New Bounds on Families Without Large Sunflowers

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Abstract

Distinct sets $F_1, F_2, \ldots, F_s$ are said to form a *sunflower* of size s and center of size i if there is an i-element set C satisfying $F_a \cap F_b = C$ for all $1 \leqslant a < b \leqslant s$. The present paper introduces the function $m_k(r_0, r_1, \ldots, r_{k-1})$, the maximum size of a collection of distinct k-sets in which for all $0 \leqslant i < k$ the maximum size of a sunflower with center of size i is at most r_i. One of the favorite open problems of Paul Erdős is whether $m_k(r, \ldots, r) < c(r)^k$ holds with some constant $c(r)$ independent of k. We present various inequalities and some exact results concerning $m_k(r_0, r_1, \ldots, r_{k-1})$. In particular we show that for k fixed and $r_0, \ldots, r_{k-1}$ simultaneously tending to infinity $m_k(r_0, \ldots, r_{k-1}) = (1 + o(1))r_0 \ldots r_{k-1}$.

Mathematics Subject Classifications: 05D05

1 Introduction

Let X be a finite set. We use $\binom{X}{k}$ to denote the family of all k-subsets of X. A subset $\mathcal{F} \subset \binom{X}{k}$ is called *a k-uniform hypergraph*, or simply *a k-graph*, on the ground set X. We say that $S_1, S_2, \ldots, S_r \in \mathcal{F}$ form *a sunflower of size r with center C* if $C = S_i \cap S_j = S_1 \cap \ldots \cap S_r$ for all $1 \leqslant i < j \leqslant r$. Define $\sigma_i(\mathcal{F})$ as the size of a largest sunflower with center of size i in $\mathcal{F}$. Define

$$m_k(r) = \max \{|\mathcal{F}| \colon \mathcal{F} \text{ is a } k\text{-graph with } \sigma_i(\mathcal{F}) \leqslant r \text{ for all } i = 0, 1, \ldots, k-1\}.$$

Since any two distinct k-sets form a sunflower, $m_k(1) = 1$.

In 1960, Erdős and Rado proved the so-called sunflower lemma.

Lemma 1.1 ([10]). For $r \geqslant 2$,

$$m_k(r) < k!r^k.$$

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The importance of this simple result lies in the fact that it is unrelated to the size of the vertex set X. It shows that sunflowers are *ubiquitous.* That is, any sufficiently large collection of k-subsets contains a relatively large sunflower.

Note that if k is relatively large with respect to r then the upper bound in Lemma 1.1 is close to $\left(\frac{kr}{e}\right)^k$. Erdős and Rado conjectured that the bound in Lemma 1.1 can be drastically improved.

Conjecture 1.2 ([10]). For $r \geqslant 2$, there exists a constant $C = C(r)$ such that

$$m_k(r) \leqslant C^k. \tag{1}$$

It has been one of the favorite open problems of Erdős (cf. [7]).

In a recent breakthrough, Alweiss, Lovett, Wu and Zhang [3] improved the bound in the sunflower lemma as follows.

Theorem 1.3 ([3]). For $r \geqslant 2$, there is some constant $C > 0$ such that

$$m_k(r) \leqslant (Cr^3 \log k \cdot \log\log k)^k.$$

For further improvements see [22], [16], [24]. Currently the best bound due to Bell, Chueluecha and Warnke [4] is $(Cr \log k)^k$ for some constant $C > 0$. However, (1) is still wide open.

Abbott, Hanson and Sauer determined $m_2(r)$.

Theorem 1.4 ([1]). For $r \geqslant 2$,

$$m_2(r) = \begin{cases} r(r+1), & r \text{ is even;} \\ \frac{1}{2}(r+1)(2r-1), & r \text{ is odd.} \end{cases} \tag{2}$$

Note that the construction for the case r even is two disjoint copies of K_{r+1}, the complete graph on $r+1$ vertices.

Abbott and Hanson gave an upper bound on $m_3(r)$.

Theorem 1.5 ([2]). For $r \geqslant 7$,

$$m_3(r) < 1.8r^3 + 9.8r^2. \tag{3}$$

In 1999, Kostochka, Rödl and Talysheva improved this bound, however only for very large r.

Theorem 1.6 ([19]). Let k be fixed and r be sufficiently large. Then there exists a large constant c_k such that

$$m_k(r) < r^k(1 + c_k r^{-2^{-k}}). \tag{4}$$

Note that (4) implies $m_k(r) = (1 + o(1))r^k$ for any fixed k and $r \to \infty$.

The late Michel Deza (cf. [6]) was the first to realize that the sunflower lemma can be used to prove results related to the Erdős-Ko-Rado Theorem ([9]). This idea was further developed by Füredi ([18]). The celebrated work of Razborov [23] on Boolean complexity is making heavy use of the sunflower lemma as well.

However, in most applications it is essential to limit the size of the sunflowers in function of the size of their center. This motivates the following definition.

Definition 1.7. Let $\emptyset \neq \mathcal{F} \subset \binom{[n]}{k}$ be a k-graph and $\vec{r} = (r_0, r_1, \dots, r_{k-1})$ be an integer vector, $r_i > 0$, $0 \leqslant i < k$. Let

$$m_k(\vec{r}) = \max \{ |\mathcal{F}| \colon \sigma_i(\mathcal{F}) \leqslant r_i \text{ for } i = 0, 1, \dots, k-1 \} .$$

Note that $m_k(r) = m_k((r, r, \dots, r))$. Obviously $m_1((r)) = r$. When it causes no confusion, we shall use $m_k(r_0, r_1, \dots, r_{k-1})$ to denote $m_k((r_0, r_1, \dots, r_{k-1}))$. For $\vec{r} = (r_0, r_1, \dots, r_{k-1})$, we also use $m_{k+1}(s, \vec{r})$ to denote $m_{k+1}(s, r_0, r_1, \dots, r_{k-1})$.

The exact value of $m_2(s, r)$ is given by

Theorem 1.8 ([5]).

$$m_2(s, r) = \begin{cases} sr + \left\lfloor \frac{s}{\lfloor \frac{r+1}{2} \rfloor} \right\rfloor \left\lfloor \frac{r}{2} \right\rfloor, & s \geqslant \frac{r}{2}; \\ sr, & s < \frac{r}{2}. \end{cases}$$

Our first result establishes a general bound on $m_k(\vec{r})$. For $\vec{r} = (r, r, \dots, r)$ it reduces to Lemma 1.1.

Proposition 1.9. For any positive integer vector $\vec{r} = (r_0, \dots, r_{k-1})$,

$$r_0 r_1 \dots r_{k-1} \leqslant m_k(\vec{r}) \leqslant k! r_0 r_1 \dots r_{k-1}. \tag{5}$$

The next results relate $m_k(1, \vec{r})$ and $m_k(s, \vec{r})$ to $m_{k-1}(\vec{r})$ under size restrictions to r_i.

Theorem 1.10. Let $\vec{r} = (r_0, \dots, r_{k-2})$ be a positive integer vector satisfying $r_0 \geqslant k+1$ and $r_0 r_1 \dots r_i \geqslant (i+2)k^{i+1}$ for each $i = 1, 2, \dots, k-2$. Then

$$m_k(1, \vec{r}) = m_{k-1}(\vec{r}). \tag{6}$$

Let $\min \vec{r}$ denote $\min\limits_{0 \leqslant i \leqslant k-1} r_i$.

Theorem 1.11. For $s \geqslant 1$, $k \geqslant 2$ and $\vec{r} = (r_0, \dots, r_{k-2})$ satisfying $\min \vec{r} \geqslant 3ks$,

$$m_k(s, \vec{r}) = s \cdot m_{k-1}(\vec{r}). \tag{7}$$

As a corollary, we determine the exact value of $m_k(\vec{r})$ for $r_{i+1} \geqslant 3(k-i) r_i$, $i = 0, 1, \dots, k-2$.

Corollary 1.12. Let $k \geqslant 2$ and $\vec{r} = (r_0, \dots, r_{k-1})$. If $r_{i+1} \geqslant 3(k-i) r_i$ holds for $i = 0, 1, \dots, k-2$, then

$$m_k(\vec{r}) = r_0 r_1 \dots r_{k-1}. \tag{8}$$

For $\min\limits_{1 \leqslant i \leqslant k-1} r_i$ sufficiently large respect to k, $m_k(\vec{r})$ is determined asymptotically.

