Bounds for Greedy B_h-sets

Kevin O'Bryant[a]

Submitted: Apr 25, 2024; Accepted: Jan 5, 2025; Published: Jun 6, 2025

Abstract

A set $\mathcal{A}$ of nonnegative integers is called a B_h-set if every solution to $a_1 + \cdots + a_h = b_1 + \cdots + b_h$, where $a_i, b_i \in \mathcal{A}$, has $\{a_1, \ldots, a_h\} = \{b_1, \ldots, b_h\}$ (as multisets). Let $\gamma_k(h)$ be the k-th positive element of the greedy B_h-set. We give a nontrivial lower bound on $\gamma_5(h)$, and a nontrivial upper bound on $\gamma_k(h)$ for $k \geqslant 5$. Specifically, $\frac{1}{8}h^4 + \frac{1}{2}h^3 \leqslant \gamma_5(h) \leqslant 0.467214h^4 + O(h^3)$, although we conjecture that $\gamma_5(h) = \frac{1}{3}h^4 + O(h^3)$. We show that $\gamma_k(h) \geqslant \frac{1}{k!}h^{k-1} + O(h^{k-2})$ for $k \geqslant 1$ and $\gamma_k(h) \leqslant \alpha_k h^{k-1} + O(h^{k-2})$, where $\alpha_6 :- 0.382978$, $\alpha_7 :- 0.269877$, and for $k \geqslant 7$, $\alpha_{k+1} :- \frac{1}{2^k k!} \sum_{j=0}^{k-1} \binom{k-1}{j}\binom{k}{j} 2^j$. This work begins with a thorough introduction and concludes with a section of open problems.

Mathematics Subject Classifications: 11B13, 05B10

1 Introduction

A set $\mathcal{A}$ of nonnegative integers is called a B_h-set if every solution to

$$a_1 + \cdots + a_h = b_1 + \cdots + b_h, \quad a_i, b_i \in \mathcal{A} \tag{1}$$

has $\{a_1, \ldots, a_h\} = \{b_1, \ldots, b_h\}$ (as multisets).

These sets (with $h = 2$) first arose in harmonic analysis [13] as a tool to create trigonometric polynomials with peculiar properties. Prompted by Sidon's work, Erdős defined B_h-sets [15] and they have become a central tool and nexus of problems in combinatorial number theory, beginning with the seminal work of Erdős & Turán in 1941 [4]. They were indepedently discovered by Babcock [1] as a means to avoid third order intermodulation of frequencies (B_2-sets) and fifth-order intermodulation (B_3-sets). Golomb rediscovered B_2-sets as a means to represent graphs (the vertices are a B_2-set, and an edge between a and b is uniquely coded as $|a - b|$), and some of his questions were written up by Martin Gardner in Scientific American [5]. See [11] for an extensive bibliography.

The primary problem of interest is to give a finite B_h-set with many elements compared to its diameter (for $\mathcal{A} \subseteq \mathbb{Z}$, we define $\operatorname{diam}(\mathcal{A}) :- 1 + \max \mathcal{A} - \min \mathcal{A}$). A natural method

[a] Department of Mathematics, City University of New York, College of Staten Island and The Graduate Center, New York City, U.S.A. (kevin.obryant@csi.cuny.edu)

to construct a B_h-set with small diameter is to simply be greedy. We set $\gamma_0 = 0, \gamma_1 = 1$, and thereafter set $\gamma_{k+1}(h)$ to be the smallest positive integer x with the property that $\{\gamma_0, \dots, \gamma_k(h), x\}$ is a B_h-set. The infinite set $\{\gamma_0, \gamma_1, \gamma_2(h), \dots\}$ is the *greedy* B_h-set; it is the lexicographically first infinite B_h-set.

For reference, we give a table of $\gamma_k(h)$ (OEIS A365515). The $h = 1$ row is trivial, as are the $k = 0$ and $k = 1$ columns. Formulas are derived for the $k = 2$ and $k = 3$ columns in [9], and a formula for $\gamma_4(h)$ is given in [10].

h	γ_0	γ_1	$\gamma_2(h)$	$\gamma_3(h)$	$\gamma_4(h)$	$\gamma_5(h)$	$\gamma_6(h)$	$\gamma_7(h)$	$\gamma_8(h)$	$\gamma_9(h)$
1	0	1	2	3	4	5	6	7	8	9
2	0	1	3	7	12	20	30	44	65	80
3	0	1	4	13	32	71	124	218	375	572
4	0	1	5	21	55	153	368	856	1424	2603
5	0	1	6	31	108	366	926	2286	5733	12905
6	0	1	7	43	154	668	2214	6876	16864	41970
7	0	1	8	57	256	1153	4181	14180	47381	115267
8	0	1	9	73	333	1822	8043	28296	102042	338447
9	0	1	10	91	500	3119	13818	59174	211135	742330

The greedy B_2-set was first considered in the literature in 1944 [8], and in the years since there has been no progress on understanding greedy B_h-sets other than the computation of more terms. The purpose of the present work is to give the first nontrivial upper bounds on $\gamma_k(h)$ for $k \geqslant 5$, and a non-trivial lower bound on $\gamma_5(h)$.

Theorem 1. *Let $\alpha_5 := 0.467214, \alpha_6 := 0.382978, \alpha_7 := 0.269877$, and for $k \geqslant 7$ set*

$$\alpha_{k+1} := \frac{1}{2^k k!} \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{k}{j} 2^j = \frac{1}{2^k k!} \, {}_2F_1(1-k, -k; 1; 2).$$

Then, for all $k \geqslant 5$,

$$\gamma_k(h) \leqslant \alpha_k h^{k-1} + O_k(h^{k-2}).$$

Theorem 2. *The fifth positive element of the greedy B_h-set satisfies $\gamma_5(h) \geqslant \frac{1}{8}h^4 + \frac{1}{2}h^3$.*

A skeptical reader (or referee) might ask whether bounding, for example, the eighth positive element of the greedy B_{100}-set is an interesting problem. First, B_h-sets are natural objects of study, and deserve answers for aesthetic reasons. Second, non-greedy methods of constructing B_h-sets require working in large finite fields. For example, to construct a 8-element B_{100} set using Singer's construction [14, 12] (or the Bose-Chowla construction [2, 12]) requires a substantial amount of arithmetic in the field with 7^{101} elements (or 8^{100} for the Bose-Chowla construction), which is implausible with current computational methods. However, finding a formula in terms of h for the 8-th positive element of the greedy B_{100} is open but plausible. Finally, the actual motivation for this project was a desire to better understand the greedy algorithm for B_2-sets (which have physical real-world applications), and greedy algorithms for additive problems in general.

A similar phenomenon to that discussed here is seen in Ulam sets [7]. The Ulam set U_h is the lexicographically least set of positive integers after $\{1, h\}$ and with the property that each element can be written as a sum of two other elements in a unique way. While U_2 is famously mysterious, polynomial patterns emerge when looking at the way U_h starts as a function of h.

Cilleruelo [3] considered a greedy algorithm for $B_h[g]$-sets (in a B_h-set, the h-fold sums don't repeat, while in a $B_h[g]$-set the h-fold sums do not repeat more than g times). He added a condition depending on g that helps the greedy process proceed more smoothly.

2 A Tour of the Greedy $\mathbf{B_h}$-sets

In this section, we repeat results that are either folklore or already in the literature. Only Lemma 6 is used in this work, but we suspect the reader will appreciate a summary of known bounds with indications of their derivations.

We start with relatively straightforward combinatorial bounds on the size of a B_h-set. These are often referenced in the B_h-set literature, but only special cases have been made explicit. The proof of Lemma 3 is simply the observation that the $\binom{k+h}{k}$ distinct possibilities for h-fold sums must all lie in the interval $[ha_0, ha_k]$, and for a B_h-set they must all be distinct.

Lemma 3. *Let $\mathcal{A} \coloneqq \{a_0 < a_1 < \cdots < a_k\}$ be a B_h-set. Then*

$$a_k \geqslant a_0 + \frac{1}{h}\left[\binom{k+h}{k} - 1\right].$$

Given a finite B_h-set, there are only finitely many integers you can union into it that will *not* give a B_h-set. Quantifying this for general h, k hasn't appeared in the literature.

