On the Off-Diagonal Unordered Erdős-Rado Numbers

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Abstract

Erdős and Rado [Journal of the London Mathematical Society 25 (1950)] introduced the Canonical Ramsey numbers $\mathrm{er}(t)$ as the minimum number n such that every edge-coloring of the ordered complete graph K_n contains either a monochromatic, rainbow, upper lexical, or lower lexical clique of order t. Richer [Journal of Combinatorial Theory Series B 80 (2000)] introduced the unordered asymmetric version of the Canonical Ramsey numbers $\mathrm{CR}(s,r)$ as the minimum n such that every edge-coloring of the (unordered) complete graph K_n contains either a rainbow clique of order r, or an orderable clique of order s.

We show that $\mathrm{CR}(s,r) = O(r^3/\log r)^{s-2}$, which, up to the multiplicative constant, matches the known lower bound and improves the previously best known bound $\mathrm{CR}(s,r) = O(r^3/\log r)^{s-1}$ by Jiang [Discrete Mathematics 309 (2009)]. We also obtain bounds on the further variant $\mathrm{ER}(m,\ell,r)$, defined as the minimum n such that every edge-coloring of the (unordered) complete graph K_n contains either a monochromatic K_m, lexical K_ℓ, or rainbow K_r.

Mathematics Subject Classifications: 05C55, 05D10, 05C15

1 Introduction

Monochromatic cliques in edge-colorings of the complete graph is one of the most studied topics in Graph Theory. Ramsey Theory studies, when the number of colors is fixed, how large a complete graph must be to ensure a monochromatic copy of a clique of given order. If we allow the number of colors to be arbitrary, we can still ask how large a complete graph must be to ensure a copy of a clique with some special coloring, which we call *canonical* coloring. Erdős and Rado [6] introduced the so-called *Canonical Ramsey Numbers*, which we denote by $\mathrm{er}(t)$, in the context of vertex-ordered graphs.

Theorem 1 (Erdős, Rado [6]). *For every integer t, there exists a minimum number $n = er(t)$ such that the following holds. For every edge-coloring $\chi : E(K_n) \to \omega$ of the complete graph on (ordered) vertex-set $[n]$, there exists a complete subgraph of order t whose coloring is one of the following:*

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- *monochromatic;*
- *rainbow;*
- *upper lexical (two edges have the same color if and only if they have the same higher endpoint);*
- *lower lexical (two edges have the same color if and only if they have the same lower endpoint).*

In terms of a quantitative bound for $\text{er}(t)$, Lefmann and Rödl [8] proved that

$$2^{c't^2} \leqslant \text{er}(t) \leqslant 2^{c''t^2 \log(t)}$$

for some constants c' and c''. Some improvements on the value of c' and c'' were made since (see, e.g. [1]) but the logarithmic gap in the exponent remains.

Richer [9] introduced an unordered variant, where (ordered) monochromatic, upper lexical and lower lexical cliques are seen as special case of the so-called *orderable* cliques in the unordered setting. These numbers, which we denote by $\text{CR}(s,r)$, were called "Unordered Canonical Ramsey Numbers."

Definition 2. A coloring $\chi : E(K_n) \to \omega$ is called *orderable* if there is an ordering $v_1, v_2, \dots, v_n$ of the vertices of K_n such that two edges have the same color if they have the same lower end, i.e., $\chi(v_iv_j) = \chi(v_iv_k)$ for every $1 \leqslant i < j < k \leqslant n$.

Remark 3. Note that in the above definition we only ask that two edges have the same color if they have the same lower end, compared to the "if and only if" requirement for the *lower lexical* coloring considered by Erdős and Rado.

Remark 4. Note that the above definition would be equivalent if one asks whether there exists ordering for which two edges have the same color if they have the same upper end.

Definition 5 (Unordered Canonical Ramsey Numbers [9])**.** We define the Unordered Canonical Ramsey Number $\text{CR}(s,r)$ as the minimum n such that every coloring $\chi : E(K_n) \to \omega$ contains one of the following:

- orderable K_o;
- rainbow K_r.

