A Variable Version of the Quasi-Kernel Conjecture

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Abstract

A quasi-kernel of a digraph D is an independent set Q such that every vertex can reach Q in at most two steps. A 48-year conjecture made by P.L. Erdős and Székely, known as the *small QK conjecture*, says that every sink-free digraph contains a quasi-kernel of size at most $n/2$. Recently, Spiro posed the *large QK conjecture*, that every digraph contains a quasi-kernel Q such that $|N^-[Q]| \geqslant n/2$, and showed that it follows from the small QK conjecture.

In this paper, we establish that the large QK conjecture implies the small QK conjecture with a weaker constant. We also show that the large QK conjecture is equivalent to a sharp version of it, answering affirmatively a question of Spiro. We formulate variable versions of these conjectures, which are still open in general.

Not many digraphs are known to have quasi-kernels of size $(1-\alpha)n$ or less. We show that digraphs with bounded dichromatic number have quasi-kernels of size at most $(1-\alpha)n$, by proving a stronger statement.

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1 Introduction

We refer readers to [2] for the standard terminology and notation not introduced in this paper. Let $D = (V(D), A(D))$ be a digraph. If $xy \in A(D)$, we say that y is an out-neighbor of x, and x is an in-neighbor of y. Let $v \in V(D)$. The open (or closed) out-neighborhood (or in-neighborhood) of v in D is defined as follows. (The subscript $_D$ is omitted if the underlying digraph is clear.)

$$N^+_D(v) = \{u \in V(D) : vu \in A(D)\},\ N^+_D[v] = N^+_D(v) \cup \{v\},$$
$$N^-_D(v) = \{u \in V(D) : uv \in A(D)\},\ N^-_D[v] = N^-_D(v) \cup \{v\}.$$

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Given vertices u, v of a digraph D, let $dist(u, v) \in \mathbb{Z}^{\geqslant 0} \cup \{\infty\}$ denote the length of a shortest directed path from u to v. With a set of vertices S, denote $dist(u, S) = \min_{v \in S} dist(u, v)$, and analogously $dist(S, v) = \min_{u \in S} dist(u, v)$. Define

$$N^+(S) = \{v \in V(D) : dist(S, v) = 1\},\ N^+[S] = \{v \in V(D) : dist(S, v) \leqslant 1\},$$
$$N^-(S) = \{u \in V(D) : dist(u, S) = 1\},\ N^-[S] = \{u \in V(D) : dist(u, S) \leqslant 1\},$$

coinciding with the earlier definitions if S is a singleton. Define

$$N^{--}(S) = \{u \in V(D) : dist(u, S) = 2\},\ N^{--}[S] = \{u \in V(D) : dist(u, S) \leqslant 2\}.$$

We call an independent set $K \subseteq V(D)$ a *kernel* of D if $N^-[K] = V(D)$, namely each vertex not in K has an out-neighbor in K. Not every digraph has a kernel (consider an odd dicycle), although every digraph without odd dicycles has one [8]. Chvátal and Lovász [3] introduced the notion of quasi-kernels. An independent set $Q \subseteq V(D)$ is said to be a *quasi-kernel* of D if $N^{--}[Q] = V(D)$. Notably, [3] proves that every digraph has a quasi-kernel. Thus, it is natural to ask if one can always find a quasi-kernel that is small (or large). P.L. Erdős and Székely made the following conjecture on the existence of small quasi-kernels.

Conjecture 1 (Small Quasi-kernel Conjecture [5], 1976)**.** If D is a sink-free digraph, then D has a quasi-kernel Q with $|Q| \leqslant \frac{1}{2}|V(D)|$.

Here, a digraph is said to be *sink-free* if it has no sinks, where *sinks* known as the vertices without out-neighbors. *Sources* and *source-free* digraphs are defined analogously, referring to the in-neighbors. The sink-free condition cannot be removed, as can be seen by considering a digraph with many sinks. The constant 1/2 is the best possible, as can be seen by considering (disjoint unions of) directed 2-cycles and 4-cycles.

