \langle\mathrm{SL}(2,2^t), L\rangle$ holds only when $z = 1$. On the other hand, since $\Gamma_2 = z_2^{-1}z_1(\Gamma_1)$ it follows that

$$\langle\mathrm{SL}(2,2^t), L\rangle = \mathrm{Aut}(\Gamma_2) = (z_2^{-1}z_1)\mathrm{Aut}(\Gamma_1)(z_2^{-1}z_1)^{-1} = (z_2^{-1}z_1)\langle\mathrm{SL}(2,2^t), L\rangle(z_2^{-1}z_1)^{-1},$$

and hence $z_2^{-1}z_1 = 1$. This gives $z_1 = z_2$, which implies that different scalar matrices do indeed give different arc-transitive Marušič-Scapellato graphs.

Let $z^{-1}(\Gamma) = X(2^t, q, \emptyset, T')$, where $T' = \{0\}$ or $T' = U_{b,i} = \{i2^{bj} : 0\leqslant j < a/b\}$. Let $z = d^k$ for some positive integer k. We need only verify that $T' = \{2k\}$ or $U_{b,i,k} = \{i2^{bj} + 2k : 0\leqslant j < a/b\}$. Now, in $z^{-1}(\Gamma)$, the neighbors of (∞, r) are $\{(y, r+u) : y\in\mathbb{F}_{2^t}, u\in U_{b,i}\}$. Considering $z(z^{-1}(\Gamma)) = \Gamma$ and applying Lemma 17, we see the neighbors of $(\infty, r+k)$ in Γ are $\{(y, r+u-k) : y\in\mathbb{F}_{2^t}, u\in U_{b,i}\}$. Equivalently, the neighbors of (∞, r) in Γ are $\{(y, r+u-2k) : y\in\mathbb{F}_{2^t}, u\in U_{b,i}\}$ and the result follows. □

We now determine the full automorphism group of any Marušič-Scapellato digraph of order a product of two distinct primes.

Theorem 22. *Let $p = 2^{2^s}+1$ be a Fermat prime, $q|(2^{2^s}-1)$ be prime, and Γ be a Marušič-Scapellato digraph of order qp. Then Γ or its complement is $X(2^{2^s}, q, S, T)$ and one of the following is true.*

1. $\mathrm{Aut}(\Gamma)$ *is primitive and*

   (a) $2^s = 2$, $qp = 15$, $S = \mathbb{Z}_3^*$ *and* $T = \{0\},\{1\}$, *or* $\{2\}$. *Then Γ is isomorphic to the line graph of K_6 and has automorphism group $d^{-1}\Sigma\mathrm{L}(2,4)d\cong S_6$ for some $d\in Z$.*

(b) $p = k^2 + 1$, $q = k + 1$, $S = \mathbb{Z}_q^*$ *and* $|T| = 1$. *Then there exists* $d \in Z/\mathcal{D}_\ell$ *such that* $\mathrm{Aut}(\Gamma) = d^{-1}\mathrm{P\Gamma Sp}(4, k)d$.

(c) $S = \mathbb{Z}_q^*$, $T = \mathbb{Z}_q$, *and* Γ *is a complete graph with automorphism group* S_{qp}.

2. $\mathrm{Aut}(\Gamma)$ *is imprimitive and*

(a) $S \subset \mathbb{Z}_q^*$, $T = \mathbb{Z}_q$, Γ *is degenerate, and* $\mathrm{Aut}(\Gamma) \cong S_p \wr \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_q, S))$.

(b) *In all other cases there exists* $L \leqslant \langle f/\mathcal{D}_\ell\rangle$ *and* $d \in Z/\mathcal{D}_\ell$ *such that*

$$\mathrm{Aut}(\Gamma) = d^{-1}\langle \mathrm{SL}(2, 2^{2^s}), L\rangle d/\mathcal{D}_\ell$$

which is isomorphic to a subgroup of $\Sigma\mathrm{L}(2, 2^{2^s})/\mathcal{D}_\ell$ *that contains* $\mathrm{SL}(2, 2^{2^s})/\mathcal{D}_\ell$.