Theorem 1.13. Suppose that $r := \min\limits_{1 \leqslant i \leqslant k-1} r_i$ is sufficiently large with respect to k. Then

$$m_k(\vec{r}) = (1 + o(1)) r_0 r_1 \dots r_{k-1}.$$

For $k = 3$, a better upper bound than (5) is established. For $s = r = p$, it improves Theorem 1.5.

Theorem 1.14. For $r \geqslant 6$,

$$m_3(s, r, p) \leqslant \frac{5}{3} s m_2(r, p) + 2sp + s. \tag{9}$$

We determine $m_3(1, r, r)$ for $r \geqslant 2$.

Theorem 1.15. For $r \geqslant 3$,

$$m_3(1, r, r) = m_2(r, r) = \begin{cases} r(r + 1), & r \text{ is even;} \\ \frac{1}{2}(r + 1)(2r - 1), & r \text{ is odd.} \end{cases} \tag{10}$$

Remark. Let us mention that $m_3(1, 2, 2) = 10$ along with $m_3(2, 2, 2) = 20$ was proved by Abbott and Hanson [2].

For $\mathcal{F} \subset \binom{X}{k}$ and $x \in X$, define

$$\mathcal{F}(x) = \{F \setminus \{x\} \colon x \in F \in \mathcal{F}\}, \ \mathcal{F}(\bar{x}) = \{F \colon x \notin F \in \mathcal{F}\}$$

and note that $|\mathcal{F}| = |\mathcal{F}(x)| + |\mathcal{F}(\bar{x})|$. For $A \subset B \subset X$, define

$$\mathcal{F}(A, B) = \{F \setminus B \colon F \in \mathcal{F}, \ F \cap B = A\}.$$

For $A = B$, we simply write $\mathcal{F}(A)$.

2 Proof of Proposition 1.9

Let us first prove a recursive lower bound on $m_k(\vec{r})$.

Proposition 2.1. Let $\vec{p} = (p_0, \ldots, p_{\ell-1})$ and $\vec{r} = (r_0, \ldots, r_{k-1})$ be positive integer vectors of respective length ℓ and k. Then

$$m_{\ell+k}(\vec{p}, \vec{r}) \geqslant m_\ell(\vec{p}) m_k(\vec{r}). \tag{11}$$

Proof. Set $m_\ell = m_\ell(\vec{p})$ and $m_k = m_k(\vec{r})$. Let $\mathcal{H}$ ($\mathcal{G}$) be an ℓ-graph (k-graph) with m_ℓ (m_k) edges, respectively showing the exactness of the corresponding sunflower bounds.

For each edge $H \in \mathcal{H}$ let $\mathcal{G}_H$ be an isomorphic copy of $\mathcal{G}$. The important requirement is that $\mathcal{G}_H$ is vertex disjoint to $\mathcal{H}$ and all the $\mathcal{G}_H$ are pairwise vertex disjoint for distinct edges H, H'.

Now define

$$\mathcal{F} = \{H \cup G \colon H \in \mathcal{H}, \ G \in \mathcal{G}_H\}.$$

The fact that $|\mathcal{F}| = |\mathcal{H}||\mathcal{G}|$ is evident. Let us show that it has no unwanted sunflowers.

Suppose first that $H_1 \cup G_1, \ldots, H_q \cup G_q$ form a sunflower of size q and with center of size i, $0 \leqslant i < \ell$. Then $H_1, \ldots, H_q$ have to be distinct. Consequently, for $1 \leqslant a < b \leqslant q$,

$G_a \cap G_b = \emptyset$. I.e., $(H_a \cup G_a) \cap (H_b \cup G_b) = H_a \cap H_b$. We infer that $H_1, \dots, H_q$ form a sunflower of size q with center of size i. Thus $q \leqslant p_i$.

Note that for $H_a \neq H_b$, $(H_a \cup G_a) \cap (H_b \cup G_b) = H_a \cap H_b$ implies that the size of the intersection is less than ℓ. Hence if $H_1 \cup G_1, \dots, H_q \cup G_q$ form a sunflower with size i and $i \geqslant \ell$ then $H_1 = \ldots = H_q$. Thus $G_1, \dots, G_q$ are from the same copy of $\mathcal{G}$ and form a sunflower with center of size $i - \ell$. This proves $q \leqslant r_{i-\ell}$ as desired. □

Applying (11) with $\ell = 1$, $m_1(s) = s$ yields

$$m_{k+1}(s, r_0, \dots, r_{k-1}) \geqslant s m_k(\vec{r}). \tag{12}$$

Starting with $m_1(r_{k-1}) = r_{k-1}$, applying (12) consecutively $k-1$ times with $s = r_{k-1}, \dots, s = r_0$ leads to the lower bound of (5).

Recall that T is called a *transversal* of $\mathcal{F}$ if $T \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The *transversal number* $\tau(\mathcal{F})$ is defined as the minimum size of a transversal of $\mathcal{F}$. If T is a transversal then

$$|\mathcal{F}| \leqslant \sum_{x \in T} |\mathcal{F}(x)|. \tag{13}$$

If $F_1, \dots, F_s \in \mathcal{F}$ are pairwise disjoint and $\nu(\mathcal{F}) = s$, then $F_1 \cup \ldots \cup F_s$ is a transversal of $\mathcal{F}$. Thus,

$$m_k(\vec{r}) \leqslant k r_0 \cdot m_{k-1}(r_1, \dots, r_{k-1}). \tag{14}$$

Therefore,

$$m_k(\vec{r}) \leqslant k! r_0 r_1 \dots r_{k-1}. \tag{15}$$

Thus Proposition 1.9 is proven.

3 Proof of Theorems 1.10 and 1.11

Let $\mathcal{F}$ be a k-graph and let $\mathcal{G}$ be a t-graph, $t \leqslant k$. We say that $\mathcal{F}$ is *covered* by $\mathcal{G}$ if for any $F \in \mathcal{F}$ there exists $G \in \mathcal{G}$ such that $G \subset F$.

By applying the branching process method developed by the first author in [11], we show that any intersecting k-graph with $\tau(\mathcal{F}) = t$ can be covered by a t-graph with at most tk^{t-1} edges. Using this fact and more detailed analysis, we prove Theorem 1.10.

Proof of Theorem 1.10. Let $\mathcal{F}$ be an intersecting k-graph with $\sigma_i(\mathcal{F}) \leqslant r_{i-1}$, $1 \leqslant i \leqslant k-1$, $k \geqslant 4$ and $|\mathcal{F}| = m_k(1, \vec{r})$. Set $t = \tau(\mathcal{F})$. If $t = 1$ then $\mathcal{F}$ is a star with center x for some x. It follows that

$$|\mathcal{F}| = |\mathcal{F}(x)| \leqslant m_{k-1}(\vec{r}).$$

Assume that $3 \leqslant t \leqslant k$ and let us form a branching process of t stages. Let $X = \cup_{F \in \mathcal{F}} F$. A sequence $S = (x_1, x_2, \dots, x_\ell)$ is an ordered sequence of distinct elements of X

and we use $\widehat{S}$ to denote the underlying unordered set $\{x_1, x_2, \dots, x_\ell\}$. Fix a transversal T of $\mathcal{F}$ with $|T| = t$. Let (x_1) be a sequence of length 1, $x_1 \in T$.

If $(x_1, \dots, x_p)$ is a sequence and $p < t$ then choose an arbitrary $F_p \in \mathcal{F}$ with $F_p \cap \{x_1, \dots, x_p\} = \emptyset$ ($\tau(\mathcal{F}) = t$ guarantees the existence of F_p). Make k new sequences $(x_1, \dots, x_p, x_{p+1})$ with $x_{p+1} \in F_p$. If $p = t$ then stop. Eventually we construct tk^{t-1} sequences.