Conjecture 1 is wide open: the best bound that works for all sink-free digraphs appears to be $|Q| \leqslant n - \sqrt{n \log n}/4$, where n is the number of vertices [9]. However, there have been substantial results that confirm that Conjecture 1 holds on certain classes of digraphs. Heard and Huang [6] showed that a sink-free digraph D has two disjoint quasi-kernels if D is semicomplete multipartite, quasi-transitive, or locally semicomplete. As a consequence, Conjecture 1 is true for these three classes of digraphs. Van Hulst [10] showed that Conjecture 1 holds for all digraphs containing kernels. Kostochka, Luo and Shan [7] proved that Conjecture 1 holds for digraphs with chromatic number at most 4. Ai et al. [1] proved that Conjecture 1 holds for one-way split digraphs. We refer the interested reader to the nice survey by P.L. Erdős et al. [4] for a more thorough overview of this problem.

Since all the quasi-kernels in a tournament must be singletons, asking for a large quasi-kernel is not as interesting as asking for a small quasi-kernel. Recently, Spiro introduced a way to ask for a large quasi-kernel in general: that is, to measure "largeness" not by the size of Q but by that of $N^-[Q]$. Note the removal of the sink-free condition below.

Conjecture 2 (Large quasi-kernel conjecture, [9])**.** Every digraph D has a quasi-kernel Q such that $|N^-[Q]| \geqslant \frac{1}{2}|V(D)|$.

Interestingly, [9] shows that Conjecture 1 implies Conjecture 2, and obtains several results on both conjectures. In this paper, we show that the converse is also true to some degree: namely, Conjecture 2 implies Conjecture 1 but with a weaker constant $2/3$ instead of $1/2$. We utilize the following conjecture (scheme) to enable more extended discussions:

Conjecture 3. Fix some $0 < \alpha \leqslant 1/2$. One can conjecture the following, for all digraphs D on n vertices:

I. (Small quasi-kernel conjecture) If D is sink-free, then it has a quasi-kernel with size at most $(1-\alpha)n$.

II. (Small quasi-kernel conjecture with sources) D has a quasi-kernel with size at most $n - \alpha s$, where s is the number of sources in D that are not sinks.

III. (Large quasi-kernel conjecture) D has a quasi-kernel Q such that $|N^-[Q]| \geqslant \alpha n$.

IV. (Sharp large quasi-kernel conjecture) D has a quasi-kernel Q such that $|Q|/2 + |N^-(Q)| \geqslant \alpha n$.

To facilitate the discussions of the above conjecture schemes, we define notations Conjecture 3I($*$), Conjecture 3II($*$), Conjecture 3III($*$), and Conjecture 3IV($*$). They mean replacing the coefficient α in the conjecture corresponding to the Roman letter with the value in the second bracket to form a new conjecture. The second bracket can be omitted if its value is exactly α.

Our next result will focus on the first three statements. At a glance, the only obvious relationship among them is that Conjecture 3I implies a "sink-free version" of Conjecture 3II (i.e., assuming additionally that D is sink-free). It is shown in [9] that Conjecture 3I implies Conjecture 3III. Here we provide a clearer picture.

Proposition 4. *Conjecture 3II and Conjecture 3III are equivalent, and equivalent respectively to their sink-free version. Conjecture 3I implies Conjecture 3III, and Conjecture 3III(α) implies Conjecture 3I($\frac{\alpha}{1+\alpha}$).*

Since a general bound of the form $(1-\Theta(1))n$ is not known for Conjecture 1, Proposition 4 suggests that the large quasi-kernel conjecture is a safe but also effective target to work on: proving Conjecture 3III, for any α, would imply a breakthrough on the small quasi-kernel conjecture. It would be interesting to know whether Conjecture 3III is completely equivalent to Conjecture 3I. Of course, a negative answer would disprove Conjecture 1.

Question 5. Is Conjecture 3III equivalent to Conjecture 3I? That is, for all $0 < \alpha \leqslant 1/2$, if all digraphs on n vertices have a quasi-kernel Q with $|N^-[Q]| \geqslant \alpha n$, then all sink-free digraphs on n vertices have a quasi-kernel with size at most $(1-\alpha)n$.

Note that we are not trying to say that these conjectures are equivalent on the same digraph D. In fact, one can see that statements I and III cannot both be false on the same digraph.

