Covering Numbers of Some Irreducible Characters of the Symmetric Group

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Abstract

The covering number of a non-linear character χ of a finite group G is the least positive integer k such that every irreducible character of G occurs in χ^k. We determine the covering numbers of irreducible characters of the symmetric group S_n indexed by certain two-row partitions (and their conjugates), namely $(n-2,2)$, and $((n+1)/2,(n-1)/2)$ when n is odd. We also determine the covering numbers of irreducible characters indexed by certain hook-partitions (and their conjugates), namely $(n-2,1^2)$, the almost self-conjugate hooks $(n/2+1,1^{n/2-1})$ when n is even, and the self-conjugate hooks $((n+1)/2,1^{(n-1)/2})$ when n is odd.

Mathematics Subject Classifications: 20B30, 20D06, 20C30, 05E05

1 Introduction

Arad, Chillag, and Herzog (see [1]) introduced the notion of a covering number of a character of a finite group analogous to the notion of a covering number of a conjugacy class of a group (see [2]). Let G be a finite group, and $\mathrm{Irr}(G)$ denote the set of all irreducible characters of G. Let $\mathrm{Irr}(G)^+ := \{\chi \in \mathrm{Irr}(G) \mid \chi(1) > 1\}$. For characters χ and ρ of G, recall that the product $\chi\rho$ is the character of G for the internal tensor product of the respective representations of G that χ and ρ afford. Let $c(\chi)$ denote the set of all the irreducible constituents of a character χ of G. The covering number of χ, denoted by $\mathrm{ccn}(\chi; G)$, is the least positive integer k (if it exists) such that $c(\chi^k) = \mathrm{Irr}(G)$. Suppose that $\mathrm{ccn}(\chi; G)$ exists for all $\chi \in \mathrm{Irr}(G)^+$. Then, the character-covering-number of G, denoted by $\mathrm{ccn}(G)$, is the least positive integer m such that $c(\chi^m) = \mathrm{Irr}(G)$ for all $\chi \in \mathrm{Irr}(G)^+$. In other words, $\mathrm{ccn}(G) := \max\{\mathrm{ccn}(\chi; G) \mid \chi \in \mathrm{Irr}(G)^+\}$.

In [1, Theorem 1], it was proved that if G is a finite non-abelian simple group, then $\mathrm{ccn}(\chi; G)$ exists for all $\chi \in \mathrm{Irr}(G)^+$. Consequently, this implies that $\mathrm{ccn}(G)$ also exists. Among various results, the authors provided bounds for $\mathrm{ccn}(G)$ in terms of the number of

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conjugacy classes of G, assuming again that G is a finite non-abelian simple group. Zisser (see [35]) proved that $\mathrm{ccn}(A_5) = 3$ and $\mathrm{ccn}(A_n) = n - \lceil \sqrt{n}\, \rceil$ when $n \geqslant 6$. Very recently, Miller (see [20]) has shown that $\mathrm{ccn}(S_n)$ exists when $n \geqslant 5$ and that $\mathrm{ccn}(S_n) = n-1$. The irreducible characters of S_n are parameterized by partitions of n. For a partition $\lambda \vdash n$, let χ_λ denote the irreducible character of S_n indexed by λ. It is not difficult to see that $\mathrm{ccn}(\chi_{(n-1,1)}; S_n) = \mathrm{ccn}(\chi_{(2,1^{n-2})}; S_n) = n - 1$. It is natural to ask whether there are other irreducible characters of S_n whose covering number is $n-1$. The first result of this article shows that this is not possible.

Theorem 1. *Let $n \geqslant 5$ and λ be a partition of n. Assume that $\lambda \notin \{(n), (1^n), (n-1, 1), (2, 1^{n-2})\}$. Then $\mathrm{ccn}(\chi_\lambda; S_n) \leqslant \left\lceil \frac{2(n-1)}{3} \right\rceil$. Moreover,*

$$\mathrm{ccn}(\chi_{(n-2,2)}; S_n) = \mathrm{ccn}(\chi_{(2^2,1^{n-4})}; S_n) = \left\lceil \frac{2(n-1)}{3} \right\rceil.$$

We determine the covering number of some other irreducible characters of S_n. We have the following two theorems:

Theorem 2. *Let $n \geqslant 5$ be odd and $k = \frac{n+1}{2}$. Then $\mathrm{ccn}(\chi_{(k,k-1)}; S_n) = \mathrm{ccn}(\chi_{(2^{k-1},1)}; S_n) = \lceil \log_2 n \rceil$.*

Theorem 3. *Let $n \geqslant 5$.*

1. $\mathrm{ccn}(\chi_{(n-2,1^2)}; S_n) = \mathrm{ccn}(\chi_{(3,1^{n-3})}; S_n) = \lfloor \frac{n}{2} \rfloor$.
2. *Assume $n \geqslant 7$ and $n \neq 8$. Let $\lambda = (\frac{n+1}{2}, 1^{\frac{n-1}{2}})$ when n is odd and $\lambda = (\frac{n}{2} + 1, 1^{\frac{n}{2}-1})$ or $(\frac{n}{2}, 1^{\frac{n}{2}})$ when n is even. Then $\mathrm{ccn}(\chi_\lambda; S_n) = \lceil \log_2 \lfloor \sqrt{n} \rfloor \rceil + 1$. Moreover, $\mathrm{ccn}(\chi_{(5,1^3)}; S_8) = \mathrm{ccn}(\chi_{(4,1^4)}; S_8) = 3$.*

The above theorem yields the covering numbers of the irreducible constituents of $\mathrm{Res}^{S_n}_{A_n} \chi_\lambda$ when $\lambda = (\frac{n+1}{2}, 1^{\frac{n-1}{2}})$ is the self-conjugate hook.

Theorem 4. *Assume $n \geqslant 5$. Let $\lambda = (\frac{n+1}{2}, 1^{\frac{n-1}{2}})$ when n is odd and $\lambda = (\frac{n}{2} + 1, 1^{\frac{n}{2}-1})$ when n is even. Let χ be $\mathrm{Res}^{S_n}_{A_n} \chi_\lambda$ when n is even and one of χ^+_λ or χ^-_λ (where $\mathrm{Res}^{S_n}_{A_n} \chi_\lambda = \chi^+_\lambda + \chi^-_\lambda$) when n is odd. Then, $\mathrm{ccn}(\chi; A_n) = \lceil \log_2 \lfloor \sqrt{n} \rfloor \rceil + 1$ when $n \geqslant 9$, $\mathrm{ccn}(\chi; A_n) = 3$ when $n = 5, 7$, and $\mathrm{ccn}(\chi; A_n) = 2$ when $n = 6, 8$.*

Surprisingly, from the last two theorems, we get an irreducible character of S_n whose covering number coincides with that of its restriction to A_n. As far as we know, this is the first instance of such a property. Based on Theorem 1, Theorem 2, and observations made using SageMath ([33]), we make a conjecture on the covering number of χ_λ when λ is a two-row partition, i.e., $\lambda = (n - k, k)$, where $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$ (see Conjecture 36).

It is natural to seek irreducible characters of finite groups with small covering numbers. In this direction, a well-known conjecture of Jan Saxl states that if $\lambda \vdash n$ is the staircase partition, that is, $\lambda = (m, m - 1, \ldots, 1)$ (thus $n = \frac{m(m+1)}{2}$) where $m \geqslant 3$, then every irreducible character appears as a constituent of χ_λ^2. In other words, the Saxl conjecture states that $\mathrm{ccn}(\chi_\lambda; S_n) = 2$, when λ is the staircase partition. There have been some interesting advances in this direction (see, for example, [15, 18, 25] and references therein), and recently it has been proved by N. Harman and C. Ryba that $c(\chi_\lambda^3) = \mathrm{Irr}(S_n)$ (i.e., $\mathrm{ccn}(\chi_\lambda; S_n) \leqslant 3$) (see [13]). It is also worthwhile to mention that if G is a finite simple group of Lie type and St denotes the Steinberg character of G, then every irreducible character of G appears as a constituent of St^2, unless $G = \mathrm{PSU}_n(q)$ and $n \geqslant 3$ is co-prime to $2(q+1)$ (see [14]). The character covering number of $\mathrm{PSL}_2(q)$ has been determined in [3] recently.

The article is organized as follows: In Sect. 2, to make the article self-contained, we discuss some basic results in the ordinary representation theory of S_n. In Sect. 3, we discuss some basic properties related to powers of characters of finite groups and the Kronecker product problem, which naturally appears in this scenario. In Sect. 4, we prove Theorem 1 and Theorem 2. Theorem 35 of this section is an interesting result as well. In Sect. 5 and Sect. 6, we prove Theorem 3 and Theorem 4, respectively.