In this note we are interested in the asymptotic behavior of $\text{CR}(s,r)$ for fixed s, when r tends to infinity. A construction from Babai [4] and an upper bound from Alon, Lefmann, and Rödl [2] show that

$$\text{CR}(3,r) = \Theta\left(\frac{r^3}{\log r}\right). \tag{1.1}$$

For general $s \geqslant 4$, Jiang [7] proved that

$$\Omega\left(\frac{r^3}{\log r}\right)^{s-2} \leqslant \text{CR}(s,r) \leqslant O\left(\frac{r^3}{\log r}\right)^{s-1}.$$

The main contribution of this paper is to close the gap between the lower and upper bounds.

Theorem 6. *There is a constant $c > 0$ such that for every s and r we have*

$$CR(s,r) \leqslant \left(\frac{c \cdot r^3}{\log r}\right)^{s-2}.$$

Further, we introduce and discuss the following version of Unordered Canonical Ramsey Numbers, where we identify only the lower lexical and upper lexical cliques as *lexical* cliques.

Definition 7. A coloring $\chi : E(K_n) \to \omega$ is called *lexical* if there is an ordering $v_1, v_2, \dots, v_n$ of the vertices of K_n such that two edges have the same color if and only if they have the same lower end.

Definition 8. We define the Unordered Erdős–Rado Number $\mathrm{ER}(m, \ell, r)$ as the minimum n such that every coloring $\chi : E(K_n) \to \omega$ contains one of the following:

- monochromatic K_m;
- lexical K_ℓ;
- rainbow K_r.

The focus of this note is the asymptotic behavior of $\mathrm{ER}(m, \ell, r)$ for fixed m, ℓ, and r tending to infinity. Some other ranges of parameters have also been studied in the past (see, e.g., [3]). We now make some observations on how the known results on $\mathrm{CR}(s, r)$ translate to results on $\mathrm{ER}(m, \ell, r)$.

The asymptotic of the Unordered Erdős–Rado Number $\mathrm{ER}(3, 3, r)$ is known up to a multiplicative constant, since (1.1) implies

$$\mathrm{ER}(3,3,r) = \Theta\left(\frac{r^3}{\log r}\right).$$

The proof of the lower bound $\mathrm{CR}(s, r) \geqslant \Omega\left(r^3/\log r\right)^{s-2}$ in [7] extends the coloring from Babai [4] for $s = 3$ to arbitrary $s \geqslant 4$ via a blow-up coloring. We highlight that this coloring gives a stronger result, because it not only avoids orderable cliques of given size, but do so by also avoiding monochromatic triangles.

Theorem 9 (Adapted from [7])**.** *There is a constant $c > 0$ such that for every integers $\ell \geqslant 3$ and $r \geqslant 3$ we have*

$$ER(3,\ell,r) \geqslant \left(\frac{c \cdot r^3}{\log r}\right)^{\ell-2}.$$

We will show that this lower bound is best possible up to the constant factor, and also obtain upper bounds on $\mathrm{ER}(m, \ell, r)$ for other fixed values of m and ℓ.

Theorem 10.

(1) *There is a constant $c > 0$ such that for every integers $m \geqslant 3$ and $r \geqslant 3$ we have*

$$ER(m,3,r) \leqslant c \cdot m \cdot \frac{r^3}{\log r}.$$

(2) *There is a constant $c > 0$ such that for every integers $\ell \geqslant 3$ and $r \geqslant 3$ we have*

$$ER(3, \ell, r) \leqslant \left(\frac{c \cdot r^3}{\log r}\right)^{\ell-2}.$$

(3) *There is a constant $c > 0$ such that for every integer $r \geqslant 3$ we have*

$$ER(4, 4, r) \leqslant \frac{c \cdot r^7}{(\log r)^2}.$$

Remark 11. The bounds in (1) and (2) are best possible, up to the dependency on m in (1), and up to constant c in (2), see Theorem 9.

Remark 12. If $R(m)$ denotes the 2-colored (diagonal) Ramsey number of m, then note that $\mathrm{ER}(m, 4, 3) \geqslant R(m) > (\sqrt{2} + o(1))^m$. One cannot expect an upper bound of the form $(cmr^3/\log r)^2$ for $\mathrm{ER}(m, 4, r)$, in contrast to $\mathrm{ER}(m, 3, r) \leqslant cmr^3/\log r$.

