Rank functions and invariants of delta-matroids

Matt Larson

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Abstract

In this note, we give a rank function axiomatization for delta-matroids and study the corresponding rank generating function. We relate an evaluation of the rank generating function to the number of independent sets of the delta-matroid, and we prove a log-concavity result for that evaluation using the theory of Lorentzian polynomials.

Mathematics Subject Classifications: 52B40, 05A20

1 Introduction

Let $[n, \overline{n}]$ denote the set $\{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$, equipped with the obvious involution $\overline{(\cdot)}$. Let AdS_n be the set of *admissible subsets* of $[n, \overline{n}]$, i.e., subsets S that contain at most one of i and $\overline{i}$ for each $i \in [n]$. These are also called *partial transversals*. Set $e_{\overline{i}} := -e_i \in \mathbb{R}^n$, and for each $S \in \mathrm{AdS}_n$, set $e_S = \sum_{a \in S} e_a$.

Definition 1. A *delta-matroid* D is a non-empty collection $\mathcal{F} \subset \mathrm{AdS}_n$ of admissible sets of size n, called the *feasible sets* of D, such that the polytope

$$P(D) := \mathrm{Conv}\{e_B : B \in \mathcal{F}\}$$

has all edges parallel to e_i or $e_i \pm e_j$, for some i, j. We say that D is *even* if all edges of $P(D)$ are parallel to $e_i \pm e_j$.

Delta-matroids were introduced in [8] by replacing the usual basis exchange axiom for matroids with one involving symmetric difference. In [8], the above definition is called a *symmetric matroid*. They were defined independently in [15, 18]. For the equivalence of the definition of delta-matroids in those works with the one given above, and for general properties of delta-matroids, see [7, Chapter 4].

A delta-matroid is even if and only if all sets in $\{B \cap [n] : B \in \mathcal{F}\}$ have the same parity. Even delta-matroids enjoy nicer properties than arbitrary delta-matroids. For instance, they satisfy a version of the symmetric exchange axiom [33].

Institute for Advanced Study and Princeton University (mattlarson@princeton.edu).

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There are many constructions of delta-matroids in the literature. Two of the most fundamental come from matroids: given a matroid M on $[n]$, we can construct a delta-matroid on $[n, \overline{n}]$ whose feasible sets are the sets of the form $B \cup \overline{B^c}$, for B a basis of M. We can also construct a delta-matroid whose feasible sets are the sets of the form $I \cup \overline{I^c}$, for I independent in M. Additionally, there are delta-matroids corresponding to graphs [19], graphs embedded in surfaces [16, 17], and points of a maximal orthogonal or symplectic Grassmannian. Delta-matroids arising from points of a maximal orthogonal or symplectic Grassmannian are called *realizable*. See [22, Section 6.2] for a discussion of delta-matroids associated to points of a maximal orthogonal Grassmannian.

Given $S, T \in \mathrm{AdS}_n$, we define $S \sqcup T = \{a \in S \cup T : \overline{a} \notin S \cup T\}$. A function $f\colon \mathrm{AdS}_n \to \mathbb{R}$ is called *bisubmodular* if, for all $S, T \in \mathrm{AdS}_n$,

$$f(S) + f(T) \geqslant f(S \cap T) + f(S \sqcup T).$$

There is a large literature on bisubmodular functions, beginning with [20]. They have been studied both from an optimization perspective [25, 24] and from a polytopal perspective [26, 23]. Additionally, bisubmodular functions are closely related to jump systems [12].

For a delta-matroid D, define a function $g_D\colon \mathrm{AdS}_n \to \mathbb{Z}$ by

$$g_D(S) = \max_{B \in \mathcal{F}} (|S \cap B| - |\overline{S} \cap B|).$$

We call g_D the *rank function* of D. Note that g_D may take negative values. The collection of feasible subsets of D is exactly $\{S : g_D(S) = n\}$, so D can be recovered from g_D.

Theorem 2. *A function $g\colon \mathrm{AdS}_n \to \mathbb{Z}$ is the rank function of a delta-matroid if and only if*

1. $g(\emptyset) = 0$ *(normalization),*
2. $|g(S)| \leqslant 1$ *if* $|S| = 1$ *(boundedness),*
3. $g(S) + g(T) \geqslant g(S \cap T) + g(S \sqcup T)$ *(bisubmodularity), and*
4. $g(S) \equiv |S| \pmod 2$ *(parity).*

Furthermore, D is even if and only if

$$g_D(S) = \frac{g_D(S \cup i) + g_D(S \cup \overline{i})}{2} \text{ whenever } |S| = n - 1 \text{ and } \{i, \overline{i}\} \cap S = \emptyset.$$

The function g_D, as well as the observation that it is bisubmodular, has appeared before in the literature [9, 15]. For example, in [9, Theorem 4.1] it is shown that, if D is represented by a point of the maximal symplectic Grassmannian, then g_D can be computed in terms of the rank of a certain matrix. It was known that delta-matroids admit a description in terms of certain bisubmodular functions. Theorem 2 answers a special case of [2, Question 9.4].

In [10, 11], Bouchet gave a rank-function axiomatization of delta-matroids in the more general setting of multimatroids. His rank function differs from ours — in Section 2.2, we discuss the relationship between his results and Theorem 2. An axiomatization of delta-matroids in terms of Bouchet's rank function was given in an unpublished paper of Allys. One can deduce Theorem 2 from this result, see Corollary 15.

Basic operations on delta-matroids — like products, deletion, contraction, and projection — can be simply expressed in terms of rank functions. See Section 2.1.

One of the most important invariants of a matroid M of rank r on $[n]$ is its *Whitney rank generating function.* If rk_M is the rank function of M, then the rank generating function is defined as

$$R_M(u,v) := \sum_{A \subset [n]} u^{r-\mathrm{rk}_M(A)} v^{|A|-\mathrm{rk}_M(A)}.$$

The more commonly used normalization is the *Tutte polynomial*, which is $R_M(u-1, v-1)$. The characterization of delta-matroids in terms of rank functions allows us to consider an analogously-defined invariant.

Definition 3. Let D be a delta-matroid on $[n, \overline{n}]$. Then we define

$$U_D(u,v) = \sum_{S \in \mathrm{AdS}_n} u^{n-|S|} v^{\frac{|S|-g_D(S)}{2}}.$$

Note that the bisubmodularity of g_D implies that the restriction of g_D to the subsets of any fixed $S \in \mathrm{AdS}_n$ is submodular. The boundedness of g_D then implies that $|g_D(S)| \leqslant |S|$. Because of the parity requirement, $|S| - g_D(S)$ is divisible by 2. Therefore $U_D(u,v)$ is indeed a polynomial. The normalization $U_D(u-1, v-1)$ is more analogous to the Tutte polynomial, but it can have negative coefficients. However, the polynomial $U_D(u, v-1)$ has non-negative coefficients (as follows, e.g., from Theorem 24).

