Degree Deviation and Spectral Radius

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Abstract

For a finite, simple, and undirected graph G with n vertices, m edges, and largest eigenvalue λ, Nikiforov introduced the degree deviation of G as

$$s = \sum_{u \in V(G)} \left| d_G(u) - \frac{2m}{n} \right|.$$

Contributing to a conjecture of Nikiforov, we show $\lambda - \frac{2m}{n} \leqslant \sqrt{\frac{2s}{3}}$. For our result, we show that the largest eigenvalue of a graph that arises from a bipartite graph with $m_{A,B}$ edges by adding m_A edges within one of the two partite sets is at most

$$\sqrt{m_A + m_{A,B} + \sqrt{m_A^2 + 2m_A m_{A,B}}},$$

which is a common generalization of results due to Stanley and Bhattacharya, Friedland, and Peled.

Mathematics Subject Classifications: 05C07, 05C50

1 Introduction

We consider finite, simple, and undirected graphs and use standard notation and terminology. For a graph G with n vertices and m edges, Nikiforov [4] introduced the *degree deviation* $s(G)$ of G as $s(G) = \sum\limits_{u \in V(G)} \left| d_G(u) - \frac{2m}{n} \right|$. For the *spectral radius* $\lambda(G)$ of G, which is the largest eigenvalue of the adjacency matrix of G, he showed that $\lambda(G) - \frac{2m}{n} \leqslant \sqrt{s(G)}$ and conjectured $\lambda(G) - \frac{2m}{n} \leqslant \sqrt{\frac{s(G)}{2}}$ for sufficiently large n and m. Zhang [7] showed $\lambda(G) - \frac{2m}{n} \leqslant \sqrt{\frac{9s(G)}{10}}$.

We make further progress on Nikiforov's conjecture by showing the following.

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Theorem 1. *If G is a graph with n vertices and m edges, then*

$$\lambda(G) - \frac{2m}{n} \leqslant \sqrt{\frac{2s(G)}{3}}.$$

For the proof of Theorem 1, we establish a new bound on the spectral radius of a graph, which is a common generalization of results due to Stanley [6] and Bhattacharya et al. [1]. For a graph G with n vertices and m edges, Stanley [6] showed $\lambda(G) \leqslant \sqrt{2m}$; in fact, he showed a slightly stronger bound. Provided that G is bipartite, Bhattacharya et al. [1] showed $\lambda(G) \leqslant \sqrt{m}$, which had been shown before by Nosal [5] for triangle-free graphs.

Theorem 2. *Let G be a graph whose vertex set is partitioned into the two sets A and B. If the edge set of G consists of m_A edges with both endpoints in A and $m_{A,B}$ edges with one endpoint in A and one endpoint in B, then*

$$\lambda(G) \leqslant \sqrt{m_A + m_{A,B} + \sqrt{m_A^2 + 2m_A m_{A,B}}}.$$

Complete split graphs $CS(q,n)$ with q universal vertices and $n-q$ vertices of degree q show that Theorem 2 is essentially best possible. In fact, it is known [3] that the spectral radius of $CS(q,n)$ is $\frac{1}{2}\left(q-1+\sqrt{(4n-2)q-3q^2+1}\right)$, which asymptotically coincides with the bound in Theorem 2 for $m_A = \binom{q}{2}$ and $m_{A,B} = q(n-q)$.

The next section contains the proofs of both results and some discussion.

2 Proofs

Since Theorem 1 relies on Theorem 2, we start with the latter.

Proof of Theorem 2. For $m_{A,B} = 0$, Stanley's result implies the desired bound. Hence, we may assume that $m_{A,B} > 0$, which implies $\lambda = \lambda(G) > 0$. Let $x = (x_u)_{u \in V(G)}$ be an eigenvector for the eigenvalue λ. For every vertex u of G, we have $\lambda x_u = \sum\limits_{v:v \in N_G(u)} x_v$ and applying this identity twice, we obtain

$$\begin{aligned} \lambda^2 x_u &= \sum_{v:v\in N_G(u)} \lambda x_v = \sum_{v:v\in N_G(u)} \left(\sum_{w:w\in N_G(v)} x_w \right) \\ &= x_u d_G(u) + \sum_{v:v\in N_G(u)} \left(\sum_{w:w\in N_G(v)\setminus\{u\}} x_w \right); \qquad (1) \end{aligned}$$

this observation seems to originate from Favaron et al. [2].