Remark 13. The proof of (3) can be extended to give an upper bound for every $m \geqslant 4$. However, we do not expect the bound in (3) to be best possible, see Conjecture 21. To simplify the presentation we opt to write the proof only for the $m = 4$ case. The best known lower bound is from

$$\mathrm{ER}(m, 4, r) \geqslant \mathrm{ER}(3, 4, r) \geqslant \Omega(r^3/\log r)^2.$$

The rest of the paper is organized as follows. In Section 2, we introduce an auxiliary lemma which will be crucial to most of the proofs, prove the main result, Theorem 6, and parts (1) and (2) of Theorem 10. In Section 3, we prove part (3) of Theorem 10. Finally, in Section 4, we introduce a coloring from [7] proving Theorem 9.

Notation. We denote by $R(k)$ the 2-colored Ramsey number, so that any 2-coloring of the edges of K_n for $n \geqslant R(k)$ contains a monochromatic copy of K_k. Given a coloring $\chi : E(G) \to \omega$ and a vertex $v \in V(G)$, we denote by $N_i(v)$ the set of vertices that are joined to v by edges in color $i \in \omega$. Further, for $U \subset V(G)$, we write $G[U]$ for the induced subgraph of G on vertex-set U, and $e_i(U)$ for the number of edges with color i and both endpoints in U. For integer k, we denote by $[k]$ the set $\{1, 2, \ldots, k\}$. Similarly, for integers $a < b$, we denote by $[a, b]$ the interval $\{a, a + 1, \ldots, b\}$.

2 Colorings with bounded maximum degree

In the proof of Theorem 6, we will make use of the following lemma.

Lemma 14 (Alon, Jiang, Miller, Pritikin; Theorem 5.6 in [1]). *There is a constant $c > 0$ such that the following holds. For every integers Δ and r, if $n > c \cdot \Delta \cdot \frac{r^3}{\log r}$ and $\chi : E(K_n) \to \omega$ is a coloring, where each color-class has maximum degree at most Δ, then χ contains a rainbow clique of order r.*

In the proofs that follow we will make repetitively use of the above Lemma 14, which motivates the following notation to be used from now on.

Definition 15. A Δ-*good* coloring of a graph G is a coloring $\chi : E(G) \to \omega$ for which every color-class has maximum degree at most Δ.

Part (1) of Theorem 10 follows from Lemma 14, since a coloring avoiding monochromatic K_m and lexical K_3 is m-good. Note that Part (2) of Theorem 10 is a consequence of Theorem 6 since every orderable K_ℓ either is a lexical K_ℓ or contains a monochromatic copy of K_3. We are now ready to prove our main result, Theorem 6.

Proof of Theorem 6. The proof is by induction on $s \geqslant 3$. Let $c > 0$ be a constant larger than the constant given by Lemma 14 and such that the base case

$$\mathrm{CR}(3, r) \leqslant c \cdot \frac{r^3}{\log r}$$

holds true. Such a constant exists by (1.1).

For the inductive step, assume $s \geqslant 4$ and $n \geqslant \left(\frac{c \cdot r^3}{\log r}\right)^{s-2}$. Observe that, by Lemma 14, if χ is $\left(\frac{c \cdot r^3}{\log r}\right)^{s-3}$-good then χ contains a rainbow clique of order r and we are done. We assume then that there exists a vertex v and a color, say 1, such that the neighborhood of v in color 1 satisfies $|N_1(v)| \geqslant \left(\frac{c \cdot r^3}{\log r}\right)^{s-3}$.

By the induction hypothesis applied to the restriction of χ to $N_1(v)$, the coloring χ contains either a rainbow K_r, or an orderable K_{s-1}. If the second case happens, then, by appending v to the orderable K_{s-1}, we obtain an orderable K_s, as desired. □

We note that the proof of Theorem 6 is similar to the proof of Theorem 6 in [7]. The main differences are that we make use of Lemma 14 and the basis step starts at $s = 3$, whereas the proof of Theorem 6 in [7] starts the basis step at $s = 2$.

3 Proof of Theorem 10 (3)

In the proof of Theorem 10 (3) we will apply Lemma 14 and analyze the coloring restricted to the neighborhood of a vertex v. We start with the following claims.