Spiro proposed a stronger version of Conjecture 2 (Question 7.9 of [9]) which, unlike Conjecture 2, would be sharp, as witnessed by any disjoint union of Eulerian tournaments. We extended this to a variable version (Conjecture 3IV), and show that it is indeed equivalent to Conjecture 3III. Setting $\alpha = 1/2$, this implies that [9, Question 7.9] is equivalent to Conjecture 2, answering affirmatively a question in [9]. In addition, we observe that if the quasi-kernel requirement in Conjecture 3IV($\frac{1}{2}$) is relaxed to allow any independent set, it will be not only sharp but also true. This extends Lemma 4.2b of [9] to a tight result.

Proposition 6.

a. Conjecture 3III and Conjecture 3IV are equivalent, and equivalent respectively to their sink-free version.

b. Every digraph D contains an independent set I with $|I|/2 + |N^-(I)| \geqslant n/2$.

As noted before, there are not many classes of digraphs on which Conjecture 3I is known to hold for some α. We recall the notion of *kernel-perfect* digraphs, and introduce a digraph measure called the *kernel-perfect number*.

Definition 7. A digraph D is said to be *kernel-perfect* if every induced subdigraph of it has a kernel. In this paper, we conveniently call a vertex set $S \subseteq V(D)$ kernel-perfect if $D[S]$ is kernel-perfect. The *kernel-perfect number* of a digraph D, denoted by $kp(D)$, is the smallest k such that $V(D)$ can be partitioned into k kernel-perfect subsets.

We recall the related notions of *chromatic number* and *dichromatic number* of digraphs.

Definition 8. Let D be a digraph.

- The chromatic number $\chi(D)$ is the smallest k such that $V(D)$ can be partitioned into k subsets, each of which induces an independent set.
- The dichromatic number $\vec{\chi}(D)$ is the smallest k such that $V(D)$ can be partitioned into k subsets, each of which induces an acyclic set.

Note that $kp(D) \leqslant \vec{\chi}(D) \leqslant \chi(D)$, since independent sets are acyclic and acyclic sets are kernel-perfect. Also, $kp(D) \leqslant \lceil \chi(D)/2 \rceil$ because one can group the color classes of the underlying graph two by two, so that each group is odd-dicycle-free, hence kernel-perfect [8]. A main result of [7] suggests that Conjecture 1 holds on digraphs with kernel-perfect number at most 2, which include all digraphs with dichromatic number at most 2 or chromatic number at most 4 (hence all planar digraphs). We extend this to a variable version that applies to digraphs with any bounded kernel-perfect number, and additionally prove the large quasi-kernel analog.

Theorem 9. *Let D be a digraph and $k = \max(kp(D), 2)$. Then*

I. Conjecture 3I($\frac{1}{k}$) holds on D.

II. Conjecture 3II($\frac{1}{k}$) holds on D.

III. Conjecture 3III($\frac{1}{k}$) holds on D.

Corollary 10. *Let D be a digraph with $\vec{\chi}(D) \leqslant k$ or $\chi(D) \leqslant 2k$, where $k \geqslant 2$. Then Conjecture 3I($\frac{1}{k}$), Conjecture 3II($\frac{1}{k}$) and Conjecture 3III($\frac{1}{k}$) hold on D.*

Remark 11.

- When $kp(D) \leqslant 2$, Theorem 9I is [7, Theorem 2], except that Theorem 9I makes no claims when D has sinks. With more care, however, an analogous claim can be proven similarly.
- Theorem 9 is perhaps another indication that Question 5 probably has a positive answer.

It would be nice to know whether Conjecture 3IV($\frac{1}{k}$) also holds for D.

Question 12. Is it true that all digraphs D on n vertices have a quasi-kernel Q such that $|Q|/2 + |N^-(Q)| \geqslant n/\max(kp(D), 2)$?

A negative answer to Question 12 would disprove Conjectures 2 and 1 by Propositions 6 and 4.

2 Equivalent formulations

Now we prove Proposition 4. In the proof, the Roman numerals refer to the corresponding conjecture in Conjecture Scheme 3, and the "sf" suffix denotes the corresponding sink-free version.

