# Generalized Quaternion Groups with the $m$-DCI Property

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**Abstract**

A Cayley digraph $\mathrm{Cay}(G,S)$ of a finite group $G$ with respect to a subset $S$ of $G$ is said to be a CI-digraph if for every Cayley digraph $\mathrm{Cay}(G,T)$ isomorphic to $\mathrm{Cay}(G,S)$, there exists an automorphism $\sigma$ of $G$ such that $S^\sigma = T$. A finite group $G$ is said to have the $m$-DCI property for some positive integer $m$ if every Cayley digraph $\mathrm{Cay}(G,S)$ of $G$ with $|S| = m$ is a CI-digraph, and is said to be a DCI-group if $G$ has the $m$-DCI property for all $1 \leqslant m \leqslant |G|$. Let $\mathrm{Q}_{4n}$ be a generalized quaternion group (also called dicyclic group) of order $4n$ with an integer $n \geqslant 3$, and let $\mathrm{Q}_{4n}$ have the $m$-DCI property for some $1 \leqslant m \leqslant 2n-1$. It is shown in this paper that $n$ is odd, and $n$ is not divisible by $p^2$ for any prime $p \leqslant m-1$. Furthermore, if $n \geqslant 3$ is a power of a prime $p$, then $\mathrm{Q}_{4n}$ has the $m$-DCI property if and only if $p$ is odd, and either $n = p$ or $1 \leqslant m \leqslant p$.

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## 1 Introduction

Unless otherwise indicated, digraphs and graphs considered in this paper are finite with no parallel edges or loops, and groups are finite. For a digraph $\Gamma$, denote by $V(\Gamma)$, $E(\Gamma)$, $\mathrm{Arc}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ the vertex set, edge set, arc set, and automorphism group of $\Gamma$, respectively. If for some integer $m$, the in-valency or out-valency of every vertex of $\Gamma$ equals $m$, then we say that the digraph has *in-valency* $m$ or *out-valency* $m$, respectively. Moreover, if the in-valency and out-valency of every vertex of a digraph both equal $m$, then we say that the digraph has *valency* $m$ or is $m$*-valent*.

Let $G$ be a group and $S$ be a subset of $G$ with $1 \notin S$. A digraph with vertex set $G$ and arc set $\{(g, sg) \mid g \in G,\ s \in S\}$ is said to be a *Cayley digraph* of $G$ with respect

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to $S$, denoted by $\mathrm{Cay}(G,S)$. If $S = S^{-1}$, then both $(u,v)$ and $(v,u)$ are arcs for two adjacent vertices $u$ and $v$ in $\mathrm{Cay}(G,S)$, and $\mathrm{Cay}(G,S)$ is a graph by identifying the two arcs with one edge $\{u,v\}$. Clearly, a Cayley graph $\mathrm{Cay}(G,S)$ as well as its identifying Cayley digraph has the same valency $|S|$. Two Cayley digraphs $\mathrm{Cay}(G,S)$ and $\mathrm{Cay}(G,T)$ are said to be *Cayley isomorphic* if $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ is the automorphism group of $G$. Cayley digraphs are isomorphic if they are Cayley isomorphic, but the converse is not necessarily true. A subset $S$ of $G$ with $1 \notin S$ is said to be a *CI-subset* if $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$, for some $T \subseteq G$ $(1 \notin T)$, implies that they are Cayley isomorphic. In this case, $\mathrm{Cay}(G,S)$ is said to be a *CI-digraph*, or a *CI-graph* when $S = S^{-1}$. A group $G$ is said to be a DCI-group or a CI-group if all Cayley digraphs or Cayley graphs of $G$ are CI-digraphs or CI-graphs, respectively.

Ádám [1] conjectured that every finite cyclic group is a CI-group. Although this conjecture was disproved by Elspas and Turner [10], many researchers actively studied CI-groups and DCI-groups during the last fifty years and obtained great contributions, see [3, 4, 7, 9, 15] for example. For cyclic DCI-groups and CI-groups, the classifications were finally completed by Muzychuk[32, 33]: a cyclic group of order $n$ is a DCI-group if and only if $n/\gcd(2,n)$ is square-free, and is a CI-group if and only if either $n/\gcd(2,n)$ is square-free or $n \in \{8, 9, 18\}$. A powerful method for studying DCI-groups or CI-groups comes from Schur ring theory, which was initiated by Schur and developed by Wielandt (see [43, Chapter IV]). In particular, this method is widely used to classify the DCI-groups and CI-groups among abelian groups, especially elementary abelian groups, refer to [14, 17, 34, 39, 40, 41]. So far DCI-groups and CI-groups have been restricted to some particular families of groups (see [8, 9, 18, 27]), and it is very difficult to determine whether these groups are DCI-groups or CI-groups.

For a positive integer $m$, a group $G$ is said to have the *$m$-DCI property* or *$m$-CI property* if all $m$-valent Cayley digraphs of $G$ are CI-digraphs or all Cayley graphs of $G$ of valency $m$ are CI-graphs, respectively. Clearly, if $G$ has the $m$-DCI property then $G$ has the $m$-CI property. A group $G$ is said to be an *$m$-DCI-group* or *$m$-CI-group* if $G$ has the $k$-DCI property or $k$-CI property for every positive integer $k \leqslant m$, respectively. Evidently, a group $G$ is a DCI-group or CI-group if $G$ has the $m$-DCI property or $m$-CI property for all $m \leqslant |G|$, respectively; that is, if it is a $|G|$-DCI-group or a $|G|$-CI-group, respectively.

Considerable work has been done on the $m$-DCI property or $m$-CI property of a group, with interesting results obtained to characterize $m$-DCI-groups or $m$-CI-groups. In [11, 12, 13], Fang and Xu completely classified abelian $m$-DCI-groups for a positive integer $m$ at most 3. For an integer $n$ at least 3 and $m \in \{1,2,3\}$, the dihedral group $\mathrm{D}_{2n}$ is an $m$-DCI-group if and only if $n$ is odd (see [36]), and the generalized quaternion group $\mathrm{Q}_{4n}$ is an $m$-DCI-group if and only if $n$ is odd (see [29]). In [24], Li, Praeger and Xu classified all finite abelian groups with the $m$-DCI property for a positive integer $m$ at most 4, and they proposed a natural problem: characterize finite groups with the $m$-DCI property. For cyclic groups, Li [20] gave a necessary condition for the cyclic group of order $n$ to have the $m$-DCI property. Soon after, Li [25] proved that all Sylow subgroups of an abelian group with the $m$-DCI property are homocyclic. For more details, we refer to [21, 22, 23, 26] for example.

Recently, Xie, Feng and Kwon [44] studied dihedral groups with the $m$-DCI property: if a dihedral group $G$ of order $2n$ has the $m$-DCI property for some $1 \leqslant m \leqslant n-1$, then $n$ is odd and not divisible by the square of any prime less than $m$; moreover, the converse of this is true for prime power $n$, but in general it is unknown whether the converse is true. In this paper, we consider the $m$-DCI property of generalized quaternion groups. Following [2, (2.1)], we call

$$\mathrm{Q}_{4n} = \langle a, b \mid a^{2n} = 1,\ b^2 = a^n,\ a^b = a^{-1} \rangle$$

the *generalized quaternion group* of order $4n$. Note that a generalized quaternion group is also called a dicyclic group (see [31, Definition 1.1]). For $n = 1$, $\mathrm{Q}_4$ is the cyclic group of order 4, and hence $\mathrm{Q}_4$ is a DCI-group by [26, Theorem 7.1]. For $n = 2$, $\mathrm{Q}_8$ is the quaternion group of order 8, and $\mathrm{Q}_8$ is a DCI-group by [38, Theorem 1.1]. Thus, we may assume that $n \geqslant 3$. For a group $G$, a subset $S$ of $G \setminus \{1\}$ is a CI-subset of $G$ if and only if the complement of $S$ in $G \setminus \{1\}$ is a CI-subset of $G$. To investigate the $m$-DCI property of $\mathrm{Q}_{4n}$, it suffices to consider $m$ such that $1 \leqslant m \leqslant 2n-1$. As the first main result of this paper, we give necessary conditions for the $m$-DCI property of $\mathrm{Q}_{4n}$, which generalizes the necessary conditions for the 1-DCI property of [29, Lemma 3.1].

**Theorem 1.** *Let $G$ be the generalized quaternion group of order $4n$ with $n \geqslant 3$ such that $G$ has the $m$-DCI property for some $1 \leqslant m \leqslant 2n-1$. Then $n$ is odd, and $n$ is not divisible by $p^2$ for any prime $p \leqslant m-1$.*

Based on Theorem 1, we have the following corollary, which can also be obtained from known results: $n$ is odd by [29, Theorem 1.4] and square free by [26, Theorem 7.1].