2 Basic results on ordinary representations of S_n

In this section, we briefly discuss some basic results in the ordinary representation theory of symmetric groups. We begin with some definitions and notations. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, where $\lambda_1 \geqslant \cdots \geqslant \lambda_l$ and $\lambda_i \in \mathbb{N}$. We say λ is a *partition* of $|\lambda| = n$ (written $\lambda \vdash |\lambda| = n$), where $|\lambda| = \sum_i \lambda_i = n$ is the sum of its parts. The number of parts of λ is denoted by $l(\lambda)$. Alternatively, for a partition λ of n, we write $\lambda = \langle 1^{m_1}, \cdots, i^{m_i}, \cdots \rangle$, where m_i is the number of times i occurs as a part in λ. The conjugacy classes of S_n are parameterized by partitions of n in the following sense: for $\pi \in S_n$, let $m(\pi) = \langle 1^{m_1}, 2^{m_2}, \cdots \rangle$ denote the cycle-type of π. Here, m_i denotes the number of i-cycles in the disjoint cycle decomposition of π, whence we have $\sum_i i m_i = n$. Thus, $m(\pi)$ yields a partition of n. Suppose $\sigma, \tau \in S_n$. Then σ and τ are conjugate in S_n if and only if $m(\sigma) = m(\tau)$. This yields the desired parametrization. For $\mu = (\mu_1, \ldots, \mu_t) \vdash n$, let $w_\mu := (1, 2, \ldots, \mu_1)(\mu_1+1, \ldots, \mu_1+\mu_2) \cdots (\mu_1+\cdots+\mu_{t-1}+1, \ldots, \mu_1+\cdots+\mu_t)$ denote the standard representative of the conjugacy class of S_n parameterized by μ.

For $\lambda \vdash n$, let $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_l}$ be a Young subgroup of S_n. We write $\mathbb{1}_G$ for the trivial character of a finite group G (or just $\mathbb{1}$ when the group G is clear from the context). Let σ_λ be the character of the partition representation of S_n indexed by λ (see [26, Section 2.3]). In other words, $\sigma_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n} \mathbb{1}$. Let χ_λ denote the irreducible character of S_n indexed by λ. It is well known that $\chi_{\lambda'} = \epsilon \otimes \chi_\lambda$, where ϵ $(= \chi_{(1^n)})$ is the sign character of S_n and λ' is the conjugate partition of λ. Let T_λ denote the Young diagram of λ. Let (i, j) denote the box of T_λ that lies in its i-th row and j-th column. Given partitions $\lambda, \mu \vdash n$, a semi-standard Young tableaux (abbrv. SSYT) of shape λ and type μ is a positive integer filling of the Young diagram T_λ such that (a) i occurs μ_i times, (b) the entries increase weakly (from left to right) along each row, and (c) the entries increase strictly down each column. The following is an example of a SSYT of shape $(4, 3, 2, 2)$

and type $(3, 3, 3, 1, 1)$.

1	1	1	3
2	2	2	
3	3		
4	5		

Figure 1: A SSYT of shape $(4, 3, 2, 2)$ and type $(3, 3, 3, 1, 1)$.

The number of SSYT of shape λ and type μ is denoted by $K_{\lambda\mu}$, and these are called *Kostka numbers*. For our example above, $K_{\lambda\mu} = 2$.

Theorem 5 (Young's rule)**.** *[26, Theorem 3.3.1] For a partition $\mu \vdash n$, we have* $\sigma_\mu = \sum_{\lambda \vdash n} K_{\lambda\mu}\chi_\lambda$.

It is well known that $K_{\lambda\mu} > 0$ if and only if $\mu \trianglelefteq \lambda$, where $\trianglelefteq$ denotes the dominance order on the set of partitions of n (see [26, Lemma 3.1.12]).

2.1 Symmetric functions

In this subsection, we recall some basics of the theory of symmetric functions and its relation to the character theory of S_n. We follow the exposition in [26, Chapter 5] (see also [19, Chapter 1] and [31, Chapter 7]). Let Λ denote the algebra of symmetric functions over the field $\mathbb{Q}$, and Λ_n denotes the vector subspace of Λ of all homogeneous symmetric functions of degree n. The dimension of Λ_n is the number of partitions of n, which we denote by $p(n)$. Let $\lambda \vdash n$. There are five fundamental bases of Λ_n: (1) the basis of monomial symmetric functions m_λ, (2) the basis of elementary symmetric functions e_λ, (3) the basis of complete symmetric functions h_λ, (4) the basis of power symmetric functions p_λ, and (5) the basis of Schur functions s_λ.

The Frobenius characteristic function denoted by ch_n relates the vector space of class functions $R(S_n)$ of S_n to Λ_n. It is defined by

$$\mathrm{ch}_n(f) = \frac{1}{n!}\sum_{\pi \in S_n} f(\pi) p_{m(\pi)}.$$

Note that the above is a $\mathbb{Q}$-linear map from $R(S_n)$ to Λ_n. The algebra Λ is equipped with the Hall inner-product. It is defined by imposing that the basis of Schur functions $\{s_\lambda \mid \lambda \vdash n\}$ form an orthonormal basis of Λ_n, that is, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. It is then extended bilinearly to Λ by imposing that if f and g are homogeneous, then $\langle f, g \rangle \neq 0$ only if f and g have the same degree. Recall that $R(S_n)$ is also equipped with an inner-product which is defined by $\langle f, g \rangle = \frac{1}{n!}\sum_{\pi \in S_n} f(\pi)g(\pi^{-1})$, where $f, g \in R(S_n)$. With these forms in $R(S_n)$ and Λ_n, ch_n is an isometry, that is, for all $f, g \in R(S_n)$, $\langle \mathrm{ch}_n(f), \mathrm{ch}_n(g) \rangle = \langle f, g \rangle$. The following result gives the desired connection.

Proposition 6. *For partition $\lambda \vdash n$, we have* $\mathrm{ch}_n(\chi_\lambda) = s_\lambda$, $\mathrm{ch}_n(\sigma_\lambda) = h_\lambda$, *and* $\mathrm{ch}_n(\epsilon\sigma_\lambda) = e_\lambda$.

For partitions $\lambda, \mu \vdash n$, we recall the Murnaghan–Nakayama rule, which states that $\chi_\lambda(w_\mu) = \langle p_\mu, s_\lambda \rangle$. In other words, $p_\mu = \sum_{\lambda \vdash n} \chi_\lambda(w_\mu) s_\lambda$. Let $\mu \vdash m, \nu \vdash n$. It is a well-known result that $\mathrm{ch}_{m+n}(\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \chi_\lambda \otimes \chi_\mu) = s_\mu s_\nu$, where $\chi_\lambda \otimes \chi_\mu$ is the external tensor product of χ_λ and χ_μ. Notice that on the RHS, the product is the usual product of symmetric functions. Thus, for a partition $\lambda \vdash m+n$, we get

$$\langle s_\mu s_\nu, s_\lambda \rangle = \langle \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \chi_\mu \otimes \chi_\nu, \chi_\lambda \rangle = \langle \mathrm{Res}_{S_m \times S_n}^{S_{m+n}} \chi_\lambda, \chi_\mu \otimes \chi_\nu \rangle = c_{\mu\nu}^\lambda. \tag{1}$$

The coefficients $c_{\mu\nu}^\lambda$ are called the Littlewood–Richardson coefficients (abbrv. LR coefficients). As a consequence of Equation (1), if $f \in \Lambda_m$ and $g \in \Lambda_n$, then

$$fg = \mathrm{ch}_{m+n}(\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \chi_f \otimes \chi_g), \tag{2}$$

where $\chi_f = \mathrm{ch}_m^{-1}(f) \in R(S_m)$, $\chi_g = \mathrm{ch}_n^{-1}(g) \in R(S_n)$, and $\chi_f \otimes \chi_g \in R(S_m \times S_n)$, which is defined by $(\chi_f \otimes \chi_g)(\alpha, \beta) := \chi_f(\alpha)\chi_g(\beta)$ for all $\alpha \in S_m, \beta \in S_n$.

