# The Largest Size of $(t, t + d, t + 2d)$-Core Partitions

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**Abstract**

In 2007, Olsson and Stanton determined the largest size of $(t_1, t_2)$-core partitions. Inspired by their result, there have been considerable research on the largest size of simultaneous core partitions. In this work, we compute the largest size of $(t, t+d, t+2d)$-core partitions for any coprime positive integers $t$ and $d$. This generalizes the result of Yang, Zhong, and Zhou, who proved the largest size of $(t, t+1, t+2)$-core partitions.

**Mathematics Subject Classifications:** 05A17, 11P81

## 1 Introduction

A partition of a positive integer $n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots \lambda_\ell \geqslant 1$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. Each $\lambda_i$ is called a part of $\lambda$. We denote the size of $\lambda$ by $|\lambda| = n$, and the length of $\lambda$ by $\ell(\lambda)$, which refers to the number of parts in $\lambda$. We draw a figure corresponding to a partition $\lambda$, which is called the Ferrers diagram of $\lambda$. In the Ferrers diagram of a partition $\lambda$, we draw $\lambda_i$ boxes in the $i$th row and all the boxes are left-aligned. In each box of the Ferrers diagram of $\lambda$, we assign a number called a hook length. The hook length of a box is the sum of the number of boxes on its right, the number of boxes in the below, and 1 for itself.

When $t$ is a positive integer, a partition $\lambda$ is called a $t$-core (partition) if none of the hook lengths in $\lambda$ is a multiple of $t$. For positive integers $t_1, t_2, \dots, t_m$, we say that a partition is a $(t_1, t_2, \dots, t_m)$-core partition if it is simultaneously a $t_i$-core for all $i = 1, 2, \dots, m$. Note that the number of $(t_1, t_2, \dots, t_m)$-core partition is finite if and only if $\gcd(t_1, t_2, \dots, t_m) = 1$.

The beta-set of a partition $\lambda$ is the set of hook lengths in the first column of its Ferrers diagram. In other words, for a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, the beta-set of $\lambda$ is

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$\beta(\lambda) = \{\lambda_1 + \ell - 1, \lambda_2 + \ell - 2, \ldots, \lambda_{\ell-1} + 1, \lambda_\ell\}$. It follows from [4, Lemma 2.7.13] that $\lambda$ is a $t$-core partition if and only if $x \in \beta(\lambda)$ implies $x - t \in \beta(\lambda)$ when $x > t$. Note that $|\lambda| = S(\beta(\lambda)) - \binom{\ell(\lambda)}{2}$, where $S(A)$ is the sum of the elements in the set $A$. For example, the partition $(6, 2, 1, 1)$ is a $t$-core partition if $t = 5, 7, 8$, or $t \geqslant 10$ (see Figure 1). Thus, this partition is a $(5, 7)$-core partition and the beta-set is $\{9, 4, 2, 1\}$.

| | | | | | |
|---|---|---|---|---|---|
| 9 | 6 | 4 | 3 | 2 | 1 |
| 4 | 1 | | | | |
| 2 | | | | | |
| 1 | | | | | |

Figure 1: The Ferrers diagram of the partition $(6, 2, 1, 1)$ and its hook lengths

The studies of simultaneous core partitions emerged from the work of Anderson [2]. By hiring a poset structure and counting the number of corresponding lattice paths, Anderson showed that the number of $(t_1, t_2)$-core partitions for coprime $t_1$ and $t_2$ is equal to the rational Catalan number $\frac{1}{t_1+t_2}\binom{t_1+t_2}{t_1}$. Moreover, Olsson and Stanton [10], and Tripathi [14] independently showed that the largest size of $(t_1, t_2)$-core partitions is $(t_1^2 - 1)(t_2^2 - 1)/24$, and Johnson [5] and Wang [15] proved that the average size of $(t_1, t_2)$-core partitions is $(t_1 + t_2 + 1)(t_1 - 1)(t_2 - 1)/24$ independently.

It is natural to ask for the properties of $(t_1, \ldots, t_m)$-core partitions with $m \geqslant 3$. Due to the work of Aggarwal [1], we have a “nice” poset structure for $(t_1, t_2, t_3)$-core partitions only for some special cases. Hence, most studies of core partitions consider cases in which cores form an arithmetic progression. Let $t$ and $m$ be positive integers such that $t \geqslant 3$. Yang, Zhong, and Zhou [19] determined both the largest size and the average size of $(t, t + 1, t + 2)$-core partitions. Xiong [16] found a formula for the largest size of $(t, t + 1, t + 2, \ldots, t + m)$-core partitions. Nam and Yu [8] computed the largest size of $(t, t+1)$-core partitions under the restriction that all parts are of the same parity. Ma and Jiang [7] generalized the result of Nam and Yu to $(t, mt \pm 1)$-core partitions. Sha and Xiong [12] computed the largest size of $(t, mt-1, mt+1)$-core partitions, which was conjectured by Nath and Sellers [9]. Also, there are several results on the largest size of core partitions such that all parts are distinct. For more details, see [6, 11, 13, 17, 18, 20, 21, 22].

The main result of this paper provides the formula for the largest size of $(t, t+d, t+2d)$-core partitions for any coprime positive integers $t$ and $d$. This is the first result on the largest size of core partitions whose cores form an arithmetic progression with a common difference greater than 1.

**Theorem 1.** *Let $t$ and $d$ be coprime positive integers with $t \geqslant 3$. The largest size of $(t, t + d, t + 2d)$-core partitions is*

$$\begin{cases} \frac{1}{6} s(s+1)(s+d)(s+d+1) & \text{when } t = 2s+1, \\ \frac{1}{6}\left(s^2 + ds - \frac{d-1}{2}\right)\left(s^2 + ds + \frac{d+1}{2}\right) & \text{when } t = 2s. \end{cases}$$

Note that Theorem 1 is a generalization of the result of Yang, Zhong, and Zhou [19].

**Theorem 2.** *[19] Let $t$ be a positive integer. The size of the largest $(t, t+1, t+2)$-core partition equals*

$$\begin{cases} (s+1)\binom{s+2}{3} & \text{when } t = 2s-1, \\ (s+1)\binom{s+1}{3} + \binom{s+2}{3} & \text{when } t = 2s. \end{cases}$$

## 2 Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of nonnegative integers. For positive integers $t_1, \ldots, t_m$, we define the set $P_{(t_1,\ldots,t_m)}$ by

$$P_{(t_1,\ldots,t_m)} = \mathbb{N} \setminus \{n \in \mathbb{N} \mid n = a_1t_1 + a_2t_2 + \cdots + a_mt_m \text{ for } a_1, \ldots, a_m \in \mathbb{N}\}.$$

For two elements $x, y \in P_{(t_1,\ldots,t_m)}$, we say that $y$ covers $x$ and denote $x \prec y$ if

$$y - x \in \{a_1t_1 + a_2t_2 + \cdots + a_mt_m \mid a_1, \ldots, a_m \in \mathbb{N}\}.$$

By convention, we often omit the partial order $\prec$ to present a poset $(P_{(t_1,\ldots,t_m)}, \prec)$. For a given poset $(P_{(t_1,\ldots,t_m)}, \prec)$, an order ideal $I$ is a subset of $P_{(t_1,\ldots,t_m)}$ with the property that for any elements $x, y \in P_{(t_1,\ldots,t_m)}$, if $x \prec y$ and $y \in I$, then $x \in I$.

It follows from [19, Theorem 2.2] that a partition $\lambda$ is a $(t_1, \ldots, t_m)$-core partition if and only if $\beta(\lambda)$ is an order ideal of $P_{(t_1,\ldots,t_m)}$. In this paper, we consider the poset $P_{(t,t+d,t+2d)}$ for coprime positive integers $t$ and $d$. By convention, we denote by $P_{[t,d]} := P_{(t,t+d,t+2d)}$.

### 2.1 The structure of the Hasse diagram

We observe the elements of $P_{[t,d]}$ and categorize them.

**Lemma 3.** *For coprime positive integers $t$ and $d$ with $t \geqslant 3$,*

$$P_{[t,d]} = P_{(t,d)} \cup \bigcup_{j=0}^{\lfloor \frac{t}{2} - 1 \rfloor} L_j,$$

*where $L_j = \{id + jt \mid i, j \in \mathbb{N}, 2j + 1 \leqslant i \leqslant t - 1\}$.*

*Proof.* First, it is clear that $P_{(t,d)} \subseteq P_{[t,d]}$ because

$$x = at + b(t + d) + c(t + 2d) = (a + b + c)t + (b + 2c)d$$

for some nonnegative integers $a$, $b$, and $c$.

Next, we consider the elements of $P_{[t,d]} \setminus P_{(t,d)}$, which are of the form $id + jt$ for some nonnegative integers $i$ and $j$. When $j = 0$, $id \in P_{[t,d]}$ for each $i = 1, 2, \ldots, t-1$ since gcd$(t,d) = 1$. Now, let $i$ and $j$ be positive integers with $1 \leqslant i \leqslant t-1$. Suppose that there exist nonnegative integers $a$, $b$, and $c$ such that $id + jt = at + b(t+d) + c(t+2d)$. Then, $2j - i = 2(a+b+c) - (b+2c) = 2a + b \geqslant 0$. Therefore, $id + jt \in P_{[t,d]} \setminus P_{(t,d)}$ when $2j+1 \leqslant i \leqslant t-1$, which completes the proof. $\square$

We draw the Hasse diagram of $P_{[t,d]}$ as follows: First, we write the elements $d, 2d, \dots,$ $(t-1)d$ horizontally and call this layer $L_0$. Put other elements of $P_{[t,d]}$ in the diagram above or below the elements of $L_0$, where the elements are connected by a line segment if the difference of two elements is $t$, $t+d$, or $t+2d$. Note that, when $t$ is even, there is only one element at the top of the Hasse diagram, whereas there are two elements at the top when $t$ is odd.