Proof of Proposition 4. The directions I $\to$ IIsf $\to$ IIIsf $\to$ III are implicitly shown and used in [9, Proposition 2.7]; we prove them for completeness. In addition, we show that III $\to$ II $\to$ I($\frac{\alpha}{1+\alpha}$).

I $\to$ IIsf This is clear: assuming D is sink-free, one can simply apply Conjecture 3I on D, since we trivially have $s \leqslant n$.

IIsf $\to$ IIIsf Let D be a sink-free digraph on n vertices. Fix a large integer C, to be chosen later. Construct a digraph D' by keeping D and add, for each $v \in D$, C new vertices pointing an arc towards v. Note that all these Cn new vertices are sources but not sinks in D', so D' is still sink-free. Assuming Conjecture 3IIsf, we obtain a quasi-kernel Q' of D' with size at most $(C+1)n - \alpha Cn$. Note that $Q = Q' \cap V(D)$ is a quasi-kernel of D. Moreover, for each $v \notin N_D^-[Q]$, all the new vertices pointing to v must be included in Q'. Thus,

$$C(n - |N_D^-[Q]|) \leqslant |Q'| \leqslant (C+1)n - \alpha Cn.$$

It follows that $|N_D^-[Q]| \geqslant \alpha n - n/C$. We are done because C can be made arbitrarily large.

IIIsf → III Suppose D is a minimal counterexample for Conjecture 3III. Assuming Conjecture 3IIIsf, D must have a sink v. Let $X = V(D) \setminus N^-[v]$. Applying Conjecture 3III on the smaller digraph $D[X]$, we obtain a quasi-kernel Q_X of $D[X]$ with $|N^-_{D[X]}[Q_X]| \geqslant \alpha|X|$. Note that $Q = Q_X \cup \{v\}$ is a quasi-kernel of D with

$$|N^-_D[Q]| = |N^-_{D[X]}[Q_X] \sqcup N^-_D[v]| \geqslant \alpha|X| + (n - |X|) \geqslant \alpha n.$$

This is a contradiction.

III → II Let D be a digraph on n vertices. Let S denote the set of sources in D that are not sinks, and $X = V(D) \setminus S$. Without loss of generality, we can assume each vertex in S has exactly one out-neighbor (which must be in X), since removing an arc from S to X can only make it harder to find a small quasi-kernel. Let $s = |S|$ and $t = |X|$. Fix a large integer C, to be chosen later. For each $a \in X$, let

$$n_a = C|N^-(a) \cap S| + 1.$$

Construct a digraph B based on $D[X]$ by replacing each $a \in X$ with n_a copies of a (vertices in S are discarded). Such a construction is commonly known as a weighted blowup. Note that every maximal quasi-kernel of B naturally induces a maximal quasi-kernel of $D[X]$, and that $|V(B)| = Cs + t$. Let Q_B be a quasi-kernel of B that maximizes $|N^-[Q_B]|$, and Q_X be the induced maximal quasi-kernel of $D[X]$. Let X' denote the set of vertices in X whose copies in B are not in $N^-[Q_B]$ (these copies are either all in or all not in because Q_B is maximal). Note that

$$Q_X \cup \bigsqcup_{a \in X'} \big(N^-(a) \cap S\big)$$

is a quasi-kernel of D, and that its size is

$$\begin{aligned}
&|Q_X| + \sum_{a \in X'} \frac{n_a - 1}{C} \\
&= |Q_X| + \frac{1}{C}(|V(B) \setminus N^-[Q_B]| - |X'|) \\
&\leqslant |X| + \frac{1 - \alpha}{C}|V(B)| && \text{(assuming Conjecture 3III)} \\
&= \left(1 + \frac{1 - \alpha}{C}\right) t + (1 - \alpha)s.
\end{aligned}$$

Since C can be made arbitrarily large, there is a quasi-kernel of D with size at most $n - \alpha s$.