Claim 1. *For each $F \in \mathcal{F}$, there is a sequence S of length t with $\widehat{S} \subset F$.*

Proof. Let $S = (x_1, \dots, x_\ell)$ be a sequence of maximal length that occurred at some stage of the branching process satisfying $\widehat{S} \subset F$. Suppose indirectly that $\ell < t$. Since $T \cap F \neq \emptyset$ at the first stage, there is a sequence (x_1) with $x_1 \in T$ such that $\{x_1\} \subset F$. Thus $\ell \geqslant 1$. Since $\ell < t$, at some stage S was picked and there is some $F_\ell \in \mathcal{F}$ with $\widehat{S} \cap F_\ell = \emptyset$ being chosen. Since $F \cap F_\ell \neq \emptyset$, there is some $y \in F \cap F_\ell$. Then $(x_1, \dots, x_\ell, y)$ is a longer sequence satisfying $\{x_1, \dots, x_\ell, y\} \subset F$, contradicting the maximality of ℓ. $\square$

By Claim 1 and $r_0 r_1 \dots r_{t-2} \geqslant tk^{t-1}$,

$$|\mathcal{F}| \leqslant tk^{t-1} m_{k-t}(r_{t-1}, \dots, r_{k-2}) \overset{(12)}{\leqslant} \frac{tk^{t-1}}{r_0 r_1 \dots r_{t-2}} m_{k-1}(\vec{r}) \leqslant m_{k-1}(\vec{r}).$$

We are left with the case $t = 2$. Define

$$\mathcal{A} = \{A \colon |A| = 2,\ A \text{ is a cover of } \mathcal{F}\}.$$

Note that $\tau(\mathcal{A}) = 2$ would enable us to make a branching process in which we choose the second set also from $\mathcal{A}$. Hence instead of $2k$ we get 4 sequences of length 2 during the branching process. Since $r_0 \geqslant k + 1 \geqslant 4$, we have

$$|\mathcal{F}| \leqslant 4 m_{k-2}(r_1, \dots, r_{k-2}) \overset{(12)}{\leqslant} \frac{4}{r_0} m_{k-1}(\vec{r}) \leqslant m_{k-1}(\vec{r}).$$

Thus we may assume that $\tau(\mathcal{A}) = 1$. I.e., $\mathcal{A}$ is a star, say $|\mathcal{A}| = a$.

Case 1. $a = 1$, say $\mathcal{A} = \{(1, 2)\}$.

In the branching process we get $2k$ sequences (i, x), $i = 1$ or 2. Two of the sequences are $(1, 2)$ and $(2, 1)$; giving rise to the same unordered set $\{1, 2\}$. The remaining $2k - 2$ of them are not a transversal of $\mathcal{F}$. I.e., for $2(k-1)$ sequences of length 2, we can extend the branching process to length 3. Thus,

$$|\mathcal{F}| \leqslant m_{k-2}(r_1, \dots, r_{k-2}) + 2(k-1)k m_{k-3}(r_2, \dots, r_{k-2}) \overset{(12)}{\leqslant} \left(\frac{1}{r_0} + \frac{2(k^2 - k)}{r_0 r_1}\right) m_{k-1}(\vec{r}).$$

Since $r_0 r_1 \geqslant 3k^2$ and $r_0 \geqslant k + 1 \geqslant 4$,

$$|\mathcal{F}| < \left(\frac{1}{4} + \frac{2}{3}\right) m_{k-1}(\vec{r}) < m_{k-1}(\vec{r}).$$

Case 2. $a \geqslant 2$, $\mathcal{A} = \{(1, 1 + i) \colon 1 \leqslant i \leqslant a\}$.

Note that if $a > k$ then $\mathcal{F}$ is a star (with center 1) and we are done. Thus $2 \leqslant a \leqslant k$. Note that $\{2, 3, \dots, a+1\} \subset F$ for every $F \in \mathcal{F}(\bar{1})$. It follows that

$$|\mathcal{F}(\bar{1})| \leqslant m_{k-a}(r_{a-1}, \dots, r_{k-2}) \overset{(12)}{\leqslant} \frac{1}{r_0 r_1 \dots r_{a-2}} m_{k-1}(\vec{r}) \leqslant \frac{1}{ak^{a-1}} m_{k-1}(\vec{r}).$$

To bound $|\mathcal{F}(1)|$, we start the branching process by the sequence (1). Using $F \in \mathcal{F}(\bar{1})$, we obtain k sequences $(1, b)$ of length 2 with $b \in F$. By $\{2, 3, \dots, a+1\} \subset F$ for all $F \in \mathcal{F}(\bar{1})$, exactly a of them form a transversal of $\mathcal{F}$. For each $(1, b)$ with $b \in F$ that is not a transversal, we can extend the branching process to get a sequence of length 3. Thus,

$$\begin{aligned} |\mathcal{F}(1)| &\leqslant am_{k-2}(r_1, \dots, r_{k-2}) + (k-a)km_{k-3}(r_2, \dots, r_{k-2}) \\ &\overset{(12)}{\leqslant} \left(\frac{a}{r_0} + \frac{k(k-a)}{r_0 r_1}\right) m_{k-1}(\vec{r}) \\ &< \left(\frac{a}{k+1} + \frac{k(k-a)}{3k^2}\right) m_{k-1}(\vec{r}). \end{aligned}$$

We need

$$\frac{a}{k+1} + \frac{k-a}{3k} + \frac{1}{ak^{a-1}} < 1. \tag{16}$$

Since $2 \leqslant a \leqslant k$, we have

$$\frac{a}{k+1} + \frac{k-a}{3k} \leqslant \frac{k}{k+1}, \ \frac{1}{ak^{a-1}} \leqslant \frac{1}{2k} < \frac{1}{k+1}.$$

Thus $|\mathcal{F}| < m_{k-1}(\vec{r})$ and the theorem is proven. □

The distinct sets $F_0, \dots, F_r$ are said to form a *pseudo sunflower* of size $r+1$ and center C if $C \subsetneq F_0$ and the sets $F_i \setminus C$ are pairwise disjoint, $0 \leqslant i \leqslant r$.

Theorem 3.1 (Füredi [17], cf. also [14]). Let k, r be positive integers and let $\mathcal{F}$ be a k-graph not containing any pseudo sunflower of size $r+1$. Then

$$|\mathcal{F}| \leqslant r^k. \tag{17}$$

For $\mathcal{F} \subset 2^{[n]}$ and $0 \leqslant i \leqslant n$, define

$$\mathcal{F}^{(i)} = \{F \in \mathcal{F} \colon |F| = i\}.$$

Let $\mathcal{F}$ be a k-graph with $\nu(\mathcal{F}) \leqslant s$. We define a *basis* $\mathcal{B}(\mathcal{F})$ which is not necessarily unique by the following process. We start with $\mathcal{F}^0 = \mathcal{F}$. Note that $\mathcal{F}^0$ is an antichain. At the ith step try and find in the current family $\mathcal{F}^i$ a pseudo sunflower $F_0, F_1, \dots, F_{ks}$ (the size of F_j may be distinct). Let C_i be the center of the pseudo sunflower. Then let $\mathcal{F}^{i+1}$ be the family obtained from $\mathcal{F}^i$ by deleting all sets containing C_i and adding C_i. Clearly $\mathcal{F}^{i+1}$ is also an antichain.

Claim 2. *If $\nu(\mathcal{F}^i) \leqslant s$ then $\nu(\mathcal{F}^{i+1}) \leqslant s$.*

Proof. Suppose that $C_i, F_{i,1}, F_{i,2}, \ldots, F_{i,s} \in \mathcal{F}^{i+1}$ form a matching of size $s+1$. Clearly $F_{i,1}, F_{i,2}, \ldots, F_{i,s} \in \mathcal{F}^i$. Since $F_0, F_1, \ldots, F_{ks}$ form a pseudo sunflower in $\mathcal{F}^i$ with center C_i, there exists F_j such that $F_j, F_{i,1}, F_{i,2}, \ldots, F_{i,s}$ form a matching in $\mathcal{F}^i$, contradicting $\nu(\mathcal{F}^i) \leqslant s$. □

Continue this process until no more pseudo sunflower of size $ks+1$ can be formed. Let $\mathcal{B} := \mathcal{B}(\mathcal{F})$ be the final family. Clearly, $\mathcal{B}$ is an antichain and for all $F \in \mathcal{F}$ there exists $B \in \mathcal{B}$ with $B \subset F$. By Claim 2, we have $\nu(\mathcal{B}) \leqslant s$.

Using the defined basis and Theorem 3.1, we prove Theorem 1.11.

Proof of Theorem 1.11. Let $\mathcal{F}$ be a k-graph with $|\mathcal{F}| = m_k(s, \vec{r})$ and let $\mathcal{B} := \mathcal{B}(\mathcal{F})$ be its basis. Since $\mathcal{B}$ contains no pseudo sunflower of size $ks+1$, $\sigma_i(\mathcal{B}) \leqslant ks$ for $i = 1, 2, \ldots, k-1$.