**Example 4.** We have $P_{[8,3]} = P_{(8,3)} \cup \bigcup_{i=0}^{3} L_i$, where $P_{(8,3)} = \{1,2,4,5,7,10,13\}$, $L_0 = \{3,6,9,12,15,18,21\}$, $L_1 = \{17,20,23,26,29\}$, $L_2 = \{31,34,37\}$, and $L_3 = \{45\}$. See Figure 2 for the Hasse diagram of $P_{[8,3]}$. Figure 3 shows the Hasse diagram of $P_{[9,2]}$.

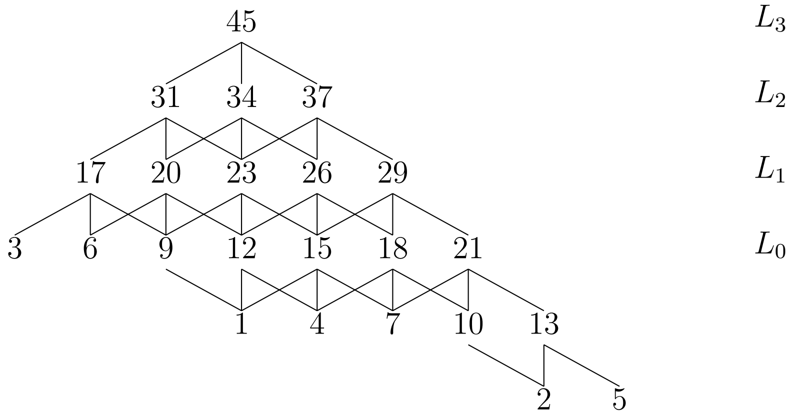


Figure 2: The Hasse diagram of $P_{[8,3]}$

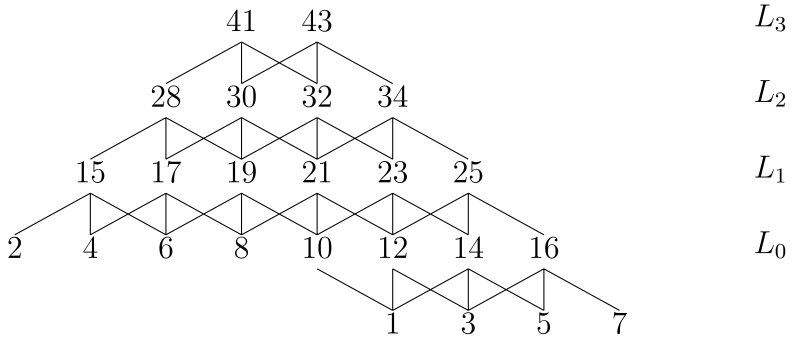


Figure 3: The Hasse diagram of $P_{[9,2]}$

Let $|A|$ be the cardinality of a set $A$. For coprime positive integers $t$ and $d$, Brown and Shiue [3] showed that the sum of the elements in $P_{(t,d)}$ is

$$S(P_{(t,d)}) = \frac{1}{12}(t-1)(d-1)(2td-t-d-1) \tag{2.1}$$

and the number of elements in $P_{(t,d)}$ is

$$|P_{(t,d)}| = (t-1)(d-1)/2. \tag{2.2}$$

Let $\kappa_{[t,d]}$ be a $(t, t+d, t+2d)$-core partition such that $\beta(\kappa_{[t,d]}) = P_{[t,d]}$. We now compute the size of $\kappa_{[t,d]}$.

**Lemma 5.** *For coprime positive integers $t$ and $d$ with $t \geqslant 3$, the size of $\kappa_{[t,d]}$ is*

$$\begin{cases} \frac{1}{6}s(s+1)(s+d)(s+d+1) & \text{when } t = 2s+1, \\ \frac{1}{6}\left(s^2+ds-\frac{d-1}{2}\right)\left(s^2+ds+\frac{d+1}{2}\right) & \text{when } t = 2s. \end{cases}$$

*Proof.* We first consider the case when $t = 2s+1$ for some positive integer $s$.

By the equations (2.1) and (2.2),

$$S(P_{(2s+1,d)}) = \frac{1}{6}s(d-1)(4sd-2s+d-2) \text{ and } |P_{(2s+1,d)}| = s(d-1).$$

The elements in $L_j$ when $0 \leqslant j \leqslant s-1$ are

$$d + (2s+2d+1)j, 2d + (2s+2d+1)j, \ldots, (2s-2j)d + (2s+2d+1)j.$$

Hence, we have

$$\begin{aligned} S\left(\bigcup_{j=0}^{s-1} L_j\right) &= \sum_{j=0}^{s-1}\{(2s-2j+1)d + (4s+4d+2)j\}(s-j) \\ &= \frac{1}{6}s(s+1)(8ds+d+4s^2-2s-2) \end{aligned}$$

and $\left|\bigcup_{j=0}^{s-1} L_j\right| = 2+4+\cdots+2s = s(s+1)$.

It follows from Lemma 3 that $S(P_{[2s+1,d]}) = \frac{1}{6}s(d-1)(4sd-2s+d-2) + \frac{1}{6}s(s+1)(8ds+d+4s^2-2s-2)$ and $|P_{[2s+1,d]}| = s(d-1) + s(s+1) = s^2+sd$.

By the definition of the beta-set, we obtain that

$$|\kappa_{[2s+1,d]}| = S(P_{[2s+1,d]}) - \binom{|P_{[2s+1,d]}|}{2} = \frac{1}{6}s(s+1)(s+d)(s+d+1).$$

We omit the proof for even $t$ since it is very similar to the proof when $t$ is odd. □

## 2.2 Properties on order ideals

In order to compute the largest size of $(t, t+d, t+2d)$-core partitions, we group the elements in the poset $P_{[t,d]}$ which are in the same diagonal from northwest to southeast. Define each group as sets $A_i$ for $i = 1, 2, \ldots, t-1$ by

$$A_i = \{id + jt \in P_{[t,d]} \mid j \in \mathbb{Z}\}.$$

Let $I$ be an order ideal of $P_{[t,d]}$. Define $a_i(I) = |A_i \setminus I|$. By convention, we abbreviate $a_i(I)$ as $a_i$ if the order ideal $I$ is obvious in the context.

**Example 6.** Consider the poset $P_{[9,2]}$. Then we have $A_1 = \{2\}$, $A_2 = \{4\}$, $A_3 = \{6, 15\}$, $A_4 = \{8, 17\}$, $A_5 = \{1, 10, 19, 28\}$, $A_6 = \{3, 12, 21, 30\}$, $A_7 = \{5, 14, 23, 32, 41\}$, and $A_8 = \{7, 16, 25, 34, 43\}$.

Let $I = \{1, 3, 5, 6, 7, 8, 10, 12, 14, 16, 21, 25\}$ be an order ideal of $P_{[9,2]}$. In Figure 4, we circle the elements of the order ideal $I$. Note that $a_5(I) = |A_5 \setminus \{1, 10\}| = 2$. Similarly, $a_1(I) = 1$, $a_2(I) = 1$, $a_3(I) = 1$, $a_4(I) = 1$, $a_6(I) = 1$, $a_7(I) = 3$, and $a_8(I) = 2$.

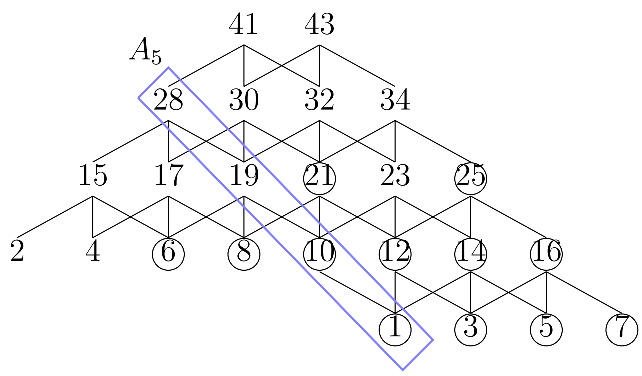


Figure 4: The Hasse diagram of $P_{[9,2]}$ and the order ideal $I$ whose elements are marked by circles

We now observe properties of $A_i$ and $a_i$. Recall that the beta-set of a $(t, t+d, t+2d)$-core partition corresponds to an order ideal of $P_{[t,d]}$.