Lemma 4. *Let $\mathcal{A} \coloneqq \{a_0 < a_1 < \cdots < a_k\}$ be a B_h-set. There are at most*

$$\sum_{r=1}^{h-1} \sum_{M=1}^{\infty} \binom{k}{M} \binom{k-M+h-r}{h-r} \left[\binom{h}{M} - \binom{r}{M}\right]$$

integers $f > a_k$ for which $\mathcal{A} \cup \{f\}$ is not a B_h-set. They are all in the interval $[ha_0, ha_k]$.

Proof. As the B_h property is translation invariant, it is without loss of generality that we assume $a_0 = 0$. There must be some equation

$$x_1 + \cdots + x_h = y_1 + \cdots + y_h, \quad x_i, y_i \in \mathcal{A}$$

involving f. Without loss of generality, we can assume that $X \coloneqq \{x_1, \dots, x_h\}$ and $Y \coloneqq \{y_1, \dots, y_h\}$ are disjoint, and that $f \in Y$ with multiplicity r. Letting m_i, n_i be the mulitplicity of a_i in X, Y, respectively, we have

$$f = \frac{1}{r} \sum_{i=0}^{k} (m_i - n_i) a_i, \quad \sum_{i=0}^{k} m_i = h, \quad \sum_{i=0}^{k} n_i = h - r, \quad m_1 n_1 = \cdots = m_k n_k = 0. \tag{2}$$

Since we are only concerned with $f > a_k$, we further know that $r < \sum_{i=1}^{k} m_i$. Let M be the number of $m_1, \ldots, m_k$ that are nonzero, and N the number of $n_1, \ldots, n_k$ that are nonzero. For a fixed r, m_0, M, N, there are

$$\binom{k}{M}\binom{k-M}{N}\binom{h-m_0-1}{N}\binom{h-r}{N}$$

possible $m_0, m_1, \ldots, m_k, n_0, n_1, \ldots, n_k$ that satisfy the conditions on Line (2) (and $r < \sum_{i=1}^{k} m_i$), by nested stars-and-bars style counting. Summing over $0 \leqslant n \leqslant \min\{h-r, k-M\}$, $1 \leqslant M \leqslant \min\{k, h-m_0\}$, $0 \leqslant m_0 \leqslant h-r-1$, and $1 \leqslant r \leqslant h-1$ and simplifying completes the proof of the first sentence of this lemma. That all of these "forbidden" f are in $[0, ha_k]$ is immediate from Line (2). □

This proof doesn't take full consideration of the requirement that $f > a_k$, nor that f is an integer. Consequently, there is room to improve, but we are skeptical that this bound can be improved by more than a factor of 1/2 without a substantial new idea.

We turn now to consider the greedy B_h-sets specifically. Lemma 3 immediately gives the following lemma.

Lemma 5. *Let k, h be positive integers. Then*

$$\gamma_k(h) \geqslant \frac{1}{h}\left[\binom{k+h}{k} - 1\right]. \tag{3}$$

Consequently,for fixed k we have $\gamma_k(h) \geqslant \dfrac{1}{k!}\, h^{k-1} + O_k(h^{k-2})$ and for fixed h we have $\gamma_k(h) \geqslant \dfrac{1}{h \cdot h!} k^h + O_h(k^{h-1})$.

The lower bound for fixed h has been improved (see [6] for improvements for general h) for all B_h-sets, not only the greedy sets. The last sentence of Lemma 4 implies that $\gamma_{k+1}(h) \leqslant h \cdot \gamma_k(h) + 1$, and from this it follows that the lower bound in Lemma 5 for fixed k has the correct order of growth.

We can set particular values for h in Lemma 4 and get explicit bounds:

$$\begin{aligned}
\gamma_k(2) &\leqslant \frac{1}{2}k^3 + \frac{1}{2}k \\
\gamma_k(3) &\leqslant \frac{1}{12}k^5 + \frac{5}{12}k^4 + \frac{3}{4}k^3 + \frac{13}{12}k^2 + \frac{2}{3}k
\end{aligned}$$

and so on. In general, one finds that

$$\gamma_k(h) \leqslant \frac{1}{h!(h-1)!}k^{2h-1} + O_h(k^{2h-2}).$$

This author is not aware of any improvement to this bound.

The following formulas for $\gamma_k(h)$ for $k \leqslant 4$ are proved in [9, 10]. We give proofs of these in Appendix A in the spirit of the lower bound we give for $\gamma_5(h)$ in Theorem 2.

Lemma 6. *We have explicit formulas for $\gamma_k(h)$ for $k \leqslant 4$: $\gamma_0 = 0$, $\gamma_1 = 1$, $\gamma_2(h) = h+1$, $\gamma_3(h) = h^2 + h + 1$, and $\gamma_4(h) = \left\lfloor \frac{h+3}{2} \right\rfloor h^2 + \left\lfloor \frac{3h+2}{2} \right\rfloor$.*

3 The Greedy $\mathbf{B_h}$-set

For the remainder of this work, we write γ_k instead of $\gamma_k(h)$.

We set $\gamma_0 = 0$, and inductively set γ_{k+1} to the smallest integer greater than γ_k such that $\{\gamma_0, \gamma_1, \ldots, \gamma_k, \gamma_{k+1}\}$ is a B_h-set. We call $\mathcal{G}(h) \coloneqq \{\gamma_0, \gamma_1, \ldots\}$ be the *greedy* B_h*-set*. It is the lexicographically first infinite B_h-set of natural numbers.

We define

$$F_r^{(k)} \coloneqq \left\{ \frac{1}{r} \sum_{i=1}^{k} \gamma_i \cdot (m_i - m_i') : m_i, m_i' \in \mathbb{N}, \sum_{i=1}^{k} m_i \leqslant h, \sum_{i=1}^{k} m_i' \leqslant h - r \right\}, \tag{4}$$

$$F^{(k)} \coloneqq \bigcup_{r=1}^{h} F_r. \tag{5}$$

We draw the reader's attention to the omission of γ_0 from the definition of $F_r^{(k)}$, and that the sum of multiplicities need not be exactly h, rather, the sum of multiplicities is at most h. Moreover, in the definition of F_r we will often assume (without loss of generality) that at least one of m_i, m_i' is 0 for each i.

Definition 7. The mex of a set is the smallest nonnegative integer that is excluded from the set. The function name "mex" is a a contraction of minimal excluded.

Lemma 8. *The* $(k+1)$*-st positive element of the greedy* B_h*-set is* $\gamma_{k+1} = \operatorname{mex} F^{(k)}$.

Proof. If $\gamma_k < x < \operatorname{mex} F$, then $x = \frac{1}{r}\sum_{i=1}^{k} \gamma_i \cdot (m_i - m_i')$ as in (4). Whence, as $\gamma_0 = 0$, we have

$$x \cdot r + \sum_{i=1}^{k} \gamma_i m_i' + \gamma_0 \cdot (h - r - \sum_{i=1}^{k} m_i') = \sum_{i=1}^{k} \gamma_i m_i + \gamma_0 \cdot (h - \sum_{i=1}^{k} m_i),$$

proving that $\{\gamma_0, \ldots, \gamma_k\} \cup \{x\}$ is not a B_h-set. On the other hand, let $x = \operatorname{mex} F$ and suppose that $\{\gamma_0, \ldots, \gamma_k\} \cup \{x\}$ is not a B_h-set, so that there is a solution to

$$x m_{k+1} + \sum_{i=0}^{k} \gamma_i m_i = x m_{k+1}' + \sum_{i=0}^{k} \gamma_i m_i'$$

with $\sum_{i=0}^{k+1} m_i = \sum_{i=0}^{k+1} m_i' = h$. As (inductively) $\{\gamma_0, \ldots, \gamma_k\}$ is a B_h-set, it must be that at least one of m_{k+1}, m_{k+1}' is positive, say $m_{k+1}' \geqslant m_{k+1} \geqslant 0$. If $m_{k+1}' = m_{k+1}$, then

$$\sum_{i=1}^{k} \gamma_i m_i + \gamma_0 \cdot (m_0 + m_{k+1}) = \sum_{i=1}^{k} \gamma_i m_i' + \gamma_0 \cdot (m_0' + m_{k+1}'),$$

contradicting that $\{\gamma_0, \ldots, \gamma_k\}$ is a B_h-set. Ergo, we may set $r \coloneqq m_{k+1}' - m_{k+1} > 0$. We have

$$\sum_{i=1}^{k} \gamma_i m_i = xr + \sum_{i=1}^{k} \gamma_i m_i',$$

which we can rearrange to

$$x = \frac{1}{r}\sum_{i=1}^{k} \gamma_i \cdot (m_i - m_i'),$$

with $\sum_{i=1}^{k} m_i = h - m_0 - m_{k+1} \leqslant h$ and $\sum_{i=1}^{k} m_i' = h - (m_0' + m_{k+1}') \leqslant h - r$. □

4 Formulas for γ_k

The formula $\gamma_0 = 0$ is by definition, and the formula $\gamma_1 = 1$ is immediate. In [9], Nathanson provides detailed proofs of the formulas $\gamma_2 = h + 1$ and $\gamma_3 = h^2 + h + 1$. In [10], Nathanson and the author prove that

$$\gamma_4 = h^2 \left\lfloor \frac{h+3}{2} \right\rfloor + \left\lfloor \frac{3h+2}{2} \right\rfloor.$$

Computation (OEIS A369818) of γ_5 for $h \leqslant 47$ has yielded a conjectural formula.