The U-polynomial of a delta-matroid was introduced by Eur, Fink, Spink, and the author in [22, Definition 1.4] in terms of a Tutte polynomial-like recursion; see Proposition 17 for a proof that Definition 3 agrees with the recursive definition considered there. The specialization $U_D(0, v)$ is the *interlace polynomial* of D, which was introduced in [3] for graphs and in [14] for general delta-matroids. See [30] for a survey on the properties of the interlace polynomial.

Various Tutte polynomial-like invariants of delta-matroids have been considered in the literature, such as the Bollobás–Riordan polynomial and its specializations [6]. In [28], a detailed analysis of delta-matroid polynomials which satisfy a deletion-contraction formula is carried out. Set $\sigma_D(A) = \frac{|A|}{2} + \frac{g_D(A)+g_D(\bar{A})}{4}$ for $A \subset [n]$. Then in [28], the polynomial

$$\sum_{A \subset [n]} (x-1)^{\sigma_D([n])-\sigma_D(A)} (y-1)^{|A|-\sigma_D(A)}$$

is shown to be, in an appropriate sense, the universal invariant of delta-matroids which satisfies a deletion-contraction formula. This polynomial is a specialization of the Bollobás–Riordan polynomial. In [21], it is shown that this polynomial has several nice combinatorial properties. This polynomial does not specialize to $U_D(u, v)$.

Example 4. [22, Example 5.5 and 5.6] Let M be a matroid of rank r on $[n]$, and let $S = S^+ \cup \overline{S^-} \in \mathrm{AdS}_n$ be an admissible set with $S^+, S^- \subset [n]$. Set $V = \{i \in [n] : S \cap \{i, \bar{i}\} = \emptyset\}$. Previously, we gave two examples of delta-matroids constructed from M. We now discuss their U-polynomials.

1. Let D be the delta-matroid arising from the independent sets of M. Then $g_D(S) = |S| + 2\,\mathrm{rk}_M(S^+) - 2|S^+|$, and

$$U_D(u,v) = (u+1)^{n-r} R_M\left(u+3, \frac{2u+v+2}{u+1}\right).$$

2. Let D be the delta-matroid arising from the bases of M. Then $g_D(S) = |S| - 2r + 2\,\mathrm{rk}_M(S^+ \cup V) - 2|S^+| + 2\,\mathrm{rk}_M(S^+)$, and

$$U_D(u,v) = \sum_{T \subset S \subset [n]} u^{|S \setminus T|} v^{r - \mathrm{rk}_M(S) + |T| - \mathrm{rk}_M(T)}.$$

We study the U-polynomial as a delta-matroid analogue of the rank generating function of a matroid. For a matroid M, the evaluation $R_M(u,0)$ is essentially the f-vector of the independence complex of the matroid, i.e., it counts the number of independent sets of M of a given size. The coefficients of the Tutte polynomial $R_M(u-1, v-1)$ can be interpreted as counting bases of M according to their internal and external activities, certain statistics that depend on an ordering of the ground set. See [4]. This shows that $R_M(u,-1)$, the (unsigned) characteristic polynomial of M, is essentially the f-vector of the broken circuit complex of M.

A set $S \in \mathrm{AdS}_n$ is *independent* if it is contained in a feasible set of a delta-matroid D. In [10], Bouchet gave an axiomatization of delta-matroids in terms of their independent sets. The independent sets form a simplicial complex, called the *independence complex* of D. We relate $U_D(u,0)$ to the f-vector of the independence complex of D (Proposition 20), which gives linear inequalities between the coefficients of $U_D(u,0)$. We give a combinatorial interpretation of the coefficients of $U_D(u, v-1)$ as counting the number of independent sets of D of a given size according to a delta-matroid version of activity (Theorem 24) which was introduce by Morse [31]. This shows that $U_D(u,-1)$ is essentially the f-vector of a certain simplicial complex associated to D.

Following a tradition in matroid theory (see, e.g., [29]), and inspired by the ultra log-concavity of $R_M(u,0)$ [1,13], we make three log-concavity conjectures for $U_D(u,0)$. These conjectures state the sequence of the number of independent sets of a delta-matroid of a given size satisfies log-concavity properties.

Conjecture 5. Let D be a delta-matroid on $[n, \overline{n}]$, and let $U_D(u,0) = a_n + a_{n-1}u + \cdots + a_0 u^n$. Then, for any $k \in \{1, \ldots, n-1\}$,

1. $a_k^2 \geqslant \frac{n-k+1}{n-k} a_{k+1} a_{k-1}$,

2. $a_k^2 \geqslant \frac{2n-k+1}{2n-k} \frac{k+1}{k} a_{k+1} a_{k-1}$, and

3. $a_k^2 \geqslant \frac{n-k+1}{n-k}\frac{k+1}{k} a_{k+1}a_{k-1}$.

Conjecture 5(1) follows from [22, Conjecture 1.5], and it is proven in [22, Theorem B] when D has an *enveloping matroid* (see Definition 27). This is a technical condition which is satisfied by many commonly occurring delta-matroids, including all realizable delta-matroids and delta-matroids arising from matroids (although not all delta-matroids, see [10, Section 4] and [22, Example 6.11]). The proof uses algebro-geometric methods. Here we prove a special case of Conjecture 5(2). Note that Conjecture 5(1) and Conjecture 5(2) are incomparable, and both are implied by Conjecture 5(3). Equality is attained in Conjecture 5(3) if D has a unique feasible set or if all admissible sets are independent.

Theorem 6. *Let D be a delta-matroid on $[n, \overline{n}]$ which has an enveloping matroid. Let $U_D(u, 0) = a_n + a_{n-1}u + \cdots + a_0 u^n$. Then, for any $k \in \{1, \ldots, n-1\}$ Conjecture 5(2) holds, i.e.,*

$$a_k^2 \geqslant \frac{2n-k+1}{2n-k}\frac{k+1}{k} a_{k+1}a_{k-1}.$$

Our argument uses the theory of Lorentzian polynomials [13]. We strengthen Theorem 6 by proving that a generating function for the independent sets of D is Lorentzian (Theorem 31), which implies the desired log-concavity statement. We deduce that this generating function is Lorentzian from the fact that the Potts model partition function of an enveloping matroid is Lorentzian [13, Theorem 4.10].