2.2 Littlewood–Richardson rule

We finish this section with the statement of the Littlewood–Richardson rule (abbrv. LR rule), which gives a combinatorial interpretation of the LR coefficients. This will be needed for our computations later. Let $\lambda \vdash n$. A lattice permutation of shape λ is a sequence of positive integers $a_1 a_2 \cdots a_n$, where i occurs λ_i times, and in any left factor $a_1 a_2 \cdots a_j$, the number of i's is at least the number of $(i+1)$'s (for all i). For example, a lattice permutation of shape $(3, 2, 1)$ is 121321. A reading word of a Young tableaux T is the sequence of entries of T obtained by concatenating the rows of T from bottom to top. For example, the reading word of the SSYT in Figure 1 is 45332221113. For partitions λ, μ, we say $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for all i (i.e., $T_\mu \subseteq T_\lambda$). Let $T_{\lambda/\mu}$ denote the Young diagram of the (skew) shape λ/μ. A SSYT of (skew) shape λ/μ and type ν is a filling of $T_{\lambda/\mu}$ with positive integer entries such that i occurs ν_i times, and entries along each row from left to right (resp. along each column from top to bottom) are weakly increasing (resp. strictly increasing). Figure 2 is an example of a SSYT of skew shape $(6, 5, 3, 3)/(3, 1, 1)$ and type $(4, 3, 2, 2, 1)$.

			1	1	5
	1	1	2	3	
	2	2			
3	4	4			

Figure 2: A SSYT of skew shape $(6, 5, 3, 3)/(3, 3, 1)$ and type $(4, 3, 2, 2, 1)$.

We are now ready to state the Littlewood–Richardson rule.

Theorem 7 (Littlewood–Richardson rule). *[31, Theorem A.1.3.3] For partitions $\mu \vdash n$, $\nu \vdash m$, and $\lambda \vdash m+n$, the LR coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of* SSYT *of shape λ/μ and type ν whose reverse reading word is a lattice permutation.*

A SSYT satisfying the condition of the above theorem is called a Littlewood–Richardson tableaux (abbrv. LR tableaux). For the tableaux in Figure 2, the reverse reading word (traversing top to bottom and reading right to left along a row) is 511321122443, which is not a lattice permutation. Hence, it is not a LR tableaux. We also mention two special cases of the above rule for convenience. A skew shape is called a horizontal strip of size n if there are n boxes, and each non-empty column has a single box. For example, if $\lambda = (4,2,1)$ and $\mu = (2,1)$, then λ/μ is a horizontal strip of size 4. A skew shape is called a vertical strip of size n if there are n boxes, and each non-empty row has a single box. For example, if $\lambda = (3,3,3,2)$ and $\mu = (2,2,2,1)$, then λ/μ is a vertical strip of size 4 (see the figures below).

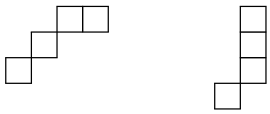


Figure 3: A horizontal strip and a vertical strip of size 4, respectively.

Corollary 8 (Pieri rule). *Let $\lambda \vdash m+n$ and $\mu \vdash m$. Then*

$$c^{\lambda}_{\mu,(n)} = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is a horizontal strip of size } n, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 9. *Let $\lambda \vdash m+n$ and $\mu \vdash m$. Then*

$$c^{\lambda}_{\mu,(1^n)} = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is a vertical strip of size } n, \\ 0 & \text{otherwise.} \end{cases}$$

We also note the branching rule, which is nothing but the Pieri rule when $n = 1$.

Theorem 10 (Branching rule). *Let $\lambda \vdash n$. Then $\mathrm{Res}^{S_n}_{S_{n-1}} \chi_\lambda = \sum_{\mu \in \lambda^-} \chi_\mu$, where λ^- is the set of all partitions of $n-1$ whose Young diagram is obtained by deleting a box from T_λ.*

By Frobenius reciprocity, it follows that $\mathrm{Ind}^{S_{n+1}}_{S_n} \chi_\lambda = \sum_{\mu \in \lambda^+} \chi_\mu$, where λ^+ is the set of all partitions of $n+1$ whose Young diagram is obtained by adding a box to T_λ.

3 Character covering in symmetric groups and Kronecker product

In this section, we discuss the main results known on the covering numbers of irreducible characters of the symmetric group. The well-known Kronecker product problem is naturally related. We discuss some results in this direction, which we will use in this article.

3.1 The Kronecker Product

Let $\times$ denote the point-wise product of two class functions in $R(S_n)$; that is, if $\varphi, \psi \in R(S_n)$, then $(\varphi \times \psi)(x) = \varphi(x)\psi(x)$ for all $x \in S_n$. We write $\varphi\psi$ to mean $\varphi \times \psi$. For $f, g \in \Lambda_n$, the Kronecker product of f and g, denoted by $f * g$, is defined by

$$f * g := \mathrm{ch}_n(\mathrm{ch}_n^{-1} f \, \mathrm{ch}_n^{-1} g). \tag{3}$$

By definition, $*$ is commutative. Further, $f * h_n = h_n * f = f$ for all $f \in \Lambda_n$. For partitions $\mu, \nu \vdash n$, from the above definition, we observe that $s_\mu * s_\nu = \mathrm{ch}_n(\chi_\mu \chi_\nu)$. Thus, if $\chi_\mu\chi_\nu = \sum_{\lambda \vdash n} g_{\mu\nu\lambda}\chi_\lambda$, then we have $s_\mu * s_\nu = \sum_{\lambda \vdash n} g_{\mu\nu\lambda} s_\lambda$. The coefficients $g_{\mu\nu\lambda} = \langle \chi_\mu\chi_\nu, \chi_\lambda \rangle = \langle s_\mu * s_\nu, s_\lambda \rangle$ are called Kronecker coefficients. As the irreducible characters of S_n are integer-valued, it follows that $g_{\mu\nu\lambda}$ is invariant under any permutation of μ, ν, λ. The Kronecker product problem asks for a combinatorial interpretation of the Kronecker coefficients and is regarded as one of the most important open problems in the theory of symmetric groups. There are many interesting results known in this direction (for example, see [4, 5, 6, 7, 8, 9, 10, 11, 12, 17, 21, 22, 23, 24, 27, 28, 29, 30, 32, 34]). [12, Section 3] discusses some elementary properties of the Kronecker product. It is convenient to extend the definition of Kronecker product to Λ by setting $f * g = 0$ if $f \in \Lambda_m, g \in \Lambda_n$, and $m \neq n$. Now we mention some results on the Kronecker product, which are required in this paper.

It is easy to describe the decomposition of the internal tensor product of two permutation (partition) characters σ_λ and σ_μ of S_n.

Definition 11. Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ and $\mu = (\mu_1, \ldots, \mu_m) \vdash n$. A $\lambda \times \mu$ matrix is an $l \times m$ matrix with non-negative integer entries such that the sum of the entries of the i-th row equals λ_i for all $1 \leqslant i \leqslant l$, and the sum of the entries of the j-th column equals μ_j for all $1 \leqslant j \leqslant m$.

Theorem 12. *[31, Exercise 7.84(b)] Let $\lambda, \mu \vdash n$. Then $h_\lambda * h_\mu = \sum_{A=(a_{ij})} \prod_{i,j} h_{a_{ij}}$ summed over all $\lambda \times \mu$ matrices A.*

Applying the inverse Frobenius characteristic function, we obtain

$$\sigma_\lambda \sigma_\mu = \sum_{A=(a_{ij})} \prod_{i,j} \sigma_{a_{ij}}, \tag{4}$$

where the product on the RHS is taken in the sense of Equation (2), that is, $\prod_{i=1}^{t} \sigma_{\alpha_i} = \mathrm{Ind}_{S_{\alpha_i} \times \cdots \times S_{\alpha_t}}^{S_{\alpha_1 + \cdots + \alpha_t}} \mathbb{1}$, where $\alpha_i \in \mathbb{N}_0$ for all $1 \leqslant i \leqslant t$. A direct representation theoretic proof of Equation (4) can be found in [16, Theorem 10]. The following basic result of Littlewood ([17, Theorem 3]) allows one to compute the Kronecker products in principle.