**Lemma 7.** *For coprime positive integers $t$ and $d$ with $t \geqslant 3$, let $\lambda$ be a $(t, t+d, t+2d)$-core partition and $a_i = a_i(\beta(\lambda))$. Then, we have the followings:*

*(1)* $\max\{A_{2i-1}\} = (2i-1)d + (i-1)t$ *for* $i = 1, 2, \ldots, \lfloor \frac{t}{2} \rfloor$

*(2)* $\max\{A_{2i}\} = 2id + (i-1)t$ *for* $i = 1, 2, \ldots, \lfloor \frac{t-1}{2} \rfloor$

*(3)* $a_{2i-1} \leqslant \frac{(t+2d)i-d-1}{t}$ *for* $i = 1, 2, \ldots, \lfloor \frac{t}{2} \rfloor$

*(4)* $a_{2i} \leqslant \frac{(t+2d)i-1}{t}$ *for* $i = 1, 2, \ldots, \lfloor \frac{t-1}{2} \rfloor$

*(5)* $a_{2i-1} \leqslant a_{2i+1}$ *for* $i = 1, 2, \ldots, \lfloor \frac{t}{2} \rfloor - 1$

*(6)* $a_{2i} \leqslant a_{2i+2}$ *for* $i = 1, 2, \dots, \lfloor \frac{t-1}{2} \rfloor - 1$

*(7)* $a_{2i} \leqslant a_{2i+1}$ *for* $i = 1, 2, \dots, \lfloor \frac{t}{2} \rfloor - 1$

*(8)* $a_{2i-1} \leqslant a_{2i} + 1$ *for* $i = 1, 2, \dots, \lfloor \frac{t-1}{2} \rfloor$

*Proof.* **(1) and (2):** Let $kd + jt \in A_k$ for a fixed positive integer $k$ and an integer $j$. By Lemma 3, we get $2j+1 \leqslant k \leqslant t-1$ or $j < 0$, which gives us that $j \leqslant \frac{k-1}{2}$. Thus, we have

$$\max\{A_k\} = kd + \left\lfloor \frac{k-1}{2} \right\rfloor t.$$

By putting $k = 2i-1$ or $k = 2i$ respectively to the above equation, we obtain the desired results.

**(3) and (4):** Since $\beta(\lambda)$ is an order ideal and $|A_k \setminus \beta(\lambda)| = a_k$, by (1) and (2), we have

$$\begin{aligned} A_{2i-1} \setminus \beta(\lambda) &= \{(2i-1)d + jt \in P_{[t,d]} \mid j \in \mathbb{Z}\} \setminus \beta(\lambda) \\ &= \{\max\{A_{2i-1}\} - jt \mid 0 \leqslant j \leqslant a_{2i-1} - 1\} \\ &= \{(2i-1)d + (i-1)t - jt \mid 0 \leqslant j \leqslant a_{2i-1} - 1\}, \\ A_{2i} \setminus \beta(\lambda) &= \{2id + (i-1)t - jt \mid 0 \leqslant j \leqslant a_{2i} - 1\}. \end{aligned}$$

Since the elements of $A_{2i-1}$ are positive integers, we get

$$(2i-1)d + (i-1)t - (a_{2i-1} - 1)t \geqslant 1 \quad \Leftrightarrow \quad a_{2i-1} \leqslant \frac{(t+2d)i - d - 1}{t}.$$

Similarly, we obtain $a_{2i} \leqslant \frac{(t+2d)i-1}{t}$.

**(5) and (6):** Suppose that $(2i+1)d + it - a_{2i+1}t \in A_{2i+1} \cap \beta(\lambda)$. If $(2i+1)d + it - a_{2i+1}t > t + 2d$, we get

$$((2i+1)d + it - a_{2i+1}t) - (t + 2d) = (2i-1)d + (i-1)t - a_{2i+1}t \in \beta(\lambda),$$

which implies that $a_{2i-1} \leqslant a_{2i+1}$. If $(2i+1)d + it - a_{2i+1}t \leqslant t + 2d$, then $\max\{A_{2i-1}\} = (2i-1)d + (i-1)t \leqslant a_{2i+1}t$. Hence, $a_{2i-1} \leqslant |A_{2i-1}| \leqslant a_{2i+1}$. Similarly, we get $a_{2i} \leqslant a_{2i+2}$.

**(7) and (8):** Suppose that $2id + (i-1)t - a_{2i}t \in A_{2i} \cap \beta(\lambda)$. If $2id + (i-1)t - a_{2i}t > t + d$, we get

$$(2id + (i-1)t - a_{2i}t) - (t + d) = (2i-1)d + (i-1)t - (a_{2i} + 1)t \in \beta(\lambda),$$

which implies that $a_{2i-1} \leqslant a_{2i} + 1$. If $2id + (i-1)t - a_{2i}t \leqslant t + d$, then $\max\{A_{2i-1}\} = (2i-1)d + (i-1)t \leqslant (a_{2i} + 1)t$. Hence, $a_{2i-1} \leqslant |A_{2i-1}| \leqslant a_{2i} + 1$. Similarly, we get $a_{2i} \leqslant a_{2i+1}$. □

We achieve the following corollary by using Lemma 7.

**Corollary 8.** *For coprime positive integers $t$ and $d$ with $t \geqslant 3$, let $\lambda$ be a $(t, t+d, t+2d)$-core partition and $a_i = a_i(\beta(\lambda))$. For each $i = 1, 2, \ldots, t$,*

$$S(A_{2i-1} \setminus \beta(\lambda)) = ita_{2i-1} + (2i-1)da_{2i-1} - \frac{a_{2i-1}(a_{2i-1}+1)}{2}t,$$

$$S(A_{2i} \setminus \beta(\lambda)) = ita_{2i} + 2ida_{2i} - \frac{a_{2i}(a_{2i}+1)}{2}t.$$

*Proof.* Since $A_i \setminus \beta(\lambda) = \{\max\{A_i\}, \max\{A_i\} - t, \ldots, \max\{A_i\} - (a_i - 1)t\}$, we have

$$S(A_i \setminus \beta(\lambda)) = a_i \max\{A_i\} - \frac{a_i(a_i-1)}{2}t.$$

By (1) and (2) of Lemma 7, we obtain the desired results. □

Our goal in the next two sections is to prove that $\kappa_{[t,d]}$ gives the largest size of $(t, t+d, t+2d)$-core partitions for all coprime positive integers $t$ and $d$ with $t \geqslant 3$. In order to prove this result, we consider two cases depending on the parity of $t$. Recall that the poset structure of $P_{[t,d]}$ has one maximal element when $t$ is even and two maximal elements when $t$ is odd.

## 3 Proof of Theorem 1 for odd $t$

In this section, we compute the largest size of $(t, t+d, t+2d)$-core partitions when $t$ is an odd integer. In Lemma 9, we compute $|\kappa_{[2s+1,d]}| - |\lambda|$ in terms of $a_i$ for a $(2s+1, 2s+d+1, 2s+2d+1)$-core partition $\lambda$. Then, we show in Theorem 11 that $|\kappa_{[2s+1,d]}| - |\lambda|$ is nonnegative for any $\lambda$.

Throughout this section, for simplicity, we denote $\sum\limits_{i<j} a_i a_j := \sum\limits_{1\leqslant i<j\leqslant 2s} a_i a_j$.

**Lemma 9.** *Let $s$ and $d$ be positive integers such that $\gcd(2s+1, d) = 1$ and $\kappa_{[2s+1,d]}$ be a partition whose beta-set is $P_{[2s+1,d]}$. For any $(2s+1, 2s+d+1, 2s+2d+1)$-core partition $\lambda$, let $a_i = a_i(\beta(\lambda))$. Then, we have*

$$|\kappa_{[2s+1,d]}| - |\lambda| = \sum_{i=1}^{s}((2i-s-1)(d+s)+i)(a_{2i-1}+a_{2i}) + d\sum_{i=1}^{s} a_{2i} - s\sum_{i=1}^{2s} a_i^2 + \sum_{i<j} a_i a_j.$$

*Proof.* By Lemma 5, we have $\ell(\kappa_{[2s+1,d]}) = s(s+d)$. By the definition of the beta-set,

$$|\kappa_{[2s+1,d]}| - |\lambda| = \sum_{i=1}^{2s} S(A_i \setminus \beta(\lambda)) - \binom{s(s+d)}{2} + \binom{\ell(\lambda)}{2}.$$

Since $\ell(\lambda) = s(s+d) - \sum\limits_{i=1}^{2s} a_i$, we get

$$\binom{s(s+d)}{2} - \binom{\ell(\lambda)}{2} = s(s+d)\sum_{i=1}^{2s} a_i - \frac{1}{2}\sum_{i=1}^{2s} a_i\left(\sum_{i=1}^{2s} a_i + 1\right).$$

By putting $t = 2s+1$ in Corollary 8, we have

$$\begin{aligned}
&\sum_{i=1}^{2s} S(A_i \setminus \beta(\lambda)) - \binom{s(s+d)}{2} + \binom{\ell(\lambda)}{2} \\
&= \sum_{i=1}^{s} \left( (2s+1)ia_{2i-1} + (2i-1)da_{2i-1} - (2s+1)\frac{a_{2i-1}(a_{2i-1}+1)}{2} \right) \\
&\quad + \sum_{i=1}^{s} \left( (2s+1)ia_{2i} + 2ida_{2i} - (2s+1)\frac{a_{2i}(a_{2i}+1)}{2} \right) - s(s+d)\sum_{i=1}^{2s} a_i \\
&\quad + \frac{1}{2}\sum_{i=1}^{2s} a_i \left( \sum_{i=1}^{2s} a_i + 1 \right) \\
&= \sum_{i=1}^{s} \left( (2s+1)i + (2i-1)d - \frac{2s+1}{2} \right)(a_{2i-1}+a_{2i}) + d\sum_{i=1}^{s} a_{2i} \\
&\quad - \frac{2s+1}{2}\sum_{i=1}^{2s} a_i^2 - s(s+d)\sum_{i=1}^{2s} a_i + \frac{1}{2}\left( \sum_{i=1}^{2s} a_i \right)^2 + \frac{1}{2}\sum_{i=1}^{2s} a_i \\
&= \sum_{i=1}^{s} ((2i-s-1)(d+s)+i)(a_{2i-1}+a_{2i}) + d\sum_{i=1}^{s} a_{2i} - s\sum_{i=1}^{2s} a_i^2 + \sum_{i<j} a_i a_j. \qquad \square
\end{aligned}$$

Now, we show that $|\kappa_{[2s+1,d]}| - |\lambda| \geqslant 0$ for any $(2s+1, 2s+1+d, 2s+1+2d)$-core partition $\lambda$. The proof of this result is quite complicated, so we provide an example to give readers an intuition for the proof of Theorem 11. A partition $\lambda'$ is called the conjugate of a partition $\lambda$ if each part $\lambda'_j$ of $\lambda'$ represents the number of boxes in the column $j$ of the Ferrers diagram of $\lambda$.