We prove (7) by induction on s. The case $s = 1$ is verified by Theorem 1.10. Now assume (7) holds for $s-1$ and we prove it for s. If $\{x\} \in \mathcal{B}^{(1)}$ then $\nu(\mathcal{B}) \leqslant s$ implies $\nu(\mathcal{B}(\bar{x})) = s-1$. It follows that $\nu(\mathcal{F}(\bar{x})) = s-1$. By the induction hypothesis,

$$|\mathcal{F}| = |\mathcal{F}(x)| + |\mathcal{F}(\bar{x})| \leqslant m_{k-1}(\vec{r}) + (s-1)m_{k-1}(\vec{r}) = sm_{k-1}(\vec{r}).$$

Thus we may assume that $\mathcal{B}^{(1)} = \emptyset$.

By Claim 2 we have $\nu(\mathcal{B}^{(i)}) \leqslant s$. Let $B_1, B_2, \ldots, B_\ell$ be a maximal matching in $\mathcal{B}^{(i)}$. By (17), for $3 \leqslant i \leqslant k$ we have

$$|\mathcal{B}^{(i)}| \leqslant \sum_{x \in B_1 \cup B_2 \cup \cdots \cup B_\ell} |\mathcal{B}^{(i)}(x)| \leqslant \ell i (ks)^{i-1} \leqslant si(ks)^{i-1}.$$

By Theorem 1.8,

$$|\mathcal{B}^{(2)}| \leqslant m_2(s, ks) \leqslant s(ks) = ks^2.$$

Using (12) it follows that

$$|\mathcal{F}| \leqslant \sum_{2 \leqslant i \leqslant k} |\mathcal{B}^{(i)}| \cdot m_{k-i}(r_{i-1}, \ldots, r_{k-2}) \leqslant \frac{ks^2}{r_0} m_{k-1}(\vec{r}) + m_{k-1}(\vec{r}) \sum_{3 \leqslant i \leqslant k} si \frac{(ks)^{i-1}}{r_0 r_1 \ldots r_{i-2}}.$$

By $r_i \geqslant 3ks$ we have

$$\frac{ks}{r_0} \leqslant \frac{1}{3}, \sum_{3 \leqslant i \leqslant k} \frac{i(ks)^{i-1}}{r_0 r_1 \ldots r_{i-2}} \leqslant \sum_{3 \leqslant i \leqslant \infty} \frac{i}{3^{i-1}} = \frac{7}{12}.$$

Thus $|\mathcal{F}| < sm_{k-1}(\vec{r})$ follows. □

4 Proof of Theorem 1.13

An edge-colouring of a k-graph $\mathcal{F}$ is called *proper* if the edges of the same colour are vertex-disjoint.

Theorem 4.1 ([21]). Let k be fixed and D be sufficiently large. Then for each $\mathcal{F} \subset \binom{[n]}{k}$ with $|\mathcal{F}(x)| \leqslant D$ for all $x \in [n]$ and $|\mathcal{F}(x,y)| \leqslant o(D)$ for all $x, y \in [n]$, there exists a proper edge-coloring of $\mathcal{F}$ with $D + o(D)$ colors.

Proof of Theorem 1.13. Let $\mathcal{F}$ be a k-graph with $|\mathcal{F}| = m_k(\vec{r})$ and $\sigma_i(\mathcal{F}) \leqslant r_i$, $0 \leqslant i < k$. Let $X = \cup_{F \in \mathcal{F}} F$. Clearly, $|\mathcal{F}(x)| \leqslant m_{k-1}(r_1, \ldots, r_{k-1})$ for all $x \in X$ and $|\mathcal{F}(x,y)| \leqslant m_{k-2}(r_2, \ldots, r_{k-1}) \leqslant \frac{1}{r_1} m_{k-1}(r_1, \ldots, r_{k-1})$ for all $x, y \in X$. Then by applying Theorem 4.1 with

$$D = m_{k-1}(r_1, \ldots, r_{k-1}),$$

we infer that there is a proper edge-coloring of $\mathcal{F}$ with $(1+o(1))D$ colors. Since $\nu(\mathcal{F}) \leqslant r_0$, we conclude that

$$|\mathcal{F}| \leqslant (1 + o(1))Dr_0 = r_0(1 + o(1))m_{k-1}(r_1, \ldots, r_{k-1}).$$

Recall that $m_1(r_{k-1}) = r_{k-1}$. By induction on k the result follows. $\square$

Note that $\sigma_i(\mathcal{F}) = 1$ means that $|F \cap F'| \neq i$ for $F, F' \in \mathcal{F}$. Hence for $\vec{r} = (r_0, \ldots, r_\ell, 1, 1, \ldots, 1)$, $m_k(\vec{r})$ is the answer of the sunflower problem for families satisfying $|F \cap F'| \leqslant \ell$ for all $F, F' \in \mathcal{F}$. In the case $\ell = 1$ such families are called *linear* hypergraphs.

Theorem 4.2. Suppose that $k \geqslant 3$ is fixed and $r_0, r_1, \ldots, r_\ell$ are tending to infinity. Then for $1 \leqslant \ell \leqslant k$,

$$m_k(r_0, \ldots, r_\ell, 1, \ldots, 1) = (1 + o(1))r_0 r_1 \ldots r_\ell. \tag{18}$$

Proof. We prove (18) by induction on ℓ. Clearly, $m_k(r_0, 1, \ldots, 1) = r_0$. Now we assume that (18) holds for $\ell - 1$ and prove it for $\ell \geqslant 1$. Let $\mathcal{F}$ be a k-graph demonstrating the exactness of the value of $m_k(r_0, \ldots, r_\ell, 1, \ldots, 1)$. Then for any two distinct vertices x, y,

$$|\mathcal{F}(x)| \leqslant m_{k-1}(r_1, \ldots, r_\ell, 1, \ldots, 1) \text{ and } |\mathcal{F}(x,y)| \leqslant m_{k-2}(r_2, \ldots, r_\ell, 1, \ldots, 1).$$

Thus $\max_{x,y} |\mathcal{F}(x,y)| \leqslant m_{k-1}(r_1, \ldots, r_\ell, 1, \ldots, 1)/r_1$. By applying applying Theorem 4.1 and the induction hypothesis, we conclude that

$$\begin{aligned} m_k(r_0, \ldots, r_\ell, 1, \ldots, 1) &\leqslant (1 + o(1))r_0 m_{k-1}(r_1, \ldots, r_\ell, 1, \ldots, 1) \\ &= (1 + o(1))r_0 r_1 \ldots r_\ell. \end{aligned}$$ $\square$

Suppose that $\mathcal{F}$ is a k-graph with $\nu(\mathcal{F}) \leqslant s$. We say that $\mathcal{F}$ is *resilient* (cf. [12]) if $\nu(\mathcal{F}(\bar{x})) = s$ for all vertices x of $\mathcal{F}$. For $\mathcal{F} \subset \binom{[n]}{k}$, define $\Delta(\mathcal{F}) = \max_{i \in [n]} |\mathcal{F}(i)|$.

Theorem 4.3. For $r \geqslant \max\{\binom{k+1}{2}s, k^2(k-2)\}$,

$$m_k(s, r, 1, 1, \ldots, 1) = sr.$$

Proof. Let $\mathcal{F}$ be a linear k-graph satisfying $\nu(\mathcal{F}) = s$, $\Delta(\mathcal{F}) \leqslant r$.

Claim 3. *If $\mathcal{F}$ is resilient then $\Delta(\mathcal{F}) \leqslant ks$.*

Proof. Choose x with $|\mathcal{F}(x)| = \Delta(\mathcal{F})$ and $F_1, \ldots, F_s \in \mathcal{F}(\bar{x})$ a matching. Let $\mathcal{F}(x) = \{E_1, \ldots, E_d\}$. By linearity it is a matching. By $\nu(\mathcal{F}) = s$, $E_i \cap (F_1 \cup \ldots \cup F_s) \neq \emptyset$. We infer $\Delta(\mathcal{F}) \leqslant |F_1 \cup \ldots \cup F_s| = ks$. □

If $\mathcal{F}$ is not resilient remove z with $\nu(\mathcal{F}(\bar{z})) = s - 1$ and repeat. Eventually we get a resilient, linear k-graph, say $\mathcal{G}$ with $\nu(\mathcal{G}) = w$ and

$$|\mathcal{F}| \leqslant (s-w)r + |\mathcal{G}|. \tag{19}$$

By Claim 3 we have $\Delta(\mathcal{G}) \leqslant kw$. Let $F_1, F_2, \ldots, F_w$ be a maximal matching in $\mathcal{G}$, $Y = F_1 \cup \ldots \cup F_w$. Partition $\mathcal{G} = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A} = \{G \in \mathcal{G} \colon |G \cap Y| = 1\}$. Note

$$\sum_{y \in Y} |\mathcal{G}(y)| \geqslant |\mathcal{A}| + 2|\mathcal{B}| = |\mathcal{G}| + |\mathcal{B}| = 2|\mathcal{G}| - |\mathcal{A}|.$$