Theorem 13. *[12, Proposition 3.2] Let $f_1, \ldots, f_k$ be homogeneous symmetric functions of degree $a_1, \ldots, a_k$, respectively. Let A be any symmetric function of degree $a_1+a_2+\cdots+a_k$. Then*

$$(f_1 f_2 \cdots f_k) * A = \sum_{\alpha^{(1)} \vdash a_1} \cdots \sum_{\alpha^{(k)} \vdash a_k} \langle s_{\alpha^{(1)}} \cdots s_{\alpha^{(k)}}, A \rangle (f_1 * s_{\alpha^{(1)}}) \cdots (f_k * s_{\alpha^{(k)}}).$$

In particular,

$$(h_{a_1} h_{a_2} \cdots h_{a_k}) * A = \sum_{\alpha^{(1)} \vdash a_1} \cdots \sum_{\alpha^{(k)} \vdash a_k} \langle s_{\alpha^{(1)}} \cdots s_{\alpha^{(k)}}, A \rangle s_{\alpha^{(1)}} \cdots s_{\alpha^{(k)}}.$$

Example 14. Let $\lambda \vdash n$ and consider the product $s_\lambda * s_{(n-1,1)}$. Using the previous result, $h_{n-1}h_1 * s_\lambda = \sum_{\mu \vdash n-1} \sum_{\nu \vdash 1} \langle s_\mu s_\nu, s_\lambda \rangle s_\mu s_\nu = \sum_{\mu \vdash n-1} \langle s_\mu s_1, s_\lambda \rangle s_\mu s_1$. Through a double application of the Pieri rule, we conclude that the above sum equals $\sum_{\mu \in \lambda^-} s_\mu s_1 = \sum_{\nu \in \lambda^\pm} s_\nu$. Here, λ^- denotes the set of partitions of $n-1$ that are obtained by removing a box from T_λ, and $\lambda^\pm$ denotes the multiset of all partitions of n that are obtained by successive removal and addition of a box to T_λ. Since $h_{(n-1,1)} = s_n + s_{(n-1,1)}$, we obtain that $s_\lambda * s_{(n-1,1)} = \sum_{\mu \in \lambda^\pm} s_\mu - s_\lambda$. Thus, $\chi_\lambda \chi_{(n-1,1)} = \sum_{\mu \in \lambda^\pm} \chi_\mu - \chi_\lambda$.

3.2 Character covering number

Some basic facts on the products of characters can be found in [1] and [20]. We need the following lemmas, whose proofs are easy and hence omitted.

Lemma 15. *Let G be a finite group, and χ_i and ρ_i ($i = 1, 2$) be characters of G. Assume that $c(\chi_i) \subseteq c(\rho_i)$ for $i = 1, 2$. Then $c(\chi_1\chi_2) \subseteq c(\rho_1\rho_2)$.*

Lemma 16. *Let G be a finite group, and χ and ρ be characters of G.*

1. *If $c(\chi) = \mathrm{Irr}(G)$, then $c(\chi\rho) = \mathrm{Irr}(G)$.*
2. *If $c(\chi) \subseteq c(\rho)$ and $c(\chi^i) = c(\rho^i)$ for some $i \geqslant 1$, then $c(\chi^j) = c(\rho^j)$ for every $j \geqslant i$.*

If χ and ρ are characters of group G such that $c(\chi) \subseteq c(\rho)$, Lemma 15 implies that $\mathrm{ccn}(\rho; G) \leqslant \mathrm{ccn}(\chi; G)$. For a partition $\lambda \vdash n$, we conclude that $\mathrm{ccn}(\sigma_\lambda; S_n) \leqslant \mathrm{ccn}(\chi_\lambda; S_n)$. Moreover, since $\mathrm{ccn}(\chi_\lambda; S_n) = \mathrm{ccn}(\chi_{\lambda'}; S_n)$, it follows that $\mathrm{ccn}(\chi_\lambda; S_n) \geqslant \max\{\mathrm{ccn}(\sigma_\lambda; S_n), \mathrm{ccn}(\sigma_{\lambda'}; S_n)\}$. Thus, it is useful to compute the covering number of σ_λ. Moreover, as a direct consequence of Lemma 16 we have the following:

Lemma 17. *Let G be a finite group, and χ and ρ be characters of G. Let $c(\chi) \subseteq c(\rho)$. If $c(\chi^i) = c(\rho^i)$ for some $i \leqslant \mathrm{ccn}(\rho; G)$, then $\mathrm{ccn}(\chi; G) = \mathrm{ccn}(\rho; G)$. In particular, for $\lambda \vdash n$, if $c(\sigma_\lambda^i) = c(\chi_\lambda^i)$ for some $i \leqslant \mathrm{ccn}(\sigma_\lambda; S_n)$, then $\mathrm{ccn}(\chi_\lambda; S_n) = \mathrm{ccn}(\sigma_\lambda; S_n)$.*

Proof. The result follows immediately from Lemma 16(2). □

Theorem 12 provides us a recipe to compute $\mathrm{ccn}(\sigma_\lambda; S_n)$. Indeed, we need to apply Theorem 12 repeatedly to find the least positive integer k such that σ_μ appears as a constituent of σ_λ^{k-1} for some partition $\mu \vdash n$, and that there exists a $\lambda \times \mu$ matrix with exactly n entries 1 and all other entries 0. Notice that in this way we shall be able to find the least k such that $\sigma_{(1^n)}$ (which is the regular character of S_n) appears as a constituent of σ_λ^k. In particular, $c(\sigma_\lambda^k) = \mathrm{Irr}(S_n)$. Since $\sigma_{(1^n)}$ is the only permutation character σ_ν that has the sign character as a constituent, we conclude that $\mathrm{ccn}(\sigma_\lambda; S_n) = k$.

Although it may not be easy to find a closed formula for $\mathrm{ccn}(\sigma_\lambda; S_n)$ for an arbitrary partition λ, in some cases it is not so difficult. Let $\mu(k)$ denote the hook partition $(n-k, 1^k)$ of n.

Lemma 18. *Let* $1 \leqslant k \leqslant n-2$. *Then* $\mathrm{ccn}(\sigma_{\mu(k)}; S_n) = \lceil \frac{n-1}{k} \rceil$.

Proof. Notice that when $i \geqslant 2$, $\sigma_{\mu(k)}^i$ has $\sigma_{\mu(ki-j)}$ as a constituent for all $0 \leqslant j \leqslant k$. This shows that $\mathrm{ccn}(\sigma_{\mu(k)}) \leqslant \lceil \frac{n-1}{k} \rceil$. Since we are considering at each iteration a $\lambda \times \mu(k)$ matrix, starting with $\lambda = \mu(k)$, it follows that the first part of λ can be reduced by at most k in each iteration. This yields that $\mathrm{ccn}(\sigma_{\mu(k)}) \geqslant \lceil \frac{n-1}{k} \rceil$. □

Let $\lambda = (n-k, k)$ $(1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor)$ be a two-row partition of n. Clearly, $\max\{l(\mu) \mid c(\sigma_\mu) \subseteq c(\sigma_\lambda^i)\} \leqslant 2^i$. By the discussion before Lemma 18, $\mathrm{ccn}(\sigma_\lambda; S_n)$ is the least integer k such that $\sigma_{(1^n)}$ appears as a constituent of σ_λ^k, whence it follows that $2^k \geqslant n$. This implies that $\mathrm{ccn}(\sigma_\lambda; S_n) \geqslant \lceil \log_2 n \rceil$. We also have the following:

Lemma 19. *Let* $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$. *Then* $\mathrm{ccn}(\sigma_{(n-k,k)}; S_n) \geqslant \left\lceil \frac{2(n-1)}{k+1} \right\rceil$.

Proof. Set $\lambda = (n-k, k)$. Let $(n) = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_r = (1^n)$ be a sequence of partitions of n such that α_i is obtained by arranging the entries of a $\alpha_{i-1} \times \lambda$ matrix (which is of dimension $l(\alpha_{i-1}) \times 2$) in weakly decreasing order. This implies that $\sigma_{(1^n)}$ appears as a constituent of σ_λ^r. For $1 \leqslant i \leqslant r$, let $a_i = l(\alpha_i) - l(\alpha_{i-1})$ and $b_i = l(\alpha'_{i-1}) - l(\alpha'_i)$. Since a_i is the increment in the number of parts of α_i and b_i is the decrement of the largest part of α_i, we conclude that $\sum_{i=1}^r a_i + b_i = 2(n-1)$. Now going from the $(i-1)$-st step to the i-th step, if we decrease any one of the largest parts (at the $(i-1)$-st step) by j, where $0 \leqslant j \leqslant k$, then the decrement in the largest part is at most j, that is, $b_i \leqslant j$. Also, the maximum possible increment in the number of parts is $k - j + 1$, that is, $a_i \leqslant k - j + 1$. This shows that $a_i + b_i \leqslant k + 1$. Hence, $\sum_{i=1}^r a_i + b_i \leqslant r(k+1) \implies r \geqslant \frac{2(n-1)}{k+1}$. Since $\mathrm{ccn}(\sigma_\lambda; S_n)$ is obtained by taking the minimum length of all the above kinds of sequences of partitions, our result follows. □

The above bound is quite crude in general (see proof of Theorem 2). The next proposition shows that in certain cases, the above lower bound is in fact an equality.