**Example 10.** We compute the largest size of $(5, 5+d, 5+2d)$-core partitions for a positive integer $d$ with $\gcd(5, d) = 1$. First, it is known that $|\kappa_{[5,d]}| = (d+2)(d+3)$ by Lemma 5. By putting $s = 2$ in Lemma 9, for each $(5, 5+d, 5+2d)$-core partition $\lambda$, we obtain that

$$|\kappa_{[5,d]}| - |\lambda| = \sum_{i=1}^{2} ((2i-3)(d+2)+i)(a_{2i-1}+a_{2i}) + d\sum_{i=1}^{2} a_{2i} - 2\sum_{i=1}^{4} a_i^2 + \sum_{1 \leqslant i<j \leqslant 4} a_i a_j.$$

Since $a_1 \leqslant a_3$, $a_2 \leqslant a_4$, $a_1 \leqslant a_2+1$, and $a_3 \leqslant a_4+1$ by Lemma 7, we have

$$\begin{aligned}
&\textstyle\sum_{i=1}^{2} ((2i-3)(d+2)+i)(a_{2i-1}+a_{2i}) + d\displaystyle\sum_{i=1}^{2} a_{2i} - 2\sum_{i=1}^{4} a_i^2 + \sum_{1 \leqslant i<j \leqslant 4} a_i a_j \\
&= (-d-1)(a_1+a_2) + (d+4)(a_3+a_4) + da_2 + da_4 - 2a_1^2 - 2a_2^2 - 2a_3^2 - 2a_4^2 \\
&\quad + a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4 \\
&= a_1(a_3-a_1) + 2a_2(a_4-a_2) + a_1(a_2-a_1+1) + a_3(a_4-a_3+1) \\
&\quad - a_3^2 - 2a_4^2 - da_1 + da_3 + 2da_4 - 2a_1 - a_2 + 3a_3 + 4a_4 + a_1a_4 + a_2a_3 - a_2a_4
\end{aligned}$$

$$\begin{aligned}
&\geqslant -a_3^2-2a_4^2-da_1+da_3+2da_4-2a_1-a_2+3a_3+4a_4+a_1a_4+a_2a_3-a_2a_4\\
&=(a_3-a_1)d-(a_2-a_1)a_4-(a_3-a_2)a_3-2a_1-a_2+3a_3+2a_4(d+2-a_4)\\
&\geqslant(a_3-a_1)d-(a_2-a_1)a_4-(a_3-a_2)a_3-2a_1-a_2+3a_3,
\end{aligned}$$

where the last inequality comes from the fact that $a_4 \leqslant \frac{2(5+2d)-1}{5} < d+2$ and the equality holds when $a_4 = 0$.

We again use the inequalities $a_3 \leqslant a_4 + 1$ and $a_4 < d + 2$ to obtain the following inequalities:

$$\begin{aligned}
&(a_3-a_1)d-(a_2-a_1)a_4-(a_3-a_2)a_3-2a_1-a_2+3a_3\\
&=(a_3-a_2)(d-a_3)+(a_2-a_1)(d-a_4)-2a_1-a_2+3a_3\\
&=(a_3-a_2)(d+3-a_3)+(a_2-a_1)(d+2-a_4)\\
&\geqslant(a_3-a_2)(d+3-(a_4+1))+(a_2-a_1)(d+2-a_4)\\
&=(a_3-a_1)(d+2-a_4)\geqslant 0.
\end{aligned}$$

The equality holds when $(a_1, a_2, a_3, a_4) = (0, 0, 0, 0)$ or $(1, 0, 1, 0)$. If $(a_1, a_2, a_3, a_4) = (0, 0, 0, 0)$, then $\lambda = \kappa_{[5,d]}$. Otherwise, $\lambda = \kappa'_{[5,d]}$.

We generalize calculations of inequalities in Example 10 to prove that $|\kappa_{[2s+1,d]}|-|\lambda| \geqslant 0$ for general $s$, which is given in Theorem 11.

**Theorem 11.** *For positive integers $s$ and $d$ such that $\gcd(2s+1, d) = 1$, let $\lambda$ be a $(2s+1, 2s+d+1, 2s+2d+1)$-core partition and $a_i = |A_i \setminus \beta(\lambda)|$. Then, $|\kappa_{[2s+1,d]}|-|\lambda| \geqslant 0$. The equality holds when $\lambda = \kappa_{[2s+1,d]}$ or $\kappa'_{[2s+1,d]}$.*

*Proof.* This proof heavily depends on Lemma 7. We first divide summations into the sums of $a_{2i-1}$ and $a_{2i}$ in order to use inequalities in Lemma 7. First, we have

$$\begin{aligned}
&|\kappa_{[2s+1,d]}|-|\lambda|\\
&=\sum_{i=1}^{s}((2i-s-1)(d+s)+i)(a_{2i-1}+a_{2i})+d\sum_{i=1}^{s}a_{2i}-s\sum_{i=1}^{2s}a_i^2+\sum_{i<j}a_ia_j\\
&=\sum_{i=1}^{s}\{(2i-s-1)(d+s)(a_{2i-1}+a_{2i})+i(a_{2i-1}+a_{2i})+da_{2i}\}-\sum_{i=1}^{2s}sa_i^2+\sum_{i<j}a_ia_j\\
&=\sum_{i=1}^{s}\{(2i-s-1)(d+s)(a_{2i-1}+a_{2i})+ia_{2i-1}\}+s(d+s)a_{2s}-sa_{2s}^2\\
&\quad-\sum_{i=1}^{s}\{(s-i)a_{2i}+(a_{2s}-a_{2i})(d+s-a_{2s})-a_{2i}a_{2s}\}-\sum_{i=2}^{s}(i-1)(a_{2i-1}^2+a_{2i}^2)\\
&\quad-\sum_{i=1}^{s}(s-i+1)(a_{2i-1}^2+a_{2i}^2)+\sum_{i=2}^{s}\sum_{k=1}^{i-1}(a_{2i-1}a_{2k}+a_{2k-1}a_{2i})
\end{aligned}$$

$$+\sum_{i=1}^{s} a_{2i-1}a_{2i} + \sum_{i=1}^{s}\sum_{k=i+1}^{s}(a_{2i-1}a_{2k-1} + a_{2i}a_{2k}),$$

where

$$\begin{aligned}
&\sum_{i=1}^{s} ia_{2i} + d\sum_{i=1}^{s} a_{2i}\\
&= s(d+s)a_{2s} - \sum_{i=1}^{s}\{(s-i)a_{2i} + (a_{2s}-a_{2i})(d+s)\}\\
&= s(d+s)a_{2s} - \sum_{i=1}^{s}\{(s-i)a_{2i} + (a_{2s}-a_{2i})(d+s-a_{2s}) - a_{2i}a_{2s}\} - sa_{2s}^2
\end{aligned}$$

and

$$\sum_{i<j} a_i a_j = \sum_{i=2}^{s}\sum_{k=1}^{i-1}(a_{2i-1}a_{2k} + a_{2k-1}a_{2i}) + \sum_{i=1}^{s} a_{2i-1}a_{2i} + \sum_{i=1}^{s}\sum_{k=i+1}^{s}(a_{2i-1}a_{2k-1} + a_{2i}a_{2k}).$$

Let

$$\begin{aligned}
X &= \sum_{i=1}^{s}(2i-s-1)(d+s)(a_{2i-1}+a_{2i}) + \sum_{i=2}^{s}\sum_{k=1}^{i-1}(a_{2i-1}a_{2k} + a_{2k-1}a_{2i})\\
&\quad - \sum_{i=2}^{s}(i-1)(a_{2i-1}^2 + a_{2i}^2) - \sum_{k=1}^{s-1}(a_{2s}-a_{2k})(d+s-a_{2s}),\\
Y &= \sum_{i=1}^{s}\sum_{k=i+1}^{s}(a_{2i-1}a_{2k-1} + a_{2i}a_{2k}) - \sum_{i=1}^{s}(s-i+1)(a_{2i-1}^2 + a_{2i}^2)\\
&\quad + \sum_{i=1}^{s}(a_{2i-1}a_{2i} + a_{2i}a_{2s}).
\end{aligned}$$

