Signed circuit 6-covers of signed K_4-minor-free graphs

You Lu[a] Rong Luo[b] Zhengke Miao[c] Cun-Quan Zhang[b]

Submitted: Nov 20, 2023; Accepted: Mar 10, 2025; Published: Apr 11, 2025

Abstract

Bermond, Jackson and Jaeger [*J. Combin. Theory Ser. B*, 35: 297–308, 1983] proved that every bridgeless ordinary graph G has a circuit 4-cover and Fan [*J. Combin. Theory Ser. B*, 54: 113–122, 1992] showed that G has a circuit 6-cover which together implies that G has a circuit k-cover for every even integer $k \geqslant 4$. The only left case when $k = 2$ is the well-known circuit double cover conjecture. For signed circuit k-cover of signed graphs, it is known that for every integer $k \leqslant 5$, there are infinitely many coverable signed graphs without signed circuit k-cover and there are signed eulerian graphs that admit nowhere-zero 2-flow but don't admit a signed circuit 1-cover. Fan conjectured that every coverable signed graph has a signed circuit 6-cover. This conjecture was verified only for signed eulerian graphs and for signed graphs whose bridgeless-blocks are eulerian. In this paper, we prove that this conjecture holds for signed K_4-minor-free graphs. The 6-cover is best possible for signed K_4-minor-free graphs.

Mathematics Subject Classifications: 05C22, 05C70

1 Introduction

Graphs or signed graphs considered in this paper are finite and may have multiple edges or loops. For terminology and notations not defined here we follow [5, 9, 21, 27].

A *signed graph* is a graph G with a mapping $\sigma : E(G) \mapsto \{1, -1\}$. The mapping σ, called *signature*, is sometimes implicit in the notation of a signed graph and will be

[a] Research & Development Institute of Northwest Polytechnical University in Shenzhen, Shenzhen, Guangdong 518063, China; School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, China (`luyou@nwpu.edu.cn`).

[b] Department of Mathematics, West Virginia University, Morgantown, WV 26505, USA (`rluo@mail.wvu.edu`, `cqzhang@mail.wvu.edu`).

[c] School of Mathematics and Statistics & Key Laboratory of Analytical Mathematics and Applications (Ministry of Education), Fujian Normal University, Fuzhou, Fujian 350007, China; Research Institute of Mathematical Science and School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, Jiangsu 221116, China (`zkmiao@jsnu.edu.cn`).

specified when needed. An edge e is *positive* if $\sigma(e) = 1$, and otherwise it is *negative.* An *ordinary graph* is a signed graph without negative edges and a *circuit* is a connected 2-regular graph. A circuit of length k is called a *k-circuit.* A circuit in a signed graph is *balanced* if it has an even number of negative edges and otherwise it is *unbalanced.* A *signed circuit* is a signed graph of one of the following three types: (1) a balanced circuit; (2) a long barbell, the union of two disjoint unbalanced circuits with a path (called the *barbell-path*) that meets the circuits only at its ends; (3) a short barbell, the union of two unbalanced circuits that meet at a single vertex (also called the *barbell-path*, for technical reasons). A *barbell* is either a long barbell or a short barbell. The edges of a signed circuit in a signed graph correspond to a minimal dependent set in the signed graphic matroid (see [30]).

Let G be a signed graph. A family $\mathcal{F}$ of signed circuits of G is called a *signed circuit cover* of G if every edge is contained in some member of $\mathcal{F}$ and is called a *signed circuit k-cover* if each edge is contained in precisely k members of $\mathcal{F}$. A signed graph is *coverable* if it has a signed circuit cover. Given a coverable signed graph G, the minimum length of a signed circuit cover of G is denoted by $SSC(G)$.

Note that an ordinary graph contains no unbalanced circuit and thus no barbell. The circuit covers of ordinary graphs are closely related to some mainstream areas in graph theory, such as, Tutte's integer flow theory [1, 4, 13, 16, 19, 24, 31], Fulkerson conjecture [14], snarks and graph minors [2, 17]. Thus the circuit cover of ordinary graphs has been studied extensively.

It is proved by Bermond, Jackson and Jaeger [4] that every ordinary graph admitting a nowhere-zero 4-flow has $SCC(G) \leqslant \frac{4}{3}|E(G)|$. By applying Seymour's 6-flow theorem [26] or Jaeger's 8-flow theorem [18], Alon and Tarsi [1], and Bermond, Jackson and Jaeger [4] proved that every bridgeless ordinary graph G has $SCC(G) \leqslant \frac{5}{3}|E(G)|$. One of the most famous open problems in this area was proposed by Alon and Tarsi [1] that every bridgeless ordinary graph G has $SCC(G) \leqslant \frac{7}{5}|E(G)|$.

Bermond, Jackson and Jaeger [4] proved that every bridgeless ordinary graph G has a circuit 4-cover and Fan [12] showed that G has a circuit 6-cover which together implies that G has a circuit k-cover for every even integer $k \geqslant 4$. The only left case when $k = 2$ is the well-known circuit double cover conjecture.

For signed graphs, Máčajová, Raspaud, Rollová and Škoviera [23] presented the first upper bound of $SSC(G)$. They showed that $SSC(G) \leqslant 11|E(G)|$ if G is coverable and the upper bound was improved by Lu et al. [22] to $\frac{14}{3}|E(G)|$. More improvements were obtained later in [7, 20, 25, 28, 29].

For k-cover of signed graphs, Fan [15] showed that for every integer $k \leqslant 5$, there are infinitely many coverable signed graphs that have no signed circuit k-cover and he proposed the following conjecture.

Conjecture 1. (Fan [15]) Every coverable signed graph has a signed circuit 6-cover.

The conjecture was verified for signed eulerian graphs in [3] and for signed graphs whose bridgeless-blocks are eulerian in [8].

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by edge contractions, edge deletions and vertex deletions; if not, G is H*-minor-free.* The class of K_4-minor-free graphs, which includes all series-parallel graphs and outerplanar graphs, is a very important family of graph class and has been studied by many researchers for various graph theory problems (for example see [10, 11]). In this paper we study the signed circuit k-cover for signed K_4-minor-free graphs and confirm Conjecture 1 for this family of signed graphs.

Theorem 2. *Every coverable signed K_4-minor-free graph has a signed circuit 6-cover.*

Note that if a coverable signed graph G containing four distinct degree 3 vertices x_1, x_2, y_1, y_2 such that $G[\{x_1, x_2\}]$ is a balanced 2-circuit and $G[\{y_1, y_2\}]$ is an unbalanced 2-circuit, then G has no signed circuit k-cover for any $1 \leqslant k \leqslant 5$. Thus the 6-cover in Theorem 2 is tight.

Before proceeding, it is worth pointing out that the problems of flow and signed circuit cover in signed graphs are significantly more challenging than their counterparts in ordinary graphs. For instance, while ordinary Eulerian graphs trivially allow for a nowhere-zero 2-flow and a 1-cover, signed Eulerian graphs can have flow values of 2, 3, or even 4, as shown in [25]. Additionally, there are signed Eulerian graphs that admit nowhere-zero 2-flow but don't have a 1-cover, as demonstrated in [3]. Unlike ordinary graphs, coverable signed graphs may have bridges. The intricate structures of signed graphs, such as barbells, bridges, and negative loops, add to their complexity in comparison to ordinary graphs.

This paper is organized as follows. In Section 2, we introduce more notations and terminology. Some simple cases and reduction lemmas needed in the proof of Theorem 2 are presented in Section 3. In Section 4, we prove Theorem 2 by contradiction.

2 Preliminaries

Let G be a graph. A vertex x is called a *cut-vertex* of G if $G-x$ has more components than G. A graph is 2*-connected* if it is connected and has no cut-vertex. A *block* of G is either a maximal 2-connected subgraph, or a cut-edge (with its ends), or an isolated vertex. An *end-block* of G is a block containing exactly one cut-vertex. Let L_x represent a loop at x and $L(G)$ be the set of all loops of G. Let $N_G(x)$ and $d_G(x)$ denote the *neighborhood* and the *degree* of x in G, respectively, where each loop at x contributes 2 to $d_G(x)$. A *d-vertex* is a vertex with degree d. For two subsets $X, Y \subseteq V(G)$ (not necessarily disjoint), denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y. A path with ends x and y is called an *xy-path.*

Let G be a signed graph. For an edge subset or a subgraph S of G, denote the set of all negative edges of S by $E_N(S)$ and define the *sign* of S to be $\sigma(S) = \Pi_{e \in S}\sigma(e)$. A path P in G is *positive* if $\sigma(P) = 1$, and *negative* otherwise. The path P is called a *subdivided edge* of G if every internal vertex of P is a 2-vertex of G. The *suppressed graph* of G, denoted by $\overline{G}$, is the signed graph obtained from G by replacing each maximal subdivided edge P with a single edge e and assigning $\sigma(e) = \sigma(P)$.

Given a signed graph G, *switching* at a vertex x is the inversion of the signs of all edges incident with x. A signed graph G' is said to be *equivalent* to G if G' can be obtained from G via a sequence of switchings and is denoted by $G' \sim G$. Define the *negativeness* of G by $\epsilon(G) = \min\{|E_N(G')| : G' \sim G\}$. A signed graph is *balanced* if its negativeness is 0 and otherwise *unbalanced*. That is, a balanced signed graph is equivalent to an *all-positive* signed graph, i.e. an ordinary graph. It is easy to see that a signed graph is balanced if and only if all of its circuits are balanced.

For two integers $n_1 \leqslant n_2$, let $[n_1, n_2]$ denote the set of integers between n_1 and n_2 inclusive. A *tadpole* at a vertex x is the union of an xy-path P and an unbalanced circuit C with $V(P) \cap V(C) = \{y\}$. The vertex x is called a *tail* and the path P is called a *tadpole-path*. Note that it is possible that $x = y$. In this case, the tadpole-path of the tadpole is a single vertex.

Definition 3. Let $\mathcal{F}$ be a family of signed subgraphs of a signed graph G. Let $t \in [0, 3]$ be an integer and x, y be two distinct vertices of G.

(1) For each $e \in E(G)$, $\mathcal{F}(e)$ denotes the number of members in $\mathcal{F}$ containing e.

(2) For an edge subset or a subgraph S of G, $\mathcal{F}$ is a *signed subgraph k-cover* of S if $\mathcal{F}(e) = k$ for each edge e in S. In particular, $\mathcal{F}$ is a *signed circuit k-cover* of G if every member of $\mathcal{F}$ is a signed circuit.

(3) A $\Psi_{xy}(t)$*-cover* is a signed subgraph 6-cover that consists of t positive xy-paths, t negative xy-paths, t tadpoles at x, $6 - 2t$ tadpoles at y, and some signed circuits.

(4) Let xy be an edge. A $\Psi^*_{xy}(2)$*-cover* is a $\Psi_{xy}(2)$-cover such that for each $u \in \{x, y\}$, one tadpole at u doesn't contain the vertex in $\{x, y\} \setminus \{u\}$, and the tadpole-path of the other tadpole at u contains the edge xy.

Signed circuit cover and flows are closely related. It is known that a signed graph G is coverable if and only if it admits a nowhere-zero k-flow for some integer $k \geqslant 2$. Refining the results in [6], we have the following characterization.

Proposition 4. *A connected signed graph G is coverable if and only if $\epsilon(G) \neq 1$ and there is no cut-edge b such that $G - b$ has a balanced component.*

3 $\Psi_{xy}(t)$-covers of two-terminal signed graphs

A *two-terminal signed graph* $H(x, y)$ is a connected signed nonempty graph H with two specified vertices, a *source terminal* x and a *target terminal* y. In particular, if $x = y$, $H(x, x)$ is defined to be a negative loop, i.e., a two-terminal signed graph with the source and target terminals same is just one vertex with a negative loop. For short, we abbreviate $H(x, y)$ to H if the terminals are understood from the context.