By the Perron-Frobenius Theorem and by normalizing the eigenvector x, we may assume that x has no negative entry and that $\max\{x_u : u \in V(G)\} = 1$. Let the vertex u' be such that $x_{u'} = 1$ and let $\alpha = \max\{x_u : u \in B\}$.

If $\alpha = 1$, then we may assume $u' \in B$ and applying (1) with $u = u'$ implies

$$\begin{aligned}
\lambda^2 &= d_G(u') + \underbrace{\sum_{v: v \in N_G(u')} \left(\sum_{w: w \in N_G(v) \setminus \{u'\}} x_w \right)}_{(*)} \\
&\leqslant d_G(u') + 2m_A + (m_{A,B} - d_G(u')) && (2) \\
&= 2m_A + m_{A,B}, && (3)
\end{aligned}$$

where (2) follows because each of the m_A edges vw with $v, w \in A$ contributes at most $x_v + x_w \leqslant 2$ to $(*)$ and each of the $m_{A,B} - d_G(u')$ edges vw with $v \in A$ and $w \in B \setminus \{u'\}$ contributes at most $x_w \leqslant \alpha = 1$ to $(*)$.

See Figure 1 for an illustration.

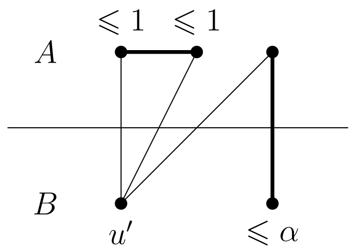

Figure 1: Two edges incident with neighbors of u' and their possible contributions to $(*)$. If one of the thin edges does not belong to G, the contribution is reduced accordingly.

Since (3) is stronger than the stated bound, the proof is complete in this case. Hence, we may assume that $\alpha < 1$, which implies that $u' \in A$.

Let u' have d_A neighbors in A and $d_{A,B}$ neighbors in B. Applying (1) with $u = u'$ implies

$$\begin{aligned}
\lambda^2 &= d_G(u') + \underbrace{\sum_{v: v \in N_G(u')} \left(\sum_{w: w \in N_G(v) \setminus \{u'\}} x_w \right)}_{(**)} \\
&\leqslant (d_A + d_{A,B}) + 2(m_A - d_A) + (1 + \alpha)(m_{A,B} - d_{A,B}) && (4) \\
&\leqslant 2m_A + (1 + \alpha)m_{A,B}, && (5)
\end{aligned}$$

where (4) follows because each of the $m_A - d_A$ edges vw with $v, w \in A \setminus \{u'\}$ contributes at most $x_v + x_w \leqslant 2$ to $(**)$ and each of the $m_{A,B} - d_{A,B}$ edges vw with $v \in A \setminus \{u'\}$ and $w \in B$ contributes at most $x_v + x_w \leqslant 1 + \alpha$ to $(**)$; recall that x has no negative entry, which implies $1 \leqslant 1 + \alpha$.

See Figure 2 for an illustration.

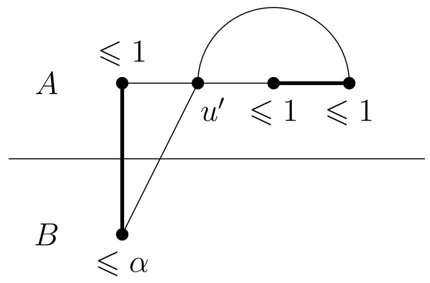

Figure 2: Two edges incident with neighbors of u' and their possible contributions to $(**)$.

If $\alpha = 0$, then (5) is stronger than the stated bound and the proof is complete in this case. Hence, we may assume that $\alpha > 0$. Note that the inequality (5) is strict if $d_A > 0$ or $\alpha d_{A,B} > 0$, that is, there is a tiny room for improvement.