Then, we have $|\kappa_{[2s+1,d]}| - |\lambda| = X + Y - \sum_{i=1}^{s-1}(s-i)a_{2i} + \sum_{i=1}^{s} ia_{2i-1} + s(d+s)a_{2s} - sa_{2s}^2$. Since $s(d+s)a_{2s} - sa_{2s}^2 = sa_{2s}(d+s-a_{2s}) \geqslant 0$, we need to show that $X + Y \geqslant \sum_{i=1}^{s-1}(s-i)a_{2i} - \sum_{i=1}^{s} ia_{2i-1}$. Let

$$X_i = \sum_{k=1}^{i-1}\{(a_{2k}-a_{2k-1})(d+s-a_{2i}) + (a_{2i-1}-a_{2k})(d+s-a_{2i-1}) + (a_{2i}-a_{2k})(d+s-a_{2i})\}$$

for $i = 2, 3, \ldots, s-1$, and

$$X_s = \sum_{k=1}^{s-1}\{(a_{2k}-a_{2k-1})(d+s-a_{2s}) + (a_{2s-1}-a_{2k})(d+s-a_{2s-1})\}.$$

Then, we obtain that $\sum\limits_{i=2}^{s} X_i = X$ because

$$\begin{aligned}
\sum_{i=2}^{s} X_i =& \sum_{i=2}^{s}\sum_{k=1}^{i-1}\{(a_{2k}-a_{2k-1})(d+s-a_{2i})+(a_{2i-1}-a_{2k})(d+s-a_{2i-1})\} \\
&+\sum_{i=2}^{s-1}\sum_{k=1}^{i-1}(a_{2i}-a_{2k})(d+s-a_{2i}) \\
=& \sum_{i=2}^{s}\sum_{k=1}^{i-1}\{(a_{2i-1}+a_{2i}-a_{2k-1}-a_{2k})(d+s)-(a_{2i-1}-a_{2k})a_{2i-1}\} \\
&-\sum_{i=2}^{s}\sum_{k=1}^{i-1}(a_{2i}-a_{2k-1})a_{2i}-\sum_{k=1}^{s-1}(a_{2s}-a_{2k})(d+s-a_{2s}) \\
=& \sum_{i=2}^{s}\sum_{k=1}^{i-1}\{(a_{2i-1}+a_{2i})(d+s)-(a_{2k-1}+a_{2k})(d+s)+a_{2i-1}a_{2k}\} \\
&+\sum_{i=2}^{s}\sum_{k=1}^{i-1}a_{2k-1}a_{2i}-\sum_{i=2}^{s}(i-1)(a_{2i-1}^2+a_{2i}^2)-\sum_{k=1}^{s-1}(a_{2s}-a_{2k})(d+s-a_{2s}) \\
=& \sum_{i=1}^{s}\{(i-1)(a_{2i-1}+a_{2i})(d+s)-(s-i)(a_{2i-1}+a_{2i})(d+s)\}+\sum_{i=2}^{s}\sum_{k=1}^{i-1}a_{2i-1}a_{2k} \\
&+\sum_{i=2}^{s}\sum_{k=1}^{i-1}a_{2k-1}a_{2i}-\sum_{i=2}^{s}(i-1)(a_{2i-1}^2+a_{2i}^2)-\sum_{k=1}^{s-1}(a_{2s}-a_{2k})(d+s-a_{2s}) \\
=& X.
\end{aligned}$$

Since $a_{2i} \leqslant \frac{(2s+2d+1)i-1}{2s+1} = i + \frac{2di-1}{2s+1} < s+d$ and $a_{2i-1} \leqslant a_{2i}+1$ by Lemma 7, we have

$$\begin{aligned}
X_i =& \sum_{k=1}^{i-1}\{(a_{2k}-a_{2k-1})(d+s-a_{2i})+(a_{2i-1}-a_{2k})(d+s-a_{2i-1})\} \\
&+\sum_{k=1}^{i-1}(a_{2i}-a_{2k})(d+s-a_{2i}) \\
\geqslant& \sum_{k=1}^{i-1}\{(a_{2k}-a_{2k-1})(d+s-a_{2i})+(a_{2i-1}-a_{2k})(d+s-1-a_{2i})\} \qquad (3.1)\\
&+\sum_{k=1}^{i-1}(a_{2i}-a_{2k})(d+s-a_{2i}) \\
=& \sum_{k=1}^{i-1}\{(a_{2i-1}+a_{2i}-a_{2k-1}-a_{2k})(d+s-a_{2i})-(a_{2i-1}-a_{2k})\} \qquad (3.2)
\end{aligned}$$

$$\geqslant -\sum_{k=1}^{i-1}(a_{2i-1}-a_{2k}) = -(i-1)a_{2i-1} + \sum_{k=1}^{i-1} a_{2k}$$

and, similarly, we also obtain that

$$X_s \geqslant -(s-1)a_{2s-1} + \sum_{k=1}^{s-1} a_{2k}.$$

Summing all the inequalities of $X_i$ gives that

$$\begin{aligned}\sum_{i=2}^{s} X_i &\geqslant \sum_{i=2}^{s-1}\left(-(i-1)a_{2i-1} + \sum_{k=1}^{i-1} a_{2k}\right) + \left(-(s-1)a_{2s-1} + \sum_{k=1}^{s-1} a_{2k}\right) \\ &= \sum_{i=1}^{s-1}(s-i)a_{2i} - \sum_{i=2}^{s}(i-1)a_{2i-1}.\end{aligned}$$

Now, we evaluate an inequality for $Y$. Let

$$\begin{aligned}Y_i &= a_{2i-1}\left(a_{2i} + \sum_{k=i+1}^{s} a_{2k-1}\right) - (s-i+1)a_{2i-1}^2, \\ Z_i &= a_{2i}\left(a_{2s} + \sum_{k=i+1}^{s} a_{2k}\right) - (s-i+1)a_{2i}^2.\end{aligned}$$

We obtain that $\sum_{i=1}^{s} Y_i + \sum_{i=1}^{s} Z_i = Y$ since

$$\begin{aligned}&\sum_{i=1}^{s} Y_i + \sum_{i=1}^{s} Z_i \\ &= \sum_{i=1}^{s}\left(a_{2i-1}\left(a_{2i} + \sum_{k=i+1}^{s} a_{2k-1}\right) - (s-i+1)a_{2i-1}^2\right) \\ &\quad + \sum_{i=1}^{s}\left(a_{2i}\left(a_{2s} + \sum_{k=i+1}^{s} a_{2k}\right) - (s-i+1)a_{2i}^2\right) \\ &= \sum_{i=1}^{s}\{a_{2i-1}a_{2i} - (s-i+1)a_{2i-1}^2\} + \sum_{i=1}^{s}\sum_{k=i+1}^{s}\{a_{2i-1}a_{2k-1} + a_{2i}a_{2k}\} \\ &\quad + \sum_{i=1}^{s} a_{2i}a_{2s} - \sum_{i=1}^{s}(s-i+1)a_{2i}^2 = Y.\end{aligned}$$

Again, since $a_{2i-1} \leqslant a_{2i+1}$, $a_{2i} \leqslant a_{2i+2}$, and $a_{2i-1} \leqslant a_{2i} + 1$ for $i = 1, 2, \ldots, s-1$ by Lemma 7, we have

$$Y_i = a_{2i-1}\left(a_{2i} + \sum_{k=i+1}^{s} a_{2k-1}\right) - (s-i+1)a_{2i-1}^2$$

$$\geqslant a_{2i-1}(a_{2i-1}-1+(s-i)a_{2i-1})-(s-i+1)a_{2i-1}^2=-a_{2i-1},$$

$$Z_i = a_{2i}\left(a_{2s}+\sum_{k=i+1}^{s} a_{2k}\right)-(s-i+1)a_{2i}^2$$

$$\geqslant a_{2i}(a_{2i}+(s-i)a_{2i})-(s-i+1)a_{2i}^2=0.$$

Thus, we obtain that

$$Y=\sum_{i=1}^{s}(Y_i+Z_i)\geqslant -\sum_{i=1}^{s} a_{2i-1}.$$

Adding the inequalities of $X$ and $Y$, we have

$$X+Y\geqslant \sum_{i=1}^{s-1}(s-i)a_{2i}-\sum_{i=2}^{s}(i-1)a_{2i-1}-\sum_{i=1}^{s} a_{2i-1}$$
$$=\sum_{i=1}^{s-1}(s-i)a_{2i}-\sum_{i=1}^{s} ia_{2i-1},$$

so we get the result.

Lastly, we check the equality condition. We first have that $s(d+s)a_{2s}-sa_{2s}^2 = sa_{2s}(d+s-a_{2s})=0$ when $a_{2s}=0$. For $i=1,\dots,s-1$, the equality $X_i=-(i-1)a_{2i-1}+\sum_{k=1}^{i-1} a_{2k}$ holds when $a_{2i-1}=a_{2k}$ or $a_{2i-1}=a_{2i}+1$ by (3.1) in the inequality of $X_i$, and $a_{2i-1}+a_{2i}-a_{2k-1}-a_{2k}=0$ for $k=1,\dots,i-1$ by (3.2) in the inequality of $X_i$. Since $a_{2k-1}\leqslant a_{2i-1}$ and $a_{2k}\leqslant a_{2i}$ by (5) and (6) of Lemma 7, we get $a_{2k-1}=a_{2i-1}$ and $a_{2k}=a_{2i}$ for all $k=1,\dots,i-1$. Similarly, we have $X_s=-(s-1)a_{2s-1}+\sum_{k=1}^{s-1} a_{2k}$ when $a_{2k-1}=a_{2s-1}$ for all $k=1,\dots,s-1$. Moreover, for $i=1,\dots,s$, we obtain that $Y_i=-a_{2i-1}$ when $a_{2i-1}=0$, or $a_{2i-1}=a_{2i}+1$ and $a_{2k-1}=a_{2i-1}$ for $k=1,\dots,i-1$. We also have that $Z_i=0$ for $i=1,\dots,s$ when $a_{2i}=0$ or $a_{2i}=a_{2k}$ for $k=i+1,\dots,s$.