When D is the delta-matroid arising from the independent sets of a matroid, Conjecture 5(3) follows from the ultra log-concavity of the number of independent sets of that matroid [1, 13]. When D is the delta-matroid arising from the bases of a matroid M on $[n]$, which has an enveloping matroid by [22, Proposition 6.10], Theorem 6 gives a new log-concavity result. If we set

$$a_k = |\{T \subset S \subset [n] : T \text{ independent in } M \text{ and } S \text{ spanning in } M,\ |S \setminus T| = n-k\}|,$$

then Theorem 6 gives that $a_k^2 \geqslant \frac{2n-k+1}{2n-k}\frac{k+1}{k} a_{k+1}a_{k-1}$ for $k \in \{1, \ldots, n-1\}$.

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2 Rank functions of delta-matroids

The proof of Theorem 2 goes by way of a polytopal description of normalized bisubmodular functions, which we now recall. To a function $f \colon \mathrm{AdS}_n \to \mathbb{R}$ with $f(\emptyset) = 0$, we associate the polytope

$$P(f) = \{x : \langle e_S, x \rangle \leqslant f(S) \text{ for all non-empty } S \in \mathrm{AdS}_n\}.$$

By [12, Theorem 4.5] (or [2, Theorem 5.2]), $P(f)$ has all edges parallel to e_i or $e_i \pm e_j$ if and only if f is bisubmodular. In this case, $P(f)$ is a lattice polytope if and only if f is integer-valued. For a normalized (i.e., $f(\emptyset)=0$) bisubmodular function f, we can recover f from $P(f)$ via the formula

$$f(S) = \max_{x\in P(f)} \langle e_S, x\rangle.$$

Under this dictionary, the bisubmodular function corresponding to the dilate $kP(f)$ is kf, and the bisubmodular function corresponding to the Minkowski sum $P(f)+P(g)$ is $f+g$.

Proof of Theorem 2. By the polyhedral description of normalized bisubmodular functions, for each delta-matroid D there is a unique normalized bisubmodular function g such that $P(D)=P(g)$. We show that the conditions on a normalized bisubmodular function g for $P(g)$ to have all vertices in $\{-1,1\}^n$ are exactly those given in Theorem 2, namely that $|g(S)|\leqslant 1$ when $|S|=1$ and $g(S)\equiv |S| \pmod 2$.

The polytope $P(g)$ has all vertices in $\{\pm 1\}^n$ if and only if $\frac{1}{2}(P(g)+(1,\ldots,1))$ is a lattice polytope which is contained in $[0,1]^n$. The normalized bisubmodular function h corresponding to the point $(1,\ldots,1)$ takes value $h(S)=|S^+|-|S^-|$ on an admissible set of the form $S=S^+\cup\overline{S^-}$, with $S^+,S^-\subset[n]$. The polytope $\frac{1}{2}(P(g)+(1,\ldots,1))$ is $P(f)$, where f is the normalized bisubmodular function defined by $f:=\frac{1}{2}(g+h)$. We note that $P(f)$ is a lattice polytope which is contained in $[0,1]^n$ if and only if

1. $f(i)\in\{0,1\}$ and $f(\bar{i})\in\{-1,0\}$, and
2. f is integer-valued.

A normalized bisubmodular function f satisfies these conditions if and only if g satisfies the conditions of Theorem 2, giving the characterization of rank functions of delta-matroids.

By [2, Example 5.2.3], the polytope $P(g_D)=P(D)$ has all edges parallel to $e_i\pm e_j$ if and only if g_D satisfies the condition

$$g_D(S) = \frac{g_D(S\cup i)+g_D(S\cup\bar{i})}{2} \text{ whenever } |S|=n-1 \text{ and } \{i,\bar{i}\}\cap S=\emptyset.$$

This gives the characterization of even delta-matroids. □

2.1 Compatibility with delta-matroid operations

In this section, we consider several operations on delta-matroids, and we show that the rank function behaves in a simple way under these operations. First we consider minor operations on delta-matroids — contraction, deletion, and projection.

Definition 7. Let D be a delta-matroid on $[n,\overline{n}]$ with feasible sets $\mathcal{F}$, and let $i\in[n]$. We say that i is a *loop* of D if no feasible set contains i, and we say that i is a *coloop* if every feasible set contains i.

1. If i is not a loop of D, then the *contraction* D/i is the delta-matroid with feasible sets $B\setminus i$, for $B\in\mathcal{F}$ containing i.

2. If i is not a coloop of D, then the *deletion* $D \setminus i$ is the delta-matroid with feasible sets $B \setminus \bar{i}$, for $B \in \mathcal{F}$ containing $\bar{i}$.

3. The *projection* $D(i)$ is the delta-matroid with feasible sets $B \setminus \{i, \bar{i}\}$ for $B \in \mathcal{F}$.

4. If i is a loop or coloop, then set $D/i = D \setminus i = D(i)$.

For $A \subset [n]$, we define $D/A, D \setminus A$, and $D(A)$ to be the delta-matroids on $[n, \bar{n}] \setminus (A \cup \bar{A})$ obtained by successively contracting, deleting, or projecting away from all elements of A. Contractions, deletions, and projections at disjoint sets commute with each other, so this is well defined. If A and B are disjoint subsets of $[n]$, then $D/A \setminus B$ is the delta-matroid obtained by contracting A and then deleting B, which is the same as first deleting B and then contracting A.

First we describe the rank function of projections. The formula is analogous to the formula for the rank function of a matroid deletion.

Proposition 8. *Let D be a delta-matroid on $[n, \overline{n}]$, and let $A \subset [n]$. For each $S \in \mathrm{AdS}_n$ disjoint from $A \cup \overline{A}$, $g_{D(A)}(S) = g_D(S)$.*

Proof. As S is disjoint from $A \cup \overline{A}$, $|B \cap S| - |B \cap \overline{S}|$ depends only on $B \setminus (A \cup \overline{A})$. The feasible sets of $D(A)$ are given by $B \setminus (A \cup \overline{A})$ for B a feasible set of D. □

The rank functions of the contractions and deletions are described by the following result. The formula is analogous to the formula for the rank function of a matroid contraction.

Proposition 9. *Let D be a delta-matroid on $[n, \overline{n}]$. Let $A, B \subset [n]$ be disjoint subsets, and let $S \in \mathrm{AdS}_n$ be disjoint from $A \cup B \cup \overline{A} \cup \overline{B}$. Then $g_{D/A \setminus B}(S) = g_D(S \cup A \cup \overline{B}) - g_D(A \cup \overline{B})$.*

Before proving this, we will need the following property of delta-matroids. It follows, for instance, from the greedy algorithm description of delta-matroids in [12].