Conjecture 9.

$$\gamma_5 = \frac{h^4}{3} + \frac{1}{6} \cdot \begin{cases} 5h^3 + 7h^2 + 7h + 9, & \text{if } h \equiv 1 \pmod 6 \\ 4h^3 + 10h^2 + 6h + 4, & \text{if } h \equiv 2 \pmod 6 \\ 7h^3 + 5h^2 + 9h + 3, & \text{if } h \equiv 3 \pmod 6 \\ 406, & \text{if } h = 4 \\ 6h^3 + 6h^2 + 8h + 8, & \text{if } h \equiv 4 \pmod 6 \text{ and } h \geqslant 10 \\ 6h^3 + 6h^2 + 8h + 6, & \text{if } h \equiv 5 \pmod 6 \\ 5h^3 + 8h^2 + 7h + 6, & \text{if } h \equiv 6 \pmod 6 \end{cases}$$

Paul Voutier [16] has computed γ_6 for $1 \leqslant h \leqslant 33$, generating the following sequence (OEIS A369819), for which no formula has yet been guessed.

h	γ_6	h	γ_6	h	γ_6
1	6	13	84026	25	1916949
2	30	14	109870	26	2361150
3	124	15	156474	27	2859694
4	368	16	217790	28	3467661
5	926	17	304910	29	3989744
6	2214	18	376260	30	4779270
7	4181	19	510220	31	5479857
8	8043	20	667130	32	6449983
9	13818	21	794873	33	7575912
10	23614	22	1008048	34	?
11	34825	23	1302947	35	?
12	54011	24	1629264	36	?

While all of $\gamma_0, \dots, \gamma_4$ are quasi-polynomials, and γ_5 appears to be, this author is skeptical that all γ_k are. However, it is plausible that each γ_k, if h is sufficiently large, is given by an expression built up from rational functions in h and floor functions.

In Appendix A, we give the alternate proofs of the formulas for $\gamma_2, \gamma_3, \gamma_4$ in the style of our proof in the next section that $\gamma_5 \geqslant \frac{1}{8}h^4 + \frac{1}{2}h^3$.

5 A Lower Bound on γ_5

We now proceed to prove that $\gamma_5(h) \geqslant \frac{1}{8}h^4 + \frac{1}{2}h^3$. We assume that $h \geqslant 5$, as the inequality holds by direct computation for $1 \leqslant h \leqslant 4$.

Definition 10. The "Iverson Bracket" $[\![P]\!]$ is 1 if P is true, and is 0 if P is false.

We define functions ℓ_i, u_i for $1 \leqslant i \leqslant 4$ as follows:

$\ell_4 := 1$	$u_4 := \left\lfloor \frac{h+3}{4} \right\rfloor$
$\ell_3(d_4) := d_4 - \left\lfloor \frac{h+1}{2} \right\rfloor$	$u_3(d_4) := \left\lfloor \frac{h+1}{2} \right\rfloor - d_4$
$\ell_2(d_4, d_3) := d_4 + \lvert d_3 \rvert - h$	$u_2(d_4, d_3) := h - d_4 - \lvert d_3 \rvert$
$\ell_1(d_4, d_3, d_2) := 1 - h - \sum_{i=1}^{4} d_i [\![d_i \leqslant 0]\!]$	$u_1(d_4, d_3, d_2) := h - \sum_{i=1}^{4} d_i [\![d_i \geqslant 0]\!]$

Definition 11. We denote vectors in the form $\langle d_1, d_2, d_3, d_4 \rangle$, and will dot-product vectors in the usual way. For a vector μ of integers, we define the function *last* by setting $last(\mu)$ to be the last component of μ. We define the function *drop* to be the vector μ with its last component removed. In particular, $last(\mu)$ is an integer, and $drop(\mu)$ is a vector of integers with one less component than μ.

We order distinct vectors of the same length by setting $\mu < \nu$ if either $last(\mu) < last(\nu)$ or both $last(\mu) = last(\nu)$ and $drop(\mu, -1) < drop(\nu, -1)$. Otherwise, $\mu \geqslant \nu$. That is, we order vectors lexicographically considering first their last components.

We now set

$$\Delta := \Bigg\{ \langle d_1, d_2, d_3, d_4 \rangle : d_i \in \mathbb{Z}, \quad \sum_{i=1}^{4} d_i [\![d_i \geqslant 0]\!] \leqslant h, \quad \sum_{i=1}^{4} d_i [\![d_i < 0]\!] \geqslant 1 - h, \\ \ell_4 \leqslant d_4 \leqslant u_4, \quad \ell_3(d_4) \leqslant d_3 \leqslant u_3(d_4), \\ \ell_2(d_4, d_3) \leqslant d_2 \leqslant u_2(d_4, d_3), \quad \ell_1(d_4, d_3, d_2) \leqslant d_1 \leqslant u_1(d_4, d_3, d_2) \Bigg\}.$$

For an element $\delta \in \Delta$, its *image* is the integer $\delta \cdot \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle$. Clearly the images of elements of Δ are in F_1 (defined on line (4)), and so γ_5 is at least the mex of the images of Δ:

$$\gamma_5 \geqslant \operatorname{mex} \{ \delta \cdot \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle : \delta \in \Delta \}.$$

On an automobile's odometer, the units digit climbs from its minimum to its maximum, and then resets to its minimum at the same moment the tens digit increases by one. If both the ones and tens digit are at their maximum, then the hundreds digits increments when the ones and tens digits reset to their minimums. Our ordering on Δ is in this spirit, with the least significant digits listed first. Conceptually, this is the motivation for for the remainder of the argument: we order the vectors of Δ according to their images. As one examines the elements of Δ in this order, the elements look an automobile odometer, with the last component counting up until it hits its maximum, and then "rolling over" to its minimum with the penultimate component being incremented. Unfortunately, this is not exactly what happens, and the detailed argument indicated here and detailed in Appendix B is to the effect that this is close enough to what actually happens to imply that the image of Δ is an interval of integers.

We note that both $\langle 0,0,0,1\rangle$, with image γ_4, and $\langle 0, h/2, h/4, \lfloor (h+3)/4 \rfloor\rangle$ with image $\lfloor (h+3)/4\rfloor \gamma_4 \geqslant \frac{1}{8}h^4 + \frac{1}{4}h^3$, are in Δ.

Let $\mu_0 < \mu_1 < \cdots$ be the ordered elements of Δ. We will show that the images of $\mu_0, \mu_1, \ldots$, a sequence of integers, increases by at most 1 at each term. Thus, the image of Δ is an interval. As $\langle 0,0,0,1\rangle \in \Delta$, we know that γ_4 is an image, and also

$$\left\langle 0, \left\lfloor \frac{h}{2} \right\rfloor, \left\lfloor \frac{h+1}{4} \right\rfloor, \left\lfloor \frac{h+3}{4} \right\rfloor \right\rangle \in \Delta$$

which has image

$$\left\lfloor \frac{h}{2} \right\rfloor \gamma_2 + \left\lfloor \frac{h+1}{4} \right\rfloor \gamma_3 + \left\lfloor \frac{h+3}{4} \right\rfloor \gamma_4 \geqslant \frac{1}{8}h^4 + \frac{1}{2}h^3.$$

This will prove Theorem 2.