Overall, the equality holds when $a_i=0$ for all $i=1,\dots,2s$ or $a_{2i-1}=1$ and $a_{2i}=0$ for $i=1,\dots,s$. The first case gives that $\lambda=\kappa_{[2s+1,d]}$ and the second one is when $\lambda=\kappa'_{[2s+1,d]}$. □

## 4 Proof of Theorem 1 for even $t$

In this section, we compute the largest size of $(t,t+d,t+2d)$-core partitons when $t=2s$ for some positive integer $s\geqslant 2$. Again, for simplicity, we denote $\sum_{i<j} a_ia_j := \sum_{1\leqslant i<j\leqslant 2s-1} a_ia_j$.

**Lemma 12.** *For positive integers $s$ and $d$ such that $\gcd(2s,d)=1$ and $s\geqslant 2$, let $\lambda$ be a $(2s,2s+d,2s+2d)$-core partition and $a_i=|A_i\setminus\beta(\lambda)|$. Let $\kappa_{[2s,d]}$ be a partition whose beta-set is $P_{[2s,d]}$. Then,*

$$|\kappa_{[2s,d]}|-|\lambda| =\frac{1}{2}\left\{\sum_{i=1}^{2s-1}(-2s+2i+1)da_i+\sum_{i=1}^{s-1}(-2s^2+4si)(a_{2i-1}+a_{2i})\right\}$$

$$+\frac{1}{2}\left\{2s^2a_{2s-1}-(2s-1)\sum_{i=1}^{2s-1}a_i^2+2\sum_{i<j}a_ia_j\right\}.$$

*Proof.* The proof of this lemma is similar to the proof of Lemma 9. Note that, by Lemma 5, $\ell(\kappa_{[2s,d]})=s^2+(2s-1)(d-1)/2$, and

$$|\kappa_{[2s,d]}|-|\lambda|=\sum_{i=1}^{2s-1}S(A_i\setminus I)-\binom{\ell(\kappa_{[2s,d]})}{2}+\binom{\ell(\lambda)}{2}.$$

Since $\ell(\lambda)=\ell(\kappa_{[2s,d]})-\sum_{i=1}^{2s-1}a_i$, by putting $t=2s$ in Corollary 8, we get

$$\begin{aligned}
&\sum_{i=1}^{2s-1}S(A_i\setminus I)-\binom{\ell(\kappa_{[2s,d]})}{2}+\binom{\ell(\lambda)}{2}\\
&=\sum_{i=1}^{s}(2sia_{2i-1}+(2i-1)da_{2i-1}-sa_{2i-1}(a_{2i-1}+1))\\
&\quad+\sum_{i=1}^{s-1}(2sia_{2i}+2ida_{2i}-sa_{2i}(a_{2i}+1))-\left(s^2+\frac{(2s-1)(d-1)}{2}\right)\sum_{i=1}^{2s-1}a_i\\
&\quad+\frac{1}{2}\left(\sum_{i=1}^{2s-1}a_i\right)\left(\sum_{i=1}^{2s-1}a_i+1\right)\\
&=(2s+2d)\sum_{i=1}^{s}ia_{2i-1}-d\sum_{i=1}^{s}a_{2i-1}-s\sum_{i=1}^{s}a_{2i-1}^2+(2s+2d)\sum_{i=1}^{s-1}ia_{2i}-s\sum_{i=1}^{s-1}a_{2i}^2\\
&\quad-\left(sd+s^2-\frac{d-1}{2}\right)\sum_{i=1}^{2s-1}a_i+\frac{1}{2}\left(\sum_{i=1}^{2s-1}a_i\right)\left(\sum_{i=1}^{2s-1}a_i+1\right)\\
&=(2s+2d)\sum_{i=1}^{s}ia_{2i-1}-d\sum_{i=1}^{s}a_{2i-1}-s\sum_{i=1}^{s}a_{2i-1}^2+(2s+2d)\sum_{i=1}^{s-1}ia_{2i}-s\sum_{i=1}^{s-1}a_{2i}^2\\
&\quad-\frac{2s^2+(2s-1)d}{2}\left(\sum_{i=1}^{s}a_{2i-1}+\sum_{i=1}^{s-1}a_{2i}\right)+\frac{1}{2}\sum_{i=1}^{2s-1}a_i^2+\sum_{i<j}a_ia_j\\
&=\sum_{i=1}^{s}\frac{(2i-s)(2s+2d)-d}{2}a_{2i-1}+\sum_{i=1}^{s-1}\frac{(2i-s)(2s+2d)+d}{2}a_{2i}\\
&\quad-\frac{2s-1}{2}\sum_{i=1}^{2s-1}a_i^2+\sum_{i<j}a_ia_j\\
&=\sum_{i=1}^{s}\frac{(2(2i-1)-2s+1)d}{2}a_{2i-1}+\sum_{i=1}^{s-1}\frac{(2(2i)-2s+1)d}{2}a_{2i}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{s-1}(-s^2+2si)(a_{2i-1}+a_{2i})+s^2a_{2s-1}-\frac{2s-1}{2}\sum_{i=1}^{2s-1}a_i^2+\sum_{i<j}a_ia_j\\
=&\frac{1}{2}\left\{\sum_{i=1}^{2s-1}(2i-2s+1)da_i+\sum_{i=1}^{s-1}(-2s^2+4si)(a_{2i-1}+a_{2i})+2s^2a_{2s-1}\right\}\\
&+\frac{1}{2}\left\{-(2s-1)\sum_{i=1}^{2s-1}a_i^2+2\sum_{i<j}a_ia_j\right\}. \qquad \square
\end{aligned}
$$

Similarly to Section 3, we show that $\kappa_{[2s,d]}$ is the largest $(2s, 2s+d, 2s+2d)$-core partition. Before proving the general case, we first give an example when $s = 2$.

**Example 13.** We compute the largest size of $(4, 4+d, 4+2d)$-core partitions for a positive integer $d$ with $\gcd(4, d) = 1$. By putting $s = 2$ in Lemma 12, for each $(4, 4+d, 4+2d)$-core partition $\lambda \neq \kappa_{[4,d]}$, we obtain that

$$
\begin{aligned}
&2(|\kappa_{[4,d]}|-|\lambda|)\\
&=\sum_{i=1}^{3}(2i-3)da_i+\sum_{i=1}^{1}(8i-8)(a_{2i-1}+a_{2i})+8a_3-3\sum_{i=1}^{3}a_i^2+2\sum_{1\leqslant i<j\leqslant 3}a_ia_j\\
&=-3a_1^2-3a_2^2-3a_3^2-da_1+da_2+3da_3+8a_3+2(a_1a_2+a_2a_3+a_1a_3).
\end{aligned}
$$

Since $a_1 \leqslant a_3$ and $a_2 \leqslant a_3$ by Lemma 7, we get

$$
\begin{aligned}
&-3a_1^2-3a_2^2-3a_3^2-da_1+da_2+3da_3+8a_3+2(a_1a_2+a_2a_3+a_1a_3)\\
&=(2a_1a_3+2a_2a_3-2a_1^2-2a_2^2)-a_1^2-a_2^2-3a_3^2-da_1+da_2+3da_3+8a_3+2a_1a_2\\
&\geqslant(a_2-a_1)(d-a_2+a_1)+3a_3(d+2-a_3)+2a_3.
\end{aligned}
$$

Let $A = (a_2-a_1)(d-a_2+a_1)+2a_3(d+2-a_3)$. We claim that $A \geqslant 2a_1-2a_2$. First, suppose that $a_2 \geqslant a_1$. By Lemma 7, we have $a_2 \leqslant \frac{4+2d-1}{4} < d+2$ and $a_3 \leqslant \frac{2(3+2d)-d-1}{4} \leqslant d+1$. Hence, $A \geqslant (a_2-a_1)(d-a_2+a_1) \geqslant -2(a_2-a_1) = 2a_1-2a_2$. If $a_2 < a_1$, then we have $a_2 - a_1 = -1$ by Lemma 7. Thus, we obtain $A = -(d+1)+2a_3(d+2-a_3)$, so it is enough to show that $-(d+1)+2a_3(d+2-a_3) \geqslant 2$. Since $\lambda \neq \kappa_{[4,d]}$, we get $a_3 \neq 0$. Therefore, we have $1 \leqslant a_3 \leqslant d+1$, which means that $a_3(d+2-a_3) \geqslant d+1$. Then, $A \geqslant -(d+1)+2(d+1) \geqslant 2$, so we get the claim. Hence,

$$
\begin{aligned}
&(a_2-a_1)(d-a_2+a_1)+3a_3(d+2-a_3)+2a_3\\
&\geqslant 2a_1-2a_2+a_3(d+2-a_3)+2a_3\\
&=2a_1+2(a_3-a_2)+a_3(d+2-a_3)\\
&\geqslant a_3(d+2-a_3)\\
&\geqslant d+1>0,
\end{aligned}
$$

which means that $\kappa_{[4,d]}$ is the $(4, 4+d, 4+2d)$-core partition with a unique maximum size.