Let $u'' \in B$ be such that $x_{u''} = \alpha$. Applying (1) with $u = u''$ implies

$$\begin{aligned}\lambda^2\alpha &= \alpha d_G(u'') + \sum_{v:v\in N_G(u'')}\left(\sum_{w:w\in N_G(v)\setminus\{u''\}} x_w\right)\\ &\leqslant \alpha d_G(u'') + 2m_A + \alpha(m_{A,B} - d_G(u'')) &(6)\\ &= 2m_A + \alpha m_{A,B}, &(7)\end{aligned}$$

where (6) follows similary as (2). Since $\alpha > 0$, the bound (7) implies

$$\lambda^2 \leqslant \frac{2}{\alpha}m_A + m_{A,B}. \tag{8}$$

Since the bound in (5) is increasing in α and the bound in (8) is decreasing in α, we obtain that $\lambda^2 \leqslant 2m_A + (1+\alpha^*)m_{A,B}$, where α^* is chosen such that $2m_A + (1+\alpha^*)m_{A,B} = \frac{2}{\alpha^*}m_A + m_{A,B}$. Solving this equation for α^* yields $\alpha^* = \sqrt{\left(\frac{m_A}{m_{A,B}}\right)^2 + 2\frac{m_A}{m_{A,B}}} - \frac{m_A}{m_{A,B}} \in [0,1]$. Substituting this value in $\lambda^2 \leqslant 2m_A + (1+\alpha^*)m_{A,B}$ yields

$$\lambda^2 \leqslant m_A + m_{A,B} + \sqrt{m_A^2 + 2m_A m_{A,B}},$$

which completes the proof. □

Proof of Theorem 1. Let $\lambda = \lambda(G)$, $s = s(G)$, $d = \left\lceil \frac{2m}{n} \right\rceil$, and $C = \{u \in V(G) : d_G(u) \geqslant d+1\}$.

We choose a set E_0 of edges of G with both endpoints in C such that

(i) $d_H(u) \geqslant d$ for every vertex u in C and the graph $H = G - E_0 = (V(G), E(G) \setminus E_0)$,

(ii) subject to condition (i), the number $m_0 = |E_0|$ of edges in E_0 is as large as possible, and

(iii) subject to conditions (i) and (ii), the expression

$$\sum_{u\in C}\max\{d_H(u)-(d+1),0\}$$

is as small as possible.

Let $C' = \{u \in C : d_H(u) = d\}$. Let C'' be the set of isolated vertices of the graph $\left(C', E_0 \cap \binom{C'}{2}\right)$. Let $A = C \setminus C''$ and $B = V(G) \setminus A$.

See Figure 3 for an illustration.

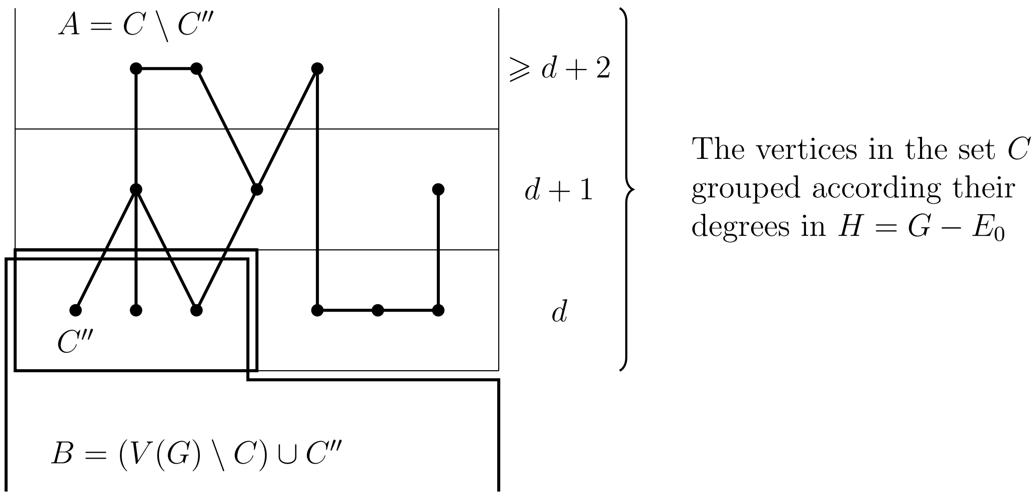

Figure 3: The partition of the vertex set of G into A and B. The edges shown within C are the edges in E_0 that are removed from G to obtain H. For the vertices in C, we consider their degrees d, $d+1$, and $\geqslant d+2$ in H.