Suppose that $\delta < \nu$ are consecutive elements of Δ, say $\delta = \langle d_1, d_2, d_3, d_4\rangle$ and $\nu = \langle v_1, v_2, v_3, v_4\rangle$. There are only a four possibilities:

(*i*) $d_1 < u_1(d_4, d_3, d_2)$, $\nu = \langle d_4, d_3, d_2, d_1 + 1\rangle$;

(*ii*) $d_1 = u_1(d_4, d_3, d_2), d_2 < u_2(d_4, d_3)$,
$\nu = \langle \ell_1(d_4, d_3, d_2 + 1), d_2 + 1, d_3, d_4\rangle$;

(*iii*) $d_1 = u_1(d_4, d_3, d_2), d_2 = u_2(d_4, d_3), d_3 < u_3(d_4)$,
$\nu = \langle \ell_1(d_4, d_3 + 1, \ell_2(d_4, d_3 + 1)), \ell_2(d_4, d_3 + 1), d_3 + 1, d_4\rangle$;

(*iv*) $d_1 = u_1(d_4, d_3, d_2), d_2 = u_2(d_4, d_3), d_3 = u_3(d_4), d_4 < u_4$,
$\nu = \langle \ell_1(d_4 + 1, \ell_3(d_4 + 1), L), L, \ell_3(d_4 + 1), d_4 + 1\rangle$, where $L = \ell_2(d_4 + 1, \ell_3(d_4 + 1))$.

In each case, the image goes from $\delta \cdot \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4\rangle$ to $\nu \cdot \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4\rangle$. Thus, our task is to show that in all cases

$$(\nu - \delta) \cdot \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4\rangle \leqslant 1.$$

Each case is a straightforward usage of the description of the case, the definitions of ℓ_i, u_i, and solving algebraic inequalities. One needs to split case (*ii*) into 4 subcases, depending on the signs of d_3, d_2, and case (*iii*) splits into 2 subcases depending on the sign of d_3. In case (*iv*), it is helpful to break into 4 subcases depending on the residue of d_4 modulo 4.

We show these details in Appendix B.

6 An upper bound on γ_k

In this section we determine a nonincreasing sequence of real numbers α_k with

$$\gamma_{k+1}(h) \leqslant \alpha_{k+1}h^k + O(h^{k-1})$$

and $\alpha_k \to 0$. We will inductively use the value of α_k to give α_{k+1}. We take $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = \frac{1}{2}$, in accordance with the formulas for γ_k, $0 \leqslant k \leqslant 4$ stated in Lemma 6, proved in [9, 10], and re-proved in the appendices to the current work. We assume henceforth that $k \geqslant 4$ and are working to bound γ_{k+1}.

Fix a real number β_k with $\frac{1}{2}\alpha_k \leqslant \beta_k \leqslant \alpha_k$. We will use

$$\beta_4 \coloneqq 0.406671, \quad \beta_5 \coloneqq 0.308672, \quad \beta_6 \coloneqq 0.203975, \quad \beta_k \coloneqq \frac{1}{2}\alpha_k, \quad (k \geqslant 7).$$

The engine of our bound (recall the definitions of $F_r^{(k)}, F_r$ on lines (4), (5)) is that

$$\begin{aligned}\gamma_{k+1} = \operatorname{mex} F^{(k)} &\leqslant 1 + \beta_k h^k + \left|F^{(k)} \cap (\beta_k h^k, \infty)\right| \\ &\leqslant 1 + \beta_k h^k + \sum_{r=1}^{h} \left|F_r^{(k)} \cap (\beta_k h^k, \infty)\right|. \qquad (6)\end{aligned}$$

For $r \geqslant 3$, (and sufficiently large h, a hypothesis we use repeatedly in this section)

$$\max F_r^{(k)} = \frac{1}{r} \cdot h\gamma_k \leqslant \frac{1}{3} \cdot h \cdot \left(\alpha_k h^{k-1} + O(h^{k-2})\right) < \frac{1}{2}\alpha_k h^k \leqslant \beta_k h^k.$$

Thus, for $r \geqslant 3$ we know that $|F_r^{(k)} \cap (\beta_k h^k, \infty)| = 0$. For $r = 2$, we arrive at

$$\max F_2^{(k)} \leqslant \beta_k h^k + O(h^{k-1}),$$

whence $|F_2^{(k)} \cap (\beta_k h^k, \infty)| = O(h^{k-1})$. The bound on line (6) simplifies to

$$\gamma_{k+1} \leqslant \beta_k h^k + O(h^{k-1}) + |F_1^{(k)} \cap (\beta_k h^k, \infty)|. \qquad (7)$$

We now stratify $F_1^{(k)}$ as $F_1(m_k, m_k')$, where at least one of m_k, m_k' is 0, and $0 \leqslant m_k \leqslant h$, and $0 \leqslant m_k' \leqslant h-1$. Specifically

$$\begin{aligned}F_1(m_k, m_k') \coloneqq (m_k - m_k')\gamma_k + \Bigg\{ &\sum_{i=1}^{k-1}(m_i - m_i')\gamma_i : \\ &\sum_{i=1}^{k-1} m_i \leqslant h - m_k, \quad \sum_{i=1}^{k-1} m_i' \leqslant h - 1 - m_k', \quad m_1 m_1' = \cdots = m_{k-1}m_{k-1}' = 0 \Bigg\}.\end{aligned}$$

The largest element of $F_1(0, m_k')$ is

$$\begin{aligned}\max F_1(0, m_k') = -m_k'\gamma_k + h\gamma_{k-1} &\leqslant h \cdot \left(\alpha_{k-1}h^{k-2} + O(h^{k-3})\right) \\ &= \alpha_{k-1}h^{k-1} + O(h^{k-2}) < \beta_k h^k,\end{aligned}$$

(for sufficiently large h), so that $F_1(0, m'_k) \cap (\beta_k h^k, \infty) = \emptyset$. We may thus assume without loss that $m_k > 0$ and $m'_k = 0$. The largest element of $F_1(m_k, 0)$ is

$$\max F_1(m_k, 0) = m_k \gamma_k + (h - m_k)\gamma_{k-1} \leqslant \alpha_k \frac{m_k}{h} h^k + O(h^{k-1}).$$

If $m_k < h\beta_k/\alpha_k$, then $\max F_1(m_k, 0)$ is less than $\beta_k h^k$ (as always in this section, for large h). Ergo, we may assume that $m_k \geqslant h\beta_k/\alpha_k$ (recall our assumption that $\beta_k \leqslant \alpha_k$).

Equation (7) now gives us

$$\gamma_{k+1} \leqslant \beta_k h^k + O(h^{k-1}) + \sum_{\substack{m_k \\ \frac{\beta_k}{\alpha_k} h \leqslant m \leqslant h}} |F_1(m_k, 0)|. \tag{8}$$

Notice that we have dropped the intersection with the interval $(\beta_k h^k, \infty)$ from our concerns. That might hurt the bound, but simplicity is its own reward.

We have

$$|F_1(m_k, 0)| \leqslant \Bigg| \Bigg\{ \big((m_1, \ldots, m_{k-1}), (m'_1, \ldots, m'_{k-1})\big) : \\ \sum_{i=1}^{k-1} m_i \leqslant h - m_k, \quad \sum_{i=1}^{k-1} m'_i \leqslant h - 1, \quad m_1 m'_1 = \cdots = m_{k-1} m'_{k-1} = 0 \Bigg\} \Bigg|.$$

Suppose that exactly j of $m'_1, \ldots, m'_{k-1}$ are nonzero. By stars-and-bars ($h - 1 - j$ stars and j bars), there are $\binom{k-1}{j}\binom{h-1}{h-1-j}$ such tuples. For each such tuple, there are $h - m_k$ stars and $(k - 1 - j)$ bars in the count of tuples $(m_1, \ldots, m_{k-1})$; that is, $\binom{h-m_k+(k-1-j)}{h-m_k}$ valid tuples $(m_1, \ldots, m_{k-1})$. Altogether, then

$$|F_1(m_k, 0)| = \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{h-1}{h-1-j} \binom{h - m_k + (k-1-j)}{h - m_k}.$$

Clearly,

$$\binom{h-1}{h-1-j} \leqslant \frac{1}{j!} h^j$$

and since $h - m_k = O(h)$ also

$$\begin{aligned}
\binom{h - m_k + (k-1-j)}{h - m_k} &\leqslant \frac{(h - m_k + k - 1 - j)^{k-1-j}}{(k-1-j)!} \\
&= \frac{1}{(k-1-j)!} \sum_{\ell=0}^{k-1-j} \binom{k-1-j}{\ell} (h - m_k)^{k-1-j-\ell} (k-1-j)^\ell \\
&= \frac{1}{(k-1-j)!} (h - m_k)^{k-1-j} + O((h - m_k)^{k-2-j}) \\
&= h^{k-1-j} \frac{(1 - m_k/h)^{k-1-j}}{(k-1-j)!} + O(h^{k-2-j}).
\end{aligned}$$