Proposition 20. *Let* $n \geqslant 5$ *and* $2 \leqslant k \leqslant \sqrt{n}$. *Then* $\mathrm{ccn}(\sigma_{(n-k,k)}; S_n) = \left\lceil \frac{2(n-1)}{k+1} \right\rceil$.

Proof. We construct a sequence of partitions as in the previous proof: $(n) = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_r = (1^n)$, where $r = \left\lceil \frac{2(n-1)}{k+1} \right\rceil$. This will prove that $\operatorname{ccn}(\sigma_\lambda) \leqslant \left\lceil \frac{2(n-1)}{k+1} \right\rceil$, and then the statement follows from the previous lemma. We break the proof into two cases.

Case 1: Assume that $k+1 \mid n-1$. Set $n-1 = (k+1)b$ where b is a positive integer. Then $n = bk+b+1$. Set $\alpha_i = (n-ik, k^i)$ when $1 \leqslant i \leqslant b$. Thus, $\alpha_b = (b+1, k^b)$. Since $k \leqslant \sqrt{n}$, $k^2 \leqslant bk+b+1 \implies (k+1)(k-1) \leqslant b(k+1) \implies b \geqslant k-1$. This implies that the α_i's are well-defined partitions. Now we construct the partition α_{b+i} from α_{b+i-1} by breaking $b-i+1$ into $b-i$ and 1 and breaking each of the largest remaining $k-1$ parts of α_{b+i-1} (say $(l_1, l_2, \ldots, l_{k-1})$) into $l_j - 1$ and 1 for every $1 \leqslant j \leqslant k-1$. By our algorithm, $\alpha_r = (1^n)$ implies that $r \geqslant 2b$. We show that $\alpha_{2b} = (1^n)$. To show this, we concentrate on the part of the algorithm that applies to $(\underbrace{k, k, \ldots, k}_{b \text{ times}})$. For $i \geqslant 0$, define a multi-set S_i inductively as follows: S_0 consists of b-many k's. S_1 consists of all those numbers that appear when the algorithm is applied to S_0 except for the $(k-1)$-many 1's that have been produced (these are the 1's that arise from parts l_j). In general, S_i consists of all those numbers that appear when the algorithm is applied to S_{i-1}, except for the $(k-1)$-many 1's that have been produced. Let i_0 be the first i such that S_i consists only of 1s and possibly 0s. Thus, i_0 is precisely the step where the b-many k's have been broken down to all 1's, that is, kb-many 1's. We claim that $i_0 = b$. In order to establish our claim, we make two observations on the multi-set S_i: (i) $|S_i| = b$ and (ii) if t_i is the maximum number present in S_i, then S_i consists only of t_i's and possibly $(t_i - 1)$'s. Both of the properties follow from the definition of our algorithm. It follows that $t_i - 1 < \frac{kb - i(k-1)}{b} \leqslant t_i$, whence $t_i = \lceil \frac{kb-i(k-1)}{b} \rceil$. Since $b \geqslant k-1$, we get that $t_{b-1} = 2$. Since $t_b = 1$, our claim is established. Since by performing b-many iterations after α_b, the number $b+1$ has been broken down to all 1's as well, we conclude that $\alpha_{2b} = (1^n)$ as desired.

Case 2: Assume $k+1 \nmid n-1$. Write $n-1 = (k+1)b + c$ where $0 < c \leqslant k$. Then $n = bk+b+c+1$. Set $\alpha_i = (n-ik, k^i)$ when $1 \leqslant i \leqslant b$. Thus, $\alpha_b = (b+c+1, k^b)$. Since $k \leqslant \sqrt{n}$, $k^2 \leqslant bk+b+c+1 \implies (k-1)(k+1) \leqslant b(k+1)+c \implies k-1 \leqslant b + \frac{c}{k+1}$. We conclude that $b \geqslant k-1$ since $c \leqslant k$. Let $\alpha_{b+1} = (b+1, k^{b-k+c}, (k-1)^{k-c}, c, 1^{k-c})$. Now for $i \geqslant 1$, α_{b+1+i} is constructed from α_{b+i} by breaking $b-i+2$ into $b-i+1$ and 1, and breaking each of the largest remaining $k-1$ parts of α_{b+i} (say $(m_1, m_2, \ldots, m_{k-1})$) into $m_j - 1$ and 1 for every $1 \leqslant j \leqslant k-1$. By our algorithm, $\alpha_r = (1^n)$ implies that $r \geqslant 2b+1$. Now, to find the exact value of r, we argue as in the previous case. For convenience, we set $\beta_i := \alpha_{b+i+1}$, where $i \geqslant 0$. Note that $\lceil \frac{2(n-1)}{k+1} \rceil$ is $2b+1$ if $0 < c \leqslant \lceil \frac{k}{2} \rceil$, and is $2b+2$ if $\lceil \frac{k}{2} \rceil < c \leqslant k$. Thus, it is enough to show that (i) $\beta_b = (1^n)$ if $0 < c \leqslant \lceil \frac{k}{2} \rceil$, and (ii) $\beta_b \neq (1^n)$ but $\beta_{b+1} = (1^n)$ if $\lceil \frac{k}{2} \rceil < c \leqslant k$. To show this, we use the same method as in the previous case. We concentrate on the part of the algorithm that applies to $(k^{b-k+c}, (k-1)^{k-c}, c)$. For $i \geqslant 0$, define a multi-set S_i inductively as follows: S_0 consists of $(b-k+c)$-many k's, $(k-c)$-many $(k-1)$'s, and c. S_1 consists of all those numbers that appear when the algorithm is applied to S_0 except for the $(k-1)$-many 1's that have been produced (these are the 1's that arise from parts m_j). In general, S_i consists of all those numbers that appear when the algorithm is applied to S_{i-1}, except for the $(k-1)$-many 1's that have been produced. Let i_0 be the first i such that S_i consists only of 1s and possibly 0s. Thus, i_0 is precisely the step where the $(k^{b-k+c}, (k-1)^{k-c}, c)$ have been broken down to all 1's, that is, $(kb-k+c)$-many 1's. We claim that $i_0 = b$ if $0 < c \leqslant \lceil \frac{k}{2} \rceil$, and $i_0 = b+1$ if $\lceil \frac{k}{2} \rceil < c \leqslant k$. We find the average of the numbers in the set S_i when $1 \leqslant i \leqslant i_0$. Clearly, since $|S_i| = b+1$ for every i, it is easy to see that the average is equal to $\frac{(kb-k+2c)-i(k-1)}{b+1}$. Also, by the definition of our algorithm, it follows that there exists z such that $1 \leqslant z < i_0$, and S_z consists entirely of the numbers $c+1$ and c. Thus, by the same argument as in the previous case, we conclude that the maximum number present in S_i is $\lceil \frac{(kb-k+2c)-i(k-1)}{b+1} \rceil$, where $z \leqslant i \leqslant i_0$. Further, this maximum number is at least 2 when $i = b-1$. $\lceil \frac{(kb-k+2c)-i(k-1)}{b+1} \rceil = \lceil \frac{b+2c-k}{b+1} \rceil$ when $i = b$. Clearly, $\lceil \frac{b+2c-k}{b+1} \rceil$ is 1 when $0 < c \leqslant \lceil \frac{k}{2} \rceil$ and it takes the value 2 if $\lceil \frac{k}{2} \rceil < c \leqslant k$. In the latter case, $\lceil \frac{(kb-k+2c)-i(k-1)}{b+1} \rceil = 1$ when $i = b+1$. This establishes our claim. Finally, starting from β_0, since the number $b+1$ has been broken down to all 1's at the b-th step, we conclude that $\beta_b = (1^n)$ when $0 < c \leqslant \lceil \frac{k}{2} \rceil$ and $\beta_{b+1} = (1^n)$ when $\lceil \frac{k}{2} \rceil < c \leqslant k$. This completes the proof. □

For clarity, we work out some examples to demonstrate the algorithm stated in the previous proof.