**Corollary 2.** *If the generalized quaternion group of order $4n$ with $n \geqslant 3$ is a DCI-group, then $n$ is odd and square-free.*

It is worth remarking that we do not know whether the converses of Theorem 1 and Corollary 2 are true in general. However, we will show that they are true when $n$ is a prime power. Note that when $n$ is a power of a prime $p$, the conclusion in Theorem 1 turns out to be that $p$ is odd and either $n = p$ or $m \leqslant p$.

The converse of Corollary 2 holds when $n = p$, as $\mathrm{Q}_{4p}$ is a DCI-group for every prime $p$ (see Lemma 11). Next let $n = 4p^\ell$ for an odd prime $p$ and an integer $\ell \geqslant 2$. Then the following theorem asserts that $\mathrm{Q}_{4n}$ has the $m$-DCI property for all $m \leqslant p$. In other words, $\mathrm{Q}_{4n}$ is a $p$-DCI-group.

**Theorem 3.** *Let $n \geqslant 3$ be a power of a prime $p$, and let $G$ be a generalized quaternion group of order $4n$. Then for $1 \leqslant m \leqslant 2n-1$, $G$ has the $m$-DCI property if and only if $p$ is odd and either $n = p$ or $m \leqslant p$.*

After this Introduction, we introduce some preliminary results in Section 2. Then Theorems 1 and 3 will be proved in Sections 3 and 4, respectively.

## 2 Preliminaries

In this section we give some basic concepts and facts that will be used later. For a positive integer $n$ and a prime $p$, denote by $n_p$ the largest $p$-power dividing $n$ and denote $n_{p'} = n/n_p$. Denote by $\mathrm{C}_n$ the undirected cycle of length $n$ and denote by $\overrightarrow{\mathrm{C}}_n$ the directed cycle of length $n$. Denote by $\mathrm{K}_n$ the complete graph with $n$ vertices in which two arbitrary vertices are adjacent, and denote by $\overline{\mathrm{K}}_n$ the empty graph with $n$ vertices in which no two vertices are adjacent. A digraph $\overrightarrow{\mathrm{K}}_{m,n}$ is called a *complete bipartite digraph* if its vertex set can be partitioned into two subsets $X$ and $Y$ such that $|X| = m$ and $|Y| = n$ and its arc set is $\{(x, y) \mid x \in X, y \in Y\}$.

Let $G$ be a group. The *commutator* of elements $x$ and $y$ in $G$ is $[x, y] = x^{-1}y^{-1}xy$. The *derived group* $G'$ of $G$ is $\langle [x, y] \mid x, y \in G\rangle$. For a subgroup $H$ of $G$, denote the normalizer and centralizer of $H$ in $G$ by $\mathrm{N}_G(H)$ and $\mathrm{C}_G(H)$, respectively. The following result is from [42, Chapter 2, Theorem 1.6].

**Proposition 4.** *Let $G$ be a $p$-group for some prime $p$ and let $H$ be a proper subgroup of $G$. Then $\mathrm{N}_G(H)$ properly contains $H$, that is, $\mathrm{N}_G(H) > H$.*

Let $p$ be a prime. A finite group $G$ is said to be *$p$-abelian* if $(xy)^p = x^p y^p$ for all $x$ and $y$ in $G$. A $p$-group $G$ is called a *regular $p$-group* if for arbitrary two elements $x$ and $y$ in $G$, there exists $c_1, c_2, \dots, c_r$ in the derived group $\langle x, y\rangle'$ of $\langle x, y\rangle$ such that $(xy)^p = x^p y^p c_1^p c_2^p \cdots c_r^p$. The following proposition is from [30, Proposition 3] and [45, Proposition 2.3].

**Proposition 5.** *Let $G$ be a $p$-group for some prime $p$. If every subgroup of $G'$ can be generated by at most $(p-1)/2$ elements, then $G$ is a regular $p$-group. Moreover, a regular $p$-group $G$ is $p$-abelian if and only if $G'$ has exponent $p$.*

Let $\mathrm{Cay}(G, S)$ be a Cayley digraph of a group $G$ with respect to $S$. For a given $g \in G$, the right multiplication $R(g)$ is a permutation on $G$ such that $x^{R(g)} = xg$ for every $x \in G$. Clearly, $R(g)$ is an automorphism of $\mathrm{Cay}(G, S)$. Let $R(G) = \{R(g) \mid g \in G\}$. Then $R(G)$ is a regular group of automorphisms of $\mathrm{Cay}(G, S)$, which is called *the right regular representation* of $G$. The following well-known Babai's criterion is from [4] (also see [27, Theorem 2.4]).

**Proposition 6.** *A Cayley digraph $\mathrm{Cay}(G, S)$ is a CI-digraph if and only if every regular subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$ isomorphic to $G$ is conjugate to $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G, S))$.*

The following result says that the $m$-DCI property of a group is hereditary by subgroups, which can be proved by the same argument as that for the $m$-CI property in [26, Lemma 8.2].

**Proposition 7.** *Suppose that a finite group $G$ has the $m$-DCI property for a positive integer $m$. Then every subgroup of $G$ has the $m$-DCI property.*

Li [20, Theorem 1.2] gave a necessary condition for cyclic groups to have the $m$-DCI property. We restate this result as follows.

**Proposition 8.** *Let $G$ be a cyclic group of order $n$ such that $G$ has the $m$-DCI property for some $p+1 \leqslant m \leqslant n-(p+2)$ with $p$ a prime. Then either $n = p^2$ and $m \equiv 0$ or $-1 \pmod p$, or $n_p$ divides $\mathrm{lcm}(4, p)$.*

For subsets of a cyclic group, we have the following result (see [25, Lemma 2.1]).

**Lemma 9.** *Let $G = \langle z \rangle$ be a cyclic group of order $n$, and let $i, j \in \{1, 2, \ldots, n-2\}$. If $\{z, z^2, \ldots, z^i\} = \{z^j, z^{2j}, \ldots . z^{ij}\}$, then $j = 1$.*

Let $G$ be a finite group. If for any two subgroups $H$ and $K$ of $G$, every isomorphism from $H$ to $K$ can be extended to an automorphism of $G$, then $G$ is called *homogeneous*. For generalized quaternion groups $\mathrm{Q}_{4n}$, the following property is shown in [29, Lemma 2.4].

**Lemma 10.** *For an odd positive integer $n$, the generalized quaternion group $\mathrm{Q}_{4n}$ is homogeneous.*

We see from [6, Corollary 29] that $\mathrm{Q}_{4p}$ is a DCI-group for each prime $p \geqslant 5$. Since it can be verified by Magma [5] that $\mathrm{Q}_8$ and $\mathrm{Q}_{12}$ are also DCI-groups, we obtain the following result.

**Lemma 11.** *For every prime $p$, the generalized quaternion group $\mathrm{Q}_{4p}$ is a DCI-group.*

From [44, Lemma 3.1], we have the following lemma, which provides a fairly general way to construct isomorphic Cayley digraphs.

**Lemma 12.** *Let $G$ be a finite group with $L \trianglelefteq G$ and $L \leqslant M \leqslant G$. Suppose that $A$ and $B$ are subsets of $M \setminus \{1\}$ such that $A^\gamma = B$ for some $\gamma \in \mathrm{Aut}(M)$ and $\gamma$ fixes every coset of $L$ in $M$, and that $C \subseteq G \setminus L$ is a union of some cosets of $L$ in $G$. Then $\mathrm{Cay}(G, A \cup C) \cong \mathrm{Cay}(G, B \cup C)$.*

Let $\Gamma$ be a digraph and let $X \subseteq V(\Gamma)$. The *induced subdigraph* $[X]$ of $\Gamma$ by $X$ is the digraph whose vertex set is $X$ and arc set is $\{(u, v) \mid u, v \in X, (u, v) \in \mathrm{Arc}(\Gamma)\}$. Let $N$ be a subgroup of $\mathrm{Aut}(\Gamma)$. Denote by $u^N$ the orbit of $N$ containing $u \in V(\Gamma)$, and by $\Gamma^+(u)$ the out-neighborhood of $u$ in $\Gamma$. The *quotient digraph* $\Gamma_N$ of $\Gamma$ induced by $N$ is defined as the digraph whose vertex set is the set of $N$-orbits in $V(\Gamma)$ such that $(u^N, v^N)$ is an arc of $\Gamma_N$, where $u^N$ and $v^N$ are distinct orbits of $N$, if and only if $(x, y)$ is an arc of $\Gamma$ for some $x \in u^N$ and $y \in v^N$. The digraph $\Gamma$ is said to be an *$N$-cover* of $\Gamma_N$, if for every $u \in \Gamma$, the out-valency of $u$ in $\Gamma$ is the same as the out-valency of $u^N$ in $\Gamma_N$, is said to be *$G$-locally primitive* if $G_u$ acts primitively on $\Gamma^+(u)$ for every $u \in V(\Gamma)$, and is said to be *strongly connected* if there exists a directed path from $u$ to $v$ for each pair of vertices $u$ and $v$. To avoid trivial cases, a digraph with one vertex is also called strongly connected. It well known that every finite connected vertex-transitive digraph is strongly connected (see [16, Lemma 2.6.1] for instance). For convenience, the complete graph on two vertices is also viewed as a directed cycle.