In particular,

$$|\mathcal{G}| \leqslant \frac{1}{2} \sum_{y \in Y} |\mathcal{G}(y)| + \frac{1}{2}|\mathcal{A}|. \tag{20}$$

We divide Y into $Y_1 \cup Y_2$ where

$$Y_1 = \cup \{F_i \colon 1 \leqslant i \leqslant w,\ \mathcal{A}(x) \neq \emptyset \text{ holds for at most one } x \in F_i\},\ Y_2 = Y \setminus Y_1.$$

Set $\frac{|Y_j|}{k} = w_j$, $w_1 + w_2 = w$. For Y_1 we have

$$\sum_{y \in Y_1} |\mathcal{A}(y)| \leqslant \frac{|Y_1|}{k} \Delta(\mathcal{G}) = w_1 k w. \tag{21}$$

Let $F_i = \{x_1, x_2, \ldots, x_k\} \subset Y_2$. Note that $\mathcal{A}(x_1), \ldots, \mathcal{A}(x_k)$ are pairwise cross-intersecting $(k-1)$-graphs that are all matchings. Since at least two of $\mathcal{A}(x_1), \ldots, \mathcal{A}(x_k)$ are non-empty, we infer that $|\mathcal{A}(x)| \leqslant k - 1$ for all $x \in F_i$. Thus,

$$\sum_{y \in Y_2} |\mathcal{A}(y)| \leqslant \frac{|Y_2|}{k} k(k-1) = w_2 k(k-1). \tag{22}$$

Note that

$$\sum_{y \in Y} |\mathcal{G}(y)| \leqslant kw\Delta(\mathcal{G}) \leqslant k^2 w^2. \tag{23}$$

If $w \leqslant k - 2$ then

$$|\mathcal{G}| \leqslant \sum_{y \in Y} |\mathcal{G}(y)| \leqslant k^2 w^2 \leqslant k^2 w(k-2).$$

By $r \geqslant k^2(k-2)$, it follows that

$$|\mathcal{F}| \leqslant (s-w)r + |\mathcal{G}| \leqslant sr - w(r - k^2(k-2)) \leqslant sr.$$

If $w \geqslant k-1$ then by (21) and (22)

$$|\mathcal{A}| \leqslant \sum_{y \in Y} |\mathcal{A}(y)| \leqslant w_1 kw + w_2 k(k-1) \leqslant w_1 kw + w_2 kw \leqslant kw^2.$$

By (20) we infer that

$$|\mathcal{G}| \leqslant \frac{1}{2}(k^2w^2 + kw^2) = \frac{k(k+1)}{2} w^2.$$

Using $r \geqslant \binom{k+1}{2} s \geqslant \binom{k+1}{2} w$, we conclude that

$$|\mathcal{F}| \leqslant (s-w)r + |\mathcal{G}| \leqslant sr - w\left(r - \frac{k(k+1)}{2} w\right) \leqslant sr. \qquad \square$$

5 Proof of Theorem 1.14

Unless otherwise stated, throughout this section $\mathcal{T}$ is a 3-graph satisfying $\nu(\mathcal{T}) = s$, $\sigma_1(\mathcal{T}) \leqslant r$ and $\sigma_2(\mathcal{T}) \leqslant p$.

Without loss of generality we suppose that $\{3i-2, 3i-1, 3i\}$, $1 \leqslant i \leqslant s$ form a maximal matching in $\mathcal{T}$. The maximality implies $T \cap [3s] \neq \emptyset$ for all $T \in \mathcal{T}$. This permits to partition $\mathcal{T}$ according to $|T \cap [3s]|$:

$$\mathcal{A} = \{A \in \mathcal{T} \colon |A \cap [3s]| = 1\},\ \mathcal{B} = \{B \in \mathcal{T} \colon\ |B \cap [3s]| = 2\},\ \mathcal{C} = \{C \in \mathcal{T} \colon C \subset [3s]\}.$$

For convenience we assume that the elements of $[3s]$ are ordered to satisfy

$$|\mathcal{A}(3i-2)| \leqslant |\mathcal{A}(3i-1)| \leqslant |\mathcal{A}(3i)|,\ 1 \leqslant i \leqslant s \text{ and} \tag{24}$$

$$|\mathcal{A}(3)| \geqslant |\mathcal{A}(6)| \geqslant \ldots \geqslant |\mathcal{A}(3s)|. \tag{25}$$

Let us note that the maximality of the matching implies that the three families $\mathcal{A}(3i-2), \mathcal{A}(3i-1), \mathcal{A}(3i)$ are pairwise cross-intersecting. Indeed if, say, $A \in \mathcal{A}(3i-1)$, $A' \in \mathcal{A}(3i)$ and $A \cap A' = \emptyset$ then replacing $\{3i-2, 3i-1, 3i\}$ by the disjoint edges $A \cup \{3i-1\}$ and $A' \cup \{3i\}$ we would get a larger matching. This implies

Claim 4. *If $\mathcal{A}(3i) > 2p$ then $\mathcal{A}(3i-1) = \mathcal{A}(3i-2) = \emptyset$.*

Proof. Suppose that $\{x, y\} \in \mathcal{A}(3i-1)$. By the cross-intersecting property, $A \cap \{x, y\} \neq \emptyset$ for all $A \in \mathcal{A}(3i)$. Now $\sigma_2(\mathcal{T}) \leqslant p$ implies $|\mathcal{A}(3i)| \leqslant 2p$. $\square$

Claim 5. *If $\mathcal{A}(3i) > 6p$ then for any vertex set T with $|T| \leqslant 6$ there exists $A \in \mathcal{A}(3i)$ such that $A \cap T = \emptyset$.*

Proof. Note that $\mathcal{A}(3i)$ is a simple graph with maximum degree at most p. The number of edges in $\mathcal{A}(3i)$ that intersect T is at most $6p$. By $\mathcal{A}(3i) > 6p$ we infer that there exists $A \in \mathcal{A}(3i)$ such that $A \cap T = \emptyset$. □

Lemma 5.1. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be simple graphs with with maximum degree at most p, $|\mathcal{P}| \geqslant |\mathcal{Q}| \geqslant |\mathcal{R}|$, that are pairwise cross-intersecting. If $|\mathcal{P}| \geqslant |\mathcal{Q}| > 1$ then

$$|\mathcal{P}| + |\mathcal{Q}| + |\mathcal{R}| \leqslant \max\{3p, 9\}. \tag{26}$$

Proof. Arguing indirectly assume $|\mathcal{P}| \geqslant p + 1$ and $|\mathcal{P}| \geqslant 4$. These imply $\nu(\mathcal{P}) \geqslant 2$. Since $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are pairwise cross-intersecting, $\mathcal{Q} \neq \emptyset$ implies $\nu(\mathcal{P}) \leqslant 2$. Thus $\nu(\mathcal{P}) = 2$.

If $p = 1$ then $\nu(\mathcal{P}) = |\mathcal{P}| \leqslant 2$ and $|\mathcal{P}| + |\mathcal{Q}| + |\mathcal{R}| \leqslant 6$ follows. Let $p \geqslant 2$ and assume $(1, 2), (3, 4) \in \mathcal{P}$. Distinguish two cases:

Case 1. $\mathcal{P} \subset \binom{[4]}{2}$.

Note that by the cross-intersecting property, $\mathcal{Q}, \mathcal{R}$ are also contained in $\binom{[4]}{2}$. Decompose $\binom{[4]}{2}$ into three matchings $\{(1, 2), (3, 4)\}$, $\{(1, 3), (2, 4)\}$, $\{(1, 4), (2, 3)\}$. Let $\mathcal{M}$ be any of these matchings. By the cross-intersecting property,

$$|\mathcal{P} \cap \mathcal{M}| + |\mathcal{Q} \cap \mathcal{M}| + |\mathcal{R} \cap \mathcal{M}| \leqslant 3.$$

Thus $|\mathcal{P}| + |\mathcal{Q}| + |\mathcal{R}| \leqslant 9$ follows.

Case 2. Using $\nu(\mathcal{P}) = 2$, without loss of generality assume that $(1, 2), (3, 4), (3, 5) \in \mathcal{P}$.