Let $H_i = H_i(x_i, y_i)$ be a two-terminal signed graph for each $i \in [1, n]$. When $x_i \neq y_i$ for each i, the *parallel connection* $\mathcal{P}(H_1, \ldots, H_n)$ of $H_1, \ldots, H_n$ is the two-terminal signed graph obtained from $H_1 \cup \cdots \cup H_n$ by identifying $x_1, \ldots, x_n$ into a source terminal and identifying $y_1, \ldots, y_n$ into a target terminal. When $x_1 \neq y_n$, the *series connection* $\mathcal{S}(H_1, \ldots, H_n)$ of $H_1, \ldots, H_n$ is the two-terminal signed graph with source terminal x_1 and target terminal y_n obtained from $H_1 \cup \cdots \cup H_n$ by identifying y_{i-1} and x_i for each $i \in [2, n]$. If G is a series connection of $H_1, \ldots, H_n$ and n is maximum with this property, then we call every H_i a *part* of G. Let $\mathcal{B}(G) = \{H_1, \ldots, H_n\}$ be the set of all parts of G. Obviously, $\mathcal{B}(G)$ can be partitioned into three subsets as follows:

$$\begin{aligned}
\mathcal{B}_0(G) &= \{H_i \in \mathcal{B}(G) : x_i = y_i\}, \\
\mathcal{B}_1(G) &= \{H_i \in \mathcal{B}(G) : x_i \neq y_i, |E(H_i)| = 1\}, \\
\mathcal{B}_2(G) &= \{H_i \in \mathcal{B}(G) : x_i \neq y_i, |E(H_i)| \geqslant 2\}.
\end{aligned}$$

Note that every member of $\mathcal{B}_0(G)$ is a negative loop and every member of $\mathcal{B}_1(G)$ is a positive or negative K_2. A series connection is shown in Fig. 1.

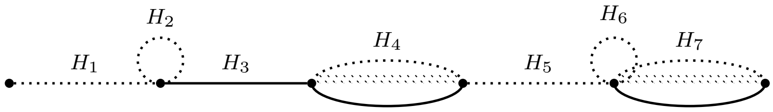

Figure 1: A series connection G with $\mathcal{B}_0(G) = \{H_2, H_6\}$, $\mathcal{B}_1(G) = \{H_1, H_3, H_5\}$ and $\mathcal{B}_2(G) = \{H_4, H_7\}$. Solid lines are positive; dotted lines are negative.

The next lemma will be applied in the reduction.

Lemma 5. *Let $H_i = H_i(x_{i-1}, x_i)$ for each $i \in [1, n]$ and $G = \mathcal{S}(H_1, \ldots, H_n)$ with $n = |\mathcal{B}(G)|$ and $|\mathcal{B}_2(G)| \geqslant 1$. Let $\theta^* = (1, 1, -1, -1)$ if $|\mathcal{B}_2(G)| = 1$, and $\theta^* = (1, 1, -1, -1)$ or $(-1, -1, -1, -1)$ if $|\mathcal{B}_2(G)| \geqslant 2$. If every $H_i \in \mathcal{B}_2(G)$ has a $\Psi_{x_{i-1}x_i}(2)$-cover, then G has a signed subgraph 6-cover*

$$\mathcal{F}_0 \cup 2\mathcal{B}_0(G) \cup \{P_1, P_2, P_3, P_4\} \cup \{T_1, T_2, T_3, T_4\},$$

where

- ▷ *$\mathcal{F}_0$ is a family of signed circuits;*
- ▷ *P_1, P_2, P_3, P_4 are four $x_0 x_n$-paths of G and $(\sigma(P_1), \sigma(P_2), \sigma(P_3), \sigma(P_4)) = \theta^*$;*
- ▷ *T_1, T_2 are two tadpoles of G at x_0 whose unbalanced circuits are in the part of $\mathcal{B}_0(G) \cup \mathcal{B}_2(G)$ with minimum subscript.*
- ▷ *T_3, T_4 are two tadpoles of G at x_n whose unbalanced circuits are in the part of $\mathcal{B}_0(G) \cup \mathcal{B}_2(G)$ with maximum subscript.*

Proof. Denote $I_j = \{i : H_i \in \mathcal{B}_j(G)\}$ for each $j \in [0, 2]$, and for each $i \in I_2$, let

$$\mathcal{F}_i = \mathcal{C}_i \cup \{P_{i1}, P_{i2}, P_{i3}, P_{i4}\} \cup \{T_{i1}, T_{i2}, T_{i3}, T_{i4}\},$$

be an arbitrary $\Psi_{x_{i-1}x_i}(2)$-cover of H_i, where $\mathcal{C}_i$ is a family of signed circuits, P_{i1}, P_{i2} are two positive $x_{i-1}x_i$-paths, P_{i3}, P_{i4} are two negative $x_{i-1}x_i$-paths, T_{i1}, T_{i2} are two tadpoles at x_{i-1}, and T_{i3}, T_{i4} are two tadpoles at x_i. Note that every part in $\mathcal{B}_0(G)$ is a negative loop and every part in $\mathcal{B}_1(G)$ is a positive or negative K_2.

Let $\mathcal{G}_1 = (\cup_{i\in I_2}\{P_{i1}, P_{i2}, P_{i3}, P_{i4}\}) \cup 4\mathcal{B}_1(G)$. Then $\mathcal{G}_1$ can be expressed as a family $\mathcal{P}$ consisting of 4 x_0x_n-paths P_1, P_2, P_3, P_4 such that $(\sigma(P_1), \sigma(P_2), \sigma(P_3), \sigma(P_4)) = \theta^*$ and $\mathcal{G}_1(e) = \mathcal{P}(e)$ for each $e \in E(G)$.

Let $\mathcal{G}_2 = (\cup_{i\in I_2}\{T_{i1}, T_{i2}, T_{i3}, T_{i4}\}) \cup 4\mathcal{B}_0(G) \cup 2\mathcal{B}_1(G)$. For the sake of convenience, let $T_{ij} = H_i$ for each $i \in I_0$ and each $j \in [1, 4]$ since H_i is a tadpole at x_{i-1} $(=x_i)$, and $I_0 \cup I_2 = \{i_1, i_2, \ldots, i_\ell\}$ with $0 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_\ell \leqslant n$. For each $j \in [1, 2]$, we construct a tadpole T_j at x_0, a tadpole T_{j+2} at x_n, and some barbells as follows:

$$\begin{aligned}
T_j &= (x_0x_1 \cdots x_{i_1-1}) \cup T_{i_1j},\\
T_{j+2} &= T_{i_\ell(j+2)} \cup (x_{i_\ell} \cdots x_{n-1}x_n),\\
B_{kj} &= T_{i_k(j+2)} \cup (x_{i_k}x_{i_k+1} \cdots x_{i_{k+1}-1}) \cup T_{i_{k+1}j}, \forall\, k \in [1, \ell-1].
\end{aligned}$$

Let $\mathcal{T} = \{T_1, T_2, T_3, T_4\}$ and $\mathcal{C} = \cup_{k=1}^{\ell-1}\{B_{k1}, B_{k2}\}$. Obviously, $\mathcal{G}_2(e) = (\mathcal{T} \cup \mathcal{C})(e)$ for each $e \in E(G)$. Therefore, $(\cup_{i\in I_2}\mathcal{C}_i) \cup \mathcal{C} \cup 2\mathcal{B}_0(G) \cup \mathcal{P} \cup \mathcal{T}$ is a desired signed subgraph 6-cover of G. □

By the definition and Lemma 5, the following result is straightforward and its proof is omitted.

Lemma 6. *Let $H_i = H_i(x_{i-1}, x_i)$ for each $i \in [1, n]$ and $G = \mathcal{S}(H_1, \ldots, H_n)$ with $n = |\mathcal{B}(G)| \geqslant 2$ and $|\mathcal{B}_2(G)| \geqslant 1$. If $\mathcal{B}_0(G) = \emptyset$ and every $H_i \in \mathcal{B}_2(G)$ has a $\Psi_{x_{i-1}x_i}(2)$-cover, then exactly one of the following statements holds.*

(1) *G has a $\Psi_{x_0x_n}(2)$-cover whose tadpoles at x_0 and x_n don't contain x_n and x_0, respectively;*

(2) *$\mathcal{B}_2(G) = \{H_1\}$ and any $\Psi_{x_0x_1}(2)$-cover of H_1 has a tadpole at x_1 containing x_0;*

(3) *$\mathcal{B}_2(G) = \{H_n\}$ and any $\Psi_{x_{n-1}x_n}(2)$-cover of H_n has a tadpole at x_{n-1} containing x_n.*

The next lemma is another reduction technique in the proof of the main result.

Lemma 7. *Let H_1, H_2, H_2' be three two-terminal signed graphs with source terminal x and target terminal y, where H_2 and H_2' satisfy one of the following conditions.*

(1) *H_2 has a $\Psi_{xy}(t)$-cover for each $t \in [0, 3]$ in which no tadpole at x contains y; H_2' is the signed graph $D_1(x, y)$ in Fig. 2.*

(2) *H_2 has a $\Psi_{xy}(2)$-cover whose tadpoles at x and y don't contain y and x, respectively; H_2' is the signed graph $D_2(x, y)$ in Fig. 2.*

If $\mathcal{P}(H_1, H_2')$ has a signed circuit 6-cover, then so does $\mathcal{P}(H_1, H_2)$.

Figure 2: Two two-terminal signed graphs with terminals x and y.

Proof. Denote $G = \mathcal{P}(H_1, H_2)$ and $G' = \mathcal{P}(H_1, H_2')$. Let $\mathcal{F}$ be a signed circuit 6-cover of G'.

We only prove the case when H_2 satisfies (1) since the augment for the other case is very similar.

As shown in Fig. 2, it follows from the structure of the signed graph $D_1(x, y)$ that, for any signed circuit $C \in \mathcal{F}$,

$$E(C) \cap E(H_2') \in \{\emptyset, \{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_3\}, \{e_1, e_2, e_3\}\}.$$

Denote by $\mathcal{F}_1$ (resp., $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$) the set of signed circuits $C \in \mathcal{F}$ with $E(C) \cap E(H_2') = \{e_1, e_2\}$ (resp., $= \{e_3, e_2\}$, $= \{e_1, e_3\}$, $= \{e_1, e_2, e_3\}$). Since $\mathcal{F}(e_1) = \mathcal{F}(e_2) = \mathcal{F}(e_3) = 6$,

$$|\mathcal{F}_1| + |\mathcal{F}_3| + |\mathcal{F}_4| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_4| = |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_4| = 6.$$

Thus there is an integer $t \in [0, 3]$ such that $|\mathcal{F}_1| = |\mathcal{F}_2| = |\mathcal{F}_3| = t$ and $|\mathcal{F}_4| = 6 - 2t$. Let $\mathcal{F}_i = \{C_{i1}, \ldots, C_{it}\}$ for each $i \in [1, 3]$ and $\mathcal{F}_4 = \{C_{41}, \ldots, C_{4(6-2t)}\}$. On the other hand, by assumption, H_2 has a $\Psi_{xy}(t)$-cover

$$\mathcal{C}_0 \cup \{P_{11}, \ldots, P_{1t}\} \cup \{P_{21}, \ldots, P_{2t}\} \cup \{P_{31}, \ldots, P_{3t}\} \cup \{P_{41}, \ldots, P_{4(6-2t)}\},$$

such that no tadpole at x contains y, $\mathcal{C}_0$ is a family of signed circuits, each P_{1j} (resp., P_{2j}) is a positive (resp., negative) xy-path, and each P_{3j} (resp., P_{4j}) is a tadpole at x (resp., y). One can easily check that the family

$$\mathcal{C}_0 \cup \left(\mathcal{F} \setminus (\cup_{i=1}^4 \mathcal{F}_i)\right) \cup \left(\cup_{i=1}^4 \cup_{j=1}^{|\mathcal{F}_i|} \{(C_{ij} - E(H_2')) \cup P_{ij}\}\right)$$

is a signed circuit 6-cover of $G = \mathcal{P}(H_1, H_2)$. $\square$

Throughout this paper, we use $R_0, R_1, \ldots, R_5$ to denote the six signed graphs shown in Fig. 3.

Observation 8. (1) *R_2 has a $\Psi_{yx}(t)$-cover for each $t \in [0, 3]$, and a $\Psi_{xy}(2)$-cover in which exactly one tadpole at y doesn't contain x.*
(2) *R_3 has a $\Psi^*_{xy}(2)$-cover.*
(3) *Both R_4 and R_5 have a $\Psi_{xy}(2)$-cover $\mathcal{F}$ satisfying that $y \notin V(T_1) \cup V(T_2)$, $x \notin V(T_3)$ and xy is in the tadpole-path of T_4, where $\{T_1, T_2\}$ and $\{T_3, T_4\}$ are the sets of tadpoles of $\mathcal{F}$ at x and y, respectively.*

For any $H = H(u, v)$, the notation $H = R_i(x, y)$ (resp., $H \sim R_i(x, y)$) means that G is isomorphic (resp., equivalent) to H, u and v correspond to x and y, respectively.

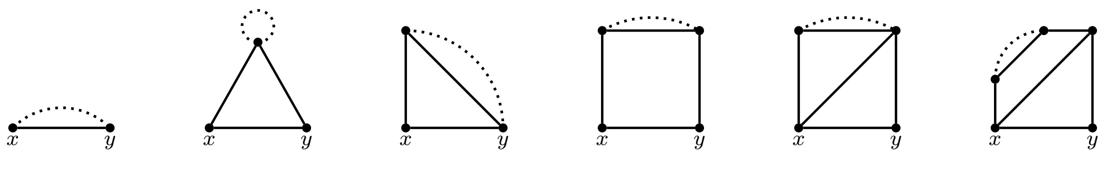

Figure 3: Six small signed graphs with two specified vertices x and y.