The following result generalizes [35, Theorem 4.1] by Praeger to digraphs, with the proof closely following her approach and incorporating minor adjustments.

**Lemma 13.** *Let $\Gamma$ be a finite connected $G$-vertex-transitive digraph, where $G \leqslant \mathrm{Aut}(\Gamma)$, and let $N$ be a normal subgroup of $G$ with at least two orbits on $V(\Gamma)$. Then the following statements hold:*

(a) *If $\Gamma$ is $G$-arc-transitive, then there are no arcs in the induced subdigraph of any orbit of $N$ in $\Gamma$.*

(b) *If $\Gamma$ is $G$-locally primitive, then either $N$ is the kernel of $G$ on $V(\Gamma_N)$ acting semiregularly on $V(\Gamma)$, and $\Gamma$ is an $N$-cover of $\Gamma_N$ with $|V(\Gamma_N)| \geqslant 3$, or $\Gamma_N$ is a directed cycle.*

*Proof.* To prove part (a), let $\Gamma$ be $G$-arc-transitive and suppose on the contrary that the induced subdigraph of some orbit of $N$ has an arc. Since $\Gamma$ is $G$-vertex-transitive, it follows that the induced subdigraph of every orbit of $N$ has an arc. By the connectivity of $\Gamma$, there is an arc between some distinct orbits of $N$, say $O_1$ and $O_2$. Let $(u, v)$ be an arc of $\Gamma$ with $u \in O_1$ and $v \in O_2$. Since $O_1$ has an arc and $N$ is transitive on $O_1$, there is an arc $(u, w)$ of $\Gamma$ with $w \in O_1$. Since $\Gamma$ is $G$-arc-transitive, there exists $g \in G$ such that $u^g = u$ and $v^g = w$. However, such an element $g$ does not preserve the set of $N$-orbits as $u, w \in O_1$ and $v \in O_2$. This contradicts the fact that $N$ is normal in $G$, completing the proof of part (a).

In following we prove part (b). Let $\Gamma$ be $G$-locally primitive. Since $\Gamma$ is connected, there exists an arc between some distinct orbits of $N$, say $O_1$ and $O_2$. Let $(u, v)$ be an arc of $\Gamma$ with $u \in O_1$ and $v \in O_2$.

First assume that the out-neighbors of $u$ are contained in the same orbit of $N$. Then $\Gamma^+(u) \subseteq O_2$ as $v \in O_2$. Since $N$ is transitive on both $O_1$ and $O_2$, we have $\Gamma^+(x) \subseteq O_2$ for all $x \in O_1$. Since $G$ has an element mapping $O_1$ to $O_2$, the out-neighborhood of each vertex in $O_2$ is a subset of some orbit of $N$. Repeating this argument, we see that the out-neighborhood of each vertex in every orbit of $N$ is a subset of some orbit of $N$. Then we conclude from the connectivity of $\Gamma$ that $\Gamma_N$ is a directed cycle.

Now assume that the out-neighbors of $u$ are not contained in the same orbit of $N$. Let $\mathcal{O} = \{O_1, O_2, \ldots, O_n\}$ be the set of orbits of $N$ and assume that the out-neighborhood of $O_1$ in $\Gamma_N$ is $\{O_2, O_3, \ldots, O_d\}$. Then $|V(\Gamma_N)| = n \geqslant d \geqslant 3$, and $|\Gamma^+(u) \cap O_i| \geqslant 1$ for each $i \in \{2, \ldots, d\}$. The hypothesis of part (b) implies that $\Gamma$ is strongly connected and $G$-arc-transitive, whence $G_u$ is transitive on $\Gamma^+(u)$. Moreover, the conclusion of part (a) implies that

$$\{\Gamma^+(u) \cap O_2, \Gamma^+(u) \cap O_3, \ldots, \Gamma^+(u) \cap O_d\}$$

is a partition of $\Gamma^+(u)$. Since $N$ is normal in $G$, it follows that this partition is preserved by $G_u$. Then we conclude from the $G$-local-primitivity of $\Gamma$ that $|\Gamma^+(u) \cap O_i| = 1$ for each $i \in \{2, \ldots, d\}$. Hence $u$ has the same out-valency as $O_1$ in $\Gamma_N$, which means that $\Gamma$ is an $N$-cover of $\Gamma_N$. Let $K$ be the kernel of $G$ acting on $\mathcal{O}$. Then the stabilizer $K_u$ fixes $\Gamma^+(u)$ pointwise because $|\Gamma^+(u) \cap O_i| = 1$ for each $i \in \{2, \ldots, d\}$. This implies that $K_u = K_w$ for every $w \in \Gamma^+(u)$. Then the strong connectivity of $\Gamma$ implies that $K_x = 1$ for all $x \in V(\Gamma)$, that is, $K$ is semiregular on $V(\Gamma)$. Noting $N \leqslant K$, we deduce by the Frattini argument that $K = NK_x = N$. This shows that $N$ is the kernel of $G$ acting on $V(\Gamma_N)$ and is semiregular on $V(\Gamma)$. □

## 3 Proof of Theorem 1

By [29, Lemma 3.1], if $Q_{4n}$ ($n \geqslant 3$) has the 1-DCI property, then $n$ is odd. This is true for every $1 \leqslant m \leqslant 2n-1$ as the following lemma states.

**Lemma 14.** *Let $G$ be a generalized quaternion group of order $4n$ with $n \geqslant 3$ such that $G$ has the $m$-DCI property for some $1 \leqslant m \leqslant 2n-1$. Then $n$ is odd.*

*Proof.* Suppose for a contradiction that $n$ is even. Then $n \geqslant 4$ as $n \geqslant 3$. Let $G = Q_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, a^b = a^{-1} \rangle$. Then $|a| = 2n$, where $|a|$ is the order of $a$, and hence $|a^2| = n$. Note that $\langle a^2 \rangle$ is a characteristic subgroup of $G$ of index 4 and $b^2 = a^n \in \langle a^2 \rangle$. Furthermore, $G = \langle a^2 \rangle \cup b\langle a^2 \rangle \cup a\langle a^2 \rangle \cup ba\langle a^2 \rangle$. Define

$$\varphi : x \mapsto x \text{ for } x \in \langle a^2 \rangle \cup b\langle a^2 \rangle \text{ and } x \mapsto bx \text{ for } x \in a\langle a^2 \rangle \cup ba\langle a^2 \rangle.$$

Then $\varphi$ fixes every element in $\langle a^2 \rangle \cup b\langle a^2 \rangle$ and acts on $a\langle a^2 \rangle \cup ba\langle a^2 \rangle$ the same as the restriction of the left multiplication of $b$ on $G$. Thus, $\varphi$ is a permutation of order 4 on $G$, and interchanges $a\langle a^2 \rangle$ and $ba\langle a^2 \rangle$. First we prove a claim.

**Claim:** Let $H \subseteq \langle a^2 \rangle$ and $K \subseteq a\langle a^2 \rangle$ such that $H^{-1} = H$, $K^{-1} = K$ and $a^n K = K$. Then $\varphi$ is an isomorphism from $\Gamma = \mathrm{Cay}(G, bH \cup K)$ to $\Sigma = \mathrm{Cay}(G, bH \cup bK)$.

Let $(u, v)$ be an arc of $\Gamma$. Then $v = su$ for some $s \in bH \cup K$. First assume that $s \in bH$. Note that $bH \subseteq b\langle a^2 \rangle$. If $u \in \langle a^2 \rangle \cup b\langle a^2 \rangle$, then $v = su \in \langle a^2 \rangle \cup b\langle a^2 \rangle$. It follows that $u^\varphi = u$ and $v^\varphi = v = su$, which implies that $(u^\varphi, v^\varphi)$ is an arc of $\Sigma$ because $s \in bH$. If $u \in a\langle a^2 \rangle \cup ba\langle a^2 \rangle$, then $v = su \in a\langle a^2 \rangle \cup ba\langle a^2 \rangle$, and so $u^\varphi = bu$ and $v^\varphi = bv = bsu = bsb^{-1}(bu)$. Since $H = H^{-1} \subseteq \langle a^2 \rangle$, we have $bsb^{-1} \in b(bHb^{-1}) = bH^{-1} = bH$, which implies that $(u^\varphi, v^\varphi)$ is an arc of $\Sigma$. Next assume that $s \in K$. Note that $K \subseteq a\langle a^2 \rangle$. If $u \in \langle a^2 \rangle \cup b\langle a^2 \rangle$, then $v = su \in a\langle a^2 \rangle \cup ba\langle a^2 \rangle$, which implies that $u^\varphi = u$ and $v^\varphi = bv = bsu$. Since $bs \in bK$, it follows that $(u^\varphi, v^\varphi)$ is an arc of $\Sigma$. If $u \in a\langle a^2 \rangle \cup ba\langle a^2 \rangle$, then $v = su \in \langle a^2 \rangle \cup b\langle a^2 \rangle$, which implies that $u^\varphi = bu$ and $v^\varphi = v = su = sb^{-1}bu = b^{-1}s^{-1}bu$ as $s \in \langle a \rangle$. Since $K^{-1} = K$ and $a^n K = K$, we have that $b^{-1}s^{-1} = ba^n s^{-1} \in ba^n K^{-1} = bK$, and so $(u^\varphi, v^\varphi)$ is an arc of $\Sigma$. Thus, in every case, $(u^\varphi, v^\varphi)$ is an arc of $\Sigma$, and hence $\varphi$ is an isomorphism from $\Gamma$ to $\Sigma$, as claimed.