By (ii) in the choice of E_0, the set $C \setminus C' = \{u \in C : d_H(u) \geqslant d+1\}$ is independent in H. If $uv \in E(H)$ with $d_H(u) \geqslant d+2$ and $v \in C' \setminus C''$, then E_0 contains an edge vw with $w \in C' \setminus C''$ and $E_0' = (E_0 \setminus \{vw\}) \cup \{uv\}$ yields a contradiction to the condition (iii) in the choice of E_0. Hence, in the graph H, the vertices in $\{u \in C : d_H(u) \geqslant d+2\}$ have all their neighbors in B. Let $E_A = E_0 \cap \binom{A}{2}$. Note that all edges in $E_0 \setminus E_A$ are between A and B. Let $E_{A,B}$ arise from $E_0 \setminus E_A$ by adding, for every vertex $u \in C$ with $d_H(u) \geqslant d+2$, exactly $d_H(u)-(d+1)$ edges incident with u. By construction, all edges in E_A have both their endpoints in $A \subseteq C$ and every edge in $E_{A,B}$ connects a vertex from A to a vertex from B. Furthermore, the graph $G' = G - (E_A \cup E_{A,B})$ has maximum degree at most $d+1$ and $d_{G'}(u) \geqslant d$ holds for every vertex $u \in A$. Let $m_A = |E_A|$, $m_{A,B} = |E_{A,B}|$, and $G'' = (V(G), E_A \cup E_{A,B})$.

Since $\sum\limits_{u\in V(G)} \left(d_G(u) - \frac{2m}{n}\right) = 0$, we have

$$2m_A + m_{A,B} \leqslant \sum_{u\in A}(d_G(u)-d) \leqslant \sum_{u\in C}(d_G(u)-d) \leqslant \sum_{u\in C}\left(d_G(u) - \frac{2m}{n}\right) \leqslant \frac{s}{2}. \tag{9}$$

Since G is the edge-disjoint union of the graphs G' and G'', we obtain using the maximum degree bound for G' and Theorem 2 for G'' that

$$\begin{aligned} \lambda &\leqslant \lambda(G') + \lambda(G'') && (10) \\ &\leqslant d + 1 + \sqrt{m_A + m_{A,B} + \sqrt{m_A^2 + 2m_A m_{A,B}}}. && (11) \end{aligned}$$

Since (11) is increasing in $m_{A,B}$, it follows using (9) that

$$\begin{aligned} \lambda &\leqslant d + 1 + \max\left\{ \sqrt{x + y + \sqrt{x^2 + 2xy}} : x, y \geqslant 0 \text{ and } 2x + y = \frac{s}{2} \right\} \\ &\leqslant d + 1 + \max\left\{ \sqrt{\frac{s}{2} - x + \sqrt{x(s - 3x)}} : 0 \leqslant x \leqslant \frac{s}{4} \right\}. && (12) \end{aligned}$$

A simple calculation shows that $x = \frac{s}{12}$ solves the maximization problem in (12) and we obtain

$$\lambda \leqslant d + 1 + \sqrt{\frac{s}{2} - \frac{s}{12} + \sqrt{\frac{s}{12}\left(s - 3\frac{s}{12}\right)}} = d + 1 + \sqrt{\frac{2s}{3}}.$$

At this point, we have $\lambda - \frac{2m}{n} \leqslant \lambda - d + 1 \leqslant \sqrt{\frac{2s}{3}} + 2$. Now, Nikiforov's blow-up argument (cf. proof of Theorem 8 in [4]), replacing every vertex of G by an independent set of order t and letting t tend to infinity, implies $\lambda - \frac{2m}{n} \leqslant \sqrt{\frac{2s}{3}}$, which completes the proof. □

We believe that the estimate (10) is the crucial point within the above proof that is too weak to establish Nikiforov's conjecture.

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