Proposition 10. *Let D be a delta-matroid on $[n, \overline{n}]$, and let $S \subset T \in \mathrm{AdS}_n$. Let $\mathcal{F}_S$ be the collection of feasible sets B of D that maximize $|S \cap B|$, i.e., have $|S \cap B| = \max_{B' \in \mathcal{F}} |S \cap B'|$. Then*

$$\max_{B \in \mathcal{F}_S} |T \cap B| = \max_{B \in \mathcal{F}} |T \cap B|.$$

First we consider the case when we delete or contract a single element.

Lemma 11. *Let D be a delta-matroid on $[n, \overline{n}]$, and let $i \in [n]$. Then*

1. *If i is not a loop, then $g_{D/i}(S) = g_D(S \cup i) - 1$,*

2. *If i is not a coloop, then $g_{D \setminus i}(S) = g_D(S \cup \bar{i}) - 1$.*

Proof. We do the case of contraction; the case of deletion is identical. Assume that i is not a loop, and let $\mathcal{F}_i$ denote the set of feasible sets of D which contain i. Note that $\mathcal{F}_i$ is non-empty, so it is the collection of feasible sets B of D which maximize $|\{i\} \cap B|$. For any $S \in \mathrm{AdS}_n$ with $S \cap \{i, \bar{i}\} = \emptyset$, by Proposition 10 we have that

$$\max_{B \in \mathcal{F}} |(S \cup i) \cap B| = \max_{B \in \mathcal{F}_i} |(S \cup i) \cap B|.$$

For any B, $|(S \cup i) \cap B| - |\overline{(S \cup i)} \cap B| = 2|(S \cup i) \cap B| - |S \cup i|$, so we see that

$$\max_{B \in \mathcal{F}} (|(S \cup i) \cap B| - |\overline{(S \cup i)} \cap B|) = \max_{B \in \mathcal{F}_i} (|(S \cup i) \cap B| - |\overline{(S \cup i)} \cap B|).$$

The left-hand side is equal to $g_D(S \cup i)$, and the right-hand side is equal to $g_{D/i}(S) + 1$. $\square$

Proof of Proposition 9. First note that $g_D(i) = 1$ if i is not a loop and is -1 if i is a loop, and similarly $g_D(\bar{i}) = 1$ if i is not a coloop and is -1 is i is a coloop. So Lemma 11 implies the result holds when $|A \cup B| = 1$.

We induct on the size of $A \cup B$. We consider the case of adding an element $i \in [n]$ to A; the case of adding it to B is identical. We compute:

$$\begin{aligned} g_{D/(A \cup i) \setminus B}(S) &= g_{D/A \setminus B}(S \cup i) - g_{D/A \setminus B}(i) \\ &= g_D(S \cup A \cup \overline{B} \cup i) - g_D(A \cup \overline{B}) - (g_D(A \cup \overline{B} \cup i) - g_D(A \cup \overline{B})) \\ &= g_D(S \cup (A \cup i) \cup \overline{B}) - g_D((A \cup i) \cup \overline{B}). \end{aligned}$$

$\square$

For two non-negative integers n_1, n_2, identify the disjoint union of $[n_1]$ and $[n_2]$ with $[n_1 + n_2]$. Given two delta-matroids D_1, D_2 on $[n_1]$ and $[n_2]$, let $D_1 \times D_2$ be the delta-matroid on $[n_1 + n_2]$ whose feasible sets are $B_1 \cup B_2$, for B_j a feasible set of D_j. Then we have the following description of the rank function of $D_1 \times D_2$.

Proposition 12. *Let D_1, D_2 be delta-matroids on $[n_1]$ and $[n_2]$, and let $S = S_1 \cup S_2$ be an admissible subset of $[n_1 + n_2, \overline{n_1 + n_2}]$, with $S_1 \subset [n_1, \overline{n}_1]$ and $S_2 \subset [n_2, \overline{n}_2]$. Then $g_{D_1 \times D_2}(S) = g_{D_1}(S_1) + g_{D_2}(S_2)$.*

Proof. Let B_1 be a feasible set of D_1 with $g_{D_1}(S_1) = |S_1 \cap B_1| - |\overline{S_1} \cap B_1|$, and let B_2 be a feasible set of D_2 with $g_{D_2}(S_2) = |S_2 \cap B_2| - |\overline{S_2} \cap B_2|$. Then $B_1 \cup B_2$ maximizes $B \mapsto |S \cap B| - |\overline{S} \cap B|$, and so $g_{D_1 \times D_2}(S) = |S_1 \cap B_1| - |\overline{S_1} \cap B_1| + |S_2 \cap B_2| - |\overline{S_2} \cap B_2| = g_{D_1}(S_1) + g_{D_2}(S_2)$. $\square$

We now study how the rank function behaves under the operation of *twisting*. Let W be the *signed permutation group*, the subgroup of the symmetric group on $[n, \overline{n}]$ which preserves AdS_n. In other words, W consists of permutations w such that $w(\bar{i}) = \overline{w(i)}$. As delta-matroids are collections of admissible sets, W acts on the set of delta-matroids on $[n, \bar{n}]$. This action is usually called twisting in the delta-matroid literature.

Proposition 13. *Let D be a delta-matroid on $[n, \overline{n}]$, and let $w \in W$. Then $g_{w \cdot D}(S) = g_D(w^{-1} \cdot S)$.*

Proof. Note that, for B a feasible set of D, $|S \cap (w \cdot B)| - |\overline{S} \cap (w \cdot B)| = |(w^{-1} \cdot S) \cap B| - |\overline{(w^{-1} \cdot S)} \cap B|$, which implies the result. □

Let $S \in \mathrm{AdS}_n$ be an admissible set of size n. For any delta-matroid D on $[n, \overline{n}]$, let r be the maximal value of $|S \cap B|$. Then $\{S \cap B \colon B \in \mathcal{F}, |S \cap B| = r\}$ is the set of bases of a matroid on S. When $S = [n]$, this is sometimes called the upper matroid of D. We describe the rank function of this matroid in terms of the rank function of D.