Example 21. Let $\lambda = (18, 3) \vdash 21$. Then $n-1 = (k+1)b$, where $b = 5$. We apply the algorithm demonstrated in case 1 of the proof above. We obtain the following sequence: $\alpha_0 = (21) \to \alpha_1 = (18, 3) \to \alpha_2 = (15, 3^2) \to \alpha_3 = (12, 3^3) \to \alpha_4 = (9, 3^4) \to \alpha_5 = (6, 3^5) \to \alpha_6 = (5, 3^3, 2^2, 1^3) \to \alpha_7 = (4, 3, 2^4, 1^6) \to \alpha_8 = (3, 2^4, 1^{10}) \to \alpha_9 = (2^3, 1^{15}) \to \alpha_{10} = (1^{21})$.

Example 22. Let $\lambda = (40, 6) \vdash 46$. Then $n-1 = (k+1)b+c$, where $b = 6, c = 3$. Note that $c \leqslant \frac{k+1}{2}$. We apply the algorithm demonstrated in case 2 of the proof above. We obtain the following sequence: $\alpha_0 = (46) \to \alpha_1 = (40, 6) \to \alpha_2 = (34, 6^2) \to \alpha_3 = (28, 6^3) \to \alpha_4 = (22, 6^4) \to \alpha_5 = (16, 6^5) \to \alpha_6 = (10, 6^6) \to \alpha_7 = (7, 6^3, 5^3, 3, 1^3) \to \alpha_8 = (6, 5^4, 4^2, 3, 1^9) \to \alpha_9 = (5, 4^5, 3^2, 1^{15}) \to \alpha_{10} = (4, 3^7, 1^{21}) \to \alpha_{11} = (3^3, 2^5, 1^{27}) \to \alpha_{12} = (2^5, 1^{36}) \to \alpha_{13} = (1^{46})$.

Example 23. Let $\lambda = (42, 5) \vdash 47$. Then $n-1 = (k+1)b+c$, where $b = 7, c = 4$. Note that $c > \frac{k+1}{2}$. We once again apply the algorithm demonstrated in case 2 of the proof above. We obtain the following sequence: $\alpha_0 = (47) \to \alpha_1 = (42, 5) \to \alpha_2 = (37, 5^2) \to \alpha_3 = (32, 5^3) \to \alpha_4 = (27, 5^4) \to \alpha_5 = (22, 5^5) \to \alpha_6 = (17, 5^6) \to \alpha_7 = (12, 5^7) \to \alpha_8 = (8, 5^6, 4^2, 1) \to \alpha_9 = (7, 5^2, 4^6, 1^6) \to \alpha_{10} = (6, 4^6, 3^2, 1^{11}) \to \alpha_{11} = (5, 4^2, 3^6, 1^{16}) \to \alpha_{12} = (4, 3^6, 2^2, 1^{21}) \to \alpha_{13} = (3^3, 2^6, 1^{26}) \to \alpha_{14} = (2^7, 1^{33}) \to \alpha_{15} = (2^2, 1^{43}) \to \alpha_{16} = (1^{47})$.

As mentioned earlier, Miller proved that $\mathrm{ccn}(S_n)$ is $n-1$. Using Theorem 12 and Example 14, it is easy to see that $c(\sigma^2_{\mu(1)}) = c(\chi^2_{(n-1,1)})$. Then Lemma 17 and Lemma 18 at once imply that $\mathrm{ccn}(\chi_{(n-1,1)}) = n-1$. Since $\mathrm{ccn}(\chi_\lambda) = \mathrm{ccn}(\chi_{\lambda'})$, it follows that $\mathrm{ccn}(\chi_{(2,1^{n-2})}) = n-1$. Thus, we get $\mathrm{ccn}(S_n) \geqslant n-1$. Miller's proof that $\mathrm{ccn}(S_n) \leqslant n-1$

(in other words, $\mathrm{ccn}(\chi_\lambda; S_n) \leqslant n-1$ for every $\lambda \neq (n), (1^n)$) is based on the following two lemmas:

Lemma 24. *[20, Lemma 11] Let λ be a non-rectangular partition of n. Then $c(\sigma_{\mu(2)}) \subseteq c(\chi_\lambda^2)$.*

Lemma 25. *[20, Lemma 12] Let λ be a rectangular partition of n. Then $c(\sigma_{\mu(5)}) \subseteq c(\chi_\lambda^4)$.*

We remark that Zisser in [35, Corollary 4.3] also proved Lemma 24. He computed the multiplicities of the irreducible constituents of $\sigma_{\mu(2)}$ in χ_λ^2. For a partition λ of n, let $d_t(\lambda) := |\{i \mid \lambda_i - \lambda_{i+1} \geqslant t\}|$.

Lemma 26. *[35, Corollary 4.2] Let λ be a partition of n. Then*

1. $\langle \chi_\lambda^2, \chi_{(n)} \rangle = 1$.
2. $\langle \chi_\lambda^2, \chi_{(n-1,1)} \rangle = d_1(\lambda) - 1$.
3. $\langle \chi_\lambda^2, \chi_{(n-2,2)} \rangle = d_1(\lambda)(d_1(\lambda) - 2) + d_2(\lambda) + d_2(\lambda')$.
4. $\langle \chi_\lambda^2, \chi_{(n-2,1^2)} \rangle = (d_1(\lambda) - 1)^2$.

It is easy to see that Lemma 24 directly follows from the above result.

4 Proof of Theorem 1 and Theorem 2

The main step towards proving Theorem 1 is an improvement of Lemma 24 and Lemma 25 by removing the cases $\lambda = (n-1,1), (2,1^{n-2})$ (see Lemma 32 and Lemma 33). The next lemma is a well-known property of induced characters.

Lemma 27. *Let G be a finite group and H be a subgroup of G. Suppose χ and φ are class functions of G and H, respectively. Then $\chi \mathrm{Ind}_H^G \varphi = \mathrm{Ind}_H^G(\varphi \mathrm{Res}_H^G \chi)$.*

Lemma 28. *Let $\lambda, \mu \vdash n$. Then $\langle \sigma_\mu, \chi_\lambda^2 \rangle = \langle \mathrm{Res}_{S_\mu}^{S_n} \chi_\lambda, \mathrm{Res}_{S_\mu}^{S_n} \chi_\lambda \rangle$.*

Proof. Use Lemma 27 and Frobenius reciprocity. □

The following is easy to compute using Young's rule (Theorem 5).

Lemma 29. *The following relations hold:*

1. $\sigma_{(n-3,3)} = \chi_{(n-3,3)} + \chi_{(n-2,2)} + \chi_{(n-1,1)} + \chi_{(n)}$.
2. $\sigma_{(n-3,2,1)} = \chi_{(n-3,2,1)} + \chi_{(n-3,3)} + \chi_{(n-2,1^2)} + 2\chi_{(n-2,2)} + 2\chi_{(n-1,1)} + \chi_{(n)}$.
3. $\sigma_{(n-3,1^3)} = \chi_{(n-3,1^3)} + 2\chi_{(n-3,2,1)} + \chi_{(n-3,3)} + 3\chi_{(n-2,1^2)} + 3\chi_{(n-2,2)} + 3\chi_{(n-1,1)} + \chi_{(n)}$.

Now we are ready to state and prove the main lemma of this section.

Lemma 30. *Let $n \geqslant 6$ and $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$. We have the following:*

1. $\langle \chi_{(n-3,3)}, \chi_\lambda^2 \rangle = d_3(\lambda) + d_3(\lambda') + d_2(\lambda)(2d_1(\lambda) - 3) + d_2(\lambda')(2d_1(\lambda') - 3) + d_1(\lambda)(d_1(\lambda) - 1)(d_1(\lambda) - 3) + k + l$,

2. $\langle \chi_{(n-3,2,1)}, \chi_\lambda^2 \rangle = d_2(\lambda)(3d_1(\lambda) - 4) + d_2(\lambda')(3d_1(\lambda') - 4) + d_1(\lambda)(2d_1(\lambda)^2 - 8d_1(\lambda) + 7) + k + l$,

3. $\langle \chi_{(n-3,1^3)}, \chi_\lambda^2 \rangle = d_2(\lambda)(d_1(\lambda) - 1) + d_2(\lambda')(d_1(\lambda') - 1) + (d_1(\lambda) - 1)(d_1(\lambda)^2 - 3d_1(\lambda) + 1) + k + l$,

where $k = |\{i : \lambda_i = \lambda_{i+1}, \lambda_{i+1} - \lambda_{i+2} \geqslant 2\}|$ *and* $l = |\{i : \lambda_i - \lambda_{i+1} = 1, \lambda_{i+1} - \lambda_{i+2} \geqslant 1\}|$.