Proof. As in the previous result, we will abuse notation and drop the $\mathcal{D}_\ell$s from our notation. The case when $\mathrm{Aut}(\Gamma) = S_{qp}$ is trivial. The other Marušič-Scapellato graphs of order qp with primitive automorphism group were calculated up to isomorphism in [17] and their automorphism groups were computed to be isomorphic to either $\Sigma\mathrm{L}(2, 4)$ or $\mathrm{P\Gamma Sp}(4, k)$ in [24]. We observe that $\Sigma\mathrm{L}(2, 2^t)$ is contained in $\mathrm{Aut}(\Gamma)$, and so the only possible isomorphisms with other Marušič-Scapellato graphs are with elements of Z by Lemma 20. That the elements of Z give different graphs follows as they normalize $\mathrm{SL}(2, 2^t)$ but are not contained in $\mathrm{Aut}(\Gamma)$.

If $\mathrm{Aut}(\Gamma)$ is imprimitive and Γ is degenerate, then $T = \emptyset$ or $\mathbb{Z}_q$ and as $\mathrm{Aut}(\Gamma)$ is imprimitive, $S \neq \emptyset$ or $\mathbb{Z}_q^*$ respectively, as otherwise $\Gamma = K_{qp}$ has a primitive automorphism group. If $T = \mathbb{Z}_q$, then as $X(2^t, q, \emptyset, \mathbb{Z}_q) \cong K_p \wr \bar{K}_q$, $\Gamma \cong K_p \wr \Gamma[B]$, where B is a nontrivial block of Γ. As a nontrivial block of Γ has order q, which is a prime, $\Gamma[B]$ is isomorphic to $\mathrm{Cay}(\mathbb{Z}_q, S)$ by [28]. Then $\Gamma \cong K_p \wr \mathrm{Cay}(\mathbb{Z}_q, S)$ and by [9, Theorem 5.7] $\mathrm{Aut}(\Gamma) \cong S_p \wr \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_q, S))$.

If $\mathrm{Aut}(\Gamma)$ is imprimitive and Γ is non-degenerate, then Γ is a generalized orbital digraph of $\mathrm{Aut}(\Gamma)$. We write $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_r$ where each Γ_i is an orbital digraph of $\mathrm{Aut}(\Gamma)$. Note that as $\mathrm{Aut}(\Gamma)$ is imprimitive, some orbital digraph of $\mathrm{Aut}(\Gamma)$ is disconnected. Also, each connected orbital digraph of $\mathrm{Aut}(\Gamma)$ is either arc-transitive or 1/2-transitive (that is, is edge but not arc-transitive), and as each orbital digraph of Γ is a generalized orbital digraph of $\mathrm{SL}(2, 2^t)$, we see each connected orbital digraph of $\mathrm{Aut}(\Gamma)$ is arc-transitive as each connected orbital digraph of $\mathrm{SL}(2, 2^t)$ is arc-transitive.

If there exist connected orbital digraphs of $\mathrm{Aut}(\Gamma)$ that are subdigraphs of Γ whose automorphism groups are contained in $d^{-1}\Sigma\mathrm{L}(2, 2^t)d$ and $e^{-1}\Sigma\mathrm{L}(2, 2^t)e$ for $d \neq e$ both in Z, then

$$\mathrm{SL}(2, 2^t) \leqslant \mathrm{Aut}(\Gamma) \leqslant d^{-1}\Sigma\mathrm{L}(2, 2^t)d \cap e^{-1}\Sigma\mathrm{L}(2, 2^t)e \leqslant d^{-1}\Sigma\mathrm{L}(2, 2^t)d.$$

Thus $\mathrm{Aut}(\Gamma)$ is a subgroup of $d^{-1}\Sigma\mathrm{L}(2, 2^t)d$ that contains $\mathrm{SL}(2, 2^t)$. Now let $\mathrm{SL}(2, 2^t) \leqslant K \leqslant \Sigma\mathrm{L}(2, 2^t)$. As $\mathrm{SL}(2, 2^t) \triangleleft \Sigma\mathrm{L}(2, 2^t)$, every element of $\Sigma\mathrm{L}(2, 2^t)$, and consequently every element of K, can be written as gf^c for some $g \in \mathrm{SL}(2, 2^t)$ and integer c. As $\mathrm{SL}(2, 2^t) \leqslant K$, we have $gf^c \in K$ if and only if f^c in K. We conclude $K = \langle \mathrm{SL}(2, 2^t), L\rangle$, where $L \leqslant \langle f\rangle$ consists of all powers of f contained in K. Then $d^{-1}\mathrm{SL}(2, 2^t)d \leqslant \mathrm{Aut}(\Gamma) \leqslant d^{-1}\Sigma\mathrm{L}(2, 2^t)d$