Lemma 9. *Let $H = H(x, y)$ and $G = \mathcal{P}(H \cup yz, xz)$ such that $xy \in E(H)$ and $xyzx$ is an unbalanced triangle. If $H \sim R_i(x, y)$ for some $i \in \{2, 4, 5\}$ or H has a $\Psi^*_{xy}(2)$-cover, then G has a $\Psi^*_{xz}(2)$-cover.*

Proof. With possible switching, assume that $\sigma(xy) = 1$. If $H \sim R_i(x, y)$ for some $i \in \{2, 4, 5\}$, then G is a small signed graph and thus it is easy to find a $\Psi^*_{xz}(2)$-cover of G. Now we assume that H has a $\Psi^*_{xy}(2)$-cover $\mathcal{F}_H$. By the definition of $\Psi^*_{xy}(2)$-cover, let

$$\mathcal{F}_H = \mathcal{C}_0 \cup \{P_1, P_2\} \cup \{Q_1, Q_2\} \cup \{T_{x1}, xy \cup T_{y2}\} \cup \{T_{y1}, yx \cup T_{x2}\},$$

where $\mathcal{C}_0$ is a family of signed circuits, P_1, P_2 (resp., Q_1, Q_2) are two positive (resp., negative) xy-paths, T_{u1}, T_{u2} are the two tadpoles at u not containing the vertex in $\{x, y\} \setminus \{u\}$ for each $u \in \{x, y\}$.

Let $e_0 = xy$, $e_1 = xz$ and $e_2 = zy$. Since $xyzx$ is unbalanced and $\sigma(e_0) = 1$, WLOG, assume that $\sigma(e_1) = -1$ and $\sigma(e_2) = 1$. From G and $\mathcal{F} \setminus \mathcal{C}_0$, we construct an auxiliary signed graph G' shown in Fig. 4. Observe that the family

$$\begin{aligned}\mathcal{F}_{G'} = &\{e_3 \cup e_2, e_3 \cup e_2\} \cup \{e_1, e_1\} \cup \{e_5, e_1 \cup e_2 \cup e_6\} \cup \{e_2 \cup e_6, e_1 \cup e_0 \cup e_4\} \\ &\cup \{e_1 \cup e_2 \cup e_0 \cup e_5, e_1 \cup e_2 \cup e_4\}\end{aligned}$$

covers $\{e_1, e_2\}$ 6 times and $E(G') \setminus \{e_1, e_2\}$ twice. Let $\mathcal{F}_G$ be the family obtained from $\mathcal{F}_{G'}$ by replacing two e_3s with P_1, P_2, two e_4s with Q_1, Q_2, two e_5s with T_{x1}, T_{x2}, two e_6s with T_{y1}, T_{y2}. One can easily check that $\mathcal{F}_G \cup \mathcal{C}_0$ is a $\Psi^*_{xz}(2)$-cover of G. $\square$

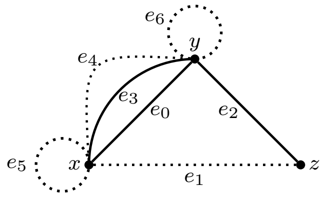

Figure 4: An auxiliary signed graph G'.

By Observation 8, each of $\{R_2, R_4, R_5\}$ has a $\Psi_{xy}(2)$-cover in which at least one tadpole at y doesn't contain x. By this fact and a similar method of the proof of Lemma 9, we obtain the following lemma.

Lemma 10. *Let $H_i = H_i(x, y_i)$ for each $i \in [1, 2]$, $G = \mathcal{P}(H_1 \cup y_1 z, H_2 \cup y_2 z)$ and G' be the signed graph obtained from G by adding a new negative loop at x.*
(1) If $H_1 = R_0$, and either $H_2 \sim R_j(x, y)$ for some $j \in \{0, 2, 4, 5\}$ or H_2 has a $\Psi^*_{xy_2}(2)$-cover, then both G and G' have a signed circuit 6-cover. Moreover, G has a $\Psi_{xz}(t)$-cover for each $t \in [0, 3]$.
(2) If either $H_i \sim R_j(x, y)$ for some $j \in \{2, 4, 5\}$ or H_i has a $\Psi^*_{xy_i}(2)$-cover for each $i \in [1, 2]$, then both G and G' have a signed circuit 6-cover. Moreover, G has a $\Psi_{xz}(2)$-cover in which no tadpole at z contains x.

4 Proof of Theorem 2

In this section, we will complete the proof of Theorem 2 by contradiction.

Let G be a counterexample to Theorem 2 with minimum $|E(G)|$. Then G is unbalanced since every coverable graph has a circuit 6-cover (see [12]). By the minimality, G contains no 2-vertices and can't be decomposed into two coverable signed subgraphs. The latter implies that G is connected and contains no positive loops.

4.1 Properties of the smallest counterexample G

In this subsection, we will present some properties of G. For two sets X and Y, the *symmetric difference* of X and Y is

$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X).$$

For two signed subgraphs H_1 and H_2 of a signed graph G, the *symmetric difference* of H_1 and H_2, denoted by $H_1 \triangle H_2$, is the signed subgraph of G induced by $(E(H_1) \setminus E(H_2)) \cup (E(H_2) \setminus E(H_1))$.

A two-terminal signed graph $H = H(x, y)$ is said to be a *piece* of G at $\{x, y\}$ if there is another two-terminal signed graph $H' = H'(x, y)$ such that $G = \mathcal{P}(H, H') = H \cup H'$.

Claim 11. *The following statements hold.*

(1) *No two negative loops share a common vertex.*

(2) *G is 2-connected.*

(3) *Every balanced piece of G is a positive or negative K_2.*

(4) *If R_1 is a piece of G at $\{x, y\}$, then $d_G(x) \geqslant 4$ and $d_G(y) \geqslant 4$.*

(5) *G contains no balanced subgraph $H = K_4 - y_1 y_2$, where K_4 is the complete graph on vertices x, y_1, y_2, y_3 and x is a 3-vertex of G.*

(6) *$G - L(G)$ contains no adjacent 2-vertices.*

Proof. (1) Suppose to the contrary that e_1, e_2 are two negative loops at a vertex. Since $C_0 = e_1 \cup e_2$ is a short barbell, $G - \{e_1, e_2\}$ is not coverable and so $G - e_1$ is coverable. By the minimality of G, $G - e_1$ has a signed circuit 6-cover $\mathcal{F}$. Pick three signed circuits C_1, C_2, C_3 from $\mathcal{F}$ containing e_2. Then the family

$$(\mathcal{F} \setminus \{C_1, C_2, C_3\}) \cup \{C_1 \triangle C_0, C_2 \triangle C_0, C_3 \triangle C_0\} \cup 3\{C_0\}$$

is a signed circuit 6-cover of G, a contradiction. This proves (1).

(2) Suppose to the contrary that there are two subgraphs H_1, H_2 in G such that $G = H_1 \cup H_2$ and $V(H_1) \cap V(H_2) = \{x\}$. Since the minimum degree of G is at least three, $|E(H_i)| \geqslant 2$ for each $i \in [1, 2]$. Note that if H_i is balanced, then it is coverable and thus both H_1 and H_2 are coverable, a contradiction to the minimality of G. Hence neither H_1 nor H_2 is balanced. Therefore for each $i \in [1, 2]$, the signed graph obtained from H_i by adding a new negative loop L_i at x is also coverable and thus has a signed circuit 6-cover $\mathcal{F}_i$ by the minimality of G again. Let $\mathcal{C}_i = \{C_{i1}, \ldots, C_{i6}\}$ be the six signed circuits in $\mathcal{F}_i$ containing L_i. Then the family

$$(\mathcal{F}_1 \setminus \mathcal{C}_1) \cup (\mathcal{F}_2 \setminus \mathcal{C}_2) \cup (\cup_{j=1}^{6} \{(C_{1j} \setminus \{L_1\}) \cup (C_{2j} \setminus \{L_2\})\})$$

is a signed circuit 6-cover of G, a contradiction. This proves (2).

(3) Suppose to the contrary that there are two pieces H_1, H_2 of G at $\{x, y\}$ such that $G = H_1 \cup H_2$ and H_2 is balanced and of size at least 2. Then at least one of H_1 and H_2 is not coverable.

We first show that H_1 is unbalanced. Suppose not. Then either H_1 or H_2 is not 2-edge-connected and for any cut-edge b of H_1 or H_2, $G - b$ is balanced, contradicting that G is coverable by Proposition 4. Hence H_1 is unbalanced.

WLOG, assume that H_2 has a positive xy-path. Then all xy-paths in H_2 are positive since H_2 is balanced.

For each $i = 1, 2$, let H_i' be the graph obtained from H_i by adding a new positive edge e_i connecting x and y. Then $|E(H_1')| < |E(G)|$ and H_2' is balanced. Moreover, both H_1' and H_2' are 2-connected and K_4-minor-free. Obviously, H_2' has a balanced circuit 6-cover, denoted by $\mathcal{F}_2$.

We now show that H_1' is coverable. Suppose not. Since H_1' is 2-connected, by Proposition 4, there is an edge e in H_1' such that $H_1' - e$ is balanced. Note that $e \neq e_1$ since $H_1 = H_1' - e_1$ is unbalanced. Since e_1 is a positive edge, every xy-path in $H_1 - e$ is positive. Thus $G - e = (H_1 - e) \cup H_2$ is balanced, for otherwise there is an unbalanced circuit C in $G - e$ such that $x, y \in V(C)$, and hence exactly one of segments xCy and yCx is a negative xy-path in $H_1 - e$ or H_2, a contradiction. Since G is unbalanced, it is not coverable by Proposition 4, a contradiction. Therefore H_1' is coverable.

By the minimality of G, H_1' has a signed circuit 6-cover $\mathcal{F}_1$. For each $i = 1, 2$, let $\mathcal{C}_i = \{C_{i1}, \cdots, C_{i6}\}$ be the six members of $\mathcal{F}_i$ containing e_i. Since every member of $\mathcal{C}_2$ is a balanced circuit, the family

$$(\mathcal{F}_1 \setminus \mathcal{C}_1) \cup (\mathcal{F}_2 \setminus \mathcal{C}_2) \cup \left(\cup_{i=j}^{6} \{(C_{1j} \setminus \{e_1\}) \cup (C_{2j} \setminus \{e_2\})\}\right)$$

is a signed circuit 6-cover of G, a contradiction. This proves (3).

(4) By symmetry, we only need to show that $d_G(y) \geqslant 4$. Suppose by contradiction that $d_G(y) = 3$. Let $H = H(x, y)$ be a piece of G such that $G = H \cup R_1$. As shown in Fig. 3, denote $C_0 = xyzx$ and $R_1 = C_0 \cup L_z$. Clearly $G - xy$ is 2-connected and coverable. Thus it has a signed circuit 6-cover $\mathcal{G}$ by the minimality of G. For each $C \in \mathcal{G}$, $E(C) \cap \{L_z, xz, yz\}$ is either $\{xz, L_z\}$, or $\{yz, L_z\}$, or $\{xz, yz\}$, or $\{xz, yz, L_z\}$. Denote by $\mathcal{G}_1$ (resp., $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$) the set of signed circuits $C \in \mathcal{G}$ with $E(C) \cap E(R_1) = \{xz, L_z\}$ (resp., $= \{yz, L_z\}$, $= \{xz, yz\}$, $= \{xz, yz, L_z\}$). Since $\mathcal{G}(xz) = \mathcal{G}(yz) = \mathcal{G}(L_z) = 6$, we have

$$|\mathcal{G}_1| + |\mathcal{G}_3| + |\mathcal{G}_4| = |\mathcal{G}_2| + |\mathcal{G}_3| + |\mathcal{G}_4| = |\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_4| = 6.$$

Thus there is an integer $t \in [0, 3]$ such that $|\mathcal{G}_1| = |\mathcal{G}_2| = |\mathcal{G}_3| = t$ and $|\mathcal{G}_4| = 6 - 2t$. Let $\mathcal{G}_i = \{C_{i1}, \ldots, C_{it}\}$ for $i \in [1, 3]$ and $\mathcal{G}_4 = \{C_{41}, \ldots, C_{4(6-2t)}\}$. Then the family

$$\begin{cases} (\mathcal{G} \setminus \{C_{41}, C_{42}, C_{43}, C_{44}\}) \cup \{C_{41} \triangle zxy, C_{42} \triangle zxy, C_{43} \triangle xyz, C_{44} \triangle xyz\} \cup 2\{C_0\} & \\ & \text{if } t \in [0, 1]; \\ (\mathcal{G} \setminus \{C_{11}, C_{31}, C_{43}, C_{44}\}) \cup \{C_{11} \triangle C_0, C_{31} \triangle C_0, C_{43} \triangle xyz, C_{44} \triangle xyz\} \cup 2\{C_0\} & \\ & \text{if } t = 2; \\ (\mathcal{G} \setminus \{C_{31}, C_{32}, C_{33}\}) \cup \{C_{31} \triangle C_0, C_{32} \triangle C_0, C_{33} \triangle C_0\} \cup 3\{C_0\} & \text{if } t = 3 \end{cases}$$

is a signed circuit 6-cover of G, a contradiction. This proves (4).