Proof. We compute $\langle \sigma_{(n-3,3)}, \chi_\lambda^2 \rangle$, $\langle \sigma_{(n-3,2,1)}, \chi_\lambda^2 \rangle$, and $\langle \sigma_{(n-3,1^3)}, \chi_\lambda^2 \rangle$ one by one. This is equivalent to computing $\langle \mathrm{Res}^{S_n}_{S_{n-3}\times S_3}\chi_\lambda, \mathrm{Res}^{S_n}_{S_{n-3}\times S_3}\chi_\lambda \rangle$, $\langle \mathrm{Res}^{S_n}_{S_{n-3}\times S_2}\chi_\lambda, \mathrm{Res}^{S_n}_{S_{n-3}\times S_2}\chi_\lambda \rangle$ and $\langle \mathrm{Res}^{S_n}_{S_{n-3}}\chi_\lambda, \mathrm{Res}^{S_n}_{S_{n-3}}\chi_\lambda \rangle$, respectively (see Lemma 28). We use the LR rule (Theorem 7) for this computation. There are nine possible skew shapes with three boxes, which are given in Figure 4. We remark that although the skew shapes need not share corners, we can attach their corners so that they are one of Figure 4(i)-(ix). For example, the skew shape $(4,2)/(3)$ (after connecting the corner) is Figure 4(ii).

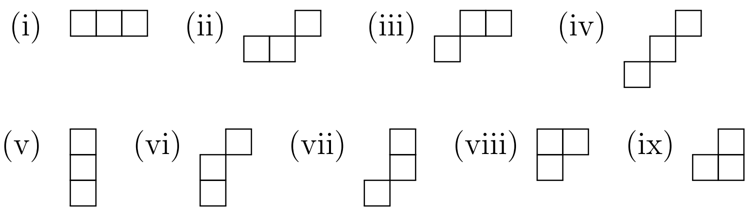


Figure 4: Possible skew shapes with three boxes.

Given $\lambda \vdash n$, we compute the number of partitions $\mu \vdash n-3$ such that λ/μ is given by each of these shapes. Clearly, the number of $\mu's$ for shape (i) is $d_3(\lambda)$ and that of shape (v) is $d_3(\lambda')$. The number of $\mu's$ such that λ/μ has shape given by one of (ii) or (iii) is $d_2(\lambda)(d_1(\lambda)-1)$. Since shapes in (vi) and (vii) are conjugates of (ii) and (iii), respectively, the number of $\mu's$ such that λ/μ is given by one of (vi) or (vii) is $d_2(\lambda')(d_1(\lambda') - 1)$. The number of $\mu's$ for shape (iv) is clearly $\binom{d_1(\lambda)}{3}$. Finally, it is not difficult to see that the number of $\mu's$ for the shape in (viii) and (ix) is given by l and k, respectively.

We start with the evaluation of $\langle \mathrm{Res}^{S_n}_{S_{n-3}\times S_3}\chi_\lambda, \mathrm{Res}^{S_n}_{S_{n-3}\times S_3}\chi_\lambda \rangle$. We have,

$$\mathrm{Res}^{S_n}_{S_{n-3}\times S_3}\chi_\lambda = \sum_{\mu \vdash n-3} c^\lambda_{\mu,(3)}(\chi_\mu \otimes \chi_{(3)}) + \sum_{\mu \vdash n-3} c^\lambda_{\mu,(2,1)}(\chi_\mu \otimes \chi_{(2,1)}) + \sum_{\mu \vdash n-3} c^\lambda_{\mu,(1^3)}(\chi_\mu \otimes \chi_{(1^3)}),$$

where the characters on the RHS are irreducible characters of $S_{n-3} \times S_3$. Using the Pieri rule (see Corollary 8), $c^\lambda_{\mu,(3)} = 1$ if λ/μ is a horizontal strip, that is, λ/μ has shape given by one of (i)-(iv), otherwise $c^\lambda_{\mu,(3)} = 0$. Thus, $\sum\limits_{\mu \vdash n-3} [c^\lambda_{\mu,(3)}]^2 = d_3(\lambda) + d_2(\lambda)(d_1(\lambda) - 1) + \binom{d_1(\lambda)}{3}$. Similarly, by Corollary 9, $c^\lambda_{\mu,(1^3)} = 1$ if λ/μ is a vertical strip, that is, λ/μ has shape given by one of (iv)-(vii), otherwise $c^{\lambda}_{\mu,(1^3)} = 0$. Thus, $\sum_{\mu\vdash n-3} [c^{\lambda}_{\mu,(1^3)}]^2 = d_3(\lambda') + d_2(\lambda')(d_1(\lambda') - 1) + \binom{d_1(\lambda)}{3}$. Now we compute $c^{\lambda}_{\mu,(2,1)}$. Using the LR rule, if μ is such that λ/μ is one of the shapes in (i)-(ix), then $c^{\lambda}_{\mu,(2,1)}$ is the number of SSYTs of that shape and type $(2,1)$ such that the reverse reading word is a lattice permutation. We have the following:

$$c^{\lambda}_{\mu,(2,1)} = \begin{cases} 1 & \text{if } \lambda/\mu \text{ has the shape as in (ii), (iii), or (vi)-(ix),} \\ 2 & \text{if } \lambda/\mu \text{ has the shape in (iv),} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\sum_{\mu\vdash n-3} [c^{\lambda}_{\mu,(2,1)}]^2 = d_2(\lambda)(d_1(\lambda) - 1) + d_2(\lambda')(d_1(\lambda') - 1) + 4\binom{d_1(\lambda)}{3}$+k+l. We conclude that

$$\begin{aligned} \langle \sigma_{(n-3,3)}, \chi^2_{\lambda} \rangle &= \sum_{\mu\vdash n-3} [c^{\lambda}_{\mu,(3)}]^2 + [c^{\lambda}_{\mu,(2,1)}]^2 + [c^{\lambda}_{\mu,(1^3)}]^2 \\ &= d_3(\lambda) + d_3(\lambda') + 2d_2(\lambda')(d_1(\lambda') - 1) + 2d_2(\lambda)(d_1(\lambda) - 1) + 6\binom{d_1(\lambda)}{3} + k \\ &\quad +l. \end{aligned}$$

By Lemma 29, $\chi_{(n-3,3)} = \sigma_{(n-3,3)} - \chi_{(n-2,2)} - \chi_{(n-1,1)} - \chi_{(n)}$, and hence (1) follows from Lemma 26.

Now we compute $\langle \mathrm{Res}^{S_n}_{S_{n-3}\times S_2}\chi_\lambda, \mathrm{Res}^{S_n}_{S_{n-3}\times S_2}\chi_\lambda \rangle$. Let $\mathrm{Res}^{S_n}_{S_{n-3}\times S_2}\chi_\lambda = \sum_{\mu\vdash n-3} a_\mu(\chi_\mu \otimes \chi_{(2)}) + \sum_{\mu\vdash n-3} b_\mu(\chi_\mu \otimes \chi_{(1^2)})$. At first, we compute a_μ. The skew shapes with two boxes (after connecting corners, if needed) are given by Figure 5.

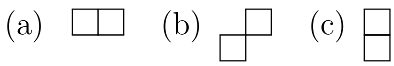
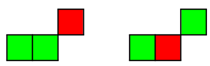


Figure 5: Possible skew shapes with two boxes.

Applying the branching rule and Pieri rule, we note that given a μ such that λ/μ is one of the skew shapes in (i)-(ix) of Figure 4, the value of a_μ is equal to the number of partitions $\nu \vdash n-1$ such that $\mu \subset \nu \subset \lambda$ with λ/ν being a single box and ν/μ being a horizontal strip with two boxes, that is, the skew shapes (a) and (b) in Figure 5. For example, if λ/μ is the skew shape in Figure 4(ii), then the number of distinct ν's with the above property is two, whence $a_\mu = 2$. The following colored diagrams explain this.