Note that every element in $G \setminus \langle a \rangle$ has order 4 and has the form $ba^i$ with $1 \leqslant i \leqslant 2n$. Since $\langle a \rangle$ is a characteristic subgroup of $G$, we obtain that

$$a^\alpha \in \langle a \rangle \text{ and } (ba^i)^\alpha \notin \langle a \rangle, \text{ for every } \alpha \in \mathrm{Aut}(G) \text{ and } 1 \leqslant i \leqslant 2n. \tag{1}$$

By hypothesis, $G$ has the $m$-DCI property for some $1 \leqslant m \leqslant 2n-1$. Since $n$ is even, we have $m \neq 1$ by [29, Lemma 3.1], and thus $2 \leqslant m \leqslant 2n-1$.

Suppose $m = 2$. Take $S = \{b, b^{-1}\}$ and $T = \{a^{n/2}, a^{3n/2}\}$. It is not difficult to see that $\mathrm{Cay}(G, S) \cong nC_4 \cong \mathrm{Cay}(G, T)$, where $nC_4$ is a disjoint union of $n$ 4-cycles. Then the 2-DCI property of $G$ implies that there is an automorphism of $G$ mapping $S$ to $T$, contradicting (1).

Suppose $m = 3$. Take $S = \{b, b^{-1}, b^2\}$ and $T = \{a^{n/2}, a^{3n/2}, a^n\}$. Then $\mathrm{Cay}(G, S) \cong nK_4 \cong \mathrm{Cay}(G, T)$, where $nK_4$ is a disjoint union of $n$ copies of $K_4$. The 3-DCI property of $G$ gives an automorphism of $G$ that maps $S$ to $T$, contradicting (1).

Suppose $m = 4, 5, 6, 7$. It follows from $|a| = 2n \geqslant 8$ that $a^2 \neq a^{-2}$. Take $K = \{a, a^{-1}, a^{n+1}, a^{n-1}\}$, and $H = \emptyset$, $\{1\}$, $\{1, a^n\}$ or $\{1, a^2, a^{-2}\}$, respectively. Then we derive from the Claim that $\mathrm{Cay}(G, bH \cup bK) \cong \mathrm{Cay}(G, bH \cup K)$. Since $G$ has the $m$-DCI property, there exists an automorphism of $G$ mapping $bH \cup bK$ to $bH \cup K$, contradicting (1).

Now we may assume that $8 \leqslant m \leqslant 2n - 1$. Write $m = 8k + j$, where $0 \leqslant j \leqslant 7$ and $k \geqslant 1$. It follows that $4k < n$. Set

$$H_1 = \{a^2, a^4, \ldots, a^{2k}, a^{2n-2}, a^{2n-4}, \ldots, a^{2n-2k}\},$$
$$K_1 = \{a, a^3, \ldots, a^{2k-1}, a^{2n-1}, a^{2n-3}, \ldots, a^{2n-(2k-1)}\}.$$

Then $H_1^{-1} = H_1 \subseteq \langle a^2 \rangle$, $K_1^{-1} = K_1 \subseteq a\langle a^2 \rangle$,

$$a^n H_1 = \{a^{n+2}, a^{n+4}, \ldots, a^{n+2k}, a^{n-2}, a^{n-4}, \ldots, a^{n-2k}\},$$
$$a^n K_1 = \{a^{n+1}, a^{n+3}, \ldots, a^{n+2k-1}, a^{n-1}, a^{n-3}, \ldots, a^{n-(2k-1)}\},$$

$(a^n H_1)^{-1} = a^n H_1 \subseteq \langle a^2 \rangle$, and $(a^n K_1)^{-1} = a^n K_1 \subseteq a\langle a^2 \rangle$. Moreover, we observe from $4k < n$ that $|H_1| = |K_1| = |a^n H_1| = |a^n K_1| = 2k$. Suppose $H_1 \cap a^n H_1 \neq \emptyset$. Let $x \in H_1 \cap a^n H_1$. Note that

$$(H_1 \cap a^n H_1)^{-1} = H_1^{-1} \cap (a^n H_1)^{-1} = H_1 \cap a^n H_1.$$

Since $x \in H_1$, we may assume $x = a^{2e}$ for some $e \in \{1, \ldots, k\}$, and then we derive from $x \in a^n H_1$ that $a^{2e} = x = a^{n-2f}$ for some $f \in \{1, \ldots, k\}$. This implies that $a^{n+2(e+f)} = 1$, which is impossible because $n + 2(e + f) \leqslant n + 4k < 2n$. Thus, $H_1 \cap a^n H_1 = \emptyset$, and so $|H_1 \cup a^n H_1| = 4k$. Similarly, if $K_1 \cap a^n K_1 \neq \emptyset$, then we can obtain $a^{n+2(e+f-1)} = 1$ for some $e, f \in \{1, \ldots, k\}$, which is also impossible because $n + 2(e + f - 1) \leqslant n + 4k < 2n$. Thus, $K_1 \cap a^n K_1 = \emptyset$, and so $|K_1 \cup a^n K_1| = 4k$.

Note that the results of the above paragraph are proved under the condition $4k < n$, which is a consequence of the assumption. If further $4k + 2 < n$, then we set

$$H_2 = H_1 \cup \{a^{2(k+1)}, a^{2n-2(k+1)}\} \text{ and } K_2 = K_1 \cup \{a^{2(k+1)-1}, a^{2n-2k-1}\}.$$

Then a similar argument to the above paragraph implies that $H_2^{-1} = H_2 \subseteq \langle a^2 \rangle$, $K_2^{-1} = K_2 \subseteq a\langle a^2 \rangle$, $(a^n H_2)^{-1} = a^n H_2 \subseteq \langle a^2 \rangle$, $(a^n K_2)^{-1} = a^n K_2 \subseteq a\langle a^2 \rangle$, $|H_2| = |K_2| = |a^n H_2| = |a^n K_2| = 2k + 2$, $|H_2 \cup a^n H_2| = |K_2 \cup a^n K_2| = 4k + 4$. Recall that $m = 8k + j$ with $k \geqslant 1$ and $0 \leqslant j \leqslant 7$. We now discuss several cases according to $j$.

For $j = 0$, write $H = H_1 \cup a^n H_1$ and $K = K_1 \cup a^n K_1$. We deduce from the Claim that $\mathrm{Cay}(G, bH \cup bK) \cong \mathrm{Cay}(G, bH \cup K)$. Then the $m$-DCI property of $G$ provides an automorphism of $G$ mapping $bH \cup bK$ to $bH \cup K$, contradicting (1). For $j = 1$ or $j = 2$, we have the same contradiction by taking $K = K_1 \cup a^n K_1$ and $H = H_1 \cup a^n H_1 \cup \{1\}$ or $H_1 \cup a^n H_1 \cup \{1, a^n\}$, respectively. For $j = 3$, we have $m = 8k + 3 \leqslant 2n - 1$ and hence $4k + 2 \leqslant n$. If $4k + 2 = n$, then we take $H = H_1 \cup a^n H_1 \cup \{1\}$ and $K = a\langle a^2 \rangle$, and if $4k + 2 < n$, then we take $H = H_2 \cup a^n H_1 \cup \{1\}$ and $K = K_1 \cup a^n K_1$. Similarly, the same contradiction for (1) occurs.

For $j = 4, 5, 6, 7$, we have $4k+2 < n$ as $m = 8k+j \leqslant 2n-1$. We take $K = K_2 \cup a^n K_2$, and $H = H_1 \cup a^n H_1$, $H_1 \cup a^n H_1 \cup \{1\}$, $H_2 \cup a^n H_1$ or $H_2 \cup a^n H_1 \cup \{1\}$, respectively. By the Claim, $\mathrm{Cay}(G, bH \cup bK) \cong \mathrm{Cay}(G, bH \cup K)$, and then the $m$-DCI property implies that there is an automorphism of $G$ mapping $bH \cup bK$ to $bH \cup K$, contradicting (1). □

For the group $G = \mathrm{Q}_{4n}$ with the $m$-DCI property and a prime divisor $p$ of $n$ such that $p + 1 \leqslant m \leqslant 2n - 1$, we have the following result.