By the cross-intersecting property, $\mathcal{Q}, \mathcal{R} \subset \{(1, 3), (2, 3)\}$. If $|\mathcal{R}| \leqslant |\mathcal{Q}| = 1$ then $|\mathcal{P}| \leqslant 2p$ and

$$|\mathcal{P}| + |\mathcal{Q}| + |\mathcal{R}| \leqslant 2p + 2 < \max\{3p, 9\}.$$

If $\mathcal{Q} = \{(1, 3), (2, 3)\}$ then $|\mathcal{P}| \leqslant 1 + p$. Whence

$$|\mathcal{P}| + |\mathcal{Q}| + |\mathcal{R}| \leqslant 2 + 2 + (1 + p) = 5 + p < \max\{3p, 9\}.$$ □

Note the trivial inequality

$$|\mathcal{T}| \leqslant \sum_{x \in [3s]} |\mathcal{T}(x)|. \tag{27}$$

Note further that on the right hand side of (27) members of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are counted once, twice and three times, respectively. Consequently

$$|\mathcal{T}| = \sum_{x \in [3s]} \left(|\mathcal{A}(x)| + \frac{|\mathcal{B}(x)|}{2} + \frac{|\mathcal{C}(x)|}{3} \right). \tag{28}$$

Using $|\mathcal{T}(x)| = |\mathcal{A}(x)| + |\mathcal{B}(x)| + |\mathcal{C}(x)| \leqslant m_2(r, p)$,

$$|\mathcal{T}| \leqslant \frac{1}{2} \sum_{x \in [3s]} |\mathcal{A}(x)| + \frac{3s m_2(r, p)}{2} - \frac{1}{6} \sum_{x \in [3s]} |\mathcal{C}(x)|$$

$$= \frac{|\mathcal{A}| - |\mathcal{C}|}{2} + \frac{3sm_2(r,p)}{2}. \tag{29}$$

Recall that $rp \leqslant m_2(r,p) \leqslant r(p+1)$. Since $m_2(r,p) \geqslant rp \geqslant 6p$ for $r \geqslant 6$, by (26) we infer that $|\mathcal{A}| \leqslant sm_2(r,p)$ for $r \geqslant 6$. Thus,

$$|\mathcal{T}| \leqslant \frac{sm_2(r,p)}{2} + \frac{3sm_2(r,p)}{2} = 2sm_2(r,p). \tag{30}$$

Let us note a simple fact.

Fact 6. *Let $\{i,j,\ell\} \in \binom{[s]}{3}$ and assume $|\mathcal{A}(3i)| > 0$, $|\mathcal{A}(3j)| > 2p$, $|\mathcal{A}(3\ell)| > 4p$. Then one can choose pairwise disjoint sets $A_i \in \mathcal{A}(3i)$, $A_j \in \mathcal{A}(3j)$, $A_\ell \in \mathcal{A}(3\ell)$.*

Proof. First fix $A_i \in \mathcal{A}(3i)$ arbitrarily. Since $|A_i| = 2$, there are at most $2p$ sets in $\mathcal{A}(3j)$ that intersect A_i. Thus we can choose $A_j \in \mathcal{A}(3j)$ with $A_i \cap A_j = \emptyset$. Now there are at most $4p$ members of $\mathcal{A}(3\ell)$ intersecting $A_i \cup A_j$. Thus there exists $A_\ell \in \mathcal{A}(3\ell)$ that is disjoint to both A_i and A_j. □

By $\nu(\mathcal{T}) = s$ the following fact is almost evident.

$$\text{If } \{3i-2, 3i-1, x\} \in \mathcal{B} \text{ for some } x \notin [3s] \text{ then } |\mathcal{A}(3i)| \leqslant p,\ 1 \leqslant i \leqslant s. \tag{31}$$

Fact 7. *Let $\{b, b', x\} \in \mathcal{B}$ with $b \in \{3i-2, 3i-1\}$, $b' \in \{3j-2, 3j-1\}$ and $x \notin [3s]$, then either $|\mathcal{A}(3i)| \leqslant p$ or $|\mathcal{A}(3j)| \leqslant 3p$.*

Proof. Suppose that $|\mathcal{A}(3i)| > p$ and $|\mathcal{A}(3j)| > 3p$. Then there are at most p sets in $\mathcal{A}(3i)$ containing x. Thus we can choose $A_i \in \mathcal{A}(3i)$ such that $x \notin A_i$. Now there are at most $3p$ members of $\mathcal{A}(3j)$ intersecting $A_i \cup \{x\}$. Thus there exists $A_j \in \mathcal{A}(3j)$ that is disjoint to A_i and $\{x\}$. Now by replacing $\{3i-2, 3i-1, 3i\}$, $\{3j-2, 3j-1, 3j\}$ with $\{b, b', x\}$, $A_i \cup \{3i\}$, $A_j \cup \{3j\}$ we get a matching of size $s+1$, a contradiction. □

Proof of Theorem 1.14. If $|\mathcal{A}| \leqslant \frac{1}{3}sm_2(r,p) + 4sp + 2s$ then (9) follows from (29). Thus we assume $|\mathcal{A}| > \frac{1}{3}sm_2(r,p) + 4sp + 2s$.

Assume that $|\mathcal{A}(3)| \geqslant |\mathcal{A}(6)| \geqslant \ldots \geqslant |\mathcal{A}(3t)| > 6p$ and $|\mathcal{A}(3(t+1))| \leqslant 6p$. By Claim 4,

$$|\mathcal{A}(3i-2)| + |\mathcal{A}(3i-1)| + |\mathcal{A}(3i)| \leqslant m_2(r,p) \text{ for all } i = 1, 2, \ldots, t. \tag{32}$$

By Claim 4 and Lemma 5.1, we infer that

$$|\mathcal{A}(3i-2)| + |\mathcal{A}(3i-1)| + |\mathcal{A}(3i)| \leqslant \max\{6p, 9\} \leqslant 6p + 3 \text{ for all } i = t+1, \ldots, s. \tag{33}$$

It follows that

$$|\mathcal{A}| \leqslant tm_2(r,p) + (s-t)(6p+3). \tag{34}$$

Since $|\mathcal{A}| > \frac{1}{3}sm_2(r,p) + 4sp + 2s$, by $r \geqslant 6$ we have $t > \frac{s}{3}$. Let $W = [3t+1, 3s]$ and $Y = [3t] \setminus \{3, 6, \ldots, 3t\}$.

Claim 8. *$W \cup \{3,6,\dots,3t\}$ is a transversal of $\mathcal{T}$.*

Proof. Suppose that there exists $T \in \mathcal{T}$ such that $T \cap (W \cup \{3,6,\dots,3t\}) = \emptyset$. Since $[3s]$ is a transversal, we infer that $T \cap Y \neq \emptyset$. By Claim 4 we see that $\mathcal{A}(y) = \emptyset$ for all $y \in Y$. Thus $|T \cap Y| \geqslant 2$. By (31) and Fact 7 we infer that $T \subset Y$. Assume that $T \subset \{3i-2, 3i-1, 3j-2, 3j-1, 3\ell-2, 3\ell-1\}$. By Fact 6, there exist pairwise disjoint sets $A_i \in \mathcal{A}(3i)$, $A_j \in \mathcal{A}(3j)$, $A_\ell \in \mathcal{A}(3\ell)$. Then by replacing $\{3i-2,3i-1,3i\}, \{3j-2,3j-1,3j\}, \{3\ell-2,3\ell-1,3\ell\}$ with $T, A_i \cup \{3i\}, A_j \cup \{3j\}, A_\ell \cup \{3\ell\}$ we obtain a larger matching, a contradiction. □

Set $\mathcal{T}_0 = \mathcal{T}(\overline{\{3,6,\dots,3t\}})$. By Claim 8, W is a transversal for $\mathcal{T}_0$ and

$$\sum_{1 \leqslant i \leqslant t} |\mathcal{T}(3i)| \leqslant t m_2(r,p). \tag{35}$$

Partition $\mathcal{T}_0$:

$$\mathcal{T}_1 = \{T \in \mathcal{T}_0 \colon |T \cap W| = 1\}, \ \mathcal{T}_2 = \{T \in \mathcal{T}_0 \colon |T \cap W| \geqslant 2\}.$$