Proposition 14. *Let $S \in \mathrm{AdS}_n$ be an admissible set of size n, and let D be a delta-matroid on $[n, \overline{n}]$ with $r = \max_{B \in \mathcal{F}} |S \cap B|$. The matroid M on S whose bases are $\{S \cap B \colon B \in \mathcal{F}, |S \cap B| = r\}$ has rank function*

$$\mathrm{rk}_M(T) = \frac{g_D(T) + |T|}{2}.$$

Proof. Let $\mathcal{F}_S$ be the collection of feasible sets B with $|S \cap B| = r$. Then we have that

$$\mathrm{rk}_M(T) = \max_{B \in \mathcal{F}_S} |T \cap B| \leqslant \max_{B \in \mathcal{F}} |T \cap B| = \frac{g_D(T) + |T|}{2}.$$

On the other hand, by Proposition 10 there is a feasible set B which maximizes $|T \cap B|$ and has $|S \cap B| = r$, so we have equality. □

2.2 An alternative normalization

The results of the previous section, particularly Proposition 14, suggest that an alternative normalization of the rank function of a delta-matroid has nice properties. Set

$$h_D(S) := \frac{g_D(S) + |S|}{2}.$$

The function $h_D(S)$ is integer-valued and bisubmodular, and the polytope it defines is $P(h_D) = \frac{1}{2}(P(D) + \square)$, where $\square = [-1, 1]^n$ is the cube and the sum is Minkowski sum. This is because the bisubmodular function corresponding to $\square$ is $S \mapsto |S|$. Note that the function h_D is non-negative and increasing, in the sense that if $S \subset T \in \mathrm{AdS}_n$, then $h_D(S) \leqslant h_D(T)$. Theorem 2 implies the following characterization of the functions arising as h_D for some delta-matroid D. In [11, Theorem 2.16], this characterization of the functions h_D is stated with a reference to an unpublished paper of Allys.

Corollary 15. *A function $h \colon \mathrm{AdS}_n \to \mathbb{Z}$ is equal to h_D for some delta-matroid D if and only if*

1. $h(\emptyset) = 0$,
2. $h(S) \in \{0, 1\}$ *if* $|S| = 1$,
3. $h(S) + h(T) \geqslant h(S \cap T) + h(S \sqcup T) + |S \cap \overline{T}|$.

Indeed, these are exactly the conditions we need for $g(S) := 2h(S) - |S|$ to satisfy the conditions in Theorem 2. Note that one can also deduce Theorem 2 from Corollary 15.

The function h_D was studied by Bouchet in [10, 11] in the more general setting of multimatroids. The following alternative characterization of the functions h_D follows from [10, Proposition 4.2]:

Proposition 16. *A function $h\colon \mathrm{AdS}_n \to \mathbb{Z}$ is equal to h_D for some delta-matroid D if and only if*

1. $h(\emptyset) = 0$,
2. $h(S) \leqslant h(S \cup a) \leqslant h(S) + 1$ *if $S \cup a$ is admissible,*
3. $h(S) + h(T) \geqslant h(S \cap T) + h(S \cup T)$ *if $S \cup T$ is admissible, and*
4. $h(S \cup i) + h(S \cup \bar{i}) \geqslant 2h(S) + 1$ *if $S \cap \{i, \bar{i}\} = \emptyset$.*

3 The U-polynomial

We now study the U-polynomial of delta-matroids. We prove the following recursion for $U_D(u, v)$, which was the original definition of the U-polynomial in [22, Definition 1.4].

Proposition 17. *If $n = 0$, the $U_D(u, v) = 1$. For any $i \in [n]$, the U-polynomial satisfies*

$$U_D(u,v) = \begin{cases} U_{D/i}(u,v) + U_{D\setminus i}(u,v) + uU_{D(i)}(u,v), & i \text{ is neither a loop nor a coloop} \\ (u+v+1)\cdot U_{D\setminus i}(u,v), & i \text{ is a loop or a coloop.} \end{cases}$$

First we study the behavior of the U-polynomial under products.

Lemma 18. *Let D_1, D_2 be delta-matroids on $[n_1, \overline{n}_1]$ and $[n_2, \overline{n}_2]$. Then $U_{D_1 \times D_2}(u, v) = U_{D_1}(u, v)U_{D_2}(u, v)$.*

Proof. We compute:

$$\begin{aligned} U_{D_1}(u,v)U_{D_2}(u,v) &= \left(\sum_{S_1 \in \mathrm{AdS}_{n_1}} u^{n_1 - |S_1|} v^{\frac{|S_1| - g_{D_1}(S_1)}{2}}\right)\left(\sum_{S_2 \in \mathrm{AdS}_{n_2}} u^{n_2 - |S_2|} v^{\frac{|S_2| - g_{D_2}(S_2)}{2}}\right) \\ &= \sum_{(S_1, S_2)} u^{n_1 + n_2 - |S_1| - |S_2|} v^{\frac{|S_1| + |S_2| - g_{D_1}(S_1) - g_{D_2}(S_2)}{2}} \\ &= \sum_{(S_1, S_2)} u^{n_1 + n_2 - |S_1| - |S_2|} v^{\frac{|S_1| + |S_2| - g_{D_1 \times D_2}(S_1 \cup S_2)}{2}} \\ &= U_{D_1 \times D_2}(u,v), \end{aligned}$$

where the third equality is Proposition 12. ☐

Proof of Proposition 17. If $n = 0$, then the only admissible subset of $[n, \overline{n}]$ is the empty set, and $g_D(\emptyset) = 0$, so $U_D(u, v) = 1$. Now choose some $i \in [n]$.

First suppose that i is neither a loop nor a coloop. The admissible subsets of $[n, \overline{n}]$ are partitioned into sets containing i, sets containing $\overline{i}$, and sets containing neither i nor $\overline{i}$. If S contains i, then $u^{n-|S|}v^{\frac{|S|-g_D(S)}{2}} = u^{n-1-|S\setminus i|}v^{\frac{|S\setminus i|-g_{D/i}(S\setminus i)}{2}}$. If S contains $\overline{i}$, then $u^{n-|S|}v^{\frac{|S|-g_D(S)}{2}} = u^{n-1-|S\setminus i|}v^{\frac{|S\setminus \overline{i}|-g_{D\setminus i}(S\setminus \overline{i})}{2}}$. If S contains neither i not $\overline{i}$, then $u^{n-|S|}v^{\frac{|S|-g_D(S)}{2}} = u \cdot u^{n-1-|S|}v^{\frac{|S|-g_{D(i)}(S)}{2}}$. Adding these up implies the recursion in this case.

If i is a loop or a coloop, then D is the product of $D \setminus i$ with a delta-matroid on 1 element with 1 feasible set. We observe that U-polynomial of a delta-matroid on 1 element with 1 feasible set is $u + v + 1$, and so Lemma 18 implies the recursion in this case. ☐

3.1 The independence complex of a delta-matroid

In this section, we introduce the independence complex of a delta-matroid and use it to study the U-polynomial.

Definition 19. We say that $S \in \mathrm{AdS}_n$ is *independent* in D if $g_D(S) = |S|$, or, equivalently, if S is contained in a feasible subset of D. The *independence complex* of D is the simplicial complex on $[n, \overline{n}]$ whose facets are given by the feasible sets of D.

Let $S \in \mathrm{AdS}_n$, and let $T = \{i \in [n] : S \cap \{i, \overline{i}\} = \emptyset\}$. Note S is independent if and only if S is a feasible set of $D(T)$.
The following result is immediate from the definition of $U_D(u, 0)$.