Claim 16. *Let $m \geqslant 3$ and $\chi : E(K_n) \to \omega$ be a coloring. Assume there are vertices u, v and distinct colors $i \neq j$ such that $u \in N_i(v)$ and $|N_j(u) \cap N_i(v)| \geqslant R(m-1)$. Then there exists in $\{u, v\} \cup (N_j(u) \cap N_i(v))$ either*

- *a lexical K_4 containing u and v;*
- *a monochromatic K_m containing either u or v.*

Proof of the claim: Any edge of color different from i and j in $N_j(u) \cap N_i(v)$ leads to a lexical K_4 (together with u and v). If all edges in $N_j(u) \cap N_i(v)$ have color i or j, then as $|N_j(u) \cap N_i(v)| \geqslant R(m-1)$, there is a monochromatic copy of K_{m-1} in $N_j(u) \cap N_i(v)$. By adding either u or v we create a monochromatic copy of K_m. ■

Claim 17. *Let $\chi : E(K_n) \to \omega$ be a coloring. Assume there are vertices u, v such that $\chi(uv) = i$ and $|N_i(u) \cap N_i(v)| \geqslant ER(4, \ell-1, r)$. Then there exists in $\{u, v\} \cup (N_i(u) \cap N_i(v))$ either a monochromatic K_4, a rainbow K_r, or a lexical K_ℓ.*

Proof of the claim: Let $V := N_i(u) \cap N_i(v)$. Since $|V| \geqslant \mathrm{ER}(4, \ell-1, r)$, to avoid rainbow K_r and monochromatic K_4 inside V, we must have a lexical $K_{\ell-1}$ inside V. Note that if there is any edge xy in color i inside V, then $\{u, v, x, y\}$ induces a monochromatic K_4 in color i. We may assume then that there is no edge in color i inside V. Thus, any lexical $K_{\ell-1}$ inside V induces a lexical K_ℓ together with either u or v. ■

Similar to the proof of Lemma 14 in [1], we will show that given the upper bounds on color degrees given by Claims 16 and 17, with positive probability a random subset of size $\Theta(r)$ contains very few color repetitions. By deleting a vertex from each edge with a repeated color, we obtain a rainbow K_r.

Proof of Theorem 10 (3). First, we note that we can assume that r is sufficiently large. Indeed, by changing the constant if needed the result follows for small values of r. Let $c > 0$ be a constant larger than the constant given by Lemma 14. Note that $\mathrm{ER}(4, 3, r) \leqslant 4c \cdot r^3/\log r$.

Let $n := 60r \cdot \left(c \cdot \frac{r^3}{\log r}\right)^2$ and $\chi : E(K_n) \to \omega$ be a coloring. We will show that χ contains either a monochromatic K_4, lexical K_4, or rainbow K_r.

If χ is $60r \cdot \left(c \cdot \frac{r^3}{\log r}\right)$-good, then, by Lemma 14, there exists a rainbow clique of order r under χ. Let us assume then that there is a vertex v and a color, say 1, such that neighborhood of v in color 1 has size $|N_1(v)| \geqslant 60r \cdot \left(c \cdot \frac{r^3}{\log r}\right)$.

Let $V := N_1(v)$. By Claims 16 and 17, we may assume that every vertex $u \in V$ satisfies

$$|N_i(u) \cap V| < 6 \text{ for every } i \neq 1. \tag{3.1}$$

$$|N_1(u) \cap V| < 4c \cdot \left(\frac{r^3}{\log r}\right). \tag{3.2}$$

Let $N := |V|$. For any subset $S \subset V$, let $X(S)$ be the number of monochromatic $K_{1,2}$ in S in a color different from 1; $Y(S)$ be the number of monochromatic $2K_2$ in S in a color different from 1; and $Z(S) = e_1(S)$ be the number of edges with color 1 in S. Let $T \subset V$ with $|T| = 3r$ be chosen uniformly at random. We will show that all random variables $X(T), Y(T), Z(T)$ are small in expectation. For a particular instance of such subset where X, Y, Z are small, we will be able to find a rainbow copy of K_r.

First, we upper bound the number of monochromatic structures in colors different from 1 inside V using (3.1). Note that there are at most $5N/2$ edges of each given color $i \neq 1$. Hence,

$$X(V) = \sum_{u \in V} \sum_{i \in \omega,\ i \neq 1} \binom{|N_i(u) \cap V|}{2} \leqslant N \cdot \frac{N}{5} \cdot \binom{5}{2} = 2N^2 \quad \text{and}$$

$$Y(V) \leqslant \frac{1}{2}\binom{N}{2} \cdot \frac{5N}{2} \leqslant \frac{5N^3}{8}.$$