Thus,

$$\begin{aligned}|F_1(m_k,0)| &\leqslant \sum_{j=0}^{k-1}\binom{k-1}{j}\frac{h^j}{j!}\left(\frac{h^{k-1-j}(1-m_k/h)^{k-1-j}}{(k-1-j)!}+O(h^{k-2-j})\right)\\ &= h^{k-1}\left(\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(1-m_k/h)^{k-1-j}}{j!(k-1-j)!}\right)+O(h^{k-2}).\end{aligned}$$

Since

$$\sum_{\substack{m_k\\ \frac{\beta_k}{\alpha_k}h\leqslant m\leqslant h}}(1-m_k/h)^{k-1-j} = h\int_{\beta_k/\alpha_k}^{1}(1-x)^{k-1-j}\,dx+O(1) = h\frac{(1-\beta_k/\alpha_k)^{k-j}}{k-j}+O(1),$$

we have

$$\sum_{\substack{m_k\\ \frac{\beta_k}{\alpha_k}h\leqslant m\leqslant h}}|F_1(m_k,0)| = h^k\left(\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(1-\beta_k/\alpha_k)^{k-j}}{j!(k-j)!}\right)+O(h^{k-1})$$

Line (8) gives us $\gamma_{k+1}\leqslant\alpha_{k+1}h^k+O(h^{k-1})$ provided that

$$\beta_k+\left(\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(1-\beta_k/\alpha_k)^{k-j}}{j!(k-j)!}\right)\leqslant\alpha_{k+1}. \tag{9}$$

Recall our earlier assumption that $\beta_k\leqslant\frac{1}{2}\alpha_k$, which (it turns out) has no slack for $k\geqslant 7$.

We proceed by choosing β_k to minimize this expression, giving us the smallest possible value of α_{k+1}.

$$\beta_4 \coloneqq 0.406671,\quad \beta_5\coloneqq 0.308672,\quad \beta_6\coloneqq 0.203975,\quad \beta_k\coloneqq\frac{1}{2}\alpha_k,\quad (k\geqslant 7).$$

We use β_k and α_k to compute α_{k+1}, and for $k\geqslant 6$ we then use α_{k+1} to compute β_{k+1}.

k	β_k	α_k
1		1
2		1
3		1
4	0.406671	1/2
5	0.308672	0.467214,
6	0.203975	0.382978,
7		0.269877,

Now set $\beta_k = \frac{1}{2}\alpha_k$ for $k \geqslant 7$. We have

$$\beta_k + \left(\sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(1-\beta_k/\alpha_k)^{k-j}}{j!(k-j)!}\right) = \frac{\alpha_k}{2} + \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{2^{-(k-j)}}{j!(k-j)!}$$
$$= \frac{\alpha_k}{2} + \frac{1}{2^k k!} \sum_{j=0}^{k-1} \binom{k-1}{j}\binom{k}{j} 2^j.$$

Thus, we can set for $k \geqslant 7$

$$\alpha_{k+1} = \frac{\alpha_k}{2} + \frac{1}{2^k k!} \sum_{j=0}^{k-1} \binom{k-1}{j}\binom{k}{j} 2^j.$$

We comment that

$$\alpha_{k+1} \leqslant \frac{\alpha_k}{2} + \frac{1}{2k \cdot k!} \sum_{j=0}^{k} \binom{k}{j}^2 = \frac{\alpha_k}{2} + \frac{1}{2k \cdot k!}\binom{2k}{k} \leqslant \frac{\alpha_k}{2} + \frac{4^k}{2k \cdot k!},$$

and so we have $(2-\epsilon)^k \alpha_k = o(1)$ for every $\epsilon > 0$.

7 Problems That Have Not Been Solved

The first problem is to find and prove formulas for γ_k for as many values of k as possible. A useful start to this would be a faster algorithm for computing $\mathcal{G}(h)$, or at least a faster implementation.

Lacking formulas for γ_k, we hope for upper bounds superior to that proved in Theorem 1, and for lower bounds applicable for $k \geqslant 6$ and superior to that in Theorem 2 and Lemma 3.

All of the known and conjectured formulas for γ_k are eventually quasi-polynomials in h: there is a modulus m and polynomials $p_1, \ldots, p_m$ and $\gamma_k(h) = p_{h \bmod m}(h)$ (for sufficiently large h). Can one show that all γ_k are quasi-polynomials? Moreover, all of the coefficients in all of the quasi-polynomials are nonnegative. No explanation is known.

Quasi-polynomials arise as Ehrhart polynomials. Is there a more concrete connection? Is there a region whose Ehrhart polynomial is γ_4 or γ_5?

In [10], it is noted that we can't even prove that $\gamma_k(h) < \gamma_k(h+1)$.

Are the elements of $\mathcal{G}(h)$ uniformly distributed in congruence classes? Experiments with the 25 000 known terms of $\mathcal{G}(2)$ show an unbalance modulo 27 (and even more modulo 221) that is unlikely to be coincidental.

Is there a solution to $33 = \gamma_k(2) - \gamma_\ell(2)$? There is a solution with $0 \leqslant \ell < k < 25\,000$ for every positive integer from 1 to 87 except 33.

In [17] it is shown that the maximum value of

$$\sum_{k=0}^{\infty} \frac{1}{a_k + 1}$$

over all B_2 sets $\mathcal{A} = \{0 = a_0 < a_1 < \dots\}$ is not obtained by $\mathcal{A} = \mathcal{G}(2)$. Is this also the case for $h \geqslant 3$? What is infimum of those s for which

$$\sum_{k=0}^{\infty} \frac{1}{(\gamma_k + 1)^s} = \sup_{\mathcal{A} \text{ a } B_h\text{-set}} \sum_{a \in \mathcal{A}} \frac{1}{(a+1)^s}.$$

References

[1] Wallace C. Babcock. Intermodulation interference in radio systems. *Bell System Technical Journal*, 32(1):63–73, 1953.

[2] R. C. Bose and S. Chowla. Theorems in the additive theory of numbers. *Comment. Math. Helv.*, 37:141–147, 1962.

[3] Javier Cilleruelo. A greedy algorithm for $B_h[g]$ sequences. *J. Combin. Theory Ser. A*, 150:323–327, 2017.

[4] P. Erdős and P. Turán. On a problem of Sidon in additive number theory, and on some related problems. *J. London Math. Soc.*, 16:212–215, 1941.

[5] Martin Gardner. Mathematical games. *Scientific American*, 226(3):108–113, March 1972.

[6] Ben Green. The number of squares and $B_h[g]$ sets. *Acta Arith.*, 100(4):365–390, 2001.

[7] J. Hinman, B. Kuca, A. Schlesinger, and A. Sheydvasser. The unreasonable rigidity of Ulam sequences. *J. Number Theory*, 194:409–425, 2019.

[8] Abdul Majid Mian and S. Chowla. On the B_2 sequences of Sidon. *Proc. Nat. Acad. Sci. India Sect. A*, 14:3–4, 1944.

[9] Melvyn B. Nathanson. The third positive element in the greedy B_h-set. *Palestine Journal of Mathematics*, 14:213–216, 2025.

[10] Melvyn B. Nathanson and Kevin O'Bryant. The fourth positive element in the greedy B_h-set. *J. Integer Seq.*, 27(7):Art. 24.7.3, 10, 2024.

[11] Kevin O'Bryant. A complete annotated bibliography of work related to Sidon sequences. *Electron. J. Combin.*, #DS11:39, 2004.

[12] Kevin O'Bryant. Constructing thick B_h-sets. *J. Integer Seq.*, 27(1):Paper No. 24.1.2, 17, 2024.

[13] S. Sidon. Ein Satz über trigonomietrische Polynome und seine Anwendungen in der Theorie der Fourier-Reihen. *Math. Annalen*, 106:536–539, 1932.

[14] James Singer. A theorem in finite projective geometry and some applications to number theory. *Trans. Amer. Math. Soc.*, 43(3):377–385, 1938.

[15] Alfred Stöhr. Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe. i, ii. *J. Reine Angew. Math.*, 194:40–65, 111–140, 1955.

[16] Paul Voutier. Personal communication, December 2023.