Proposition 20. *Let $f_i(D)$ be the number of i-dimensional faces of the independence complex of D. Then $U_D(u, 0) = f_{n-1}(D) + f_{n-2}(D)u + \cdots + f_{-1}(D)u^n$.*

Note that the f-vector of a pure simplicial complex, like the independence complex of a delta-matroid, is a *pure O-sequence*. Then [27] gives the following inequalities.

Corollary 21. *Let $U_D(u, 0) = a_n + a_{n-1}u + \cdots a_0u^n$. Then $(a_0, \ldots, a_n)$ is the f-vector of a pure simplicial complex. In particular, $a_i \leqslant a_{n-i}$ for $i \leqslant n/2$ and $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{\lfloor \frac{n+1}{2} \rfloor}$.*

Proposition 20 is a delta-matroid analogue of the fact that, for a matroid M, the coefficients of $R_M(u, 0)$, when written backwards, are the face numbers of the independence complex of M. The independence complex of a matroid is shellable [5], which is reflected in the fact that $R_M(u - 1, 0)$ has non-negative coefficients. The independence complex of a delta-matroid is not in general shellable or Cohen–Macaulay, and $U_D(u - 1, 0)$ can have negative coefficients.

Recall that $\square = [-1, 1]^n$ is the cube. The map $S \mapsto e_S$ induces a bijection between AdS_n and lattice points of $\square$. We use this to give a polytopal description of the independent sets of D, which will be useful in the sequel.

Proposition 22. *The map $S \mapsto e_S$ induces a bijection between independent sets of D and lattice points in $\frac{1}{2}(P(D) + \square)$.*

Proof. If S is independent in D, then there is $T \in \mathrm{AdS}_n$ such that $S \cup T \in \mathcal{F}$. Then $e_S = \frac{1}{2}(e_{S \cup T} + e_{S \cup \overline{T}})$, so e_S lies in $\frac{1}{2}(P(D) + \square)$.

The correspondence between normalized bisubmodular functions and polytopes gives that

$$\frac{1}{2}(P(D) + \square) = \left\{ x : \langle e_S, x \rangle \leqslant \frac{g_D(S) + |S|}{2} \right\}.$$

If S is not independent, then e_S violates the inequality $\langle e_S, e_S \rangle \leqslant \frac{g_D(S)+|S|}{2}$, so e_S does not lie in $\frac{1}{2}(P(D) + \square)$. □

3.2 The activity expansion of the U-polynomial

We now discuss an expansion of $U_D(u, v-1)$ in terms of a statistic associated to each independent set of a delta-matroid D, similar to the expansion of the Tutte polynomial of a matroid in terms of basis activities. We rely heavily on the work of Morse [31], who gave such an expansion for the interlace polynomial $U_D(0, v-1)$. Throughout we fix the ordering $1 < 2 < \cdots < n$ on $[n]$. For $S \in \mathrm{AdS}_n$, let $\underline{S} \subset [n]$ denote the unsigned version of S, i.e., the image of S under the quotient of $[n, \overline{n}]$ by the involution $\overline{(\cdot)}$.

Definition 23. Let B be a feasible set in a delta-matroid D. We say that $i \in [n]$ is *B-orientable* if the symmetric difference $B \Delta \{i, \bar{i}\}$ is not a feasible set of D. We say that i is *B-active* if i is B-orientable and there is no $j < i$ such that $B \Delta \{i, j, \bar{i}, \bar{j}\}$ is a feasible set of D. For an independent set I of D, we say that $i \in \underline{I}$ is *I-active* if i is I-active in the projection $D([n] \setminus \underline{I})$. Let $a(I)$ denote the number of $i \in \underline{I}$ which are I-active.

Theorem 24. *Let D be a delta-matroid on $[n, \bar{n}]$. Then*

$$U_D(u, v-1) = \sum_{I \text{ independent in } D} u^{n-|I|} v^{a(I)}.$$

Proof. By [31, Corollary 5.3], this holds after we evaluate at $u = 0$ for any delta-matroid D. By [22, Proposition 5.2], we have that

$$U_D(u, v-1) = \sum_{S \subset [n]} u^{n-|S|} U_{D([n] \setminus S)}(0, v-1).$$

The result follows because each independent set I is a feasible set of exactly one projection of D. □

Theorem 24 implies that the coefficient of u^{n-i} in $U_D(u, -1)$ counts the number of independent sets of size i with $a(I) = 0$. This is analogous to how the coefficient of u^{r-i} in $R_M(u, -1)$ counts the number of independent sets of external activity zero in a matroid M, which form the faces of dimension $i - 1$ in the broken circuit complex of M [4]. This interpretation in terms of a simplicial complex generalizes to delta-matroids.

Proposition 25. *The independent sets I of D with $a(I) = 0$ form a simplicial complex on $[n, \bar{n}]$.*

Proof. It suffices to check that if i is not B-active for some feasible set B of D and $S \subset [n] \setminus i$, then i is not active for $B \setminus (S \cup \bar{S})$. Because i is not B-active, either $B \Delta \{i, \bar{i}\}$ is feasible (which remains true after we project away from S), or there is $j < i$ such that $B \Delta \{i, j, \bar{i}, \bar{j}\}$ is feasible. If $j \notin S$, then this remains true after we project away from S. If $j \in S$, then i is not $B \setminus (S \cup \bar{S})$-orientable. □

This complex can be complicated; for instance, its dimension is not easy to predict. The following example shows that the complex defined above need not be pure, so we cannot use it to deduce that $U_D(u, -1)$ is pure O-sequence as in Corollary 21.

Example 26. Let D be the delta-matroid on $[3, \bar{3}]$ with feasible sets $\{1, \bar{2}, \bar{3}\}, \{\bar{1}, 2, \bar{3}\}$, and $\{\bar{1}, \bar{2}, 3\}$. Then one can check that every element of $[3, \bar{3}]$ has no active elements, the sets $\{\bar{1}, 2\}, \{\bar{1}, \bar{2}\}, \{\bar{2}, 3\}, \{\bar{2}, \bar{3}\}, \{\bar{1}, 3\}$, and $\{\bar{1}, \bar{3}\}$ are the independent sets of size 2 with no active elements, and every feasible set has an active element. The complex defined in Proposition 25 has f-vector $(1, 6, 6)$, so $U_D(u, -1) = 6u + 6u^2 + u^3$. This complex is not pure because 1 is not contained in any facet.

3.3 Enveloping matroids

We now recall the definition of an enveloping matroid of a delta-matroid, which was introduced for algebro-geometric reasons in [22, Section 6]. A closely related notion was considered in [10], see Remark 29.