and $\mathrm{Aut}(\Gamma) = d^{-1}\langle \mathrm{SL}(2,2^t), L\rangle d$ for some $L \leqslant \langle f\rangle$ as required. We thus assume that every connected orbital digraph of $\mathrm{Aut}(\Gamma)$ that is a subdigraph of Γ has automorphism group contained in $d^{-1}\Sigma\mathrm{L}(2,2^t)d$ for some scalar matrix $d \in Z$. Suppose that $\Gamma_1, \ldots, \Gamma_w$, $w \leqslant r$ are the connected orbital digraphs of $\mathrm{Aut}(\Gamma)$ that are subdigraphs of Γ. Then by Theorem 21, Γ_i has automorphism group $d^{-1}\langle \mathrm{SL}(2,2^t), L_i\rangle d$ where $L_i \leqslant \langle f\rangle$. Then $\mathrm{Aut}(\Gamma) \leqslant d^{-1}\langle \mathrm{SL}(2,2^t), L'\rangle d$ where $L' = \cap_{i=1}^{w} L_i$. Finally, let L be the subgroup of L' consisting of automorphisms of the subdigraph of Γ obtained by removing all edges between elements of $\mathrm{PG}(1,2^t)/\mathcal{D}_\ell$. Then $\mathrm{Aut}(\Gamma) = d^{-1}\langle \mathrm{SL}(2,2^t), L\rangle d$ and the result follows. □

We remark that the automorphism group of a Marušič-Scapellato digraph Γ can be calculated quite quickly:

If Γ is nondegenerate and $\mathrm{Aut}(\Gamma)$ imprimitive, then one only needs to determine the subgroup of $d^{-1}\Sigma\mathrm{L}(2,2^t)d/\mathcal{D}_\ell = \langle \mathrm{SL}(2,2^t), d^{-1}fd\rangle/\mathcal{D}_\ell$ which is $\mathrm{Aut}(\Gamma)$. In particular, one only needs to determine the maximal subgroup of $d^{-1}\langle f\rangle d$ contained in $\mathrm{Aut}(\Gamma)$. This can easily be accomplished as all such subgroups can be computed quickly. As $F(x) = x^2$, f has order 2^t, so there are $t+1$ subgroups of $\langle f\rangle$ each determined by a generator of the form f^{2^r}, $0 \leqslant r \leqslant t$. As $Z/\mathcal{D}_\ell$ has order at most 2^t, there are at most $(t+1)2^t$ maps which need to be tested as elements of $\mathrm{Aut}(\Gamma)$ in order to determine $\mathrm{Aut}(\Gamma)$.

If Γ is degenerate or $\mathrm{Aut}(\Gamma)$ is primitive, then this can be determined easily as the sets S and T are given explicitly. Again, one only needs to determine d, and this can be done as above by checking which $d^{-1}gd$ is contained in $\mathrm{Aut}(\Gamma)$.

5 Missing digraphs whose automorphism group is primitive

The first error in the literature is most probably simply an unfortunate typographical error. The misprint occurs in [15, Table 3] for the groups $\mathrm{PSL}(2,q)$ of degree $q(q^2-1)/24$ with point stabilizer A_4. In the "Comment" column, the paper literally lists "$q \equiv +3 \pmod 8, q \leqslant 19$". Of course, as written the "+" is entirely superfluous, but in reality it should be a "$\pm$". Indeed, without the $\pm$ the group $\mathrm{PSL}(2,13)$ which has A_4 as a maximal subgroup is not listed. The action of $\mathrm{PSL}(2,13)$ on right cosets of A_4 is primitive of degree $|\mathrm{PSL}(2,13)|/|A_4| = 7\cdot 13$. The authors thank Primož Potočnik for pointing out this error.