[17] Zhen Xiang Zhang. A B_2-sequence with larger reciprocal sum. *Math. Comp.*, 60(202):835–839, 1993. http://links.jstor.org/sici?sici=0025-5718%28199304%2960%3A202%3C835%3AAWLRS%3E2.0.CO%3B2-S.

A Appendix: Formulas for $\gamma_2, \gamma_3, \gamma_4$

A.1 A Formula for γ_2

We have $\gamma_0 = 0$ and $\gamma_1 = 1$, and we wish to find γ_2. By Lemma 8,

$$\gamma_2 = \operatorname{mex}\left\{\frac{1}{r}(m_1 - m_1') : 1 \leqslant r \leqslant h, \quad m_1, m_1' \in \mathbb{N}, \right.$$
$$\left. m_1 \leqslant h, \quad m_1' \leqslant h - r\right\} = \operatorname{mex}\{[-(h-1), h]\} = h + 1.$$

A.2 A Formula for γ_3

By Lemma 8,

$$\gamma_3 = \operatorname{mex}\left\{\frac{1}{r}\big((m_1 - m_1') + (h+1)(m_2 - m_2')\big) : 1 \leqslant r \leqslant h, \quad m_1, m_1', m_2, m_2' \in \mathbb{N} \right.$$
$$\left. m_1 + m_2 \leqslant h, \quad m_1' + m_2' \leqslant h - r, \quad m_1 m_1' = m_2 m_2' = 0\right\}.$$

If $m_2' > 0$, then $m_2 = 0$ and $m_1 \leqslant h$, and so

$$\frac{1}{r}\big((m_1 - m_1') + (h+1)(m_2 - m_2')\big) = \frac{1}{r}\Big((m_1 - m_1') - (h+1)m_2'\Big) \leqslant \frac{1}{r}\Big(h - (h+1)m_2'\Big) < 0.$$

Thus we can assume that $m_2' = 0$ and $m_1 \leqslant h - m_2$. We now have

$$\gamma_3 = \operatorname{mex}\left\{\frac{1}{r}\Big((m_1 - m_1') + (h+1)m_2\Big) : m_1 + m_2 \leqslant h, m_1' \leqslant h - r, m_1 m_1' = 0\right\}.$$

With $r = 1$, we have $-(h-1) \leqslant m_1 - m_1' \leqslant h - m_2$ and so

$$\big\{(m_1 - m_1') + (h+1)m_2 : m_1 + m_2 \leqslant h, \quad m_1' \leqslant h - r, \quad m_1 m_1' = 0\big\}$$
$$= \bigcup_{m_2=0}^{h}\Big((h+1)m_2 + [-(h-1), h - m_2]\Big). \quad (10)$$

But since the right endpoint of the interval $(h+1)m_2 + [-(h-1), h - m_2]$ is at least as large as 1 less than the left endpoint of the interval $(h+1)(m_2+1) + [-(h-1), h - (m_2+1)]$, the union in (10) is the interval $[1-h, h(h+1)]$.

If $r \geqslant 2$, then

$$\frac{1}{r}\big(m_1 - m_1' + (h+1)(m_2 + m_2')\big) \leqslant \frac{1}{2}\big(h(h+1)\big).$$

Thus, $\gamma_3 = \operatorname{mex}[0, h(h+1)] = h(h+1) + 1 = h^2 + h + 1$.

A.3 A Formula for γ_4

The formula for γ_4, proved in [10],is more involved, and the proof requires substantially better organization. The argument given here is not substantively different from that of [10], although the exposition is quite different.

Set

$$\Delta_r \coloneqq \left\{ \langle d_1, d_2, d_3 \rangle : d_i \in \mathbb{Z}, \quad \sum_{i=1}^{3} d_i \llbracket d_i > 0 \rrbracket \leqslant h, \quad \sum_{i=1}^{3} (-d_i) \llbracket d_i < 0 \rrbracket \leqslant h - r \right\}.$$

Then γ_4 is smallest nonnegative integer that is not equal to

$$\frac{1}{r}\, \delta \cdot \langle \gamma_1, \gamma_2, \gamma_3 \rangle$$

for any $\delta \in \Delta_r$ for any $r \in [1, h]$.

Define

$$\Delta \coloneqq \left\{ \langle m_1 - m_1', m_2 - m_2', m_3 \rangle : m_i, m_i' \in \mathbb{N}, \quad \sum_{i=1}^{3} m_i \leqslant h, \quad \sum_{i=1}^{2} m_i' \leqslant h - 1, \right.$$
$$\left. m_1 m_1' = m_2 m_2' = m_3 m_3' = 0, \quad m_2' + m_3 \leqslant h, \quad m_3 \leqslant \frac{h+1}{2} \right\},$$

which is clearly a subset of Δ_1. The cleverness in this proof, if there is any, is in the choice of the conditions $m_2' + m_3 \leqslant h, m_3 \leqslant \frac{h+1}{2}$, which we don't explain but are used below.

The values for m_3 that appear in Δ are $[0, \lfloor (h+1)/2 \rfloor]$. For each value of m_3, the quantity $m_2 - m_2'$ varies from $-(h - m_3)$ up to $h - m_3$. For each pair of values $m_2 - m_2', m_3$, the quantity $m_1 - m_1'$ varies through an interval (which depends on whether $m_2 - m_2'$ is nonnegative or negative).

We order the elements of Δ as follows. We define $\delta = \langle d_1, d_2, d_3 \rangle < \delta' = \langle d_1', d_2', d_3' \rangle$ if there is a j with $d_{3-i} = d_{3-i}'$ for $0 \leqslant i < j$ and $d_{3-j} < d_{3-j}'$. In this ordering, we define the *odometer* of Δ to be the numbering $\delta_0 \coloneqq \langle 0, 0, 0 \rangle < \delta_1 < \ldots$ of all elements of Δ that are at least δ_0. Let $\vec{\gamma} \coloneqq \langle \gamma_1, \gamma_2, \gamma_3 \rangle = \langle 1, h+1, h^2+h+1 \rangle$. The sequence $\delta_0 \cdot \vec{\gamma}, \delta_1 \cdot \vec{\gamma}, \delta_2 \cdot \vec{\gamma}, \ldots$ tends to increase, but need not do so monotonically. We call $\delta \cdot \vec{\gamma}$ the *image of* δ.

In Δ, the last component d_3 is never negative. Consecutive elements of the odometer have either first coordinate going up by 1, second coordinate going up by 1, or third coordinate going up by 1 (just like a car odometer, which always has some digit going up by 1, while less significant digits drop to their minimum possible values). Consecutive elements of the odometer have one of four shapes:

(i) $\langle d_1, d_2, d_3 \rangle < \langle d_1 + 1, d_2, d_3 \rangle$;
(ii) $d_2 < 0$ and $\langle d_1, d_2, d_3 \rangle = \langle h - m_3, -m_2', m_3 \rangle < \langle -h + m_2', -m_2' + 1, m_3 \rangle$;
(iii) $d_2 \geqslant 0$ and $\langle d_1, d_2, d_3 \rangle = \langle h - m_2 - m_3, m_2, m_3 \rangle < \langle -(h-1), m_2 + 1, m_3 \rangle$;
(iv) $\langle d_1, d_2, d_3 \rangle = \langle 0, h - m_3, m_3 \rangle < \langle -(h - m_3 - 1), -m_3, m_3 + 1 \rangle$.

The restriction $m_3 \leqslant (h+1)/2$ in Δ prevents a fifth shape.

In case (i), the image goes up by 1.

In case (ii), the image increases by

$$\big(\langle -h+m_2', -m_2'+1, m_3\rangle - \langle h-m_3, -m_2', m_3\rangle \big) \cdot \langle 1, h+1, \gamma_3\rangle = -h+m_2'+m_3+1 \leqslant 1,$$

where we have used the condition in the definition of Δ that $m_2'+m_3 \leqslant h$. While the "increase" could be negative (or zero), that is not a concern.