Let $F_j = \{3j-2, 3j-1, 3j\}$. For each F_j with $t+1 \leqslant j \leqslant s$, there are no two $U, V \in \mathcal{T}_1$ with $U \cap V = \emptyset$ and $|U \cap F_j| = |V \cap F_j| = 1$. Indeed otherwise $(U \cup V) \setminus F_j$ is a 4-set with 1 up to 4 elements in Y. Recall that $|\mathcal{A}(3i)| > 6p$ for $1 \leqslant i \leqslant t$. For each i with $(U \cup V) \cap \{3i-1, 3i-2\} \neq \emptyset$, by Claim 5 there exists $A_i \in \mathcal{A}(3i)$ such that $A_i \cap ((U \cup V) \setminus [3s]) = \emptyset$. Thus we can find a larger matching, a contradiction. Therefore, the three families $\mathcal{T}_1(3j-2), \mathcal{T}_1(3j-1), \mathcal{T}_1(3j)$ are pairwise cross-intersecting. It implies that either at most one is non-empty or by Lemma 5.1

$$\sum_{\ell=0,1,2} |\mathcal{T}_1(3j-\ell)| \leqslant \max\{3p, 9\} < rp \leqslant m_2(r,p).$$

Thus,

$$|\mathcal{T}_1| \leqslant \sum_{t+1 \leqslant j \leqslant s} \sum_{\ell=0,1,2} |\mathcal{T}_1(3j-\ell)| \leqslant (s-t) m_2(r,p).$$

Since $|\mathcal{T}_1(x)| + |\mathcal{T}_2(x)| \leqslant m_2(r,p)$ for any $x \in W$, we have

$$|\mathcal{T}_0| \leqslant \sum_{x \in W} \left(|\mathcal{T}_1(x)| + \frac{|\mathcal{T}_2(x)|}{2} \right) \leqslant \frac{|\mathcal{T}_1|}{2} + \frac{3(s-t)}{2} m_2(r,p) \leqslant 2(s-t) m_2(r,p). \tag{36}$$

By (35) and (36), we obtain that

$$|\mathcal{T}| \leqslant (2s-t)\, m_2(r,p).$$

By $t > \frac{s}{3}$, we conclude that $|\mathcal{T}| < \frac{5}{3} s m_2(r,p)$. □

6 Proof of Theorem 1.15

First we determine $m_3(1, r, r)$ for $r \geqslant 4$.

Theorem 6.1. For $r \geqslant 4$,

$$m_3(1, r, r) = m_2(r, r) = \begin{cases} r(r+1), & r \text{ is even;} \\ \frac{1}{2}(r+1)(2r-1), & r \text{ is odd.} \end{cases} \tag{37}$$

Proof. Let $\mathcal{T}$ be an intersecting 3-graph satisfying $\sigma_i(\mathcal{T}) \leqslant r$ for $i = 1, 2$. If $\mathcal{T}$ is a star with center x then $|\mathcal{T}| = |\mathcal{T}(x)|$ and $\mathcal{T}(x)$ is a graph with $\sigma_i(\mathcal{T}(x)) \leqslant r$, $i = 0, 1$, thus the bound follows from Theorem 1.4.

In the sequel $\mathcal{T}$ is a triple system, non-trivial intersecting, that is, $\cap_{T \in \mathcal{T}} T = \emptyset$, $\sigma_i(\mathcal{F}) \leqslant r$. We distinguish three cases.

Case 1. $\mathcal{T}$ is 2-intersecting, that is, $|T \cap T'| \geqslant 2$ for all $T, T' \in \mathcal{T}$.

Without loss of generality, $\{1, 2, 3\} \in \mathcal{T}$. Then

$$|\mathcal{T}| = 1 + |\mathcal{T}(\{1, 2\}, [3])| + |\mathcal{T}(\{1, 3\}, [3])| + |\mathcal{T}(\{2, 3\}, [3])|. \tag{38}$$

Note that $\mathcal{T}(\{i, j\}, [3])$ is 1-uniform for $1 \leqslant i < j \leqslant 3$ and the three families have to be pairwise cross-intersecting. Hence either $|\mathcal{T}(\{i, j\}, [3])| \leqslant 1$ always, whence the RHS of (38) is at most 4 or two out of the three 1-graphs are empty. By $\sigma_2(\mathcal{T}) \leqslant r$, $|\mathcal{T}(\{i, j\}, [3])| \leqslant r - 1$. Hence $|\mathcal{T}| \leqslant r < m_2(r, r)$ for $r \geqslant 2$.

Now we may assume that $\mathcal{T}$ is not 2-intersecting whence without loss of generality $T_1 = \{1, 2, 4\}$ and $T_2 = \{1, 3, 5\}$ are in $\mathcal{T}$. (Note that we no longer assume $\{1, 2, 3\} \in \mathcal{T}$.)

Case 2. There exists $T \in \mathcal{T}(\bar{1})$ with $|T \cap T_i| = 1$, $i = 1, 2$.

Without loss of generality, $T_3 = \{2, 3, 6\} \in \mathcal{T}$. Using $\nu(\mathcal{T}) = 1$, except possibly for $\{4, 5, 6\}$ all $T \in \mathcal{T}$ contain at least one of the six sets $A_i := [3] \setminus \{i\}$, $i = 1, 2, 3$ and $B_i = \{i, 7 - i\}$, $i = 1, 2, 3$. Note that $A_i \cap B_i = \emptyset$.

Claim 9. *For* $r \geqslant 4$*,*

$$|\mathcal{T}(A_i)| + |\mathcal{T}(B_i)| \leqslant r + 2. \tag{39}$$

Proof. Consider $x, y \notin A_i \cup B_i$, $x \neq y$. Then $\{x\} \in \mathcal{T}(A_i)$ and $\{y\} \in \mathcal{T}(B_i)$ cannot hold simultaneously. Thus $|\mathcal{T}(A_i)| \geqslant 3$ guarantees $|\mathcal{T}(B_i)| \leqslant 3$ and $|\mathcal{T}(A_i)| \geqslant 4$ guarantees $|\mathcal{T}(B_i)| \leqslant 2$. Using $\sigma_2(\mathcal{T}) \leqslant r$, we have

$$|\mathcal{T}(A_i)| + |\mathcal{T}(B_i)| \leqslant \max\{r + 2, 3 + 3\},$$

proving (39) for $r \geqslant 4$. □

By Claim 9 we have

$$|\mathcal{T}| = 1 + |\mathcal{T} \setminus \{\{4, 5, 6\}\}| \leqslant 1 + 3(r + 2) = 3r + 7 < \frac{1}{2}(r + 1)(2r - 1) \text{ for } r \geqslant 5$$

and $3r+7<(r+1)r$ for $r=4$.

Case 3. $T\subset[2,5]$ for all $T\in\mathcal{T}(\bar{1})$.

Then $|\mathcal{T}(\bar{1})|\leqslant 4$. Without loss of generality, $\{2,3,4\}\in\mathcal{T}(\bar{1})$. By $\nu(\mathcal{T})=1$,

$$|\mathcal{T}(1)|\leqslant|\mathcal{T}(1,2)|+|\mathcal{T}(1,3)|+|\mathcal{T}(1,4)|\leqslant 3r.$$

Thus for $r\geqslant 4$,

$$|\mathcal{T}|\leqslant 4+3r<\frac{1}{2}(r+1)(2r-1).$$ □

Let us recall an old result of Erdős and Lovász.

Theorem 6.2 ([8]). Let $\mathcal{T}$ be an intersecting 3-graph with $\tau(\mathcal{T})=3$. Then $|\mathcal{T}|\leqslant 10$.

Strengthening the exact result $m_3(1,2,2)=10$ (cf. [2]), let us show

Proposition 6.3.

$$m_3(1,3,3)=10.$$

Proof. Since $m_3(1,3,3)\geqslant m_3(1,2,2)=10$, we are left to show $m_3(1,3,3)\leqslant 10$. Suppose indirectly that $\mathcal{T}$ is an intersecting triple system with $\sigma_i(\mathcal{T})\leqslant 3$, $i=1,2$, and $|\mathcal{T}|\geqslant 11$. If $\mathcal{T}$ is a star, then by Theorem 1.4 we have $|\mathcal{T}|\leqslant m_2(3)=10$, a contradiction. Thus $\tau(\mathcal{T})\geqslant 2$. If $\tau(\mathcal{T})=3$, then by Theorem 6.2 we have $|\mathcal{T}|\leqslant 10$, a contradiction. Thus we may assume $\tau(\mathcal{T})=2$.