(5) Suppose that such a balanced subgraph H exists. Since G is K_4-minor-free, y_3 is in all y_1y_2-paths of $G - x$ and thus y is a cut-vertex of $G - x$. Let H_1, H_2 be two subgraphs of $G - x$ such that $G - x = H_1 \cup H_2$, $V(H_1) \cap V(H_2) = \{y_3\}$ and $y_iy_3 \in E(H_i)$ for $i \in [1, 2]$. Note that $d_{H_i}(y_i) \geqslant 2$. Since G is 2-connected, by (3), either H_i is 2-connected and unbalanced, or H_i is the union of y_iy_3 and a negative loop at y_i. Hence $G - x$ is coverable and thus has a signed circuit 6-cover $\mathcal{F}$ by the minimality of G. Pick six distinct members $C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}$ from $\mathcal{F}$ such that $y_iy_3 \in E(C_{ij})$ for $i \in [1, 2]$ and $j \in [1, 3]$. Then the family

$$(\mathcal{F} \setminus (\cup_{i=1}^{2} \cup_{j=1}^{3} \{C_{ij}\})) \cup (\cup_{i=1}^{2} \cup_{j=1}^{3} \{C_{ij} \triangle xy_iy_3x\}) \cup 3\{xy_1y_3y_2x\}$$

is a signed circuit 6-cover of G, a contradiction. This proves (5).

(6) Suppose to the contrary that x, y are two adjacent 2-vertices of $G - L(G)$. If $G - L_y$ is not coverable, then $E_N(G - L_y) = \{L_x\}$ and $G - \{L_x, L_y\}$ is balanced. Since $G - \{L_x, L_y\}$ is 2-connected, it has a balanced circuit 6-cover $\mathcal{F}$. Pick $C_1, C_2, C_3 \in \mathcal{F}$ with $xy \in E(C_i)$ for $i \in [1, 3]$. Then the family

$$(\mathcal{F} \setminus \{C_1, C_2, C_3\}) \cup (\cup_{i=1}^{3} \{L_x \cup xy \cup L_y, L_x \cup (C_i - xy) \cup L_y\})$$

is a signed circuit 6-cover of G, a contradiction. Therefore, $G - L_y$ is coverable. Let $\mathcal{F}'$ be a signed circuit 6-cover of $G - L_y$ by the minimality of G. Similar to the proof of (4), we can extend $\mathcal{F}'$ to a signed circuit 6-cover of G, a contradiction. This proves (6) and thus completes the proof of the claim. □

Claim 12. *Let H be a 2-connected piece of G. If $\epsilon(H) = 1$, then $H \sim R_i$ for some $i \in [0, 5]$.*

Proof. Let H and H' be two pieces of G at $\{x, y\}$ such that $G = H \cup H'$. WLOG, assume that $E_N(H) = \{e_0\}$ and the ends of e_0 are z_1, z_2 (possibly $z_1 = z_2$). If $|V(H)| \leqslant 3$, it is obvious that $H \in \{R_0, R_1, R_2\}$. Thus we assume that $|V(H)| \geqslant 4$.

We first show that H is outerplanar. Since H is K_4-minor-free, it is sufficient to prove that H is $K_{2,3}$-minor-free. Suppose by contradiction that H has a $K_{2,3}$-minor. Then there are two distinct vertices u, v and three internally disjoint uv-paths P_1, P_2, P_3 in H such that each $|V(P_i)| \geqslant 3$. For each $i \in [1, 3]$, let M_i be the component of $H - \{u, v\}$ containing $V(P_i) \setminus \{u, v\}$, and $M_i' = H[V(M_i) \cup \{u, v\}] - uv$. Since H is K_4-minor-free, for $\{w_1, w_2\} = \{x, y\}$ or $\{z_1, z_2\}$, there are at least two members of $\{M_1, M_2, M_3\}$ containing neither w_1 nor w_2. Therefore there is a member of $\{M_1, M_2, M_3\}$, say M_3, satisfying $\{x, y, z_1, z_2\} \cap V(M_3) = \emptyset$. This implies that M_3' is an all-positive piece of G at $\{u, v\}$. By Claim 11-(3), M_3' is a positive or negative K_2, a contradiction to $|E(M_3')| \geqslant |E(P_3)| = |V(P_i) \setminus \{u, v\}| + 1 \geqslant 2$. This proves that H is outerplanar.

Let C be an outer facial circuit of H. Since $G = H \cup H'$ is K_4-minor-free, there are two pieces H_1, H_2 of G at $\{x, y\}$ such that $H = H_1 \cup H_2$, $V(H_1) = V(xCy)$ and $V(H_2) = V(yCx)$. Since $E_N(H) = \{e_0\}$, at least one of H_1 and H_2, say H_1, is all-positive and so H_1 is a positive K_2 by Claim 11-(3). Therefore, $xy \in E(C)$. It follows that $H - e_0$ remains 2-connected for otherwise the end-block B of $H - e_0$ with $xy \notin E(B)$ and $V(B) \cap \{z_1, z_2\} \neq \emptyset$ is a positive K_2 by Claim 11-(3). Thus at least one of z_1 and z_2 is a 2-vertex of G, a contradiction. If e_0 is a loop, then $z_1 = z_2 \notin \{x, y\}$ since $H - e_0$ is not a piece of G at $\{x, y\}$. If e_0 is not a loop, then there are two pieces H_3, H_4 of G at $\{z_1, z_2\}$ such that $G = H_3 \cup \{e_0\} \cup H_4$ and $V(H') \subseteq V(H_4)$. Thus H_3 is a positive K_2 by Claim 11-(3). Denote the single edge of H_3 by e_1. Then $e_0 \cup e_1$ is an unbalanced 2-circuit and $E(C) \cap \{e_0, e_1\} \neq \emptyset$.

WLOG, assume that z_1, x, y, z_2 appear on C in the cyclic order. Let

$$P_1 = z_1 C x = u_0 u_1 \cdots u_p \text{ and } P_2 = y C z_2 = v_q \cdots v_1 v_0$$

such that $C = z_1 z_2 \cup P_1 \cup xy \cup P_2$, where $u_0 = z_1$, $u_p = x$, $v_q = y$ and $v_0 = z_2$. Because H is outerplanar and the minimum degree of G is at least 3, $E(H) \setminus E(C) \subseteq E_H[V(P_1), V(P_2)]$ by Claim 11-(3).

If $u_0 = v_0$, then e_0 is a negative loop not at x or y. Thus $u_0 u_1 v_1 u_0 \cup e_0 = R_1$. Since $|V(C)| = |V(H)| \geqslant 4$, either $d_G(u_1) \leqslant 3$ or $d_G(v_1) \leqslant 3$, contradicting Claim 11-(4). Hence $u_0 \neq v_0$.

Note that there are no two indices $i \in [0, p]$ and $j \in [0, q-2]$ satisfying $\{v_j, v_{j+1}, v_{j+2}\} \subseteq N_H(u_i)$, for otherwise $H[\{u_i, v_j, v_{j+1}, v_{j+2}\}] - e_0 = K_4 - v_j v_{j+2}$ and $d_G(v_{j+1}) = d_H(v_{j+1}) = 3$, where K_4 is the complete graph on $\{u_i, v_j, v_{j+1}, v_{j+2}\}$, contradicting Claim 11-(5). By the symmetry of P_1 and P_2, it follows that $p \geqslant 1$, $q \geqslant 1$ and

$$\begin{cases} d_H(w) \leqslant 3 & \text{if } w \in \{u_p, v_q\}; \\ d_H(w) \leqslant 4 & \text{if } w \in V(H) \setminus \{u_p, v_q\}. \end{cases} \tag{1}$$

Let $H^* = H[\{u_0, u_1, v_0, v_1\}]$. According to $d_H(u_0)$ and $d_H(v_0)$, we distinguish the following two cases.

Case 1. $d_H(u_0) = 4$ or $d_H(v_0) = 4$.

By the symmetry of P_1 and P_2, assume that $d_H(v_0) = 4$. Then $d_H(u_0) = 3$, $u_1v_0 \in E(H)$ and $H[\{u_0, u_1, v_0\}] = R_1$. If $p = 1$, then $H = H^* = R_4$ by Eq. (1). Now we assume $p \geqslant 2$.

If $d_H(u_1) = 3$, then $u_2u_1 \cup H[\{u_0, u_1, v_0\}]$ is a piece of G at $\{u_2, v_0\}$ and has a $\Psi_{v_0u_2}(t)$-cover for each $t \in [0, 3]$ in which no tadpole at v_0 contains u_2. By Lemma 7-(1) and the minimality of G, G has a signed circuit 6-cover, a contradiction.

If $d_H(u_1) = 4$, then $H^* = R_4$. Let $C_0 = u_1v_0v_1u_1$ and $G' = G - \{u_1v_0, u_1v_1\}$. Clearly, G' has a signed circuit 6-cover $\mathcal{F}'$ by the minimality of G. Note that $d_{G'}(u_1) = 2$ and $d_{G'}(u_0) = d_{G'}(v_0) = 3$. By the structure of G', there are 3 signed circuits C_1, C_2, C_3 in $\mathcal{F}'$ such that each of C_1, C_2 contains the tadpole $e_0 \cup e_1 \cup v_0v_1$ but not the vertex u_1, and C_3 contains the path $u_2u_1u_0 \cup e_1 \cup v_0v_1$. Hence the family

$$(\mathcal{F}' \setminus \{C_1, C_2, C_3\}) \cup \{C_1 \bigtriangleup C_0, C_2 \bigtriangleup C_0, C_3 \bigtriangleup C_0\} \cup 3\{C_0\}$$

is a signed circuit 6-cover of G, a contradiction.

Case 2. $d_H(u_0) = d_H(v_0) = 3$.

If $p = q = 1$, then $H = R_3$. If $p = 1$ and $q \geqslant 2$, then $d_H(v_1) = 3$. Thus $q = 2$ and $H = R_5$. By the symmetry of P_1 and P_2, we assume that $p \geqslant 2$ and $q \geqslant 2$. Then $u_1v_1 \in E(H)$.

If $d_H(u_1) = d_H(v_1) = 3$, then $H^* = R_3$. By Lemma 6-(1) $u_2u_1 \cup H^* \cup v_1v_2$ has a $\Psi_{u_2v_2}(2)$-cover satisfying the condition of Lemma 7-(2). Together with the minimality of G, we can obtain a signed circuit 6-cover of G, a contradiction.

Assume that either $d_H(u_1) \geqslant 4$ or $d_H(v_1) \geqslant 4$. WLOG assume $d_H(u_1) \geqslant 4$. Then $d_H(u_1) = 4$ by Eq. (1) and thus $d_H(v_1) = 3$. Let $G' = G - \{u_1v_1, u_1v_2\}$. By the minimality of G, G' has a signed circuit 6-cover $\mathcal{F}'$. Note that $d_{G'}(u_1) = d_{G'}(v_1) = 2$ and $d_{G'}(u_0) = d_{G'}(v_0) = 3$. It follows from the structure of G' that there are 4 signed circuits C_1, C_2, C_3, C_4 in $\mathcal{F}'$ such that both C_1 and C_2 contain the tadpole $e_0 \cup e_1 \cup v_0v_1v_2$ but not the vertex u_1 and both C_3 and C_4 contain the path $u_2u_1u_0 \cup e_1 \cup v_0v_1v_2$. Let $C_{01} = u_1v_1v_2u_1$, $C_{02} = u_1u_0 \cup e_1 \cup v_0v_1u_1$ and $C_{03} = u_1u_0 \cup e_1 \cup v_0v_1v_2u_1$. Then the family

$$(\mathcal{F}' \setminus \{C_1, C_2, C_3, C_4\}) \cup \{C_1 \bigtriangleup C_{01}, C_2 \bigtriangleup C_{01}, C_3 \bigtriangleup C_{02}, C_4 \bigtriangleup C_{03}\} \cup 2\{C_{01}\} \cup \{C_{02}, C_{03}\}$$

is a signed circuit 6-cover of G, a contradiction.