**Theorem 14.** *For positive integers $s$ and $d$ such that* $\gcd(2s, d) = 1$ *and* $s \geqslant 2$*, let* $\lambda$ *be a* $(2s, 2s+d, 2s+2d)$*-core partition and* $a_i = |A_i \setminus \beta(\lambda)|$*. Then,* $|\kappa_{[2s,d]}| - |\lambda| > 0$ *when* $\lambda \neq \kappa_{[2s,d]}$*.*

*Proof.* The proof is similar to the proof of Theorem 11, but slightly more complicated. By Lemma 7, we have $a_{2s-1} \geqslant a_{2i-1}$ and $a_{2s-1} \geqslant a_{2s-2} \geqslant a_{2i}$ for $i = 1, 2, \ldots, s-1$. Thus, if $a_{2s-1} = 0$, then $a_i = 0$ for all $i = 1, \ldots, 2s-1$. This gives that $\lambda = \kappa_{[2s,d]}$ if and only if $a_{2s-1} = 0$. Since we assume that $\lambda \neq \kappa_{[2s,d]}$, we have $a_{2s-1} \geqslant 1$. We observe that, by Lemma 12,

$$\begin{aligned}
&2(|\kappa_{[2s,d]}| - |\lambda|)\\
&= \sum_{i=1}^{2s-1}(-2s+2i+1)da_i + \sum_{i=1}^{s-1}(-2s^2+4si)(a_{2i-1}+a_{2i}) + 2s^2 a_{2s-1}\\
&\quad - (2s-1)\sum_{i=1}^{2s-1} a_i^2 + 2\sum_{i<j} a_i a_j\\
&= \sum_{i=1}^{s-1}\{2(2i-s)(d+s)(a_{2i}+a_{2i-1}) + d(a_{2i}-a_{2i-1})\} + (2s-1)da_{2s-1}\\
&\quad - \sum_{i=1}^{s-1}(2s-1)(a_{2i}^2 + a_{2i-1}^2) - (2s-1)a_{2s-1}^2 + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s-1} 2a_{2i}a_{2j} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} 2a_{2i-1}a_{2j-1}\\
&\quad + \sum_{i=2}^{s-1}\sum_{k=1}^{i-1}(a_{2k-1}a_{2i} + a_{2k}a_{2i-1}) + \sum_{i=1}^{s-1}(2a_{2i}a_{2s-1} + 2a_{2i}a_{2i-1}) + 2s^2 a_{2s-1},
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{i=1}^{2s-1}(-2s+2i+1)da_i\\
&= \sum_{i=1}^{s}(-2s+2(2i-1)+1)da_{2i-1} + \sum_{i=1}^{s-1}(-2s+2(2i)+1)da_{2i}\\
&= \sum_{i=1}^{s-1}\{2(2i-s)d(a_{2i}+a_{2i-1}) + d(a_{2i}-a_{2i-1})\} + (2s-1)da_{2s-1}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i<j} a_i a_j &= \sum_{i=2}^{s-1}\sum_{k=1}^{i-1}(a_{2k-1}a_{2i} + a_{2k}a_{2i-1}) + \sum_{i=1}^{s-1} a_{2i}a_{2i-1}\\
&\quad + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} a_{2i-1}a_{2j-1} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s-1} a_{2i}a_{2j} + \sum_{i=1}^{s-1} a_{2i}a_{2s-1}.
\end{aligned}$$

Let

$$
\begin{aligned}
U =& \sum_{i=1}^{s-1} 2(2i-s)(d+s-1)(a_{2i}+a_{2i-1}) - \sum_{i=2}^{s-1} 2(i-1)(a_{2i}^2+a_{2i-1}^2) \\
&+ \sum_{i=2}^{s-1}\sum_{k=1}^{i-1} 2(a_{2k-1}a_{2i}+a_{2k}a_{2i-1}), \\
V =& \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} 2a_{2i-1}a_{2j-1} - \sum_{i=1}^{s-1} 2(s-i)(a_{2i-1}^2+a_{2i}^2) \\
&+ \sum_{i=1}^{s-1}\sum_{j=i+1}^{s-1} 2a_{2i}a_{2j} + \sum_{i=1}^{s-1} 2a_{2i}a_{2s-1}, \\
W =& \sum_{i=1}^{s-1}\{2(2i-s)(a_{2i}+a_{2i-1}) - (a_{2i}^2+a_{2i-1}^2) + d(a_{2i}-a_{2i-1})\} \\
&+ \sum_{i=1}^{s-1} 2a_{2i}a_{2i-1} + ((2s-1)d+2s^2)a_{2s-1} - (2s-1)a_{2s-1}^2.
\end{aligned}
$$

This gives that $2(|\kappa_{[2s,d]}|-|\lambda|) = U+V+W$, so we need to show that $U+V+W>0$.

For $i=2,3,\ldots,s-1$, let

$$
U_i = \sum_{k=1}^{i-1}\{2(a_{2i}-a_{2k-1})(d+s-1-a_{2i}) + 2(a_{2i-1}-a_{2k})(d+s-1-a_{2i-1})\}.
$$

Then, we obtain $\sum_{i=2}^{s-1} U_i = U$ since

$$
\begin{aligned}
&\sum_{i=2}^{s-1} U_i \\
&= \sum_{i=2}^{s-1}\sum_{k=1}^{i-1}\{2(a_{2i}-a_{2k-1})(d+s-1-a_{2i}) + 2(a_{2i-1}-a_{2k})(d+s-1-a_{2i-1})\} \\
&= \sum_{i=2}^{s-1}\sum_{k=1}^{i-1}\{2(a_{2i}+a_{2i-1}-a_{2k}-a_{2k-1})(d+s-1) - 2(a_{2i}-a_{2k-1})a_{2i}\} \\
&\quad - \sum_{i=2}^{s-1}\sum_{k=1}^{i-1} 2(a_{2i-1}-a_{2k})a_{2i-1} \\
&= \sum_{i=2}^{s-1}\sum_{k=1}^{i-1}\{2(a_{2i}+a_{2i-1})(d+s-1) - 2(a_{2k}+a_{2k-1})(d+s-1) - 2a_{2i}^2\} \\
&\quad + \sum_{i=2}^{s-1}\sum_{k=1}^{i-1}\{2(a_{2k-1}a_{2i}+a_{2k}a_{2i-1}) - 2a_{2i-1}^2\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=2}^{s-1} 2(i-1)(a_{2i}+a_{2i-1})(d+s-1) - \sum_{k=1}^{s-2} 2(s-1-k)(a_{2k}+a_{2k-1})(d+s-1) \\
&\quad - \sum_{i=2}^{s-1} 2(i-1)(a_{2i}^2+a_{2i-1}^2) + \sum_{i=2}^{s-1}\sum_{k=1}^{i-1} 2(a_{2k-1}a_{2i}+a_{2k}a_{2i-1}) \\
&= \sum_{i=1}^{s-1} 2(2i-s)(a_{2i}+a_{2i-1})(d+s-1) - \sum_{i=2}^{s-1} 2(i-1)(a_{2i}^2+a_{2i-1}^2) \\
&\quad + \sum_{i=2}^{s-1}\sum_{k=1}^{i-1} 2(a_{2k-1}a_{2i}+a_{2k}a_{2i-1}) = U.
\end{aligned}
$$

Since $a_{2i} \geqslant a_{2k}$ and $a_{2i-1} \geqslant a_{2k-1}$ for $i > k$, and $a_{2i} \leqslant \frac{(2s+2d)i-1}{2s} = i + \frac{2di-1}{2s} < s+d$ by Lemma 7, we have

$$
\begin{aligned}
U_i &= \sum_{k=1}^{i-1} \{2(a_{2i}-a_{2k-1})(d+s-1-a_{2i}) + 2(a_{2i-1}-a_{2k})(d+s-1-a_{2i-1})\} \\
&= \sum_{k=1}^{i-1} \{2(a_{2k}-a_{2k-1})(d+s-1-a_{2i}) + 2(a_{2k+1}-a_{2k})(d+s-1-a_{2i-1})\} \\
&\quad + \sum_{k=1}^{i-1} \{2(a_{2i}-a_{2k})(d+s-1-a_{2i}) + 2(a_{2i-1}-a_{2k+1})(d+s-1-a_{2i-1})\} \\
&\geqslant \sum_{k=1}^{i-1} \{2(a_{2k}-a_{2k-1})(d+s-1-a_{2i}) + 2(a_{2k+1}-a_{2k})(d+s-1-(a_{2i}+1))\} \quad (4.1) \\
&= \sum_{k=1}^{i-1} \{2(a_{2k+1}-a_{2k-1})(d+s-a_{2i}) - 2(a_{2k+1}-a_{2k})\} \\
&\geqslant -\sum_{k=1}^{i-1} 2(a_{2k+1}-a_{2k}). \quad (4.2)
\end{aligned}
$$

Hence, we get

$$
U \geqslant -\sum_{i=2}^{s-1}\sum_{k=1}^{i-1} 2(a_{2k+1}-a_{2k}) = -2\sum_{i=1}^{s-1}(s-1-i)(a_{2i+1}-a_{2i}). \quad (4.3)
$$