II → I($\frac{\alpha}{1+\alpha}$) Let D be a sink-free digraph. We claim that it has a quasi-kernel with size at most $n/(1+\alpha)$, where α is such that Conjecture 3II(α) holds. Let Q be a minimal quasi-kernel of D, $N = N^-(Q)$ and $M = N^{--}(Q) = V(D) \setminus (Q \cup N)$. Take a maximal directed matching from N to Q. Denote by Q_1 the set of vertices in Q it covers, and $Q_2 = Q \setminus Q_1$. By the maximality of the matching, every vertex in N has an out-neighbor in Q_1. Thus, by the minimality of Q, every vertex in Q_2 has no out-neighbor in N, so its out-neighbors are all in M. Note that Q_2 is a set of sources that are not sinks in $D[Q_2 \cup M]$. Let $r = |Q_1|$, $s = |Q_2|$, $p = |N| \geqslant r$, and $m = |M|$. Assuming Conjecture 3II, there is a quasi-kernel Q' of $D[Q_2 \cup M]$ with size at most $m + (1-\alpha)s$. Note that $Q' \cup (Q_1 \setminus N^-(Q'))$ is a quasi-kernel of D (see Figure 1 for an illustration), with size at most $r + m + (1-\alpha)s$. This quantity and $|Q| = r + s$ cannot be both greater than $n/(1+\alpha)$: otherwise,

$$\begin{aligned}
n &< (1+\alpha)(r + m + (1-\alpha)s) \\
&\leqslant (1+\alpha)(p + m + (1-\alpha)s) \\
&= (1-\alpha^2)(p+m) + (\alpha+\alpha^2)(n - (r+s)) + (1+\alpha)(1-\alpha)s \\
&< (1-\alpha^2)(p+m) + (\alpha+\alpha^2)(n - n/(1+\alpha)) + (1+\alpha)(1-\alpha)s \\
&\leqslant n - (1-\alpha^2)r \\
&\leqslant n,
\end{aligned}$$

a contradiction. Hence D has a quasi-kernel with size at most $n/(1+\alpha)$. □

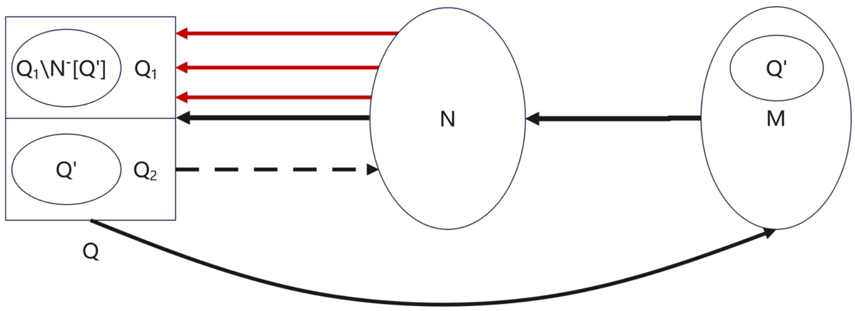

Figure 1: $Q' \cup (Q_1 \setminus N^-(Q'))$

Reusing a few of the earlier techniques, we next prove Proposition 6.

Proof of Proposition 6. For part (a), because IV $\to$ IVsf $\to$ IIIsf is trivial, and we showed in Proposition 4 that IIIsf $\to$ III, it suffices to show III $\to$ IV.

III $\to$ IV Let α be such that Conjecture 3III holds. Consider

$$B = \{\beta \in [0,1] \mid \forall \text{ digraph } D,\ \exists \text{ quasi-kernel } Q : \beta|Q| + |N^-(Q)| \geqslant \alpha|V(D)|\},$$

and observe that B is a closed interval containing 1 as when we fix a digraph D, we can get a closed interval containing 1. We are done if $1/2 \in B$, so let $b = \min(B)$; we claim that $(b+1)/3 \in B$, suggesting that $b \leqslant 1/2$ (and hence $1/2 \in B$).