This completes the proof of the claim. □

For two distinct vertices $x, y \in V(G)$, let $t_G(x, y)$ denote the maximum number of pieces $H_1, \ldots, H_t$ of G at $\{x, y\}$ such that $G = \mathcal{P}(H_1, \ldots, H_t)$.

Claim 13. *$t_G(x, y) \leqslant 3$ for any two distinct vertices $x, y \in V(G)$.*

Proof. Suppose to the contrary that there are two distinct vertices x, y such that $t = t_G(x, y) \geqslant 4$. Let $H_1, \ldots, H_t$ be t pieces of G at $\{x, y\}$ such that $G = \mathcal{P}(H_1, \ldots, H_t)$. Since G is K_4-minor-free, no H_i is 2-connected by the maximality of t.

WLOG, assume that $\epsilon(H_1) \leqslant \epsilon(H_2) \leqslant \cdots \leqslant \epsilon(H_t)$. Then by Claim 11-(3), $\epsilon(H_1 \cup H_2) \geqslant 1$ and $\epsilon(H_3) \geqslant 1$. Thus if $t \geqslant 5$, then G can be decomposed into two coverable subgraphs $H_1 \cup H_2 \cup H_3$ and $H_4 \cup \cdots \cup H_t$, a contradiction. Hence $t = 4$.

We first consider the case when $\epsilon(H_2) = 0$. Then $H_1 \cup H_2 = R_0$ by Claim 11-(3). Since G cannot not be decomposed into two coverable subgraphs, $H_i \cup H_j$ is not coverable for some $i \in \{1, 2\}$ and $j \in \{3, 4\}$. WLOG, assume that $H_1 \cup H_3$ is not coverable and thus $\epsilon(H_1 \cup H_3) = 1$. By Claim 12, for some $k \in [1, 5]$,

$$H_1 \cup H_3 \sim R_k \text{ and } H_1 \cup H_2 \cup H_3 \sim R_k \cup e, \tag{2}$$

where e is a negative edge not in R_k with ends x, y. Since $H_1 \cup H_2 \cup H_3$ is coverable, H_4 is not coverable. By Proposition 4, there is an edge b of H_4 such that $H_4 - b$ has a balanced component M. Since H_4 is not 2-connected, it follows from Claim 11-(3) that M is the single vertex x or y, say y. Then $d_{H_4}(y) = 1$. Let $b = yy'$. Then $H = H_1 \cup H_2 \cup H_3 \cup b$ is a piece of G at $\{x, y'\}$. By Eq. (2), it is easy to check that for any $t' \in [0, 3]$, H has a $\Psi_{xy'}(t')$-cover in which no tadpole at x contains y'. By Lemma 7-(1) and the minimality of G, G has a signed circuit 6-cover, a contradiction.

Now we consider the case when $\epsilon(H_2) \geqslant 1$. Since $\epsilon(H_2) \geqslant 1$, for any $\{i, j\} \subseteq [2, 4]$, $\epsilon(H_i \cup H_j) \geqslant \epsilon(H_i) + \epsilon(H_j) \geqslant 2$. Thus $H_i \cup H_j$ is coverable. This implies that for each $j \in [2, 4]$, $H_1 \cup H_j$ is not coverable and thus by Claim 12, $H_1 \cup H_j \sim R_{k_j}$ for some $k_j \in [1, 5]$. With some switchings, assume that H_1 is the positive edge xy and thus for each $j \in [2, 4]$,

$$H_j = R_{k_j} - xy.$$

One can check directly that G has a signed circuit 6-cover, a contradiction. This proves the claim. $\square$

Claim 14. *$H - L(H)$ is unbalanced for every 2-connected piece H of G.*

Proof. Prove by contradiction. Let x, y be two distinct vertices and H be a 2-connected piece of G at $\{x, y\}$ such that

(i) $H - L(H)$ is balanced;

(ii) subject to (i), $|E(H)|$ is as small as possible.

WLOG, assume that $H - L(H)$ is all-positive. Then $E_N(H) = L(H)$. By (ii), no member of $L(H)$ has its end at x or y. Denote by H' another piece of G at $\{x, y\}$ such that $G = \mathcal{P}(H, H')$.

We first show that $H = R_1$. Since G is K_4-minor-free and H is 2-connected, there are two pieces H_1, H_2 of G at $\{x, y\}$ such that $G = \mathcal{P}(H_1, H_2, H') = H_1 \cup H_2 \cup H'$. Note that G is 2-connected. By (ii), neither H_1 nor H_2 is 2-connected, and $\mathcal{B}_2(H_1) = \mathcal{B}_2(H_2) = \emptyset$. This implies that $H_i - L(H_i)$ is an xy-path for $i \in [1, 2]$. Furthermore, the length of each

$H_i - L(H_i)$ is equal to 1 or 2 by Claim 11-(6). Since G (and thus $H - L(H)$) contains no balanced 2-circuit, $H - L(H) = (H_1 - L(H_1)) \cup (H_2 - L(H_2))$ is a 3- or 4-circuit. Suppose that $H - L(H) = xz_1yz_2x$ is a 4-circuit. Then $H = xz_1yz_2x \cup \{L_{z_1}, L_{z_2}\}$ and hence any signed circuit 6-cover of $H' \cup \{L_x, L_y\}$ can be extended to a signed circuit 6-cover of G, where L_u is a new negative loop at u for each $u \in \{x, y\}$, a contradiction. Thus $H - L(H)$ is a 3-circuit and $H = R_1$.

Next we show that H' has a cut-edge. Suppose to be contrary that H' is 2-edge-connected. Since $G = \mathcal{P}(H, H')$ and H is 2-connected, $t_{H'}(x, y) = t_G(x, y) - t_H(x, y) \leqslant 1$ by Claim 13. Thus H' contains cut-vertices separating x from y. This implies that there are s ($\geqslant 2$) 2-connected subgraphs or negative loops $B_1, \dots, B_s$ such that $H' = \mathcal{S}(B_1, \dots, B_s)$ with $x \in V(B_1)$ and $y \in V(B_s)$. By Claim 11-(3), $\epsilon(B_i) \geqslant 1$ for each $i \in [1, s]$. If $s \geqslant 3$, then both $H \cup B_1$ and $B_2 \cup \cdots \cup B_s$ are coverable. If $\epsilon(B_i) \geqslant 2$ for some $i \in [1, s]$, then both $H \cup (\cup_{j \in [1,s] \setminus \{i\}} B_j)$ and B_i are coverable. In both cases, we get a contradiction that G has a decomposition into two coverable signed subgraphs. Hence $s = 2$ and $\epsilon(B_1) = \epsilon(B_2) = 1$. By Claim 12, $B_1 \sim R_{j_1}$ and $B_2 \sim R_{j_2}$ for some $j_1, j_2 \in [0, 5]$. By the structures of R_1, R_{j_1} and R_{j_2}, it is easy to find a signed circuit 6-cover of G, a contradiction. Thus H' has a cut-edge.

By the above two claims, let $H = C_0 \cup L_z$ where $C_0 = xzyx$ and uv be a cut-edge of H'. Let M_1, M_2 be the two components of $H' - uv$ with $x, u \in V(M_1)$ and $y, v \in V(M_2)$.

Let $G' = G - xy$. Then G' is 2-connected and coverable. By the minimality of G, G' has a signed circuit 6-cover. Choose a signed graph 6-cover $\mathcal{F}'$ of G' such that the number of balanced circuits and short barbells in $\mathcal{F}'$ is as large as possible.

To complete the proof, we will construct a signed circuit 6-cover $\mathcal{F}$ of G from $\mathcal{F}'$.

With a similar argument of the proof of Claim 11-(4), one can show that there is an integer $t \in [0, 3]$ and four families $\mathcal{F}_i = \{C_{i1}, \dots, C_{it_i}\}$, $i \in [1, 4]$, in $\mathcal{F}'$ such that $t_1 = t_2 = t_3 = t$, $t_4 = 6 - 2t$ and for every $C \in \mathcal{F}_1$ (resp., $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$), $E(C) \cap E(H) = \{L_z, zx\}$ (resp., $= \{L_z, yz\}$, $= \{zx, yz\}$, $= \{L_z, zx, yz\}$).

If $t \in [0, 1]$, let

$$\mathcal{F} = (\mathcal{F}' \setminus \{C_{41}, C_{42}, C_{43}, C_{44}\}) \cup \{C_{41} \triangle zxy, C_{42} \triangle zxy, C_{43} \triangle xyz, C_{44} \triangle xyz\} \cup 2\{C_0\}.$$

If $t = 3$, let $\mathcal{F} = (\mathcal{F}' \setminus \{C_{31}, C_{32}, C_{33}\}) \cup \{C_{31} \triangle C_0, C_{32} \triangle C_0, C_{33} \triangle C_0\} \cup 3\{C_0\}$.

If $t = 2$ and either $y \notin V(C_{11}) \cap V(C_{12})$ or $x \notin V(C_{21}) \cap V(C_{22})$, say $y \notin V(C_{11})$, let

$$\mathcal{F} = (\mathcal{F}' \setminus \{C_{11}, C_{31}, C_{41}, C_{42}\}) \cup \{C_{11} \triangle C_0, C_{31} \triangle C_0, C_{41} \triangle xyz, C_{42} \triangle xyz\} \cup 2\{C_0\}.$$

In each of the above cases, we obtain a signed circuit 6-cover of G, a contradiction.

Finally we consider the case that $t = 2$, $y \in V(C_{11}) \cap V(C_{12})$ and $x \in V(C_{21}) \cap V(C_{22})$.

Then $uv \in \cap_{j=1}^{2} (E(C_{1j}) \cap E(C_{2j}) \cap E(C_{4j}))$ but $uv \notin E(C_{31}) \cup E(C_{32})$. For each $j \in [1, 2]$, denote by P_{1j} (resp., T_{2j}, P_{4j}^1), the segment of C_{1j} (resp., C_{2j}, C_{4j}) in M_1, and by T_{1j} (resp., P_{2j}, P_{4j}^2) the segment of C_{1j} (resp., C_{2j}, C_{4j}) in M_2. Thus

$$C_{1j} = L_z \cup zx \cup P_{1j} \cup uv \cup T_{1j},\ C_{2j} = L_z \cup zy \cup P_{2j} \cup vu \cup T_{2j},\ C_{4j} = L_z \cup zx \cup P_{4j}^1 \cup uv \cup P_{4j}^2 \cup yz.$$

Clearly P_{1j} and P_{4j}^1 are xu-paths, P_{2j} and P_{4j}^2 are vy-paths, and T_{1j} (resp., T_{2j}) is a tadpole at v (resp., u).

Since $uv \notin E(C_{31})$ and z is a cut-vertex of $G' - uv$, C_{31} is a barbell and z is in the barbell-path of C_{31}. Hence there are two barbells, denoted by C_{31}^1, C_{31}^2, in $C_{31} \cup L_z$ such that $\{C_{31}^1, C_{31}^2\}$ covers C_{31} once and L_z twice.