**Lemma 15.** *Let $G$ be a generalized quaternion group of order $4n$ with $n \geqslant 3$. If $G$ has the $m$-DCI property such that $p + 1 \leqslant m \leqslant 2n - 1$ for some prime divisor $p$ of $n$, then $p$ is odd and $n$ is not divisible by $p^2$.*

*Proof.* Let $G = \mathrm{Q}_{4n} = \langle a, b \mid a^{2n} = 1,\, b^2 = a^n,\, a^b = a^{-1}\rangle$. Suppose that $G$ has the $m$-DCI property such that $p + 1 \leqslant m \leqslant 2n - 1$ for some prime divisor $p$ of $n$. By Lemma 14, $n$ is odd, and so $p$ is odd.

Write $n' = 2n/p$, $z = a^{n'}$ and $P = \langle z\rangle$. Then $n'$ is even and $P$ is the unique subgroup of order $p$ in $G$, which implies that $P$ is characteristic in $G$. Suppose for a contradiction that $p$ divides $n'$. Note that $\langle a\rangle$ has the $m$-DCI property by Proposition 7 and by our hypothesis on $G$. Then it follows from Proposition 8 that $2n - (p + 1) \leqslant m \leqslant 2n - 1$. Define an integer $j \in \{1, \dots, p - 2\}$ and a subset $Q$ of $G$ as follows:

$$(j, Q) = \begin{cases} (m \bmod p,\ \emptyset), & \text{if } m \not\equiv 0 \text{ or } -1 \pmod p \\ (p - 2,\ \{b\}), & \text{if } m \equiv -1 \pmod p \\ (p - 2,\ \{b, bz\}), & \text{if } m \equiv 0 \pmod p. \end{cases}$$

Then $m = kp + j + |Q|$ for some positive integer $k \leqslant n' - 1$. Write $X = \langle z, b\rangle = \langle z\rangle \rtimes \langle b\rangle$. Then $X$ has an automorphism $\gamma$ induced by $z \mapsto z^{-1}$ and $b \mapsto b$. Let $Z = \{z, \dots, z^j\}$ and let

$$S = aP \cup (baP \cup \cdots \cup ba^{k-1}P) \cup (Z \cup Q),$$
$$T = aP \cup (baP \cup \cdots \cup ba^{k-1}P) \cup (Z^\gamma \cup Q^\gamma).$$

Note that $|S| = |T| = m$. Taking $L = P$ and $M = X$ and $C = aP \cup (baP \cup \cdots \cup ba^{k-1}P)$ in Lemma 12, we obtain $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$. Since $G$ has the $m$-DCI property, we have $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(G)$. Let $x \in aP$. Then $x = az^\ell = a^{\ell n' + 1}$ for some $0 \leqslant \ell \leqslant p - 1$, which implies that $|x| = 2n/(2n, \ell n' + 1)$. Since 2 divides $n'$ and $p$ divides $n'$, we have $(2n, \ell n' + 1) = 1$, and so $|x| = 2n$. This means that every element in $aP$ has order $2n$. Note that every element in $(baP \cup \cdots \cup ba^{k-1}P) \cup Q \cup Q^\gamma$ has order 4 and every element in $Z \cup Z^\gamma$ has order $p$. We derive from $S^\sigma = T$ that $(aP)^\sigma = aP$ and $Z^\sigma = Z^\gamma$. Since $\langle a\rangle$ is characteristic in $G$, it follows that $a^\sigma = a^r$ for some integer $r$. In particular,

$$\{z^r, \dots, z^{jr}\} = Z^\sigma = Z^\gamma = \{z^{-1}, \dots, z^{-j}\}.$$

Then by Lemma 9, $r \equiv -1 \pmod p$. Note that $P^\sigma = P$ as $P$ is characteristic in $G$. We conclude that $aP = (aP)^\sigma = a^\sigma P^\sigma = a^r P$, which leads to $a^{r-1} \in P = \langle a^{n'}\rangle$. However, this together with $p$ dividing $n'$ implies that $p$ divides $r-1$, contradicting $r \equiv -1 \pmod{p}$. Thus $p$ does not divide $n'$, which means that $n$ is not divisible by $p^2$, completing the proof. $\square$

Now we are ready to prove Theorem 1.

*Proof of Theorem 1.* Let $G$ be a generalized quaternion group of order $4n$ with $n \geqslant 3$ such that $G$ has the $m$-DCI property for some $1 \leqslant m \leqslant 2n-1$. Then $n$ is odd as Lemma 14 asserts. Furthermore, for any prime $p \leqslant m-1$, according to Lemma 15 we have that $n$ is not divisible by $p^2$. This completes the proof. $\square$

## 4 Proof of Theorem 3

In this section, we prove Theorem 3. Besides being important ingredients of the proof of Theorem 3, the following two lemmas are of their own interest as well.

**Lemma 16.** *Let $G \leqslant A \leqslant \mathrm{Sym}(\Omega)$ with $G$ regular on $\Omega$, and let $H$ be a normal subgroup of odd order $n$ in $G$. Suppose $G = H \rtimes \langle b \rangle$ for some $b \in G$ with $|b| \in \{2, 4\}$ such that either $G = H \times \langle b \rangle$ or $h^b = h^{-1}$ for all $h \in H$. Then for a regular subgroup $X$ of $A$ isomorphic to $G$, the subgroups $X$ and $G$ are conjugate in $A$ if and only if $H$ and the unique subgroup of order $n$ of $X$ are conjugate in $A$.*

*Proof.* By the assumption of the lemma, there exists $r = \pm 1$ such that $h^b = h^r$ for all $h \in H$. Let $X$ be a subgroup of $A$ isomorphic to $G$. Then $X$ has a unique subgroup of order $n$, say $Y$, and we may write $X = Y \rtimes \langle c \rangle$ such that $|b| = |c|$ and $y^c = y^r$ for all $y \in Y$. We need to prove that $G$ and $X$ are conjugate in $A$ if and only if $H$ and $Y$ are conjugate in $A$. The necessity is clear because $H$ and $Y$ are the unique subgroups of order $n$ in $G$ and $X$, respectively. To finish the proof, assume that $A$ has an element $\alpha$ with $Y^\alpha = H$, and we shall show that there exists an element of $A$ conjugating $X$ to $G$.

Since $c \in \mathrm{N}_A(Y)$, we have $c^\alpha \in \mathrm{N}_A(Y^\alpha) = \mathrm{N}_A(H)$. Hence both the 2-elements $b$ and $c^\alpha$ are in $\mathrm{N}_A(H)$. Let $P$ be a Sylow 2-subgroup of $\mathrm{N}_A(H)$ such that $b \in P$. By Sylow Theorem, there exists $\beta \in \mathrm{N}_A(H)$ such that $(c^\alpha)^\beta \in P$. Then

$$X^{\alpha\beta} = (Y \rtimes \langle c \rangle)^{\alpha\beta} = Y^{\alpha\beta} \rtimes \langle c^{\alpha\beta} \rangle = H^\beta \rtimes \langle c^{\alpha\beta} \rangle = H \rtimes \langle c^{\alpha\beta} \rangle.$$

Let $d = c^{\alpha\beta} \in P$. Then $|d| = |c| = |b|$ and $h^d = h^r$ for all $h \in H$ as $y^c = y^r$ for all $y \in Y$.

Write $m = |b|$. Then $m = 2$ or 4. The regularity of $G$ on $\Omega$ implies $|\Omega| = |G| = |b||H| = mn$. Since $H$ is a normal subgroup of $G$, it follows that $H$ has $m$ orbits on $\Omega$, say $\Omega_1, \Omega_2, \dots, \Omega_m$, where $|\Omega_i| = n$ for every $i \in \{1, \dots, m\}$. Moreover, since $G = H \rtimes \langle b \rangle$ with $|b| = m$, the element $b$ permutes the set $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ cyclicly. Similarly, $d$ permutes $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ cyclicly because $X^{\alpha\beta} = H \rtimes \langle d \rangle$ with $|d| = m$.

Note that $P$ is a 2-group and $b, d \in P$. Every orbit of $P$ on $\Omega$ has length 2-power that is at least $m$, where $m = 2$ or 4. If every orbit of $P$ on $\Omega$ has length greater than $m$, then every orbit of $P$ on $\Omega$ has length divisible by $2m$, and so $|\Omega|$ is divisible by $2m$, which is impossible because $|\Omega| = mn$ with $n$ odd. Thus $P$ has an orbit of length $m$, say $\Delta$. In particular, both $\langle b\rangle$ and $\langle d\rangle$ are regular on $\Delta$. Write $\Delta = \{\delta_1, \delta_2, \dots, \delta_m\}$. Since $b$ permutes $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ cyclicly, we have $|\Delta \cap \Omega_i| = 1$, say $\delta_i \in \Omega_i$, for every $i \in \{1, \dots, m\}$.