For $S \subseteq [n, \bar{n}]$, let u_S denote the corresponding indicator vector in $\mathbb{R}^{[n,\bar{n}]}$. For a matroid M on $[n, \bar{n}]$, let $P(M) = \mathrm{Conv}\{u_B : B \text{ basis of } M\}$, and let $IP(M) = \mathrm{Conv}\{u_S : S \text{ independent in M}\}$.

Definition 27. Let $\mathrm{env}\colon \mathbb{R}^{[n,\overline{n}]} \to \mathbb{R}^n$ be the map given by $(x_1, \ldots, x_n, x_{\bar{1}}, \ldots, x_{\overline{n}}) \mapsto (x_1 - x_{\bar{1}}, \ldots, x_n - x_{\overline{n}})$. Let D be a delta-matroid on $[n, \overline{n}]$, and let M be a matroid on $[n, \overline{n}]$. We say that M is an *enveloping matroid* for D if $\mathrm{env}(P(M)) = P(D)$.

Note that enveloping matroids necessarily have rank n. In [22, Section 6.3], it is shown that many different types of delta-matroids have enveloping matroids, such as realizable delta-matroids, delta-matroids arising from the independent sets or bases of a matroid, and delta-matroids associated to graphs or embedded graphs. We will need the following property of enveloping matroids.

Proposition 28. *Let M be an enveloping matroid for a delta-matroid D on $[n, \overline{n}]$. Let $S \in \mathrm{AdS}_n$ be an admissible set. Then S is independent in M if and only if it is independent in D.*

Proof. If $S \in \mathrm{AdS}_n$, then $\mathrm{env}(u_S) = e_S$, and S is the only admissible set with this property. Furthermore, if $S \in \mathrm{AdS}_n$ has size n, then u_S is the only indicator vector of a subset of $[n, \bar{n}]$ of size n which is a preimage of e_S under env. Because $\mathrm{env}(P(M)) = P(D)$, we see that if B is a feasible set of D, then B is a basis for M. This implies that the independent sets in D are independent in M.

By [22, Lemma 7.6], $\mathrm{env}(IP(M)) = \frac{1}{2}(P(D) + \square)$. If S is admissible and independent in M, then $\mathrm{env}(u_S) = e_S \in \frac{1}{2}(P(D)+\square)$, so by Proposition 22, S is independent in D. $\square$

Remark 29. A matroid M on $[n, \bar{n}]$ is a *sheltering matroid* for a delta-matroid D if every independent set of D is independent in M. Equivalently, M is sheltering if the restriction of the rank function of M to AdS_n is h_D. The proof of Proposition 28 shows that if M is an enveloping matroid for D, then it is also a sheltering matroid. The converse is false, see [22, Remark 6.7].

3.4 Lorentzian polynomials

For a multi-index $\mathbf{m} = (m_0, m_1, \dots)$, let $w^{\mathbf{m}} = w_0^{m_0} w_1^{m_1} \cdots$. A homogeneous polynomial $f(w_0, w_1, \dots)$ of degree d with real coefficients is said to be *strictly Lorentzian* if all its coefficients are positive, and the quadratic form obtained by taking $d-2$ partial derivatives is nondegenerate with exactly one positive eigenvalue. We say that f is *Lorentzian* if it is a coefficient-wise limit of strictly Lorentzian polynomials. Lorentzian polynomials enjoy strong log-concavity properties, and the class of Lorentzian polynomials is preserved under many natural operations.

The following lemma is a special case of [32, Proposition 3.3]. Alternatively, it can be deduced from the proof of [13, Corollary 3.5]. We thank Nima Anari for discussing this lemma with us.

Lemma 30. *For a polynomial $f(w_0, w_1, \dots) = \sum_{\boldsymbol{m}} c_{\boldsymbol{m}} w^{\boldsymbol{m}}$, let*

$$\overline{f}(w_0, w_1, \dots) = \sum_{\boldsymbol{m}: m_i \leqslant 1 \text{ for } i \neq 0} c_{\boldsymbol{m}} w^{\boldsymbol{m}}.$$

If f is Lorentzian, then $\overline{f}$ is Lorentzian.

For $S \in \mathrm{AdS}_n$, recall that $\underline{S} \subset [n]$ denotes the unsigned version of S. For a set T, let $w^T = \prod_{a \in T} w_a$. We now state a strengthening of Theorem 6.

Theorem 31. *Let D be a delta-matroid on $[n, \overline{n}]$ which has an enveloping matroid. Then the polynomial*

$$\sum_{S \text{ independent in } D} w_0^{2n-|S|} w^{\underline{S}} \in \mathbb{R}[w_0, w_1, \dots, w_n]$$

is Lorentzian.

Remark 32. In [22, Theorem 8.1], it is proven that if D has an enveloping matroid, then the polynomial

$$\sum_{S \text{ independent in } D} \frac{w_0^{|S|}}{|S|!} w^{[n] \setminus \underline{S}} \in \mathbb{R}[w_0, w_1, \dots, w_n]$$

is Lorentzian.

Proof of Theorem 6. By [13, Theorem 2.10], the specialization

$$\sum_{S \text{ independent in } D} w_0^{2n-|S|} y^{|S|} = \sum_{i=0}^{n} f_{i-1}(D) w_0^{2n-i} y^i$$

is Lorentzian. By [13, Example 2.26], the coefficients of a Lorentzian polynomial in two variables of degree $2n$ are log-concave after dividing the coefficient of $w_0^{2n-i}y^i$ by $\binom{2n}{i}$, which implies the result. □

Proof of Theorem 31. Let M be an enveloping matroid of D. By [13, Proof of Theorem 4.14], the polynomial

$$\sum_{S \text{ independent in } M} w_0^{2n-|S|} w^S \in \mathbb{R}[w_0, w_1, \ldots, w_n, w_{\overline{1}}, \ldots, w_{\overline{n}}]$$

is Lorentzian. Setting $w_{\overline{i}} = w_i$, by [13, Theorem 2.10] the polynomial

$$\sum_{S \text{ independent in } M} w_0^{2n-|S|} w^{S \cap [n]} w^{\overline{S \cap [\overline{n}]}} \in \mathbb{R}[w_0, w_1, \ldots, w_n]$$

is Lorentzian. A term $w_0^{2n-|S|} w^{S \cap [n]} w^{\overline{S \cap [\overline{n}]}}$ has degree at most 1 in each of the variables $w_1, \ldots, w_n$ if and only if S is admissible, in which case it is equal to $w^{\underline{S}}$. Therefore, by Lemma 30, the polynomial

$$\sum_{S \in \mathrm{AdS}_n \text{ independent in } M} w_0^{2n-|S|} w^{\underline{S}} \in \mathbb{R}[w_0, w_1, \ldots, w_n]$$

is Lorentzian. By Proposition 28, this polynomial is equal to the polynomial in Theorem 31. □

Remark 33. Let (U, Ω, r) be a multimatroid [10], i.e., U is a finite set, Ω is a partition of U, and r is a function on partial transversals of Ω satisfying certain conditions. An *independent set* is a partial transversal S of Ω with $r(S) = |S|$. A multimatroid is called *shelterable* if r can be extended to the rank function of a matroid on U. Then the argument used to prove Theorem 6 shows that, if a_k is the number of independent sets of a shelterable multimatroid of size k, then

$$a_k^2 \geqslant \frac{|U| - k + 1}{|U| - k} \frac{k+1}{k} a_{k+1} a_{k-1}.$$

In particular, the proof of Theorem 6 only requires D to be shelterable.