If $\sigma(P_{1j_1})\sigma(P_{2j_2}) \neq \sigma(P_{41}^1)\sigma(P_{41}^2)$ for some $j_1, j_2 \in [1,2]$, then $C_1 = zx \cup P_{1j_1} \cup uv \cup P_{2j_2} \cup yz$ is a balanced circuit. Let

$$\mathcal{F}'' = (\mathcal{F}' \setminus \{C_{1j_1}, C_{2j_2}, C_{31}\}) \cup \{C_1, C_{31}^1, C_{31}^2, T_{1j_1} \cup uv \cup T_{2j_2}\}.$$

If $\sigma(P_{1j_1})\sigma(P_{2j_2}) = \sigma(P_{41}^1)\sigma(P_{41}^2)$ for any $j_1, j_2 \in [1,2]$, then both $C_2 = zx \cup P_{11} \cup uv \cup P_{21} \cup yz$ and $C_3 = zx \cup P_{12} \cup uv \cup P_{22} \cup yz$ are unbalanced circuits. Let

$$\mathcal{F}'' = (\mathcal{F}' \setminus \{C_{11}, C_{12}, C_{21}, C_{22}, C_{31}\}) \cup \{C_2 \cup L_z, C_3 \cup L_z, C_{31}^1, C_{31}^2, T_{11} \cup vu \cup T_{21}, T_{12} \cup vu \cup T_{22}\}.$$

In both cases, $\mathcal{F}''$ is a signed circuit 6-cover of G' which has a larger number of balanced circuits and short barbells than $\mathcal{F}'$, a contradiction to the choice of $\mathcal{F}'$. This completes the proof of the claim. $\square$

Claim 15. *Every balanced 3-circuit is in a piece H of G with $H \sim R_i$ for some $i \in \{2,4,5\}$.*

Proof. Let $C = xyzx$ be a balanced 3-circuit. With possible switchings, assume that C is all-positive. For any $\{u,v\} \subseteq V(C)$,

$$V_{uv} = \{w \in V(G) \setminus V(C) : \text{ there is a } uv\text{-path containing } w \text{ but not } V(C) \setminus \{u,v\} \text{ in } G\}.$$

Since G is 2-connected and K_4-minor-free, $\{V_{xy}, V_{xz}, V_{yz}\}$ is a partition of $V(G) \setminus V(C)$. Let $G_{uv} = G[V_{uv} \cup \{u,v\}]$, where every loop at $V(C)$ belongs to exactly one of $\mathcal{G} = \{G_{xy}, G_{xz}, G_{yz}\}$. Then G_{uv} is a piece of G at $\{u,v\}$ and

$$G = \mathcal{P}(\mathcal{S}(G_{xz}, G_{zy}), G_{xy}) = G_{xz} \cup G_{zy} \cup G_{xy}.$$

WLOG, assume that $\epsilon(G_{xy}) \geqslant \epsilon(G_{yz}) \geqslant \epsilon(G_{xz})$. Note that, by the definition and Claim 11-(3), every $G_{uv} \in \mathcal{G}$ is a positive edge if $\epsilon(G_{uv}) = 0$ and is 2-connected if $\epsilon(G_{uv}) \geqslant 1$.

If $\epsilon(G_{xz}) \geqslant 1$, then $\epsilon(G_{xy}) = \epsilon(G_{yz}) = \epsilon(G_{xz}) = 1$; otherwise G can be decomposed into two coverable subgraphs G_{xy} and $G_{xz} \cup G_{yz}$, a contradiction. By Claim 12, every $G_{uv} \in \mathcal{G}$ is equivalent to R_i for some $i \in [0,5]$. One can check easily that G has a signed circuit 6-cover, a contradiction. Therefore $\epsilon(G_{xz}) = 0$. By Claim 11-(3), $G_{xz} = xz$.

Note that $G_{yz} \neq yz$ otherwise z is a 2-vertex of G. Thus $\epsilon(G_{yz}) \geqslant 1$.

If $\epsilon(G_{yz}) \geqslant 2$, then $\epsilon(G_{xy}) \geqslant \epsilon(G_{yz}) \geqslant 2$. This implies that both G_{xy} and G_{yz} are coverable. By the minimality of G, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two signed circuit 6-covers of G_{xy} and G_{yz}, respectively. For each $i \in [1,2]$, pick three members C_{i1}, C_{i2}, C_{i3} from $\mathcal{F}_i$ such that $xy \in E(C_{1j})$ and $yz \in E(C_{2j})$ for $j \in [1,3]$. Then

$$\cup_{i=1}^2 \left((\mathcal{F}_i \setminus \{C_{i1}, C_{i2}, C_{i3}\}) \cup \{C_{i1} \triangle C, C_{i2} \triangle C, C_{i3} \triangle C\}\right)$$

is a signed circuit 6-cover of G, a contradiction. Therefore $\epsilon(G_{yz}) = 1$.

Recall that G_{yz} is 2-connected. By Claims 12 and 14, $G_{yz} \sim R_i$ for some $i \in \{0,2,3,4,5\}$. By the structure of R_i, there is an edge $e \in E(G_{yz}) \setminus \{yz\}$ such that $G_{yz} - e$ is balanced. Since C is balanced, $(G_{yz} - e) \cup C$ is also balanced. Thus $\epsilon(G_{yz} \cup C) = 1$. By Claims 12 and 14 again, $G_{yz} \cup C \sim R_j$ for some $j \in \{2,4,5\}$. This completes the proof of the claim. $\square$

Claim 16. *Let $B_i = B_i(x_{i-1}, x_i)$ for $i \in [1,h]$ and $H = \mathcal{S}(B_1, \ldots, B_h)$ be a piece of G at $\{x_0, x_h\}$ such that $h = |\mathcal{B}(H)| \geqslant 2$, $\epsilon(G - E(H)) \geqslant 1$, and every $B_i \in \mathcal{B}_2(H)$ has a $\Psi_{x_{i-1}x_i}(2)$-cover.*

(1) *If $\mathcal{B}_0(H) = \emptyset$, then either $H \sim D_2(x,y)$ in Fig. 2, or $h = 2$ and $\mathcal{B}_2(H) \in \{\{B_1\}, \{B_2\}\}$. Furthermore, when $\mathcal{B}_2(H) = \{B_1\}$, the following statements hold.*

(1a) *Every $\Psi_{x_0x_1}(2)$-cover of B_1 has a tadpole at x_1 containing x_0;*

(1b) *B_1 has no $\Psi_{x_0x_1}(t)$-cover for some $t \in \{0,1,3\}$;*

(1c) *If B_1 has a $\Psi^*_{x_0x_1}(2)$-cover and $e = x_0x_h \in E(G - E(H))$, then either $H \cup e$ has a $\Psi^*_{x_0x_2}(2)$-cover, or $H \cup e$ is equivalent to one of $R_2(y,x)$, $R_4(x,y)$ and $R_5(x,y)$.*

(2) *If $h \geqslant 3$, $\mathcal{B}_2(H) = \{B_k\}$ and $\mathcal{B}_0(H) = \{B_{k+1}\}$ for some $k \in [1, h-2]$, then B_k has no $\Psi^*_{x_{k-1}x_k}(2)$-cover and B_k is not equivalent to R_i for each $i \in \{2,4,5\}$.*

Proof. Let H' be a piece of G at $\{x_0, x_2\}$ such that $G = \mathcal{P}(H, H')$. Then $E(H') = E(G - E(H))$.

(1) Assume that $H \not\sim D_2(x,y)$. If H has a $\Psi_{x_0x_h}(2)$-cover whose tadpoles at x_0 and x_h don't contain x_h and x_0, respectively, then we have $|E(H)| \geqslant 5$. Since $\epsilon(H') \geqslant 1$, $\mathcal{P}(D_2(x,y), H')$ has a signed circuit 6-cover by the minimality of G. By Lemma 7-(2), G has a signed circuit 6-cover, a contradiction. Hence H has no such $\Psi_{x_0x_h}(2)$-cover. Since $\mathcal{B}_0(H) = \emptyset$, $h = 2$ and $\mathcal{B}_2(H) \in \{\{B_1\}, \{B_2\}\}$ by Lemma 6.

Assume $\mathcal{B}_1(H_1) = \{B_1\}$. Clearly, (1a) follows from Lemma 6 and (1b) follows from Lemma 7-(1).

We now prove (1c). Suppose to the contrary that $H \cup e$ has no $\Psi^*_{x_0x_2}(2)$-cover and $H \cup e$ is not equivalent to any of $R_2(y,x)$, $R_4(x,y)$, and $R_5(x,y)$. Furthermore since x_2 is a 2-vertex of $H \cup e$, $H \cup e \not\sim R_i$ for each $i \in \{2,4,5\}$. Since B_1 has a $\Psi^*_{x_0x_1}(2)$-cover, $x_0x_1 \in E(B_1)$ by the definition. With some switchings, assume that x_0x_1 is positive. By Lemma 9, $C = x_0x_1x_2x_0$ is a balanced 3-circuit. Note that x_1 is a 2-vertex of $H' \cup C$. By Claim 15, $(H' \cup C)(x_0, x_1)$ is equivalent to $R_2(y,x)$ or $R_i(x,y)$ for some $i \in \{4,5\}$, and thus $H' \sim R_i(x,y)$ for some $i \in \{0,2,3\}$. Since $G = \mathcal{P}(B_1 \cup x_1x_2, H')$, by Lemma 10 and Observation 8-(2), G has a signed circuit 6-cover, a contradiction. This proves (1c).

(2) Suppose to the contrary that either B_k has a $\Psi^*_{x_{k-1}x_k}(2)$-cover or $B_k \sim R_i$ for some $i \in \{2,4,5\}$. Since B_{k+1} is a negative loop at x_k $(= x_{k+1})$, $B_k \cup B_{k+1}$ has a $\Psi_{x_{k-1}x_{k+1}}(2)$-cover in which no tadpole at x_{k+1} contains x_{k-1}. Since $k \leqslant h-2$, we have $B_h \in \mathcal{B}_1(H)$. Thus H has a $\Psi_{x_0x_h}(2)$-cover in which no tadpole at x_0 (resp, x_h) contains x_h (resp., x_0). Since $\epsilon(H') \geqslant 1$, by Lemma 7-(2) and the minimality of G, G has a signed circuit 6-cover, a contradiction. This prove (2) and thus completes the proof of the claim. $\square$

Claim 17. *Suppose that $G = \mathcal{P}(H_1, H_2, H_3)$ where each $H_i = H_i(x, y)$ and $\epsilon(H_3) \geqslant 1$. If $H = H_1 \cup H_2 \not\sim R_i$ for any $i \in \{0, 2, 4, 5\}$ and contains no negative loop at $\{x, y\}$, then the following statements hold.*

(1) *If either $H_1 \sim D_1(x, y)$ or $H_2 \sim D_1(x, y)$, then H has a $\Psi_{xy}(t)$-cover for each $t \in [0, 3]$, where $D_1(x, y)$ is the two-terminal signed graph in Fig. 2.*

(2) *If $xy \in E(H)$, then H has a $\Psi^*_{xy}(2)$-cover.*

(3) *If $xy \notin E(H)$ and neither H_1 nor H_2 is equivalent to $D_1(x, y)$, then H has a $\Psi_{xy}(2)$-cover in which no tadpole at y contains x.*

Proof. Suppose that H is a counterexample to the claim with minimum $|E(H)|$. Recall that G is 2-connected and contains no positive loop. By the definition, let $B_i = B_i(x_{i-1}, x_i)$, $i \in [1, s]$, such that

$$H_1(x, y) = \mathcal{S}(B_1, \ldots, B_h) = B_1 \cup \cdots \cup B_h, \quad H_2(y, x) = \mathcal{S}(B_{h+1}, \ldots, B_s) = B_{h+1} \cup \cdots \cup B_s$$

and s is maximum with this property, where $x = x_0 = x_s \in V(B_1) \cap V(B_s)$ and $y = x_h \in V(B_h) \cap V(B_{h+1})$. Then, for any $B \in \mathcal{B}_2(H_1) \cup \mathcal{B}_2(H_2)$ with terminals u and v, B is 2-connected by the maximality of s, and $B - L(B)$ is unbalanced by Claim 14. Furthermore, it follows from the minimality of H that B has either a $\Psi_{uv}(t)$-cover for each $t \in [0, 3]$, or a $\Psi^*_{uv}(2)$-cover, or a $\Psi_{uv}(2)$-cover in which no tadpole at v contains u, unless $B \sim R_i$ for some $i \in \{0, 2, 4, 5\}$. By this fact and Observation 8, B has a $\Psi_{uv}(2)$-cover.

We will find a desired $\Psi_{xy}(2)$-cover of H, contradicting that H is a counterexample to the claim. To do this, when H_i, $i \in [1, 2]$, is not a single edge (that is, $|\mathcal{B}_0(H_i)| + |\mathcal{B}_2(H_i)| \geqslant 1$), we apply Lemma 5 to construct a signed subgraph 6-cover $\mathcal{F}^*_i$ of H_i as follows:

$$\mathcal{F}^*_i = \mathcal{F}_{i0} \cup 2\mathcal{B}_0(H_i) \cup \{P_{i1}, P_{i2}, P_{i3}, P_{i4}\} \cup \{T_{i1}, T_{i2}, T_{i3}, T_{i4}\},$$

where

- ▷ $\mathcal{F}_{i0}$ is a subfamily of signed circuits of H_i;
- ▷ P_{i1} and P_{i2} (resp., P_{i3} and P_{i4}) are two positive (resp., negative) xy-paths of H_i if $|\mathcal{B}_2(H_i)| \geqslant 1$, and otherwise $P_{i1} = P_{i2} = P_{i3} = P_{i4} = H_i - \mathcal{B}_0(H_i)$;
- ▷ T_{i1}, T_{i2} (resp., T_{i3}, T_{i4}) are two tadpoles of H_i at x (resp., y) such that the unbalanced circuit in $T_{i(2i-1)}$ (resp., $T_{i(2i)}$, $T_{i(5-2i)}$, $T_{i(6-2i)}$) is in the part in $\mathcal{B}_0(H_i) \cup \mathcal{B}_2(H_i)$ with minimum (resp., minimum, maximum, maximum) subscript.