Case (iii) can only arise if $\langle -(h-1), m_2+1, m_3\rangle$ is in Δ, so that we have $m_3+m_2+1 \leqslant h$. The image increases by

$$\big(\langle -(h-1), m_2+1, m_3\rangle - \langle h-m_2-m_3, m_2, m_3\rangle \big) \cdot \langle 1, h+1, \gamma_3\rangle = -h+2+m_2+m_3 \leqslant 1.$$

Case (iv) only arises if $\langle -(h-m_3-1), -m_3, m_3+1\rangle$ is in Δ, so that we may assume that $m_3+1 \leqslant h$. Also, the condition $m_3 \leqslant (h+1)/2$ in Δ insures that

$$\langle -(h-m_3-1), -m_3, m_3+1\rangle$$

is in Δ. The image increases by

$$\begin{aligned}\big(\langle -(h-m_3-1), -m_3, m_3+1\rangle - \langle 0, h-m_3, m_3\rangle \big) \cdot \big\langle 1, h+1, h^2+h+1\big\rangle \\ = m_3+1-h+1 \leqslant 1.\end{aligned}$$

Thus, as we move through the odometer, the images start at 0 and increase by at most 1. As Δ contains $\big\langle 0, \lfloor\frac{h}{2}\rfloor, \lfloor\frac{h+1}{2}\rfloor\big\rangle$, which has image $\lfloor\frac{h}{2}\rfloor(h+1) + \lfloor\frac{h+1}{2}\rfloor(h^2+h+1)$, we have shown that the image of Δ contains the interval $[0, \lfloor\frac{h}{2}\rfloor(h+1) + \lfloor\frac{h+1}{2}\rfloor(h^2+h+1)]$. And since $\Delta \subseteq \Delta_1$, we have shown that the image of Δ_1 contains every natural number in the interval

$$\left[0, \left\lfloor\frac{h}{2}\right\rfloor(h+1) + \left\lfloor\frac{h+1}{2}\right\rfloor(h^2+h+1)\right]. \tag{11}$$

Moreover, Δ_1 also contains

$$\left\{ \left\langle d_1, -\left\lfloor\frac{h+1}{2}\right\rfloor, \left\lfloor\frac{h+3}{2}\right\rfloor \right\rangle : 1-h+\left\lfloor\frac{h+1}{2}\right\rfloor \leqslant d_1 \leqslant h - \left\lfloor\frac{h+3}{2}\right\rfloor \right\}$$

which has image

$$\left[\big(h^2+h+1\big)\left\lfloor\frac{h+3}{2}\right\rfloor - h\left\lfloor\frac{h+1}{2}\right\rfloor - h+1, h^2\left\lfloor\frac{h+3}{2}\right\rfloor + \left\lfloor\frac{3h}{2}\right\rfloor \right]. \tag{12}$$

The intervals in (11) and (12) contain every natural number up to and including $h^2\lfloor(h+3)/2\rfloor + \lfloor 3h/2\rfloor$, which proves that $\gamma_4 \geqslant h^2\lfloor(h+3)/2\rfloor + \lfloor 3h/2\rfloor + 1$.

Now suppose that $h^2\lfloor(h+3)/2\rfloor + \lfloor 3h/2\rfloor + 1$ is the image of $\langle d_1, d_2, d_3\rangle \in \Delta_1$:

$$d_1 + d_2(h+1) + d_3(h^2+h+1) = h^2\lfloor(h+3)/2\rfloor + \lfloor 3h/2\rfloor + 1. \tag{13}$$

If $d_3 \leqslant (h+1)/2$, then

$$d_1 + d_2(h+1) + d_3(h^2+h+1) \leqslant (h - \lfloor \tfrac{h+1}{2} \rfloor)(h+1) + \lfloor \tfrac{h+1}{2} \rfloor (h^2+h+1) \\ < h^2 \lfloor (h+3)/2 \rfloor + \lfloor 3h/2 \rfloor + 1.$$

If $d_3 \geqslant (h+4)/2$, then

$$d_1 + d_2(h+1) + d_3(h^2+h+1) \geqslant (1-h)(h+1) + \lceil \tfrac{h+4}{2} \rceil (h^2+h+1) \\ > h^2 \lfloor (h+3)/2 \rfloor + \lfloor 3h/2 \rfloor + 1.$$

Thus, $\frac{h+2}{2} \leqslant d_3 \leqslant \frac{h+3}{2}$, i.e., $d_3 = \lfloor (h+3)/2 \rfloor$. Equation (13) becomes

$$d_2(h+1) + d_1 + (h+1) \left\lfloor \frac{h+1}{2} \right\rfloor = \left\lfloor \frac{h}{2} \right\rfloor .$$

Reducing this modulo $h+1$ reveals that $d_1 = \lfloor h/2 \rfloor$ or $d_1 = \lfloor h/2 \rfloor - (h+1)$, but in the first case $d_1 + d_3 = \lfloor h/2 \rfloor + \lfloor (h+3)/2 \rfloor > h$, and so $\langle d_1, d_2, d_3 \rangle$ is not in Δ_1. In the second case, we find that $d_2 = 1 - \lfloor (h+1)/2 \rfloor$ and so $(-d_1) + (-d_2) = (h+1) - \lfloor h/2 \rfloor + \lfloor (h+1)/2 \rfloor - 1 > h-1$, and again $\langle d_1, d_2, d_3 \rangle \notin \Delta_1$. Thus no element of Δ_1 has $h^2 \lfloor (h+3)/2 \rfloor + \lfloor 3h/2 \rfloor + 1$ has an image.

The largest image of an element of Δ_r, for $r \geqslant 2$, is

$$\frac{1}{r} \langle 0, 0, h \rangle \cdot \langle \gamma_1, \gamma_2, \gamma_3 \rangle \leqslant \frac{1}{2} h \cdot (h^2+h+1) \leqslant h^2 \lfloor (h+3)/2 \rfloor + \lfloor 3h/2 \rfloor.$$

Thus, $h^2 \lfloor (h+3)/2 \rfloor + \lfloor 3h/2 \rfloor + 1$ is not $\frac{1}{r}\delta \cdot \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ for any r and any $\delta \in \Delta_r$. Thus, $\gamma_4 = h^2 \lfloor (h+3)/2 \rfloor + \lfloor 3h/2 \rfloor + 1$.

B The Details in the Proof that $\gamma_5 \geqslant \frac{1}{8}h^4 + \frac{1}{2}h^3$

We now handle the four possibilities one at a time.

(i) $d_1 < u_1(d_4, d_3, d_2)$, $\nu = \langle d_4, d_3, d_2, d_1 + 1 \rangle$.

We have $(\nu - \delta) \cdot \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle = \gamma_1 = 1$, This ends the easiest case.

(ii) $d_1 = u_1(d_4, d_3, d_2), d_2 < u_2(d_4, d_3)$,
$\nu = \langle \ell_1(d_4, d_3, d_2+1), d_2+1, d_3, d_4 \rangle$;

We first note that since $\delta \in \Delta$, we know that $d_2 \geqslant \ell_2(d_4, d_3)$, and since $\nu \in \Delta$, we know that $d_2 + 1 \leqslant u_2(d_4, d_3)$. Combined, we have

$$d_4 + |d_3| - h \leqslant d_2 \leqslant h - d_4 - |d_3| - 1. \tag{14}$$

We have

$$(\nu - \delta) \cdot \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle = \big(\ell_1(d_4, d_3, d_2+1) - u_1(d_4, d_3, d_2)\big)\gamma_1 + \gamma_2 \\ = \ell_1(d_4, d_3, d_2+1) - u_1(d_4, d_3, d_2) + h + 1.$$

As $d_4 \geqslant 1$, we have just 4 sub-cases to consider based on the sign of d_2 and d_3.

d_3	d_2	$\ell_1(d_4, d_3, d_2 + 1)$	$u_1(d_4, d_3, d_2)$
negative	negative	$1 - h - d_3 - (d_2 + 1)$	$h - d_4$
negative	nonnegative	$1 - h - d_3$	$h - d_4 - d_2$
nonnegative	negative	$1 - h - (d_2 + 1)$	$h - d_4 - d_3$
nonnegative	nonnegative	$1 - h$	$h - d_4 - d_3 - d_2$

d_3	d_2	$\ell_1(d_4, d_3, d_2 + 1) - u_1(d_4, d_3, d_2) + h + 1$
negative	negative	$1 - h + d_4 - d_3 - d_2$
negative	nonnegative	$2 - h + d_4 - d_3 + d_2$
nonnegative	negative	$1 - h + d_4 + d_3 - d_2$
nonnegative	nonnegative	$2 - h + d_4 + d_3 + d_2$

If $d_2 < 0$, then, we have

$$\ell_1(d_4, d_3, d_2 + 1) - u_1(d_4, d_3, d_2) + h + 1 \leqslant 1 - h + d_4 + |d_3| - d_2.$$

From line (14), we have $-d_2 \leqslant h - d_4 - |d_3|$ and so

$$1 - h + d_4 + |d_3| - d_2 \leqslant 1 - h + d_4 + |d_3| + h - d_4 - |d_3| = 1.$$

Similarly, if $d_2 \geqslant 0$ we have

$$\begin{aligned}\ell_1(d_4, d_3, d_2 + 1) - u_1(d_4, d_3, d_2) + h + 1 &\leqslant 2 - h + d_4 + |d_3| + d_2 \\ &\leqslant 2 - h + d_4 + |d_3| + (h - 1 - d_4 - |d_3|) = 1.\end{aligned}$$

Thus, in all 4 sub-cases the image increments by at most 1, and case (*ii*) is ended.