Lemma 23. *Let* $\mathrm{PSL}(2,13)$ *act transitively on* $7\cdot 13$ *points with point-stabilizer* A_4.

Then there are 3 *self-paired orbitals of size* 4 *all of which are* 2*-arc-transitive. No other orbital digraphs are* 2*-arc-transitive. Two of the graphs corresponding to these self-paired orbitals are isomorphic with automorphism group* $\mathrm{PSL}(2,13)$. *The graph corresponding to the union of these orbitals is arc-transitive and has automorphism group* $\mathrm{PGL}(2,13)$. *The graph corresponding to the remaining orbital has automorphism group* $\mathrm{PGL}(2,13)$ *and is arc-transitive.*

There is 1 *self-paired orbital of size* 6 *whose corresponding graph is arc-transitive and has automorphism group* $\mathrm{PGL}(2,13)$.

There are 2 *non self-paired orbitals of size* 12 *whose corresponding digraphs have automorphism group* $\mathrm{PSL}(2,13)$, *and whose union corresponds to an arc-transitive graph with automorphism group* $\mathrm{PGL}(2,13)$.

There are 4 self-paired orbitals of size 12 *that are all arc-transitive, two of which correspond to graphs that are isomorphic with automorphism group* $\mathrm{PSL}(2, 13)$*. Their union corresponds to a graph that has automorphism group* $\mathrm{PGL}(2, 13)$ *and is arc-transitive. The remaining two self-paired orbitals correspond to graphs that are non-isomorphic and have automorphism group* $\mathrm{PGL}(2, 13)$ *and are arc-transitive.*

Any other digraph of order 91 *that contains* $\mathrm{PSL}(2, 13)$ *as a transitive subgroup and is not complete or the complement of a complete graph is a union of the above digraphs and is not arc-transitive. It will have automorphism group either* $\mathrm{PSL}(2, 13)$ *or* $\mathrm{PGL}(2, 13)$*, depending upon whether or not it can be written as a union of graphs all of whose automorphism groups are* $\mathrm{PGL}(2, 13)$*. If this is possible, then it has automorphism group* $\mathrm{PGL}(2, 13)$*; otherwise, its automorphism group will be* $\mathrm{PSL}(2, 13)$*.*

Proof. The information about the orbital digraphs of $\mathrm{PSL}(2, 13)$, including whether or not they are self-paired and their automorphism groups and whether they are 2-arc-transitive or arc-transitive, was obtained using `magma`. So was information about the automorphism groups of unions of exactly two orbital digraphs. It thus remains to determine the automorphism group of any other digraph of order 91 that contains $\mathrm{PSL}(2, 13)$ and is not complete or its complement.

Let Γ be such a digraph. Then $\mathrm{Aut}(\Gamma) \neq S_{91}$, and $\mathrm{Aut}(\Gamma)$ is 2-closed. There is only one other socle of a primitive but not 2-transitive subgroup of S_{91}, namely $\mathrm{PSL}(3, 9)$ by [5, Table B.2]. However, $\mathrm{PSL}(3, 9)$ contains no subgroup isomorphic to $\mathrm{PSL}(2, 13)$ by [2]. Thus $\mathrm{soc}(\mathrm{Aut}(\Gamma)) = \mathrm{PSL}(2, 13)$ and so $\mathrm{Aut}(\Gamma) = \mathrm{PSL}(2, 13)$ or $\mathrm{PGL}(2, 13)$. Clearly, if Γ can be written as a union of graphs whose automorphism group is $\mathrm{PGL}(2, 13)$, then $\mathrm{Aut}(\Gamma) = \mathrm{PGL}(2, 13)$. Otherwise, by the first part of this lemma, Γ is a union of digraphs one of which has automorphism group $\mathrm{PSL}(2, 13)$ but is not invariant under $\mathrm{PGL}(2, 13)$ and its different image under $\mathrm{PGL}(2, 13)$ is not a subdigraph of Γ. Hence $\mathrm{Aut}(\Gamma) = \mathrm{PSL}(2, 13)$. □