Without loss of generality, assume that $\{1,2\}$ is a transversal. By $\sigma_2(\mathcal{T})\leqslant 3$ we have $|\mathcal{T}(1,2)|\leqslant 3$. Thus,

$$|\mathcal{T}|=|\mathcal{T}(1,2)|+|\mathcal{T}(1,\bar{2})|+|\mathcal{T}(\bar{1},2)|\leqslant 3+|\mathcal{T}(1,\bar{2})|+|\mathcal{T}(\bar{1},2)|.$$

It follows that

$$|\mathcal{T}(1,\bar{2})|+|\mathcal{T}(\bar{1},2)|\geqslant 8. \tag{40}$$

Clearly, $\mathcal{T}(1,\bar{2})$, $\mathcal{T}(\bar{1},2)$ are non-empty cross-intersecting 2-graphs. Assuming $|\mathcal{T}(1,\bar{2})|\geqslant|\mathcal{T}(\bar{1},2)|$, $|\mathcal{T}(1,\bar{2})|\geqslant 4$ follows. Hence $\mathcal{T}(1,\bar{2})$ is not a triangle. Either it is a star or non-intersecting. In the first case $|\mathcal{T}(1,\bar{2})|\geqslant 4$ implies $\sigma_2(\mathcal{T})\geqslant 4$, a contradiction.

Thus without loss of generality we may assume $\{3,4\},\{5,6\}\in\mathcal{T}(1,\bar{2})$. It follows that $\mathcal{T}(\bar{1},2)\subset\{3,4\}\times\{5,6\}$. If $\nu(\mathcal{T}(\bar{1},2))\neq 1$ then without loss of generality $\{3,5\},\{4,6\}\in\mathcal{T}(\bar{1},2)$ and $\mathcal{T}(1,\bar{2})\subset\{3,5\}\times\{4,6\}$. In particular, $\mathcal{T}(1,\bar{2}),\mathcal{T}(\bar{1},2)\subset\binom{\{3,4,5,6\}}{2}$. By the cross-intersecting property,

$$|\mathcal{T}(1,\bar{2})|+|\mathcal{T}(\bar{1},2)|\leqslant\binom{4}{2}=6,$$

contradicting (40). Thus $\mathcal{T}(\bar{1},2)$ is intersecting.

If $|\mathcal{T}(\bar{1},2)| = 1$, without loss of generality assume $\mathcal{T}(\bar{1},2) = \{\{3,5\}\}$, then by the cross-intersecting property and $\sigma_2(\mathcal{T}) \leqslant 3$,

$$|\mathcal{T}(1,\bar{2})| \leqslant |\mathcal{T}(1,\bar{2},3)| + |\mathcal{T}(1,\bar{2},5)| \leqslant 3+3=6.$$

It follows that $|\mathcal{T}(1,\bar{2})| + |\mathcal{T}(\bar{1},2)| \leqslant 7$, contradicting (40).

If $|\mathcal{T}(\bar{1},2)| = 2$, without loss of generality assume $\mathcal{T}(\bar{1},2) = \{\{3,5\},\{3,6\}\}$, then by the cross-intersecting property, $\{5,6\}$ is the only possible member of $\mathcal{T}(1,\bar{2},\bar{3})$. It follows that

$$|\mathcal{T}(1,\bar{2})| + |\mathcal{T}(\bar{1},2)| = |\mathcal{T}(1,\bar{2},3)| + |\mathcal{T}(1,\bar{2},\bar{3})| + 2 \leqslant 6,$$

contradicting (40) again. □

Now Theorem 1.15 follows from Theorem 6.1 and Proposition 6.3.

7 Concluding remarks

Let us first point out a connection to a famous open problem. Define

$$el(k) := \max\{|\mathcal{F}| \colon \mathcal{F} \text{ is an intersecting } k\text{-graph with } \tau(\mathcal{F}) = k\}.$$

Erdős and Lovász [8] proved that $\lfloor k!(e-1)\rfloor \leqslant el(k) \leqslant k^k$ and $el(3) = 10$. Lovász [20] conjectured that $\lfloor k!(e-1)\rfloor$ is the exact bound. In [15], Lovász's conjecture was disproved for $k \geqslant 4$ and the lower bound of $el(k)$ was improved to $(1+o(1))\left(\frac{k}{2}\right)^k$.

Proposition 7.1. *$el(k) \leqslant m_k(1,k,\ldots,k)$.*

Proof. Let $\mathcal{F}$ be an intersecting k-graph with $\tau(\mathcal{F}) = k$. We claim that $\sigma_i(\mathcal{F}) \leqslant k$ for $i = 1,2,\ldots,k-1$. Indeed, if there is a sunflower of size $k+1$ with center C. Then C is a cover of $\mathcal{F}$ of size less than k, contradicting $\tau(\mathcal{F}) = k$. Thus $el(k) \leqslant m_k(1,k,\ldots,k)$ follows. □

However the complete k-partite k-graph with partite sets of size $1,k,k,\ldots,k$ shows that $m_k(1,k,\ldots,k) \geqslant k^{k-1}$. Since by now there are upper bounds (cf. [13], [25]) implying $el(k) = o(k^{k-1})$, except possibly for small values of k, Proposition 7.1 is not suitable to provide useful bounds.

Consider a vector $\vec{r} = (r_0,\ldots,r_{k-1})$ and a k-graph $\mathcal{F}$ with $\sigma_i(\mathcal{F}) \leqslant r_i$, $0 \leqslant i < k$, satisfying $|\mathcal{F}| = m_k(\vec{r})$. For an arbitrary integer ℓ, $\ell \geqslant 2$, replacing each vertex by a distinct ℓ-subset (keeping these ℓ-sets pairwise disjoint) produces a $k\ell$-graph $\mathcal{F}_\ell$ in which the intersection of any two edges has size a multiple of ℓ. Setting $\vec{s} = (s_0,\ldots,s_{k\ell-1})$ where $s_i = 1$ for $\ell \nmid i$ and $s_{j\ell} = r_j$ for $0 \leqslant j < k$ we infer

$$m_{k\ell}(\vec{s}) \geqslant m_k(\vec{r}). \tag{41}$$

It would be very interesting to know how large the ratio of the two sides might be.

With all the new problems that we considered in this paper we still feel that the most exciting open question is the original one proposed by Erdős and Rado, in particular the conjecture

$$m_k(2, 2, \ldots, 2) < c^k \text{ for some absolute constant } c. \tag{42}$$

Let us conclude this paper by a generalization of (42).

Conjecture 7.2. Let $\vec{r} = (r_0, \ldots, r_p)$ be a vector with $r_i \in \{1, 2\}$ for each i and suppose that $r_i = 2$ for exactly k values of i where $k > p/10$. There is an absolute constant c such that

$$m_p(\vec{r}) < c^k. \tag{43}$$

The factor $1/10$ is arbitrary but the next example shows that some restriction on k is necessary.

Let $q = pk$ and let $X_1, \ldots, X_k$ be disjoint $(p+1)$-sets. For each sequence $(x_1, \ldots, x_k)$, $x_1 \in X_1, \ldots, x_k \in X_k$ define the q-set

$$F(x_1, x_2, \ldots, x_k) = X_1 \cup X_2 \cup \ldots \cup X_k \setminus \{x_1, x_2, \ldots, x_k\}.$$

Then $\mathcal{F} = \{F(x_1, x_2, \ldots, x_k) \colon x_1 \in X_1, \ldots, x_k \in X_k\}$ is a q-graph, $|\mathcal{F}| = (p+1)^k$. Obviously $|F \cap F'| \geqslant q - k$ for $F, F' \in \mathcal{F}$.

Let us prove

$$\sigma_j(\mathcal{F}) = 2 \text{ for } q - k \leqslant j < q. \tag{44}$$

Suppose indirectly that $F, G, H \in \mathcal{F}$ form a sunflower. Since $F \neq G$ we may suppose by symmetry that $F \cap X_1 \neq G \cap X_1$. Say $F \cap X_1 = X_1 \setminus \{x\}$, $G \cap X_1 = X_1 \setminus \{y\}$. Then $F \cap G = F \cap H$ forces $X_1 \setminus \{x, y\} \subset H$. As $|H \cap X_1| = |X_1| - 1$, either $F \cap X_1 = H \cap X_1$ or $G \cap X_1 = H \cap X_1$, contradicting $F \cap H = G \cap H$.

The above example shows

$$m_{pk}(1, \ldots, 1, \overbrace{2, \ldots, 2}^{k \text{ times}}) \geqslant (p+1)^k > (q/k)^k, \text{ which is greater than } c^k \text{ for } q > ck.$$

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