(*iii*) $d_1 = u_1(d_4, d_3, d_2), d_2 = u_2(d_4, d_3), d_3 < u_3(d_4)$,
$\nu = \langle \ell_1(d_4, d_3 + 1, \ell_2(d_4, d_3 + 1)), \ell_2(d_4, d_3 + 1), d_3 + 1, d_4\rangle$;

We have

$$\begin{aligned}\delta &= \langle u_1(d_4, d_3, d_2), h - d_4 - |d_3|, d_3, d_4\rangle \\ \nu &= \langle \ell_1(d_4, d_3 + 1, d_4 + |d_3 + 1| - h), d_4 + |d_3 + 1| - h, d_3 + 1, d_4\rangle \\ \nu - \delta &= \left\langle \ell_1(d_4, d_3 + 1, d_4 + |d_3 + 1| - h) - u_1(d_4, d_3, d_2), 2d_4 + 2\left|d_3 + \tfrac{1}{2}\right| - 2h, 1, 0\right\rangle\end{aligned}$$

The image increments by

$$\ell_1(d_4, d_3 + 1, d_4 + |d_3 + 1| - h) - u_1(d_4, d_3, d_2) + (2d_4 + 2\left|d_3 + \tfrac{1}{2}\right| - h)(h + 1) + 1 \quad (15)$$

Moreover, from $\delta \in \Delta$ we know that $d_3 \geqslant \ell_3(d_4)$, and from $\nu \in \Delta$ we know that $d_3 + 1 \leqslant u_3(d_4)$. Thus,

$$d_4 - \left\lfloor \frac{h+1}{2} \right\rfloor \leqslant d_3 \leqslant \left\lfloor \frac{h+1}{2} \right\rfloor - d_4 - 1.$$

We have two subcases depending on whether d_3 is negative or nonnegative. Suppose first that $d_3 < 0$. We have

$$\begin{aligned} d_2 = u_2(d_4, d_3) = h - d_4 + d_3 > 0 \\ |d_3 + 1| = -d_3 - 1 \\ u_2 = d_4 + |d_3 + 1| - h \leqslant d_4 + \left\lfloor \frac{h+1}{2} \right\rfloor - d_4 - 1 - h = -\left\lfloor \frac{h+2}{2} \right\rfloor < 0 \\ \ell_1(d_4, d_3 + 1, u_2) = 1 - h - (d_3 + 1) - (d_4 - d_3 - 1 - h) = 1 - d_4 \\ u_1(d_4, d_3, d_2) = h - d_4 - d_2 = h - d_4 - (h - d_4 + d_3) = -d_3. \end{aligned}$$

The image increments, substituting into (15), by

$$2 - d_4 + d_3 + (2d_4 + 2\left|d_3 + \tfrac{1}{2}\right| - h)(h + 1).$$

Since $d_3 \leqslant -1$, we have $2|d_3 + 1/2| = -2d_3 - 1$, giving an increment of

$$\begin{aligned} 2 - d_4 + d_3 &+ (2d_4 - 2d_3 - 1 - h)(h + 1) \\ &\leqslant 2 - d_4 + \left\lfloor \frac{h+1}{2} \right\rfloor - d_4 - 1 + \left(2d_4 - 2d_4 - 2\left\lfloor \frac{h+1}{2} \right\rfloor - 1 - h\right)(h + 1) \\ &\leqslant 1 + \left\lfloor \frac{h+1}{2} \right\rfloor - 2 - \left(2\left\lfloor \frac{h+1}{2} \right\rfloor + 1 + h\right)(h + 1) \\ &\leqslant \frac{h}{2} - (2h + 1)(h + 1), \end{aligned}$$

which is negative, for all $h \geqslant 0$.

Now suppose that $d_3 \geqslant 0$.We have

$$\begin{aligned} d_2 = u_2(d_4, d_3) = h - d_4 - d_3 > 0 \\ |d_3 + 1| = d_3 + 1 \leqslant \left\lfloor \frac{h+1}{2} \right\rfloor - d_4 \\ u_2 = d_4 + |d_3 + 1| - h \leqslant \left\lfloor \frac{h+1}{2} \right\rfloor - h < 0 \\ \ell_1(d_4, d_3 + 1, u_2) = 1 - h - u_2 = -d_4 - d_3 \\ u_1(d_4, d_3, d_2) = h - d_4 - d_3 - d_2. \end{aligned}$$

The image increments, substituting into (15), by

$$\begin{aligned} 1 - d_4 - d_3 &- (h - d_4 - d_3 - d_2) + (2d_4 + 2d_3 + 1 - h)(h + 1) + 1 \\ &= 2 - h + d_2 + (2d_4 + 2d_3 + 1 - h)(h + 1) \\ &\leqslant 2 - h + h - d_4 - d_3 + \left(2d_4 + 2(\left\lfloor \tfrac{h+1}{2} \right\rfloor - d_4 - 1) + 1 - h\right)(h + 1) \\ &= 2 - d_4 - d_3 + \left(2\left\lfloor \frac{h+1}{2} \right\rfloor - 1 - h\right)(h + 1) \\ &\leqslant 1 + 0 \cdot (h + 1) \leqslant 1. \end{aligned}$$

Thus, in both sub-cases, the increment is at most 1, and this ends case (*iii*).

(*iv*) $d_1 = u_1(d_4, d_3, d_2), d_2 = u_2(d_4, d_3), d_3 = u_3(d_4), d_4 < u_4$,
$\nu = \langle \ell_1(d_4 + 1, \ell_3(d_4 + 1), L), L, \ell_3(d_4 + 1), d_4 + 1\rangle$, where $L \coloneqq \ell_2(d_4+1, \ell_3(d_4+1))$.

As $d_4 < u_4$, we have $d_4 \leqslant \lfloor \frac{h-1}{4} \rfloor$. Also, $d_3 = u_3(d_4) = \lfloor \frac{h+1}{2} \rfloor - d_4$, $d_2 = h - \lfloor \frac{h+1}{2} \rfloor$ and $d_1 = 0$.

Thus,

$$\begin{aligned} \nu &= \left\langle -d_4, -\left\lfloor \frac{h}{2} \right\rfloor, d_4 - \left\lfloor \frac{h-1}{2} \right\rfloor, d_4 + 1 \right\rangle \\ \delta &= \left\langle 0, \left\lfloor \frac{h}{2} \right\rfloor, \left\lfloor \frac{h+1}{2} \right\rfloor - d_4, d_4 \right\rangle \\ \nu - \delta &= \left\langle -d_4, -2\left\lfloor \frac{h}{2} \right\rfloor, 2d_4 - 2\left\lfloor \frac{h+1}{2} \right\rfloor + 1, 1 \right\rangle. \end{aligned}$$

The increment is then

$$-d_4 - 2(h+1)\left\lfloor \frac{h}{2} \right\rfloor + \left(2d_4 + 1 - 2\left\lfloor \frac{h+1}{2} \right\rfloor\right)(h^2 + h + 1) + \left\lfloor \frac{h+3}{2} \right\rfloor h^2 + \left\lfloor \frac{3h+2}{2} \right\rfloor,$$

which is more profitably written as

$$d_4\left(2h^2 + 2h + 1\right) - 2(h+1)\left\lfloor \frac{h}{2} \right\rfloor + \left(1 - 2\left\lfloor \frac{h+1}{2} \right\rfloor\right)(h^2+h+1) + \left\lfloor \frac{h+3}{2} \right\rfloor h^2 + \left\lfloor \frac{3h+2}{2} \right\rfloor.$$

We can now use $d_4 \leqslant \lfloor \frac{h-1}{4} \rfloor$. Splitting into 4 sub-cases depending the residue of h modulo 4, we have the routine problem (for each sub-case) of solving a *quadratic* inequality.