For any fixed copy H of a monochromatic $K_{1,2}$ in V in a color different from 1, the probability that T contains all three vertices of H is at most $(3r/N)^3$. Similarly, for any fixed copy H of a monochromatic $2K_2$ in V in a color different from 1, the probability that T contains all four vertices of H is at most $(3r/N)^4$. Therefore, by linearity of expectation,

$$\mathbb{E}[X(T)+Y(T)] \leqslant 2N^2\left(\frac{3r}{N}\right)^3+\frac{5N^3}{8}\left(\frac{3r}{N}\right)^4=\frac{432r^3+405r^4}{8N}<\frac{r}{3}.$$

By Markov's Inequality, we conclude

$$\mathbb{P}(X(T)+Y(T)\geqslant r)\leqslant\frac{1}{3}.$$

From (3.2), the number of edges in color 1 inside V, $e_1(V)$, is at most

$$e_1(V)<2c\cdot\left(\frac{r^3}{\log r}\right)N\leqslant\frac{N^2}{30r}.$$

For each edge of color 1 in V, the probability that T contains both of its endpoints is at most $(3r/N)^2$, then

$$\mathbb{E}[Z(T)]<e_1(V)\cdot\frac{(3r)^2}{N^2}<\frac{N^2}{30r}\cdot\frac{9r^2}{N^2}<\frac{r}{3}$$

and, by Markov's inequality, we have

$$\mathbb{P}\left(Z(T)\geqslant r\right)\leqslant\frac{1}{3}.$$

Therefore, with positive probability, there exists a $3r$-subset $T\subset V$ that contains at most r edges of color 1, and at most r monochromatic $2K_2$ and $K_{1,2}$ in colors different from 1. By deleting one vertex from each edge of color 1, and one vertex from each monochromatic $2K_2$ and $K_{1,2}$ from T, we obtain a subset $T'\subset T$ of order at least r containing no monochromatic $2K_2$ and $K_{1,2}$. Since such T' induces a rainbow clique, the result follows. □

4 Lower bound for $\mathrm{ER}(3,\ell,r)$

In this section we will show the following.

Claim 18. *For every integers $\ell\geqslant 3$ and $r\geqslant 3$, we have that*

$$ER(3,\ell+1,r)\geqslant(ER(3,\ell,r)-1)\cdot(ER(3,3,r)-1)+1.$$

The case $\ell=3$ in Theorem 9, namely $\mathrm{ER}(3,3,r)\geqslant\Omega(r^3/\log r)$, follows from (1.1). Theorem 9 then follows immediately from the claim. Indeed, by induction on $\ell\geqslant 3$, the bound $\mathrm{ER}(3,\ell,r)-1\geqslant(cr^3/\log r)^{\ell-2}$ implies

$$\mathrm{ER}(3,\ell+1,r)-1\geqslant\left(\frac{cr^3}{\log r}\right)^{\ell-2}\cdot\left(\frac{cr^3}{\log r}\right)=\left(\frac{cr^3}{\log r}\right)^{\ell-1}.$$

Proof of the claim: Let $N_\ell:=\mathrm{ER}(3,\ell,r)-1$. By the definition of the number $\mathrm{ER}(m,\ell,r)$, there exists a coloring $\chi_\ell:\binom{[N_\ell]}{2}\to\omega$ avoiding monochromatic K_3, lexical K_ℓ, and rainbow K_r. In what follows we consider colorings χ_3 and χ_ℓ that use disjoint sets of colors[1].

We will define a coloring $\chi : \binom{[N_\ell \cdot N_3]}{2} \to \omega$ as follows. Partition the vertices $[N_\ell \cdot N_3]$ into sets $V_1, \ldots, V_{N_3}$ of equal size $|V_i| = N_\ell$ by setting

$$V_i \coloneqq [(i-1)N_\ell + 1, iN_\ell] \text{ for } 1 \leqslant i \leqslant N_3.$$

For vertices $u \in V_i$ and $v \in V_j$ when $i \neq j$ we set

$$\chi(uv) \coloneqq \chi_3(ij).$$

For every $i \in [N_3]$ and vertices $u, v \in V_i$ we write $u = (i-1) \cdot N_\ell + u'$, $v = (i-1) \cdot N_\ell + v'$, and set

$$\chi(uv) \coloneqq \chi_\ell(u'v').$$

We claim χ contains no monochromatic K_3, lexical $K_{\ell+1}$, or rainbow K_r. First, χ contains no monochromatic K_3. Indeed, if $a, b, c \in V_i$ then abc is not monochromatic since χ_ℓ contains no monochromatic K_3. If $a, b \in V_i$ and $c \in V_j$ for $i \neq j$ then $\chi(ab) \neq \chi(ac)$ since the colors from χ_3 and χ_ℓ are disjoint. Finally, if $a \in V_i$, $b \in V_j$, $c \in V_k$ for distinct i, j, k, then abc is not monochromatic since χ_3 does not contain monochromatic K_3.