This leads to the next error in the literature, which is also mainly typographical. Namely, in [24, Table II] the entries for $\mathrm{PSL}(2, p)$ require $p \geqslant 11$ (this restriction is found in the last paragraph of page 262). So the primes $p = 5$ and 7 were not considered. For $p = 5$, $\mathrm{PSL}(2, 5)$ is 2-transitive in its representation of degree 6, and so any digraph of order 6 whose automorphism group contains $\mathrm{PSL}(2, 5)$ is necessarily complete or has no arcs and has automorphism group S_6. For $\mathrm{PSL}(2, 7) \cong \mathrm{PSL}(3, 2)$, we see from Theorem 12 that there are other digraphs that are not graphs that are not listed in [24, Table II]. The error here is one of omission rather than a mistake in the proof - in [24] the proofs are for vertex-transitive digraphs and graphs of order at least $5p$ (see for example [24, Table IV]), $p \geqslant 7$, as the case when $p = 3$ was already considered in [29] - but [29] did not consider digraphs that were not graphs, and so these examples were overlooked in [24, Table II].

The next error involves M_{23} in its actions on $11 \cdot 23$ points. There are two actions of M_{23} on $253 = 11 \cdot 23$ points. One is on pairs taken from a set of 23 elements, while the other is on the heptads (sets of size 7) in the Steiner system $S(4, 7, 23)$ [4]. The action on pairs gives M_{23} as a transitive subgroup of $\mathrm{Aut}(T_{23})$, the triangle graph, whose automorphism group is S_{23}, and this graph is listed in the row corresponding to A_{23}. The action of M_{23} on heptads was not considered in [24].

Lemma 24. *The action of M_{23} on heptads (sets of size 7) in the Steiner system $S(4,7,23)$ of degree $11 \cdot 23$ has two orbital digraphs which are graphs of valency 112 and 140. Both of these graphs are Cayley graphs of the nonabelian group of order $11 \cdot 23$ and so are also isomorphic to metacirculant graphs. Both graphs have automorphism group M_{23} and neither is 2-arc-transitive.*

Proof. The action on heptads gives M_{23} as a transitive subgroup of $\mathrm{Aut}(M_{23})$. By [25] the suborbits are of length 112 and 140, and by [4] there is a maximal subgroup H of M_{23} of order 253, and H is isomorphic to the Frobenius group of order 253. By order arguments no element of H is contained in the stabilizer of a heptad, and so H must be semiregular. By the Orbit-Stabilizer Theorem we see that H is regular. As Sabidussi showed [26] that a graph is isomorphic to a Cayley graph of the group G if and only if it contains a regular subgroup isomorphic to G, each of the two orbital graphs of M_{23} are Cayley graphs. As every Cayley graph of order qp is a metacirculant graph, these two graphs are also metacirculant.

Turning to the automorphism groups of the two orbital digraphs Γ_1 and Γ_2 of M_{23}, they are complements of each other and so $\mathrm{Aut}(\Gamma_1) = \mathrm{Aut}(\Gamma_2)$. Also, with respect to the 2-closure of this action of M_{23} (which can be defined as the intersection of the automorphism groups of its orbital digraphs), we have $M_{23}^{(2)} = \mathrm{Aut}(\Gamma_1) \cap \mathrm{Aut}(\Gamma_2) = \mathrm{Aut}(\Gamma_1)$. By [14, Theorem 1] we have $M_{23} \triangleleft \mathrm{Aut}(\Gamma_1)$. By [5, Table B.2] we have $\mathrm{Aut}(\Gamma_1) = M_{23}$.

Finally, in order to be 2-arc-transitive, $d(d-1)$ must divide the order of the stabilizer of a point in M_{23} where d is the valency of Γ_1 or Γ_2, and this stabilizer has order $2^7 \cdot 3^2 \cdot 5 \cdot 7$. So neither Γ_1 nor Γ_2 is 2-arc-transitive. □

Theorem 25. *Let Γ be an arc-transitive graph of order qp, where q and p are distinct primes, whose automorphism group G is simply primitive. Then $\mathrm{soc}(G)$ is given in Table 1. There is a boldface entry in the column "Valency" if and only if there is a 2-arc-transitive graph of that valency. The superscipt symbols in the table have the following meanings:*