References

[1] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials III: Mason's ultra-log-concavity conjecture for independent sets of matroids. arXiv:1807.00929v2.

[2] Federico Ardila, Federico Castillo, Christopher Eur, and Alexander Postnikov. Coxeter submodular functions and deformations of Coxeter permutahedra. *Adv. Math.*, 365:107039, 2020.

[3] Richard Arratia, Béla Bollobás, and Gregory B. Sorkin. The interlace polynomial of a graph. *J. Combin. Theory Ser. B*, 92(2):199–233, 2004.

[4] Spencer Backman. Tutte polynomial activities. arXiv:1906.02781v2.

[5] Anders Björner. The homology and shellability of matroids and geometric lattices. In *Matroid applications*, volume 40 of *Encyclopedia Math. Appl.*, pages 226–283. Cambridge Univ. Press, Cambridge, 1992.

[6] Béla Bollobás and Oliver Riordan. A polynomial invariant of graphs on orientable surfaces. *Proc. London Math. Soc. (3)*, 83(3):513–531, 2001.

[7] Alexandre V. Borovik, Israel Gelfand, and Neil White. *Coxeter matroids*, volume 216 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2003.

[8] André Bouchet. Greedy algorithm and symmetric matroids. *Math. Programming*, 38(2):147–159, 1987.

[9] André Bouchet. Representability of $\triangle$-matroids. In *Combinatorics (Eger, 1987)*, volume 52 of *Colloq. Math. Soc. János Bolyai*, pages 167–182. North-Holland, Amsterdam, 1988.

[10] André Bouchet. Multimatroids. I. Coverings by independent sets. *SIAM J. Discrete Math.*, 10(4):626–646, 1997.

[11] André Bouchet. Multimatroids. II. Orthogonality, minors and connectivity. *Electron. J. Combin.*, 5:#R8, 25, 1998.

[12] André Bouchet and William H. Cunningham. Delta-matroids, jump systems, and bisubmodular polyhedra. *SIAM J. Discrete Math.*, 8(1):17–32, 1995.

[13] Petter Brändén and June Huh. Lorentzian polynomials. *Ann. of Math. (2)*, 192(3):821–891, 2020.

[14] Robert Brijder and Hendrik Jan Hoogeboom. Interlace polynomials for multimatroids and delta-matroids. *European J. Combin.*, 40:142–167, 2014.

[15] R. Chandrasekaran and Santosh N. Kabadi. Pseudomatroids. *Discrete Math.*, 71(3):205–217, 1988.

[16] Carolyn Chun, Iain Moffatt, Steven D. Noble, and Ralf Rueckriemen. Matroids, delta-matroids and embedded graphs. *J. Combin. Theory Ser. A*, 167:7–59, 2019.

[17] Carolyn Chun, Iain Moffatt, Steven D. Noble, and Ralf Rueckriemen. On the interplay between embedded graphs and delta-matroids. *Proc. Lond. Math. Soc. (3)*, 118(3):675–700, 2019.

[18] Andreas Dress and Timothy F. Havel. Some combinatorial properties of discriminants in metric vector spaces. *Adv. in Math.*, 62(3):285–312, 1986.

[19] Alain Duchamp. Delta matroids whose fundamental graphs are bipartite. *Linear Algebra Appl.*, 160:99–112, 1992.

[20] F. D. J. Dunstan and Dominic Welsh. A greedy algorithm for solving a certain class of linear programmes. *Math. Programming*, 5:338–353, 1973.

[21] Joanna A. Ellis-Monaghan, Andrew J. Goodall, Iain Moffatt, Steven D. Noble, and Lluís Vena. Irreducibility of the Tutte polynomial of an embedded graph. *Algebr. Comb.*, 5(6):1337–1351, 2022.

[22] Christopher Eur, Alex Fink, Matt Larson, and Hunter Spink. Signed permutohedra, delta-matroids, and beyond. *Proc. Lond. Math. Soc. (3)*, 128(3):Paper No. e12592, 54, 2024.

[23] Satoru Fujishige. Bisubmodular polyhedra, simplicial divisions, and discrete convexity. *Discrete Optim.*, 12:115–120, 2014.

[24] Satoru Fujishige. Parametric bisubmodular function minimization and its associated signed ring family. *Discrete Appl. Math.*, 227:142–148, 2017.

[25] Satoru Fujishige and Satoru Iwata. Bisubmodular function minimization. *SIAM J. Discrete Math.*, 19(4):1065–1073, 2005.

[26] Satoru Fujishige and Sachin Patkar. The box convolution and the Dilworth truncation of bisubmodular functions. Report No. 94823, Research Institute for Discrete Mathematics, University of Bonn, 1994.

[27] Takayuki Hibi. What can be said about pure O-sequences? *J. Combin. Theory Ser. A*, 50(2):319–322, 1989.

[28] Thomas Krajewski, Iain Moffatt, and Adrian Tanasa. Hopf algebras and Tutte polynomials. *Adv. in Appl. Math.*, 95:271–330, 2018.

[29] John H. Mason. Matroids: unimodal conjectures and Motzkin's theorem. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, pages 207–220. Inst. Math. Appl., Southend-on-Sea, 1972.

[30] Ada Morse. The interlace polynomial. In *Graph polynomials*, Discrete Math. Appl. (Boca Raton), pages 1–23. CRC Press, Boca Raton, FL, 2017.

[31] Ada Morse. Interlacement and activities in delta-matroids. *European J. Combin.*, 78:13–27, 2019.

[32] Julius Ross, Hendrik Süss, and Thomas Wannerer. Dually Lorentzian polynomials. arXiv:2304.08399.

[33] Walter Wenzel. Δ-matroids with the strong exchange conditions. *Appl. Math. Lett.*, 6(5):67–70, 1993.