Note that $P_{11} \cup P_{21}$ is a circuit and every part in $\mathcal{B}_0(H_1) \cup \mathcal{B}_0(H_2)$ is a negative loop. When $\mathcal{B}_0(H_1) \cup \mathcal{B}_0(H_2) \neq \emptyset$, the signed graph $P_{11} \cup P_{21} \cup \mathcal{B}_0(H_1) \cup \mathcal{B}_0(H_2)$ has a family

$$\mathcal{C}_0 \cup \{T'_1, T'_2\}$$

which covers $P_{11} \cup P_{21}$ once and $\mathcal{B}_0(H_1) \cup \mathcal{B}_0(H_2)$ twice, where $\mathcal{C}_0$ is a set of barbells and T'_1, T'_2 are two tadpoles at x.

(1) WLOG, assume that $H_2 = D_1(x,y)$. Then $h \geqslant 2$ since $H \not\sim R_2$. Let $t \in [0,3]$.

If $\mathcal{B}_2(H_1) = \emptyset$, then $H_1 = xx_1y \cup L_{x_1}$ by Claim 11-(6), and thus it is easy to check that $H = H_1 \cup H_2$ has a $\Psi_{xy}(t)$-cover.

If $h = 2$ and $B_2 \in \mathcal{B}_1(H_1)$, then $\mathcal{B}_0(H_1) = \emptyset$ and $\mathcal{B}_2(H_1) = \{B_1\}$. By (1a) and (1b) of Claim 16, B_1 has a $\Psi^*_{x_0x_1}(2)$-cover. Thus H has a $\Psi_{xy}(t)$-cover by Lemma 10-(1).

Next assume that either $h \geqslant 3$ and $\mathcal{B}_2(H_1) \neq \emptyset$, or $h = 2$ and $B_2 \in \mathcal{B}_2(H_1)$. Then $x \notin V(T_{13}) \cup V(T_{14})$. We construct a family $\mathcal{F}^*$ as follows.

$$\begin{aligned}
&\mathcal{F}^* = \mathcal{F}_{10} \cup \mathcal{F}_{20} \cup \{P_{11} \cup P_{21}, T_{11} \cup T_{21}\} \cup \\
&\begin{cases}
\{P_{14} \cup P_{24}, T_{12} \cup T_{22}\} \cup \{P_{12} \cup P_{23}, P_{13} \cup P_{22}, T_{13}, T_{14}, T_{23}, T_{24}\} & \text{if } t = 0; \\
\{P_{14} \cup P_{24}, T_{12} \cup T_{22}\} \cup \{P_{12}, P_{23}\} \cup \{P_{13} \cup P_{22}, T_{13}, T_{14}, T_{23}, T_{24}\} & \text{if } t = 1; \\
\{P_{14} \cup P_{24}, T_{13} \cup T_{23}\} \cup \{P_{12}, P_{22}, P_{13}, P_{23}\} \cup \{T_{12}, T_{22}, T_{14}, T_{24}\} & \text{if } t = 2; \\
\{T_{12} \cup T_{22}\} \cup \{P_{12}, P_{22}, P_{24} \triangle B_s, P_{13}, P_{14}, P_{23}\} \cup \{B_s, T_{13} \cup P_{24}, T_{14} \cup P_{24}\} & \text{if } t = 3.
\end{cases}
\end{aligned}$$

When $|\mathcal{B}_0(H_1)| = 0$, let $\mathcal{F} = \mathcal{F}^*$. When $\mathcal{B}_0(H_1) = \{B_i\}$ for some $i \in [2, h-1]$, let $\mathcal{C}_0 = \emptyset$ and

$$\mathcal{F} = \begin{cases}
(\mathcal{F}^* \setminus \{P_{11} \cup P_{21}, P_{14} \cup P_{24}\}) \cup \{P_{11} \cup P_{24} \cup B_i, P_{14} \cup P_{21} \cup B_i\} & \text{if } t \in [0,2]; \\
(\mathcal{F}^* \setminus \{P_{11} \cup P_{21}, B_s\}) \cup \{B_s \cup T_1'\} \cup \{T_2'\} & \text{if } t = 3.
\end{cases}$$

When $|\mathcal{B}_0(H_1)| \geqslant 2$, let $\mathcal{F} = (\mathcal{F}^* \setminus \{P_{11} \cup P_{21}\}) \cup \mathcal{C}_0 \cup \{T_1' \cup T_2'\}$. In each case, one can easily check that $\mathcal{F}$ is a $\Psi_{xy}(t)$-cover of H by the structure of $H_2 = D_1(x,y)$.

(2) WLOG, assume that $H_2 = xy$ is positive. Then $h \geqslant 2$ since $H \not\sim R_0$.

If $\mathcal{B}_2(H_1) = \emptyset$, then $H_1 = xx_1y \cup L_{x_1}$ by Claim 11-(6). Thus $H = xx_1yx \cup L_{x_1}$ is a short barbell by Claim 14 and has a $\Psi^*_{xy}(2)$-cover.

If $\mathcal{B}_0(H_1) = \emptyset$, then by Claim 16-(1), either $H_1 \sim D_2(x,y)$ in Fig. 2 or $h = 2$ and $\mathcal{B}_2(H_1) = \{B_1\}$ or $\{B_2\}$. In the former case, $H \sim R_3$ and thus has a $\Psi^*_{xy}(2)$-cover. In the latter case, by the symmetry, assume that $\mathcal{B}_2(H_1) = \{B_1\}$. Thus B_1 has a $\Psi^*_{x_0x_1}(2)$-cover by (1a) and (1b) of Claim 16. Since $H \not\sim R_i$ for each $i \in \{2,4,5\}$, H has a $\Psi^*_{xy}(2)$-cover by (1c) of Claim 16.

Now we assume that $\mathcal{B}_2(H_1) \neq \emptyset$ and $\mathcal{B}_0(H_1) \neq \emptyset$. Then $h \geqslant 3$. Let B_k (resp., B_ℓ) be the part in $\mathcal{B}_2(H_1) \cup \mathcal{B}_0(H_1)$ with minimum (resp., maximum) subscript.

If $V(B_k) \cap V(B_\ell) = \emptyset$, then by the choice of $\mathcal{F}_1^*$, $V(T_{11}) \cap V(T_{13}) = \emptyset$. Thus $T_{11} \cup \{xy\} \cup T_{13}$ is a barbell. Therefore, the family

$$\mathcal{F}_{10} \cup \mathcal{C}_0 \cup \{P_{12} \cup xy, T_{11} \cup xy \cup T_{13}\} \cup \{xy, xy, P_{13}, P_{14}\} \cup \{T_1', T_2', T_{12} \cup xy, T_{14}\}$$

is a $\Psi^*_{xy}(2)$-cover of H.

If $V(B_k) \cap V(B_\ell) \neq \emptyset$, then either $\mathcal{B}_2(H_1) = \{B_k, B_{k+2}\}$ and $\mathcal{B}_0(H_1) = \{B_{k+1}\}$, or $\mathcal{B}_2(H_1) \cup \mathcal{B}_0(H_1) = \{B_k, B_{k+1}\}$. In the former case, by the proof of Lemma 5, there are 4 negative x_0x_h-paths $P_{11}', P_{12}', P_{13}', P_{14}'$ in H_1 such that $(\mathcal{F}_1^* \setminus \{P_{11}, P_{12}, P_{13}, P_{14}\}) \cup \{P_{11}', P_{12}', P_{13}', P_{14}'\}$ is a signed subgraph 6-cover of H_1 and hence the family

$$\mathcal{F}_{10} \cup \{P_{11}' \cup xy \cup B_{k+1}, P_{12}' \cup xy \cup B_{k+1}\} \cup \{xy, xy, P_{13}', P_{14}'\} \cup \{T_{11}, xy \cup T_{13}, yx \cup T_{12}, T_{14}\}$$

is a $\Psi^*_{xy}(2)$-cover of H. In the latter case, assume that $\mathcal{B}_2(H_1) = \{B_k\}$ and $\mathcal{B}_0(H_1) = \{B_{k+1}\}$ by the symmetry. Then $k = h-2 \in [1,2]$ since H has no negative loop at x_h and G contains no 2-vertex. By Claim 16-(2), B_k has no $\Psi^*_{x_{k-1}x_k}(2)$-cover and $B_k \not\sim R_i$ for each $i \in \{2,4,5\}$. Hence $k=1$; otherwise $B_2 \cup B_1$ is a piece of G at $\{x_2, x_0\}$ and thus, by (1a) and (1b) of Claim 16, B_1 has a $\Psi^*_{x_2x_1}(2)$-cover, a contradiction. Since $\mathcal{B}_0(H_1) = \{B_2\}$ and $H_2 \cup H_3$ is unbalanced, $G - E(B_1)$ is coverable. Hence B_1 is not coverable. By Claim 12, $B_1 = R_0$ and thus $H - L_{x_1} \sim R_2(y,x)$. Since H has a unique balanced 3-circuit $C = x_0x_1x_2x_0$, by Claim 15, $C \cup H_3 \sim R_i$ for some $i \in \{2,4,5\}$. Therefore, one can easily check that $G = (H - E(C)) \cup (C \cup H_3)$ has a signed circuit 6-cover, a contradiction.

(3) Since $xy \notin E(H)$, both H_1 and H_2 contain cut-vertices by Claim 13. Thus $h \geqslant 2$ and $s - h \geqslant 2$. If $|\mathcal{B}_2(H_1)| = |\mathcal{B}_2(H_2)| = 0$, then $H = x_0x_1x_2x_3x_0 \cup \{L_{x_1}, L_{x_3}\}$ by Claim 11-(6) and $H - L(H)$ is unbalanced by Claim 14. Thus one can easily find a desired $\Psi_{xy}(2)$-cover, a contradiction. Hence $|\mathcal{B}_2(H_1)| + |\mathcal{B}_2(H_2)| \geqslant 1$ and, when $|\mathcal{B}_2(H_i)| = 0$, we may assume that $H_i - L(H_i)$ is positive (with possible switchings).

By the construction, we can choose $\mathcal{F}_1^*$ and $\mathcal{F}_2^*$ such that $y \notin V(T_{i1}) \cap V(T_{i2})$ and $x \notin V(T_{i3}) \cap V(T_{i4})$ for each $i \in [1,2]$; otherwise, if either $y \in \cup_{i=1}^2 (V(T_{i1}) \cap V(T_{i2}))$ or $x \in \cup_{i=1}^2 (V(T_{i3}) \cap V(T_{i4}))$, say $y \in V(T_{11}) \cap V(T_{12})$, then $(\mathcal{B}_0(H_1), \mathcal{B}_1(H_1), \mathcal{B}_2(H_1)) = (\emptyset, \{B_1\}, \{B_2\})$ and for every $\Psi_{x_1x_2}(2)$-cover of B_2, both its tadpoles at x_1 contain x_2, contradicting that B_2 has a $\Psi^*_{x_2x_1}(2)$-cover by (1a) and (1b) of Claim 16. Therefore, WLOG, assume that $y \notin V(T_{11}) \cup V(T_{21})$ and $x \notin V(T_{14}) \cup V(T_{24})$.

If $x \notin V(T_{13})$ or $x \notin V(T_{23})$, say $x \notin V(T_{13})$, since $|\mathcal{B}_2(H_1)| + |\mathcal{B}_2(H_2)| \geqslant 1$, the family

$$\begin{aligned} \mathcal{F} = {} & \mathcal{F}_{10} \cup \mathcal{F}_{20} \cup \{T_{12} \cup T_{21}, T_{13} \cup T_{23}\} \cup \\ & \begin{cases} \mathcal{C}_0 \cup \{T_{11} \cup T_{22}\} \cup \mathcal{P} \cup \{T_1', T_2', T_{14}, T_{24}\} & \text{if } \mathcal{B}_0(H_1) \cup \mathcal{B}_0(H_2) \neq \emptyset; \\ \{P_{11} \cup P_{21}\} \cup \mathcal{P} \cup \{T_{11}, T_{22}, T_{14}, T_{24}\} & \text{if } \mathcal{B}_0(H_1) \cup \mathcal{B}_0(H_2) = \emptyset. \end{cases} \end{aligned}$$

is a desired $\Psi_{xy}(2)$-cover, where $\mathcal{P} = \{P_{12}, P_{13}, P_{22}, P_{23}\} \cup \{P_{14} \cup P_{24}\}$ if $|\mathcal{B}_2(H_1)| \geqslant 1$ and $|\mathcal{B}_2(H_2)| \geqslant 1$, and $\mathcal{P} = \{P_{13}, P_{14}, P_{23}, P_{24}\} \cup \{P_{12} \cup P_{22}\}$ otherwise.