Consider $b^\Delta$ and $d^\Delta$, namely, the permutations of $\Delta$ induced by $b$ and $d$, respectively. For $m = 2$, since both $\langle b\rangle$ and $\langle d\rangle$ are regular on $\Delta$, we have $b^\Delta = d^\Delta$ and set $x = d$. Now assume $m = 4$. It is easy to check that if a product of two elements of order 4 in $\mathrm{S}_4$ has 2-power order, then the two elements are equal or inverse to each other. Since $bd \in P$ has 2-power order, we conclude that $b^\Delta = d^\Delta$ or $b^\Delta = (d^{-1})^\Delta$. Set $x = d$ in the former case, and $x = d^{-1}$ in the latter case. Then summarizing this paragraph, we obtain $b^\Delta = x^\Delta$ with $x = d^{\pm 1}$. Consequently, $bx^{-1}$ fixes every element in $\Delta$.

Since $h^d = h^r$ for every $h \in H$ and $r = \pm 1$, we have $h^{d^{-1}} = h^r$ for every $h \in H$. This together with $x = d^{\pm 1}$ gives $h^x = h^r = h^b$ for every $h \in H$, which indicates that $bx^{-1}$ centralizes $H$. For each $i \in \{1, \dots, m\}$, since $bx^{-1}$ fixes $\delta_i$ and $\Omega_i$ is the orbit of $H$ containing $\delta_i$, it follows that $bx^{-1}$ fixes every element in $\Omega_i$. Hence $bx^{-1} = 1$, and so $\langle b\rangle = \langle x\rangle = \langle d\rangle$. As $X^{\alpha\beta} = H \rtimes \langle c^{\alpha\beta}\rangle = H \rtimes \langle d\rangle$, this shows that $X^{\alpha\beta} = H \rtimes \langle b\rangle = G$, which completes the proof. □

Based on Lemma 16, we may prove the following:

**Lemma 17.** *Let $G$ be a cyclic group of order $2^\ell n$ with $\ell \in \{0, 1, 2\}$ and $n$ odd, and let $p$ be the least prime divisor of $n$. Then every connected Cayley digraph of $G$ with valency at most $p$ is a CI-digraph.*

*Proof.* Write $G = \langle a\rangle \cong \mathbb{Z}_{2^\ell n}$. Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley digraph with $|S| \leqslant p$, and let $A = \mathrm{Aut}(\Gamma)$. If $\ell = 0$, then since $G$ is a connected $p$-DCI-group ( [21, Theorem 1.1]), $\Gamma$ is a CI-digraph, as required. Next we consider the case $\ell \in \{1, 2\}$. Denote by $A_1$ the stabilizer of 1 in $A$.

Assume that $p$ does not divide $|A_1|$. Since $\Gamma$ is connected and has valency at most $p$, each prime divisor of $|A_1|$ is at most $p$. Then as $p$ is the least prime divisor of $n$, we conclude that $|A_1|$ is coprime to $n$. Let $\pi$ be the set of prime divisor of $n$. It follows from $A = R(G)A_1$ that $\langle a^{2^\ell}\rangle$ is a Hall $\pi$-subgroup of $A$. By [37, Theorem 9.1.10], all nilpotent Hall $\pi$-subgroup of $A$ are conjugate. Hence all subgroups isomorphic to $\langle a^{2^\ell}\rangle$ are conjugate in $A$, and so all regular subgroups of $A$ isomorphic to $R(G)$ are conjugate by Lemma 16. This shows that $\Gamma$ is a CI-digraph by Proposition 6.

Assume that $p$ divides $|A_1|$. If $\Gamma$ has valency less than $p$, then the connectivity of $\Gamma$ means that $|A_1|$ is not divisible by $p$, a contradiction. Thus $\Gamma$ has valency $p$, and it further follows from $p$ dividing $|A_1|$ that $\Gamma$ is arc-transitive. Then by [28, Theorem 1.3], every connected arc-transitive Cayley digraph over a cyclic group is a CI-digraph, and hence $\Gamma$ is a CI-digraph. This completes the proof. □

Let $X$ and $Y$ be digraphs. The *lexicographic product* $X[Y]$ of $X$ and $Y$ is defined as the digraph with vertex set $V(X) \times V(Y)$ such that $((x_1, y_1), (x_2, y_2))$, where $x_1, x_2 \in V(X)$ and $y_1, y_2 \in V(Y)$, is an arc if and only if $(x_1, x_2) \in \mathrm{Arc}(X)$, or $x_1 = x_2$ and $(y_1, y_2) \in \mathrm{Arc}(Y)$. We now give some sufficient conditions for Cayley digraphs of generalized quaternion groups $\mathrm{Q}_{4n}$ to be CI-digraphs with $n \geqslant 3$ odd.

**Lemma 18.** *Let $\Gamma = \mathrm{Cay}(\mathrm{Q}_{4n}, S)$ be a connected Cayley digraph of $\mathrm{Q}_{4n}$ with $n \geqslant 3$ odd, and let $A = \mathrm{Aut}(\Gamma)$. Then the following statements hold:*

(a) *If $|A_1|$ is coprime to $n$, then $\Gamma$ is a CI-digraph.*

(b) *If $n$ is a power of a prime $p$ and $\Gamma$ is arc-transitive with $|S| = p$, then $\Gamma$ is a CI-digraph.*

*Proof.* Let $G = \mathrm{Q}_{4n} = \langle a, b \mid a^{2n} = 1,\ b^2 = a^n,\ a^b = a^{-1}\rangle$, and let $\pi$ be the set of prime divisors of $n$.

To prove part (a), suppose that $|A_1|$ is coprime to $n$. Since $A = R(G)A_1$ and $n$ is odd, we conclude that $\langle R(a^2)\rangle$ is a Hall $\pi$-subgroup of $A$. Since all nilpotent Hall $\pi$-subgroups of $A$ are conjugate by [37, Theorem 9.1.10], Lemma 16 implies that all regular subgroups of $A$ isomorphic to $R(G)$ are conjugate. Hence $\Gamma$ is a CI-digraph by Proposition 6. This proves part (a).

To prove part (b), suppose that $n = p^\ell$ for an odd prime $p$ and a positive integer $\ell$, and that $\Gamma$ is arc-transitive with $|S| = p$. If $\ell = 1$, it follows from Lemma 11 that $\mathrm{Q}_{4p}$ is a DCI-group. Hence $\Gamma$ is a CI-digraph.

From now on we assume that $\ell \geqslant 2$. Since $A_1$ acts transitively on $S$, the order $|A_1|$ is divisible by $p$, and so $|A| = |R(G)||A_1| = 4n|A_1|$ is divisible by $p^{\ell+1}$. Write

$$H = \langle a^2\rangle \quad \text{and} \quad N = \mathrm{N}_A(R(H)).$$

Since $|R(H)| = |H| = p^\ell$ and $|A|$ is divisible by $p^{\ell+1}$, it is clear that $R(H)$ is not a Sylow $p$-subgroup of $A$. By Sylow Theorem and Proposition 4, $|N|$ is divisible by $p^{\ell+1}$. Since $R(H) \trianglelefteq R(G)$, we get $R(G) \leqslant N$. It follows that $\Gamma$ is $N$-vertex-transitive, and

$$|N_u| = |N|/|R(G)| \text{ is divisible by } p \text{ for every } u \in V(\Gamma). \tag{2}$$

Hence $\Gamma$ is $N$-arc-transitive as $|S| = p$. Since $N = \mathrm{N}_A(R(H))$, we have $R(H) \trianglelefteq N$. Since $|S| = p$, it follows that $\Gamma$ is $N$-locally primitive. The orbit set of $R(H)$ on $V(\Gamma)$ is

$$\{H, bH, b^2H, b^3H\} = V(\Gamma_{R(H)}).$$

Recall that $\Gamma$ is connected. Then we have $ba^i \in S$ for some integer $i$. Note that there is an automorphism $\alpha$ of $G$ sending $a$ and $b$ to $a$ and $ba^i$, respectively. Then replacing $S$ by $S^\alpha$, we may assume that $b \in S$, whence

$$\mathrm{Arc}(\Gamma_{R(H)}) = \{(H, bH), (bH, b^2H), (b^2H, b^3H), (b^3H, H)\}. \tag{3}$$

By Lemma 13 (b), either $R(H)$ is the kernel of $N$ on $V(\Gamma_{R(H)})$ and $\Gamma$ is a $R(H)$-cover of $\Gamma_{R(H)}$, or $\Gamma_{R(H)}$ is the directed cycle $\overrightarrow{\mathrm{C}}_4$ of length 4.

Assume that $R(H)$ is the kernel of $N$ on $V(\Gamma_{R(H)})$ and $\Gamma$ is a $R(H)$-cover of $\Gamma_{R(H)}$. Then $\Gamma_{R(H)}$ has order 4 and out-valency $p \geqslant 3$. Hence $p = 3$. According to [29, Theorem 1.4], $\mathrm{Q}_{4p^\ell}$ is a 3-DCI-group. Hence $\Gamma$ is a CI-digraph, as required.