Let C_3 denote the directed triangle. For any digraph D, denote by D' the C_3-*blowup* of D, which is the digraph obtained by replacing each vertex in D by a copy of C_3, so that the arcs between different copies are as induced by D. For $v \in D$, denote by $f(v)$

the set of vertices that take the place of v in D' (so $D'[f(v)] \simeq C_3$). Since $b \in B$, there is a quasi-kernel Q' of D' such that $b|Q'| + |N^-(Q')| \geqslant \alpha|V(D')|$. Its projection onto D,

$$Q = \{v \in D : f(v) \cap Q' \neq \emptyset\},$$

is a quasi-kernel of D. For all $v \in Q$, let $g(v) = f(v) \cap Q'$, containing exactly one vertex of $D'[f(v)]$. Note that

$$\begin{aligned} Q' &= \bigcup_{v \in Q} g(v), \\ N^-(Q') &= \bigcup_{v \in Q} N^-_{D'[f(v)]}(g(v)) \quad \cup \bigcup_{v \in N^-(Q)} f(v), \end{aligned}$$

where all the unions are disjoint. Thus,

$$\begin{aligned} \alpha|V(D')| &\leqslant b|Q'| + |N^-(Q')| \\ &= b|Q| + |Q| + 3|N^-(Q)| \\ &= \frac{V(D')}{V(D)} \left(\frac{b+1}{3}|Q| + |N^-(Q)| \right). \end{aligned}$$

Since D is arbitrary, this shows $(b+1)/3 \in B$. Hence $(b+1)/3 \geqslant b$ so we are done.

Part (b) We show that every digraph D contains an independent set I with $|I|/2 + |N^-(I)| \geqslant n/2$. The proof in [9, Lemma 4.2b] can be adapted for this. Here we simplify it slightly.

Suppose D is a minimal counterexample to the statement, which must be nonempty. By the handshaking dilemma, there is $v \in D$ such that $|N^-(v)| \geqslant |N^+(v)|$. By the minimality of D, the smaller, possibly empty digraph $D' = D[V(D) \setminus (N^+[v] \cup N^-(v))]$ contains a maximal independent set I' with $|I'|/2 + |N^-_{D'}(I')| \geqslant |D'|/2 = (n - |N^+[v] \cup N^-(v)|)/2$. Note that $I = I' \cup \{v\}$ is a maximal independent set in D and $N^-(I) \supseteq N^-_{D'}(I') \cup N^-(v)$, so

$$\begin{aligned} \frac{|I|}{2} + |N^-(I)| &\geqslant \frac{|I'|+1}{2} + |N^-_{D'}(I')| + |N^-(v)| \\ &\geqslant \frac{n - |N^+[v] \cup N^-(v)|}{2} + \frac{1}{2} + |N^-(v)| \\ &\geqslant \frac{n}{2}. \end{aligned}$$

Thus, I satisfies the requirement. This is a contradiction. □

3 Digraphs with bounded kernel-perfect number

In this section, we prove Theorem 9. We start with two convenient lemmas.

Lemma 13. *Let D be a digraph. For any maximal kernel-perfect set X, $N^-[X] = V(D)$.*

Proof. Suppose not, let $v \in V(D) \backslash N^-[X]$. We claim that $X \cup \{v\}$ is also kernel-perfect, which would contradict the maximality of X. Pick an arbitrary subset $V' \subseteq X \cup \{v\}$, in which we will look for a kernel. We consider two cases. If $v \notin V'$, since $V' \subseteq X$ and X is kernel-perfect, there exists a kernel K of V'. If $v \in V'$, let $V'' = V' \setminus N^-(v)$. Since X is kernel-perfect and $V'' \subseteq X$, there exists a kernel K of V''. Note that v has no outgoing edges to X, so it is disjoint from V''. It follows that $K \cup \{v\}$ is a kernel of V'.

As V' always contains a kernel, $X \cup \{v\}$ is kernel-perfect, contradicting the maximality of X. □

Lemma 13 implies the following lemma. As singleton sets and acyclic sets are kernel-perfect, Lemma 14 directly implies [9, Lemma 3.1] and [9, Lemma 4.4].

Lemma 14. *Let D be a digraph with a kernel-perfect set $P \subseteq V(D)$. Then D has a quasi-kernel Q such that $P \subseteq N^-[Q]$ and $Q \cap N^-(P) = \emptyset$.*

Proof. Let P' be a maximal kernel-perfect set containing P in $D[V(D) \setminus N^-(P)]$. Note that $N^-(P') \supseteq N^-(P)$. By Lemma 13, we also have that $N^-[P'] \supseteq V(D) \setminus N^-(P)$. Thus, $N^-[P] = V(D)$. Take an arbitrary kernel Q of P'. It is a quasi-kernel of D and satisfies $P' \subseteq N^-[Q]$ and $Q \cap N^-(P') = \emptyset$. Since $N^-(P') \supseteq N^-(P)$, we also have $P \subseteq N^-[Q]$ and $Q \cap N^-(P) = \emptyset$.