For each $i = 1, 2, \ldots, s-1$, let

$$
V_i = 2a_{2i-1}\sum_{j=i+1}^{s} a_{2j-1} - 2(s-i)a_{2i-1}^2
$$

and

$$
V_i' = 2a_{2i}\left(\sum_{j=i+1}^{s-1} a_{2j} + a_{2s-1}\right) - 2(s-i)a_{2i}^2.
$$

Then, we have

$$\begin{aligned}\sum_{i=1}^{s-1}(V_i+V_i') &= \sum_{i=1}^{s-1}\left(2a_{2i-1}\sum_{j=i+1}^{s}a_{2j-1}-2(s-i)a_{2i-1}^2\right)\\&\quad+\sum_{i=1}^{s-1}\left(2a_{2i}\left(\sum_{j=i+1}^{s-1}a_{2j}+a_{2s-1}\right)-2(s-i)a_{2i}^2\right)\\&=\sum_{i=1}^{s-1}\sum_{j=i+1}^{s}2a_{2i-1}a_{2j-1}-\sum_{i=1}^{s-1}2(s-i)a_{2i-1}^2+\sum_{i=1}^{s-1}\sum_{j=i+1}^{s-1}2a_{2i}a_{2j}\\&\quad+\sum_{i=1}^{s-1}\{2a_{2i}a_{2s-1}-2(s-i)a_{2i}^2\}=V.\end{aligned}$$

Because $a_{2i-1}\leqslant a_{2j-1}$ for $i<j$ by Lemma 7, we have

$$V_i=2a_{2i-1}\sum_{j=i+1}^{s}a_{2j-1}-2(s-i)a_{2i-1}^2\geqslant 0. \tag{4.4}$$

Again, since $a_{2i}\leqslant a_{2j}$ for $i<j$ by Lemma 7 and from the inequality $a_{2s-1}\geqslant a_{2i+1}\geqslant a_{2i}$, we obtain that

$$V_i'=2a_{2i}\left(\sum_{j=i+1}^{s-1}a_{2j}+a_{2s-1}\right)-2(s-i)a_{2i}^2\geqslant 0. \tag{4.5}$$

Hence, we get $V\geqslant 0$. By the inequality (4.3) and $V\geqslant 0$, we get

$$U+V\geqslant -2\sum_{i=1}^{s-1}(s-1-i)(a_{2i+1}-a_{2i}). \tag{4.6}$$

Now, it is sufficient to show that $W>2\sum_{i=1}^{s-1}(s-1-i)(a_{2i+1}-a_{2i})$. Note that

$$\begin{aligned}W&=\sum_{i=1}^{s-1}\{2(2i-s)(a_{2i}+a_{2i-1})-(a_{2i}^2+a_{2i-1}^2)+d(a_{2i}-a_{2i-1})+2a_{2i}a_{2i-1}\}\\&\quad+((2s-1)d+2s^2)a_{2s-1}-(2s-1)a_{2s-1}^2\\&=-\sum_{i=1}^{s-1}\{(a_{2i}^2+a_{2i-1}^2)+2a_{2i}a_{2i-1}+d(a_{2i}-a_{2i-1})+2(2i-s)(a_{2i}+a_{2i-1})\}\\&\quad+(2s-1)a_{2s-1}(d+s-a_{2s-1})+sa_{2s-1}\\&=\sum_{i=1}^{s-1}\{(a_{2i}-a_{2i-1})(d-a_{2i}+a_{2i-1})+2(2i-s)(a_{2i}+a_{2i-1})\}\end{aligned}$$

$$
\begin{aligned}
&+ (2s-1)a_{2s-1}(d+s-a_{2s-1}) + sa_{2s-1} \\
=& \sum_{i=1}^{s-1}(a_{2i}-a_{2i-1})(d-a_{2i}+a_{2i-1}) + (2s-1)a_{2s-1}(d+s-a_{2s-1}) \\
&+ \sum_{i=1}^{s-1} 2(2i-s)(a_{2i}+a_{2i-1}) + sa_{2s-1}.
\end{aligned}
$$

For $i = 1, 2, \ldots, s-1$, let

$$W_i = (a_{2i}-a_{2i-1})(d-a_{2i}+a_{2i-1}) + 2a_{2s-1}(d+s-a_{2s-1}).$$

It follows that

$$
\begin{aligned}
\sum_{i=1}^{s-1} W_i &= \sum_{i=1}^{s-1}(a_{2i}-a_{2i-1})(d-a_{2i}+a_{2i-1}) + 2(s-1)a_{2s-1}(d+s-a_{2s-1}) \\
&= W - a_{2s-1}(d+s-a_{2s-1}) - \sum_{i=1}^{s-1} 2(2i-s)(a_{2i}+a_{2i-1}) - sa_{2s-1}.
\end{aligned}
$$

We claim that, for $i = 1, 2, \ldots, s-1$,

$$W_i \geqslant -s(a_{2i}-a_{2i-1}).$$

First, suppose that $a_{2i} \geqslant a_{2i-1}$. By Lemma 7, we have $a_{2i} \leqslant \frac{(2s+2d)i-1}{2s} = i + \frac{2di-1}{2s} < s+d$ and $a_{2s-1} \leqslant \frac{(2s+2d)s-d-1}{2s} < s+d$. Hence, $d+s-a_{2i}+a_{2i-1} > a_{2i-1} \geqslant 0$. Then, we have $W_i \geqslant -s(a_{2i}-a_{2i-1})$. If $a_{2i} < a_{2i-1}$, then we have $a_{2i}-a_{2i-1} = -1$ by Lemma 7. Thus, we obtain $W_i = -(d+1) + 2a_{2s-1}(d+s-a_{2s-1})$, so it is enough to show that $-(d+1) + 2a_{2s-1}(d+s-a_{2s-1}) \geqslant s$. Since $a_{2s-1} \leqslant \frac{(2s+2d)s-d-1}{2s} < d+s$, we get $1 \leqslant a_{2s-1} \leqslant d+s-1$. Hence, $a_{2s-1}(d+s-a_{2s-1}) \geqslant d+s-1$, which implies that

$$W_i = -(d+1) + 2a_{2s-1}(d+s-a_{2s-1}) \geqslant -(d+1) + 2(d+s-1) = d+2s-3 \geqslant s.$$

So, we get $W_i \geqslant -s(a_{2i}-a_{2i-1})$, and

$$\sum_{i=1}^{s-1} W_i \geqslant -s\sum_{i=1}^{s-1}(a_{2i}-a_{2i-1}).$$

Then,

$$W \geqslant a_{2s-1}(d+s-a_{2s-1}) + \sum_{i=1}^{s-1} 2(2i-s)(a_{2i}+a_{2i-1}) + sa_{2s-1} - s\sum_{i=1}^{s-1}(a_{2i}-a_{2i-1}).$$

Since $a_{2s-1}(d+s-a_{2s-1}) > 0$, it suffices to show that

$$\sum_{i=1}^{s-1} 2(2i-s)(a_{2i}+a_{2i-1}) + sa_{2s-1} - s\sum_{i=1}^{s-1}(a_{2i}-a_{2i-1}) \geqslant 2\sum_{i=1}^{s-1}(s-1-i)(a_{2i+1}-a_{2i}). \tag{4.7}$$

Then,

$$
\begin{aligned}
&\sum_{i=1}^{s-1} 2(2i-s)(a_{2i}+a_{2i-1}) + sa_{2s-1} - s\sum_{i=1}^{s-1}(a_{2i}-a_{2i-1}) - 2\sum_{i=1}^{s-1}(s-1-i)(a_{2i+1}-a_{2i}) \\
&= \sum_{i=1}^{s-1} 2(2i-s)(a_{2i}+a_{2i-1}) + \sum_{i=1}^{s-1}(s-2i-2)a_{2i} + \sum_{i=1}^{s} sa_{2i-1} - \sum_{i=1}^{s-1} 2(s-1-i)a_{2i+1} \\
&= \sum_{i=1}^{s-1} 2(2i-s)(a_{2i}+a_{2i-1}) + \sum_{i=1}^{s-1}(s-2i-2)a_{2i} + \sum_{i=1}^{s-1}(-s+2i+2)a_{2i+1} + sa_1 \\
&= \sum_{i=1}^{s-1}(2i-s)(a_{2i+1}+a_{2i}+2a_{2i-1}) + \sum_{i=1}^{s-1} 2(a_{2i+1}-a_{2i}) + sa_1.
\end{aligned}
$$

Since $a_1 \leqslant a_3 \leqslant \ldots \leqslant a_{2s-3}$ and $a_2 \leqslant a_4 \leqslant \ldots \leqslant a_{2s-2}$, by Chebyshev's sum inequality, we have

$$
\sum_{i=1}^{s-1}(2i-s)(a_{2i+1}+a_{2i}+2a_{2i-1}) \geqslant \frac{1}{s-1}\sum_{i=1}^{s-1}(2i-s)\sum_{i=1}^{s-1}(a_{2i+1}+a_{2i}+2a_{2i-1}) = 0.
$$

Note that $\sum_{i=1}^{s-1} 2(a_{2i+1}-a_{2i}) \geqslant 0$ by Lemma 7. Thus, we can check that the inequality (4.7) is true, so we have

$$
W > \sum_{i=1}^{s-1}(s-1-i)(a_{2i+1}-a_{2i}). \tag{4.8}
$$

Combining the inequalities (4.6) and (4.8), we get the desired result. □

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