In the rest of proof, we show that $\Gamma_{R(H)}$ cannot be the directed cycle $\overrightarrow{\mathrm{C}}_4$. For this purpose, we claim that $\Gamma \ncong \overrightarrow{\mathrm{C}}_{4p^{\ell-1}}[\overline{\mathrm{K}}_p]$. Suppose for a contradiction that $\Gamma \cong \overrightarrow{\mathrm{C}}_{4p^{\ell-1}}[\overline{\mathrm{K}}_p]$.

Then $N$ has an imprimitive block system on $V(\Gamma)$ such that each block is an independent set of size $p$ and the induced subdigraph of each two blocks is either $\overline{\mathrm{K}}_{2p}$ or $\overrightarrow{\mathrm{K}}_{p,p}$. Let $\Delta$ be an imprimitive block containing 1. Since $R(G)$ is a regular subgroup of $N$, we derive that $\Delta$ is a subgroup of $G$ and $S$ is a union of left cosets of $\Delta$ in $G$. Since $b \in S$ and $|S| = p$, it follows that $\Delta = \langle a^{2p^{\ell-1}} \rangle$ and $S = b\Delta = b\langle a^{2p^{\ell-1}} \rangle$. This implies that $\langle S \rangle \leqslant \langle a^{2p^{\ell-1}}, b \rangle \cong \mathrm{Q}_{4p}$, contradicting the condition $\langle S \rangle = G$ by the connectivity of $\Gamma$.

To complete the proof, suppose that $\Gamma_{R(H)} = \overrightarrow{\mathrm{C}}_4$. To derive a contradiction, by the above claim, we only need to show that $\Gamma \cong \overrightarrow{\mathrm{C}}_{4p^{\ell-1}}[\overline{\mathrm{K}}_p]$. Let $C = \mathrm{C}_A(R(H))$ and let $K$ be the kernel of $C$ acting on $V(\Gamma_{R(H)})$. Then $R(H) \leqslant C$, $R(b^2) \in C$, $C \trianglelefteq N$, and $C/K \leqslant \mathrm{Aut}(\Gamma_{R(H)}) = \mathrm{Aut}(\overrightarrow{\mathrm{C}}_4) \cong \mathbb{Z}_4$. Note that $b^i H$ is an orbit for both $R(H)$ and $K$. By the Frattini argument, $K = R(H)K_u$ for $u \in V(\Gamma)$. As $\Gamma_{R(H)} = \overrightarrow{\mathrm{C}}_4$, it follows that $C_u$ fixes $V(\Gamma_{R(H)})$ pointwise. Hence $C_u \leqslant K$ and $C_u = K_u$. Since $K \leqslant C = \mathrm{C}_A(R(H))$, we obtain

$$K = R(H) \times C_u \text{ for every } u \in V(\Gamma). \tag{4}$$

Consequently, $C_1 C_{b^2} \trianglelefteq K$. Noting that $R(H)$ is a $p$-group, it follows from (4) that $|K|_{p'} = |C_1|_{p'} = |C_{b^2}|_{p'} = |C_1 C_{b^2}|_{p'}$. Since $|C_1 \cap C_{b^2}| = |C_1||C_{b^2}|/|C_1 C_{b^2}|$, this implies

$$|C_1 \cap C_{b^2}|_{p'} = |K|_{p'}.$$

In this paragraph, we prove by contradiction that $K \neq R(H)$. Suppose that $K = R(H)$. Then (4) implies that $C_1 = 1$, and so $N_1$ acts faithfully on $R(H)$ by conjugation. Hence $N_1 \leqslant \mathrm{Aut}(R(H)) \cong \mathbb{Z}_{p^{\ell-1}(p-1)}$ is cyclic. This together with (2) implies that $N_1$ has a unique subgroup of order $p$, say $P$. Let $L$ be the kernel of $N$ acting on $V(\Gamma_{R(H)})$. Since $\Gamma_{R(H)} = \overrightarrow{\mathrm{C}}_4$, it follows that $N_1$ fixes $V(\Gamma_{R(H)})$ pointwise, which means that $N_1 = L_1$. Thus, by the Frattini argument, $L = R(H)N_1$. Consequently, $L/R(H)$ is cyclic. Write $M = R(H)P$. Then $M/R(H)$ is the unique subgroup of order $p$ of $L/R(H)$ and so characteristic in $L/R(H)$. Note that $L/R(H) \trianglelefteq N/R(H)$. Then $M/R(H) \trianglelefteq N/R(H)$. This implies that $R(H) \leqslant M \trianglelefteq N$, and so all orbits of $M$ on $V(\Gamma)$ have length $|R(H)|$. Clearly,

$$R(H)P = M = R(H)M_1 = R(H)M_b.$$

Since $|M| = |R(H)||P| = p|R(H)|$, we obtain $|M_1| = p = |M_b|$. Hence both $M_1$ and $M_b$ are cyclic groups of order $p$. Recall that $\mathrm{Aut}(R(H)) \cong \mathbb{Z}_{p^{\ell-1}(p-1)}$ and $\ell \geqslant 2$. The unique subgroup of order $p$ of $\mathrm{Aut}(R(H))$ is generated by the automorphism $\gamma$ of $R(H) = \langle R(a^2) \rangle$ defined by

$$\gamma : R(a^2) \mapsto R(a^2)^r = R(a^{2r}), \quad \text{where } r := p^{\ell-1} + 1.$$

Since the action of $M_1 \leqslant N_1$ by conjugation on $R(H)$ is faithful, it follows that

$$R(a^2)^\alpha = R(a^2)^\gamma = R(a^{2r}) \text{ for some generator } \alpha \text{ of } M_1.$$

For integers $i$ and $j$, since $a^2$ has order $n = p^\ell$ and $r^j \equiv jp^{\ell-1} + 1 \pmod{p^\ell}$, we have

$$R(a^{2i})^{\alpha^j} = R(a^{2ir^j}) = R(a^{2i(jp^{\ell-1}+1)}) \in R(a^{2i})\langle R(a^{2p^{\ell-1}}) \rangle. \tag{5}$$

Take arbitrary $x, y \in M$. Since $M = R(H)M_1$, we may write $x = x_1x_2$ and $y = y_1y_2$ with $x_1, y_1 \in R(H)$ and $x_2, y_2 \in M_1$. Then the commutator

$$[x, y] = [x_1x_2, y_1y_2] = (x_1x_2)^{-1}(y_1y_2)^{-1}(x_1x_2)(y_1y_2) = (x_1^{-1})^{x_2}(y_1^{-1}x_1)^{x_2y_2}(y_1)^{y_2}.$$

This together with (5) implies that $[x, y] \in \langle R(a^{2p^{\ell-1}})\rangle$. Hence the derived group

$$M' = \langle R(a^{2p^{\ell-1}})\rangle \cong \mathbb{Z}_p.$$

Since $M = M_bR(H)$, we may write $\alpha = \beta R(a^2)^t$ for some $\beta$ of $M_b$ and integer $t$. Since $|M'| = p$, we derive from Proposition 5 that $(R(a^2)^t)^p = (\beta^{-1}\alpha)^p = (\beta^{-1})^p\alpha^p = 1$. Therefore, $t$ is divisible by $p^{\ell-1}$. In particular, $t$ is divisible by $p$ as $\ell \geqslant 2$. Since

$$b^\alpha = b^{\beta R(a^2)^t} = b^{R(a^2)^t} = ba^{2t},$$

we derive for each integer $k$ that

$$(ba^{2tk})^\alpha = b^{R(a^{2tk})\alpha} = b^{\alpha R(a^{2tk})^\alpha} = b^{\alpha R(a^{2tkr})} = (ba^{2t})^{R(a^{2tkr})} = ba^{2t(1+kr)}.$$

Hence $\alpha$ stabilizes $b\langle a^{2t}\rangle$, and so $M_1 = \langle\alpha\rangle$ stabilizes $b\langle a^{2t}\rangle$. Note that the stabilizer $M_1$ is transitive or trivial on the out-neighborhood $\Gamma^+(1) = S$ of 1 in $V(\Gamma)$. If $M_1$ is trivial on $S$, then we obtain a contradiction that $M_1 = 1$ as $\Gamma$ is $N$-vertex-transitive and strongly connected. Hence $M_1$ is transitive on $S$, and so $S = b^{M_1}$ as $b \in S$. Then $S = b^{M_1} \subseteq b\langle a^{2t}\rangle$, and so $\langle S\rangle \leqslant \langle b, a^p\rangle < G$ as $p$ divides $t$. This contradicts the connectivity of $\Gamma$. Therefore, we obtain $K \neq R(H)$.

Finally, we achieve $\Gamma \cong \overrightarrow{\mathrm{C}}_{4p^{\ell-1}}[\overline{\mathrm{K}}_p]$ by discussing two cases.