Now, χ contains no lexical $K_{\ell+1}$. Indeed, let $v_1, v_2, \ldots, v_{\ell+1}$ be a potential lexical clique, meaning $\chi(v_i v_j) = \chi(v_r v_s)$ for $i < j$ and $r < s$ if and only if $i = r$. If $v_1, v_2 \in V_i$ for some i then $\chi(v_1 v_2) = \chi(v_1 u)$ only for vertices $u \in V_i$, we conclude $v_1, v_2, \ldots, v_{\ell+1} \in V_i$. As χ_ℓ contains no lexical K_ℓ, we conclude $v_1, \ldots, v_{\ell+1}$ cannot induce a lexical clique. Assume then $v_1 \in V_i$ and $v_2 \in V_j$ for $i \neq j$. As χ_3 is a proper coloring, $\chi(v_1 v_2) = \chi(v_1 u)$ only for vertices $u \in V_j$, we conclude $v_2, v_3, \ldots, v_{\ell+1} \in V_j$. Again, as χ_ℓ contains no lexical K_ℓ, we conclude $v_1, \ldots, v_{\ell+1}$ cannot induce a lexical clique.

Finally, χ contains no rainbow K_r. Indeed, if $a, b \in V_i$ and $c \in V_j$ for $i \neq j$ then $\chi(ac) = \chi(bc)$. The only two possibilities for a rainbow clique in χ are 1) if all vertices belong to same set V_i for some i, or 2) if they all belong to different sets. As χ_3 and χ_ℓ does not contain rainbow clique of order r, we conclude χ does not contain rainbow clique of order r as well. ∎

5 Concluding remarks

In this paper, we obtain upper bounds for the numbers $\mathrm{CR}(s, r)$ and $\mathrm{ER}(m, \ell, r)$. For any $s, m, \ell \geqslant 3$, we obtained the correct asymptotics for $\mathrm{CR}(s, r) = \Theta(r^3/\log r)^{s-2}$, $\mathrm{ER}(m, 3, r) = \Theta_m(r^3/\log r)$, and $\mathrm{ER}(3, \ell, r) = \Theta(r^3/\log r)^{\ell-2}$. However, there is still a gap between the known lower bounds and our upper bounds for $\mathrm{ER}(m, \ell, r)$ for general m and ℓ. We expect the known lower bounds to describe the correct asymptotic in this general case as well.

Conjecture 19. For every $m, \ell \geqslant 4$, there exists a constant $c = c(m, \ell) > 0$ such that, for every $r \geqslant 3$,

$$\mathrm{ER}(m, \ell, r) \leqslant c \cdot \left(\frac{r^3}{\log r} \right)^{\ell - 2}.$$

[1]We highlight that when $\ell = 3$ we consider two distinct colorings of $\binom{[N_3]}{2}$ using a disjoint set of colors.

Note that, for $\ell = 4$, we can show that $\mathrm{ER}(m,4,r) = O\left(r^{m+3+o(1)}\right)$ by a proof analogous to the one for $\mathrm{ER}(4,4,r) \leqslant c \cdot r^7/(\log r)^2$. However, we expect the correct behavior to be a much lower order of magnitude in the exponent of r.

Conjecture 20. For every $m \geqslant 4$, there exists a constant $c = c(m) > 0$ such that, for every $r \geqslant 3$,

$$\mathrm{ER}(m,4,r) \leqslant c \cdot \left(\frac{r^3}{\log r}\right)^2.$$

The smallest case in which we do not know the correct asymptotic and potential easiest case to deal with is when we forbid monochromatic and lexical K_4.

Conjecture 21. There exists a constant $c > 0$ such that, for every $r \geqslant 3$,

$$\mathrm{ER}(4,4,r) \leqslant c \cdot \left(\frac{r^3}{\log r}\right)^2.$$

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