If $x \in V(T_{13}) \cap V(T_{23})$, then for each $i \in [1,2]$,

$$(\mathcal{B}_0(H_i), \mathcal{B}_1(H_i), \mathcal{B}_2(H_i)) = (\emptyset, \{B_{i+1}\}, \{B_{3i-2}\}),$$

and both B_1 and B_4 have $\Psi^*_{x_{j-1}x_j}(2)$-covers by (1a) and (1b) of Claim 16. Therefore H has a desired $\Psi_{xy}(2)$-cover by Claim 10-(2). This completes the proof of the claim. $\square$

4.2 The final step

Since $\overline{G - L(G)}$ is 2-connected, loopless, K_4-minor-free, and of minimum degree at least 3, it contains a 2-circuit, denoted by $C_1 = x_0x_1x_0$. Let C_2 be the circuit of $G - L(G)$ corresponding to C_1 and let

$$B_1 = C_2 \cup \{L_z \in L(G) : z \in V(C_2) \setminus \{x_0, x_1\}\}.$$

Obviously, B_1 is a 2-connected piece of G at $\{x_0, x_1\}$. By Claims 14 and 11-(6), $C_2 = B_1 - L(B_1)$ is an unbalanced circuit of length 2 or 3 or 4, denoted by $x_0x_1x_0$ or $x_0zx_1x_0$

or $x_0z_1x_1z_2x_0$ depending on its length. Hence $B_1 = x_0x_1x_0$ or $B_1 = x_0zx_1x_0 \cup L_z$ or $B_1 = x_0z_1x_1z_2x_0 \cup \{L_{z_1}, L_{z_2}\}$. In each case, B_1 has a $\Psi_{x_0x_1}(2)$-cover

$$\mathcal{F}_1^* = \mathcal{F}_{10} \cup \{P_{11}, P_{12}, P_{13}, P_{14}\} \cup \{T_{21}, T_{22}, T_{23}, T_{24}\},$$

where $\mathcal{F}_{10}$ consists of signed circuits, P_{11} and P_{12} (resp., P_{13} and P_{14}) are two positive (resp., negative) x_0x_1-paths, and T_{11} and T_{12} (resp., T_{13} and T_{14}) are two tadpoles at x_0 (resp., x_1).

Let $H = H(x_0, x_1)$ such that $G = \mathcal{P}(B_1, H)$. Choose $B_i = B_i(x_{i-1}, x_i)$, $i \in [2, s]$, such that

$$H(x_1, x_0) = \mathcal{S}(B_2, B_3, \cdots, B_s) = B_2 \cup B_3 \cup \cdots \cup B_s$$

and s is maximum with this property, where $x_1 \in V(B_2)$ and $x_s = x_0 \in V(B_s)$. Then $|\mathcal{B}_2(H)| \geqslant 1$; otherwise, by Claim 11-(6), $H - L(H)$ is a positive or negative path with length 1 or 2, and thus one can easily find a signed circuit 6-cover of G, a contradiction. Furthermore, $|\mathcal{B}_1(H)| + |\mathcal{B}_2(H)| \geqslant 2$ by Claim 13, and every $B_i \in \mathcal{B}_2(H)$ has a $\Psi_{x_{i-1}x_i}(2)$-cover by Claim 17. Applying Lemma 5, we pick a signed subgraph 6-cover $\mathcal{F}_2^*$ of H as follows:

$$\mathcal{F}_2^* = \mathcal{F}_{20} \cup 2\mathcal{B}_0(H) \cup \{P_{21}, P_{22}, P_{23}, P_{24}\} \cup \{T_{21}, T_{22}, T_{23}, T_{24}\},$$

where $\mathcal{F}_{20}$ is a family of signed circuits, P_{21} and P_{22} (resp., P_{23} and P_{24}) are two positive (resp., negative) x_0x_1-paths, T_{21} and T_{22} (resp., T_{23} and T_{24}) are two tadpoles in H at x_0 (resp., x_1) whose unbalanced circuit is in the part in $\mathcal{B}_0(H) \cup \mathcal{B}_2(H)$ with maximum (resp., minimum) subscript.

Let $U = \bigcap_{B \in \mathcal{B}_0(H) \cup \mathcal{B}_2(H)} V(B)$. We first show $U \cap \{x_0, x_1\} = \emptyset$. Otherwise $x_0 \notin V(T_{23}) \cap V(T_{24})$ and $x_1 \notin V(T_{21}) \cap V(T_{22})$. Thus the family

$$\begin{aligned} &\mathcal{F}_{10} \cup \mathcal{F}_{20} \cup \{P_{12} \cup P_{22}, P_{13} \cup P_{23}\} \cup \{T_{11} \cup T_{21}, T_{12} \cup T_{22}, T_{13} \cup T_{23}, T_{14} \cup T_{24}\} \\ &\cup \begin{cases} \{P_{14} \cup P_{24}\} \cup \mathcal{C}_0 & \text{if } |\mathcal{B}_0(H)| \neq 1; \\ \{P_{11} \cup P_{24} \cup \mathcal{B}_0(H), P_{14} \cup P_{21} \cup \mathcal{B}_0(H)\} & \text{if } |\mathcal{B}_0(H)| = 1 \end{cases} \end{aligned}$$

is a signed circuit 6-cover of G, where $\mathcal{C}_0$ is a family of signed circuits of $P_{11} \cup P_{21} \cup \mathcal{B}_0(H)$ which covers $P_{11} \cup P_{21}$ once and $\mathcal{B}_0(H)$ twice, a contradcition. Hence $U \cap \{x_0, x_1\} \neq \emptyset$.

WLOG, assume that $x_0 \in U$. Then $\mathcal{B}_1(H) = \{B_2\} = \{x_1x_2\}$, $\mathcal{B}_2(H) = \{B_3\}$ and $\mathcal{B}_0(H) \in \{\emptyset, \{B_4\}\}$ since $|\mathcal{B}_1(H)| + |\mathcal{B}_2(H)| \geqslant 2$ and $|\mathcal{B}_2(H)| \geqslant 1$. Hence $G = B_4 \cup \mathcal{P}(B_1 \cup x_1x_2, B_3)$.

Note that $x_0z_1x_1z_2x_0 \cup \{L_{z_1}, L_{z_2}\}$ has a $\Psi_{x_0x_1}(2)$-cover in which no tadpole at x_1 contains x_0. Since $B_1 \cup x_1x_2$ is a piece of G at $\{x_0, x_2\}$, by (1a) of Claim 16, we have either $B_1 = x_0x_1x_0$ or $x_0zx_1x_0 \cup \{L_z\}$. Since $B_3 \cup x_2x_1$ is a piece of G at $\{x_0, x_1\}$, it follows from Claim 17 and (1a) and (1b) of Claim 16 that either B_3 has a $\Psi^*_{x_0x_1}(2)$-cover or $B_3 = B_3(x_0, x_2) \sim R_i(x, y)$ for some $i \in \{0, 2, 4, 5\}$. Therefore, by Lemma 7-(2), G has a signed circuit 6-cover, a contradiction. This completes the proof of Theorem 2.

Acknowledgements

We are very grateful to the anonymous referees for their useful comments and suggestions. You Lu was supported by National Natural Science Foundation of China (No. 12271438), Guangdong Basic and Applied Basic Research Foundation (No. 2023A1515012340), Natural Science Foundation of Qinghai Province (No. 2022-ZJ-753) and Shaanxi Fundamental Science Research Project for Mathematics and Physics (No. 22JSZ009). Rong Luo was supported by a grant from Simons Foundation (No. 839830). Zhengke Miao was supported by National Natural Science Foundation of China (No. 12431013). Cun-Quan Zhang was supported by an NSF grant DMS-1700218.

References

[1] N. Alon, and M. Tarsi. Covering multigraphs by simple circuits. *SIAM J. Algebraic Discrete Methods*, 6: 345–350, 1985.

[2] B. Alspach, L.A. Goddyn, and C.-Q. Zhang. Graphs with the circuit cover property. *Trans. Amer. Math. Soc.*, 344: 131–154, 1994.

[3] B. Bao, R. Chen, and G. Fan. Circuit covers of signed Eulerian graphs. *Electron. J. Comb.*, 28(1): #P1.14, 2021.

[4] J.C. Bermond, B. Jackson, and F. Jaeger. Shortest covering of graphs with cycles. *J. Combin. Theory, Ser. B*, 35: 297–308, 1983.

[5] J.A. Bondy, and U.S.R. Murty. Graph Theory, GTM, vol. 244, Springer, 2008.

[6] A. Bouchet. Nowhere-zero integral flows on a bidirected graphs. *J. Combin. Theory, Ser. B*, 34: 279–292, 1983.

[7] J. Chen, and G. Fan. Short signed circuit cover of signed graphs. *Discrete Appl. Math.*, 235: 51–58, 2018.

[8] J. Chen, and G. Fan. Circuit k-covers of signed graphs. *Discrete Appl. Math.*, 294: 41–54, 2021.

[9] R. Diestel. Graph Theory, Fourth edition, Springer-Verlag, 2010.

[10] Z. Dvořák, D. Král, and J. Teska. Toughness threshold for the existence of 2-walks in K_4-minor-free graphs. *Discrete Math.*, 310(3): 642–651, 2010.

[11] M. Ellingham, S. Shan, D. Ye, and X. Zha. Toughness and spanning trees in K_4-minor-free graphs. *J. Graph Theory*, 96(3): 379–402, 2021.

[12] G. Fan. Integer flows and cycle covers. *J. Combin. Theory, Ser. B*, 54: 113–122, 1992.

[13] G. Fan. Short cycle covers of cubic graphs. *J. Graph Theory*, 18: 131–141, 1994.

[14] G. Fan, and A. Raspaud. Fulkerson's conjecture and circuits covers. *J. Combin. Theory, Ser. B*, 61: 133-138, 1994.

[15] G. Fan. Flows and circuit covers in signed graphs. Lectures in NSFC Tianyuan Summer School, Jinhua, 2018.

[16] B. Jackson. Shortest circuit covers and postman tours of graphs with a nowhere-zero 4-flow. *SIAM J. Comput.*, 19: 659–665, 1990.

[17] B. Jackson. Shortest circuit covers of cubic graphs. *J. Combin. Theory, Ser. B*, 60: 299–307, 1994.

[18] F. Jaeger. Flows and generalized coloring theorems in graphs. *J. Combin. Theory, Ser. B*, 26: 205–216, 1979.

[19] U. Jamshy, A. Raspaud, and M. Tarsi. Short circuit covers for regular matroids with nowhere-zero 5-flow. *J. Combin. Theory, Ser. B*, 43: 354–357, 1987.

[20] T. Kaiser, R. Lukot'ka, E. Máčajová, and E. Rollová. Shorter signed circuit covers of graphs. *J. Graph Theory*, 92: 39–56, 2019.

[21] T. Kaiser, and E. Rollová. Nowhere-zero flows in signed series-parallel graphs. *SIAM J. Discrete Math.*, 30(2): 1248–1258, 2016.

[22] Y. Lu, J. Cheng, R. Luo, and C.-Q. Zhang. Shortest circuit cover of signed graphs. *J. Combin. Theory, Ser. B*, 134: 164–178, 2019.

[23] E. Máčajová, A. Raspaud, E. Rollová, and M. Škoviera. Circuit covers of signed graphs. *J. Graph Theory*, 81(2): 120–133, 2016.

[24] E. Máčajová, A. Raspaud, M. Tarsi, and X.-D. Zhu, Short cycle covers of graphs and nowhere-zero flows. *J. Graph Theory*, 68: 340–348, 2011.

[25] E. Máčajová, E. Rollová, and M. Škoviera. Circuit covers of signed eulerian graphs. *SIAM J. Discrete Math.*, 33(2): 933–957, 2019.

[26] P.D. Seymour. Nowhere-zero 6-flows. *J. Combin. Theory, Ser. B*, 30: 130–135, 1981.

[27] D.B. West. Introduction to Graph Theory, Upper Saddle River, NJ: Prentice Hall, 1996.

[28] Y. Wu, and D. Ye. Circuit covers of cubic signed graphs. *J. Graph Theory*, 89(1): 40–54, 2018.

[29] Y. Wu, and D. Ye. Minimum T-joins and signed-circuit covering. *SIAM J. Discrete Math.*, 34(2): 1192–1204, 2020.

[30] T. Zaslavsky. Signed graphs. *Discrete Appl. Math.*, 4: 47–74, 1982.

[31] C.-Q. Zhang. Minimum cycle coverings and integer flows. *J. Graph Theory*, 14: 537–546, 1990.