- $*$ *means $p \geqslant 7$,*
- † *means that these graphs are also Marušič-Scapellato graphs but in the case of A_{p+1} this is only true for A_6,*
- $**$ *means these graphs are Cayley if and only if $p \equiv 3 \pmod 4$,*
- # *means that these graphs are metacirculant graphs which are not Cayley graphs.*

Proof. Most of the information in the Table 1 is taken directly from the sources in the column "Reference", with the following exceptions. First, information about 2-arc-transitive graphs not given in Lemma 23 or 24 can be found in [22]. That the generalized orbital

G or $\mathrm{soc}(G)$	qp	Valency	Cayley	Reference
A_{qp}	qp	$0, qp-1$	Y	
A_p	$\frac{p(p-1)}{2}$	$2(p-2), \frac{(p-2)(p-3)}{2}$	Y*	[24, 3.1]
A_{p+1}	$\frac{p(p+1)}{2}$	$2(p-1), \frac{(p-1)(p-2)}{2}$	N†	[24, 3.1]
A_7	$5 \cdot 7$	$\mathbf{4}, 12, 18$	N	[24, 3.2]
$\mathrm{PSL}(4,2)$	$5 \cdot 7$	$16, 18$	N	[24, 3.3]
$\mathrm{PSL}(5,2)$	$5 \cdot 31$	$42, 112$	Y	[24, 3.3]
$\Omega^{\pm}(2d,2)$	$(2^d \mp 1)(2^d \pm 1)$	$2^{2d-2}, 2(2^{d-1} \mp 1)(2^{d-2} \pm 1)$	N	[24, 3.4]
$\mathrm{PSp}(4,k)$	$(k^2+1)(k+1)$	k^2+k, k^3, k even	N†	[24, 3.5]
$\mathrm{PSL}(2,k^2)$	$k(k^2+1)/2$	$k^2-1, \frac{k^2-k}{2}, k^2 \pm k$, $k \equiv 1 \pmod 4$	N	[24, 4.1]
$\mathrm{PSL}(2,k^2)$	$k(k^2+1)/2$	$k^2-1, \frac{k^2+k}{2}, k^2 \pm k$, $k \equiv 3 \pmod 4$	N	[24, 4.1]
$\mathrm{PSL}(2,p)$	$\frac{p(p\mp1)}{2}$	$\frac{p\pm1}{2}, p \pm 1$, or $\frac{p\pm1}{4}$ or $2(p-1)$	Y**	[24, 4.4]
$\mathrm{PGL}(2,7)$	$3 \cdot 7$	$4, 8$	Y	[29, Ex. 2.3]
$\mathrm{PGL}(2,11)$	$5 \cdot 11$	$\mathbf{4}, 6, 8, 12, 24$	Y	[23, 4.3]
$\mathrm{PSL}(2,13)$	$7 \cdot 13$	$\mathbf{4}, 6, 12, 24$	N	Lemma 23
$\mathrm{PSL}(2,19)$	$3 \cdot 19$	$\mathbf{6}, 20, 30$	Y	[23, 4.2]
$\mathrm{PSL}(2,23)$	$11 \cdot 23$	$\mathbf{4}, 6, 8, 12, 24$	Y	[23, 4.3]
$\mathrm{PSL}(2,29)$	$7 \cdot 29$	$12, 20, 30, 60$	N$^\#$	[23, 4.2]
$\mathrm{PSL}(2,59)$	$29 \cdot 59$	$\mathbf{6}, 10, 12, 20, 30, 60$	Y	[23, 4.2]
$\mathrm{PSL}(2,61)$	$31 \cdot 61$	$\mathbf{6}, 10, 12, 20, 30, 60$	N	[23, 4.2]
M_{22}	$7 \cdot 11$	$\mathbf{16}, 60$	N	[23, 3.6]
M_{23}	$11 \cdot 23$	$112, 140$	Y	Lemma 24

Table 1: The graphs of order pq with primitive automorphism groups. Bold font indicates the corresponding graph is 2-arc-transitive.

digraphs of $\mathrm{PSL}(2,29)$ are metacirculants is proven in [17, pg. 192, paragraph 3]. The vertex-transitive graphs of order pq with primitive automorphism group that are also isomorphic to nontrivial Marušič-Scapellato graphs are determine in [17] starting at the bottom of page 192. □

There is one final gap in the information about automorphism groups of vertex-transitive digraphs of order qp that we are aware of. Namely, what are the automorphism groups of such a digraph when it is not a graph and its automorphism group is primitive?