□

Somewhat curiously, we need only the first property of Q to prove Theorem 9III and Theorem 9II, and only the second property to prove Theorem 9I, not using the fact that they can be satisfied at the same time.

Proof of Theorem 9III. This is clear: Suppose $V(D) = V_1 \cup \cdots \cup V_k$ is the kernel-perfect partition. Apply Lemma 14 on the largest part to get a quasi-kernel Q with $|N^-[Q]| \geqslant |V(D)|/k$. □

Proof of Theorem 9II. We note that the proof of "III → II" in Proposition 4 can also show that Theorem 9III implies Theorem 9II. Say we want to show that Conjecture 3II($\frac{1}{k}$) holds on the digraph D. To use that reduction argument, we just need to be able to apply Conjecture 3III($\frac{1}{k}$) on a digraph B, which is a weighted blowup of some induced subdigraph of D ($D[A]$). Thus, $kp(B) = kp(D[A]) \leqslant kp(D)$, so if D has a small kernel-perfect number, so does B. Hence Theorem 9III can be correctly invoked. □

Actually, the proof of "II → I($\frac{\alpha}{1+\alpha}$)" in Proposition 4 would also follow through with the kernel-perfect number condition, concluding a weaker bound than we need for Theorem 9I. To get the precise bound, we use a standalone proof inspired by [7, Theorem 2].

Proof of Theorem 9I. Recall that $k = \max\{kp(D), 2\}$ and, in particular, since $k \geqslant kp(D)$, we assume that $V(D) = V_1 \cup \cdots \cup V_k$ is a partition (empty sets allowed) such that each V_i is kernel-perfect. Without loss of generality, assume that V_1 is maximally kernel-perfect: in particular, every $v \notin V_1$ has an out-neighbor in V_1. In other words, $N^-[V_1] = V(D)$. Let K be a kernel of $D[V_1]$, and K_0 be a minimal subset of K such that $N^-(K_0) = N^-(K)$.

Let $V_1' = N^-(K) \cup K_0$ and observe that $|K_0| \leqslant |N^-(K)|$: for all $v \in K_0$, because K_0 is minimal, there must be some $u = u(v) \in N^-(K)$ whose only out-neighbor in K_0 is v. For all $i \in \{2, \ldots, k\}$, let $V_i' = V_i \setminus N^-(K)$. Then let $V_0' = K \setminus K_0$. Note that $\{V_i'\}_{i=0,\ldots,k}$ is a partition of $V(D)$. Let

$$W = V(D) \setminus V_1' = V_0' \cup V_2' \cup \cdots \cup V_k'.$$

If for some $i \in \{2, \ldots, k\}$, $|N^-(V_i') \cap V_0'| \geqslant |W|/k$, then since V_i' is a kernel-perfect set in $D[W]$, by Lemma 14 there is a quasi-kernel Q of $D[W]$ that is disjoint with $N^-_{D[W]}(V_i')$. Observe that $Q \cup (K_0 \setminus N^-(Q))$ is a quasi-kernel of D, and

$$\begin{aligned} |Q \cup (K_0 \setminus N^-(Q))| &\leqslant |W \setminus N^-_{D[W]}(V_i')| + |K_0| \\ &\leqslant |W| - |W|/k + |V_1'|/2 \\ &\leqslant (k-1)n/k. \end{aligned}$$

Otherwise, since D is sink-free and $N^-[V_1] = V(D)$, $(N^-(W) \cap V_0') \cup K_0$ is a quasi-kernel of D, and

$$\begin{aligned} |(N^-(W) \cap V_0') \cup K_0| &\leqslant \sum_{i=2}^{k} |N^-(V_i') \cap V_0'| + |K_0| \\ &\leqslant (k-1)|W|/k + |V_1'|/2 \\ &\leqslant (k-1)n/k. \end{aligned}$$

Either way, we find a quasi-kernel of D with size at most $(k-1)n/k$. □

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