**Case 1:** $C_1 \cap C_{b^2} = 1$.

Recall that $R(H) \times C_1 = K \neq R(H)$. Then $C_1 \neq 1$, and $|C_1|_{p'} = |K|_{p'} = |C_1 \cap C_{b^2}|_{p'} = 1$. This means that $C_1$ is a $p$-group. Since $C/K \leqslant \mathbb{Z}_4$, it follows that $K = R(H) \times C_1$ is a Sylow $p$-subgroup of $C$ and thus characteristic in $C$. Note that

$$C_1 \cong C_1/(C_1 \cap C_{b^2}) \cong C_1C_{b^2}/C_{b^2} \leqslant K/C_{b^2} \cong R(H)$$

is cyclic. Hence $K$ has a characteristic subgroup $D = X \times Y \cong \mathbb{Z}_p^2$, where $\mathbb{Z}_p \cong X \leqslant R(H)$ and $\mathbb{Z}_p \cong Y \leqslant C_1$. Then $D$ is characteristic in $C$. As $C \trianglelefteq N$, we have $D \trianglelefteq N$. Since $X$ is semiregular of order $p$ and $Y$ fixes the vertex 1, we then conclude that the orbits of $D = YX$ on $V(\Gamma)$ all have length $p$. For every $u \in V(\Gamma)$, it follows that $D = XD_u$, and so $D_u \cong \mathbb{Z}_p$ is either transitive or trivial on the out-neighborhood $\Gamma^+(u)$ of $u$. If $D_u$ is trivial on $\Gamma^+(u)$, then $D_u = 1$ as $\Gamma$ is $N$-vertex-transitive and strongly connected, contradicting to $D_u \cong \mathbb{Z}_p$. Thus $D_u$ is transitive on $\Gamma^+(u)$ for every $u \in V(\Gamma)$. This implies that if $\Delta_1$ and $\Delta_2$ are two orbits of $D$ and there is an arc from some vertex of $\Delta_1$ to some vertex of $\Delta_2$, then $(x, y) \in \mathrm{Arc}(\Gamma)$ for all $x \in \Delta_1$ and $y \in \Delta_2$. Since $\Gamma$ has out-valency $p$, it follows that $\Gamma \cong \overrightarrow{\mathrm{C}}_{4p^{\ell-1}}[\overline{\mathrm{K}}_p]$, as required.

**Case 2:** $C_1 \cap C_{b^2} \neq 1$.

Recall that $\Gamma_{R(H)} = \overrightarrow{\mathrm{C}}_4$ and $b \in S$, we have $S \cap (H \cup b^2H \cup b^3H) = \emptyset$. From $C/K \leqslant \mathbb{Z}_4$ we deduce that $B := K\langle R(b^2)\rangle \trianglelefteq C$. Since $R(b^2)$ interchanges $C_1$ and $C_{b^2}$ by conjugation, we have $C_1 \cap C_{b^2} \trianglelefteq B$. Note that $H \cup b^2H$ and $bH \cup b^3H$ are the orbits of $B$ on $V(\Gamma)$. Then the orbits of $C_1 \cap C_{b^2}$ on $bH \cup b^3H$ have the same length, say $t$. Hence the valency of $\Gamma$ is a multiple of $t$. As $\Gamma$ is $p$-valent, we deduce that $t = 1$ or $p$. Recall that

$$K = R(H) \times C_1 = R(H) \times C_{b^2}.$$

The group $C_1 \cap C_{b^2}$ fixes $H \cup b^2H$ pointwise. If $t = 1$, then $C_1 \cap C_{b^2}$ fixes both $H \cup b^2H$ and $bH \cup b^3H$ pointwise, which means that $C_1 \cap C_{b^2} = 1$, a contradiction. Thus $t = p$, that is, the orbits of $C_1 \cap C_{b^2}$ on $bH \cup b^3H$ all have length $p$. Since $R(b) \in N$ normalizes $C$, it follows that $C_b \cap C_{b^3} = (C_1 \cap C_{b^2})^{R(b)}$ fixes $(H \cup b^2H)^{R(b)} = bH \cup b^3H$ pointwise and that the orbits of $C_b \cap C_{b^3}$ on $(bH \cup b^3H)^{R(b)} = H \cup b^2H$ all have length $p$. Let

$$T = (C_1 \cap C_{b^2})(C_b \cap C_{b^3}).$$

Then all orbits of $T$ on $V(\Gamma)$ have length $p$. Note that $C_1 \cap C_{b^2} \leqslant T_v$ for every $v \in H \cup b^2H$ and $C_b \cap C_{b^3} \leqslant T_w$ for every $w \in bH \cup b^3H$. Then we derive from (3) that the stabilizer $T_u$ is transitive on the out-neighbors of $u$ in $\Gamma$ for every $u \in V(\Gamma)$. This implies that if $\Delta_1$ and $\Delta_2$ are two orbits of $T$ and there exists an arc from some vertex of $\Delta_1$ to some vertex of $\Delta_2$, then $(x, y) \in \mathrm{Arc}(\Gamma)$ for all $x \in \Delta_1$ and $y \in \Delta_2$. Hence $\Gamma \cong \overrightarrow{\mathrm{C}}_{4p^{\ell-1}}[\overline{\mathrm{K}}_p]$, as required. □

Let $\Gamma$ be a connected Cayley digraph of a finite group $G$ of valency $m < p$, and let $A = \mathrm{Aut}(\Gamma)$. By the same argument as [19, Lemma 2.1] we see that every prime divisor of $|A_1|$ is less than $p$. Thus the following result is a consequence of Lemma 18.

**Lemma 19.** *Let $n$ be a power of an odd prime $p$, let $\Gamma = \mathrm{Cay}(\mathrm{Q}_{4n}, S)$ be a connected Cayley digraph of $\mathrm{Q}_{4n}$ with $|S| \leqslant p$. Then $\Gamma$ is a CI-digraph.*

Now we are ready to prove Theorem 3.

*Proof of Theorem 3.* Let $n = p^\ell$, where $p$ is a prime and $\ell$ is a positive integer, let $G = \mathrm{Q}_{4n} = \langle a, b \mid a^{2n} = 1,\, b^2 = a^n,\, a^b = a^{-1}\rangle$, and let $m$ be an integer with $1 \leqslant m \leqslant 2n - 1$.

First, we suppose that $G$ has the $m$-DCI property. By Theorem 1, $n$ is odd, and so $p$ is odd. If $\ell \geqslant 2$, then it follows from Theorem 1 that $m \leqslant p$. This shows that either $n = p$ or $m \leqslant p$, which completes the proof of the necessity.

Next, we prove the sufficiency. So suppose that $p$ is odd and either $n = p$ or $m \leqslant p$. If $n = p$, then it follows from Lemma 11 that $\mathrm{Q}_{4p}$ is a DCI-group, and so $G$ has the $m$-DCI property. Now assume $m \leqslant p$. Let $\mathrm{Cay}(G, S)$ be a Cayley digraph with $|S| = m$, and let $\mathrm{Cay}(G, T)$ be a Cayley digraph isomorphic to $\mathrm{Cay}(G, S)$. Since $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$, we have $\mathrm{Cay}(\langle S\rangle, S) \cong \mathrm{Cay}(\langle T\rangle, T)$, which implies that $|\langle S\rangle| = |\langle T\rangle|$. As $G$ is a generalized quaternion group of order $4p^\ell$ with $p$ odd prime, it follows that $\langle S\rangle \cong \langle T\rangle$. According to Lemma 10, there exists $\delta \in \mathrm{Aut}(G)$ such that $\langle T\rangle^\delta = \langle S\rangle$. Then we have

$$\mathrm{Cay}(\langle T\rangle, T) \cong \mathrm{Cay}(\langle T\rangle^\delta, T^\delta) = \mathrm{Cay}(\langle S\rangle, T^\delta),$$

and hence $\mathrm{Cay}(\langle S\rangle, S) \cong \mathrm{Cay}(\langle S\rangle, T^\delta)$. Set $\Gamma = \mathrm{Cay}(\langle S\rangle, S)$. Then $\Gamma$ is a connected $m$-valent Cayley digraph with $m = |S| \leqslant p$. As a subgroup of $\mathrm{Q}_{4n}$, we see that $\langle S\rangle$ is either a cyclic or a generalized quaternion subgroup of $\mathrm{Q}_{4n}$. Since Lemmas 17 and 19 assert that $\Gamma$ is a CI-digraph, there is an automorphism of $\langle S\rangle$ mapping $S$ to $T^\delta$. Again by Lemma 10, this automorphism can be extended to an automorphism of $G$, say $\gamma$. Then $S^\gamma = T^\delta$, and by taking $\sigma = \gamma\delta^{-1}$ we have $\sigma \in \mathrm{Aut}(G)$ and $S^\sigma = T$. This shows that $G$ has the $m$-DCI property, proving the sufficiency. □

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