Theorem 26. *Let Γ be a vertex-transitive digraph of order qp such that Γ is not a graph and $\mathrm{Aut}(\Gamma)$ is primitive with socle G. Let H be the stabilizer of a point in G. Then one of the following is true:*

1. *$qp = 7 \cdot 13$ or $11 \cdot 23$, $G \cong \mathrm{PSL}(2,13)$ or $\mathrm{PSL}(2,23)$, respectively, and $H \cong S_4$,*
2. *$qp = 29 \cdot 59$ or $31 \cdot 61$, $G \cong \mathrm{PSL}(2,59)$ or $\mathrm{PSL}(2,61)$, respectively, and $H \cong A_5$, or*
3. *$qp = p(p \pm 1)/2$, $G \cong \mathrm{PSL}(2,p)$, and $H \cong D_{p\mp 1}$.*

Furthermore, in all cases $\mathrm{Aut}(\Gamma) = G$.

Proof. Let Γ satisfy the hypothesis of the result. As Γ is a digraph, Γ must contain the arc set of an orbital digraph of G which is not a graph. Checking the cases considered in [24] where qp satisfy $q < p$ and $q \geqslant 5$, we see that the only groups with orbital digraphs which are not graphs are given in [24, Lemmas 4.2, 4.3, 4.4], and are listed in the result. There are the additional digraphs given in Lemma 23 as well as in [29, Example 2.3]. These are all the possible socles of $\mathrm{Aut}(\Gamma)$. It only remains to show that in all cases we have that $\mathrm{Aut}(\Gamma) = G$.

The case where $G = \mathrm{PSL}(2,13)$ follows from Lemma 23. For $G = \mathrm{PSL}(2,p)$ with point stabilizer $D_{p\mp 1}$, we have by [24, Lemma 4.4] that every orbital digraph of $\mathrm{PGL}(2,p)$ is a graph. As the only possibility for $\mathrm{Aut}(\Gamma)$ other than G is $\mathrm{PGL}(2,p)$, the result follows. In all of the remaining cases, suppose $\gamma \in \mathrm{Aut}(\Gamma)$. Then γ normalizes G. By [11, Exercise 1.4.16], we see that γ permutes the orbital digraphs of G. By [24, Lemmas 4.3 (b) and 4.4 (b)], we see that the orbital digraphs of G are pairwise non-isomorphic and have automorphism group G. Hence γ fixes each orbital digraph of G, and so γ is an automorphism of each orbital digraph of G whose arc set is contained in the arc set of Γ. Hence $\gamma \in G$. □

6 Other errors in the literature

To conclude this paper, we list the errors that we are aware in the literature that follow from the errors above and that are not in the original papers where the error was made.

- The statement of [8, Theorem 2.5] is missing the graphs given in Theorem 12 with imprimitive automorphism group $\mathrm{PSL}(2,11)$. This result is only used to discuss graphs of order 21, and so this error does not affect any results proven in the paper.

- The result [30, Proposition 2.5] does not list the arc-transitive graphs of valency 4 given by Lemma 23. Consequently, [30, Lemma 3.4] has a small gap which can be filled using GAP or `magma`.

- The result [22, Proposition 4.2] is missing the 2-arc-transitive graphs of valency 4 given by Lemma 23.

- The result [6, Corollary 3.3, Table 1] is missing the graphs given by Lemmas 23 and 24. Additionally, [6, Theorem 3.2] and [6, Corollary 3.3] are missing the group $\mathrm{PSL}(2, 11)$ in its imprimitive action action on 55 points. Finally, [6, Theorem 4.1(3)] is missing these same graphs.

  The result from [6, Theorem 3.2(1)] could be strengthened to digraphs by including the digraphs with simple and imprimitive automorphism groups.

- The result [13, Theorem] does not consider the action of $\mathrm{PSL}(2, 13)$ given in Lemma 23 nor the action of M_{23} given in Lemma 24.

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