In both of the diagrams above, ν is such that λ/ν is the red colored box and ν/μ is the green colored horizontal strip. More generally, we have the following:

$$a_\mu = \begin{cases} 1 & \text{if } \lambda/\mu \text{ has the shape as in (i) or (vi)-(ix)}, \\ 2 & \text{if } \lambda/\mu \text{ has the shape as in (ii) or (iii)}, \\ 3 & \text{if } \lambda/\mu \text{ has the shape as in (iv)}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\sum_{\mu \vdash n-3} a_\mu^2 = d_3(\lambda) + 4d_2(\lambda)(d_1(\lambda)-1) + d_2(\lambda')(d_1(\lambda')-1) + k + l + 9\binom{d_1(\lambda)}{3}$. Similarly, we can compute the value of b_μ. In this case, we have to take into account the vertical strips with two boxes, that is, skew shapes (b) and (c) of Figure 5. We have the following:

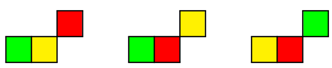
$$b_\mu = \begin{cases} 1 & \text{if } \lambda/\mu \text{ has the shape as in (ii), (iii), (v), (viii), or (ix)}, \\ 2 & \text{if } \lambda/\mu \text{ has the shape as in (vi) or (vii)}, \\ 3 & \text{if } \lambda/\mu \text{ has the shape as in (iv)}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\sum_{\mu \vdash n-3} b_\mu^2 = d_3(\lambda') + d_2(\lambda)(d_1(\lambda)-1) + 4d_2(\lambda')(d_1(\lambda')-1) + k + l + 9\binom{d_1(\lambda)}{3}$. Therefore, we conclude that

$$\begin{aligned} \langle \sigma_{(n-3,2,1)}, \chi_\lambda^2 \rangle &= \sum_{\mu \vdash n-3} a_\mu^2 + b_\mu^2 \\ &= d_3(\lambda) + d_3(\lambda') + 5d_2(\lambda')(d_1(\lambda')-1) + 5d_2(\lambda)(d_1(\lambda)-1) + 18\binom{d_1(\lambda)}{3} \\ &\quad +2k + 2l. \end{aligned}$$

By Lemma 29, $\chi_{(n-3,2,1)} = \sigma_{(n-3,2,1)} - \chi_{(n-3,3)} - \chi_{(n-2,1^2)} - 2\chi_{(n-2,2)} - 2\chi_{(n-1,1)} - \chi_{(n)}$. Thus, by Lemma 26, (2) follows.

Finally, we compute $\langle \mathrm{Res}^{S_n}_{S_{n-3}} \chi_\lambda, \mathrm{Res}^{S_n}_{S_{n-3}} \chi_\lambda \rangle$. Suppose that $\mathrm{Res}^{S_n}_{S_{n-3}} \chi_\lambda = \sum_{\mu \vdash n-3} g_\mu \chi_\mu$. By applying the branching rule, we note that if $\mu \vdash n-3$ is such that λ/μ is one of the skew shapes (i)-(ix) of Figure 4, then g_μ is the cardinality of the set of all tuples (ν, η), where $\eta \vdash n-1$, $\nu \vdash n-2$ are such that $\mu \subset \nu \subset \eta \subset \lambda$ with λ/η, η/ν, ν/μ each being a single box. For example, if μ is such that λ/μ is the skew shape in (ii), then the number of different tuples (μ, η) with the desired property is three. The following colored diagrams explain this.



In the above figure, λ/η is denoted by the red box, η/ν is denoted by the yellow box, and

ν/μ is denoted by the green box. This way of counting gives the following:

$$g_\mu = \begin{cases} 1 & \text{if } \lambda/\mu \text{ has the shape as in (i) or (v)}, \\ 2 & \text{if } \lambda/\mu \text{ has the shape as in (viii) or (ix)}, \\ 3 & \text{if } \lambda/\mu \text{ has the shape as in (ii), (iii), (vi), or (vii)}, \\ 6 & \text{if } \lambda/\mu \text{ has the shape as in (iv)}. \end{cases}$$

Thus, we conclude that

$$\begin{aligned} \langle \sigma_{(n-3,1^3)}, \chi_\lambda^2 \rangle &= \sum_{\mu \vdash n-3} g_\mu^2 \\ &= d_3(\lambda) + d_3(\lambda') + 9d_2(\lambda')(d_1(\lambda') - 1) + 9d_2(\lambda)(d_1(\lambda) - 1) + 36\binom{d_1(\lambda)}{3} \\ &\quad +4k + 4l. \end{aligned}$$

By Lemma 29, $\chi_{(n-3,1^3)} = \sigma_{(n-3,1^3)} - 2\chi_{(n-3,2,1)} - \chi_{(n-3,3)} - 3\chi_{(n-2,1^2)} - 3\chi_{(n-2,2)} - 3\chi_{(n-1,1)} - \chi_{(n)}$, and hence (3) also follows from Lemma 26. The proof is now complete. ☐

Remark 31. The previous proof can be written completely in the language of symmetric functions. We recall that for partitions μ, λ such that $\mu \subseteq \lambda$, $s_{\lambda/\mu}$ is the skew-Schur function corresponding to the skew shape λ/μ. Further, if $\lambda \vdash m+n, \mu \vdash m$, and $\nu \vdash n$ then

$$c_{\mu\nu}^\lambda = \langle s_\mu s_\nu, s_\lambda \rangle = \langle s_\mu, s_{\lambda/\nu} \rangle = \langle s_\nu, s_{\lambda/\mu} \rangle.$$

We note that $\langle \sigma_\mu, \chi_\lambda^2 \rangle$ is equal to the coefficient of s_λ in the Schur expansion of $h_\mu * s_\lambda$. A consequence of Theorem 13 is the following result (see [12, Theorem 3.1]).

$$h_\mu * s_\lambda = \sum \prod_{i \geqslant 1} s_{\lambda^i/\lambda^{i-1}}, \tag{5}$$

summed over all sequences $(\lambda^0, \lambda^1, \cdots, \lambda^l)$ of partitions such that $\emptyset = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^l = \lambda$ and $|\lambda^i/\lambda^{i-1}| = \mu_i$ for all $i \geqslant 1$. Using this decomposition and the LR rule, finding the coefficient of s_λ boils down to the same computations that we have done above.

Lemma 32. *Let $n \geqslant 6$ and λ be a non-rectangular partition of n such that $\lambda \notin \{(4,2), (2^2,1^2), (n-1,1), (2,1^{n-2})\}$. Then $c(\sigma_{\mu(3)}) \subseteq c(\chi_\lambda^2)$.*

Proof. Due to Lemma 24, it is enough to show that $\{\chi_{(n-3,3)}, \chi_{(n-3,2,1)}, \chi_{(n-3,1^3)}\} \subseteq c(\chi_\lambda^2)$. We use the multiplicity results of Lemma 30. We also use the fact that $d_1(\lambda) = d_1(\lambda')$. Since λ is non-rectangular, $d_1(\lambda) \geqslant 2$. Assume first that $d_1(\lambda) \geqslant 3$. Note that the quadratic functions $2x^2 - 8x + 7$ and $x^2 - 3x + 1$ are both strictly positive in the interval $[3, \infty)$. Therefore, it easily follows that $\langle \chi_{(n-3,2,1)}, \chi_\lambda^2 \rangle > 0$ and $\langle \chi_{(n-3,1^3)}, \chi_\lambda^2 \rangle > 0$. If $d_2(\lambda)$ or $d_2(\lambda')$ is positive, then $\langle \chi_{(n-3,3)}, \chi_\lambda^2 \rangle > 0$. Thus, we may assume that both $d_2(\lambda)$ and $d_2(\lambda')$ are zero. If $d_1(\lambda) \geqslant 4$, we once again conclude that $\langle \chi_{(n-3,3)}, \chi_\lambda^2 \rangle > 0$. Now assume that $d_1(\lambda) = 3$. It is clear that the only choice is $\lambda = (3,2,1)$ in this case. If $\lambda = (3,2,1)$, then $l = 2$. We conclude that $\langle \chi_{(n-3,3)}, \chi_\lambda^2 \rangle > 0$. Overall, we conclude that if $d_1(\lambda) \geqslant 3$, then $c(\sigma_{\mu(3)}) \subseteq c(\chi_\lambda^2)$. Now we may assume that $d_1(\lambda) = 2$. We have the following:

1. $\langle \chi_{(n-3,3)}, \chi_\lambda^2 \rangle = d_3(\lambda) + d_3(\lambda') + d_2(\lambda) + d_2(\lambda') + k + l - 2$.

2. $\langle \chi_{(n-3,2,1)}, \chi_\lambda^2 \rangle = 2d_2(\lambda) + 2d_2(\lambda') + k + l - 2$.

3. $\langle \chi_{(n-3,1^3)}, \chi_\lambda^2 \rangle = d_2(\lambda) + d_2(\lambda') + k + l - 1$.

Since $d_1(\lambda) = 2$, we write $\lambda = (a^x, b^y)$, where $x, y \geqslant 1$ and $a > b$. If $b \geqslant 2$ and $y \geqslant 2$, then both $d_2(\lambda)$ and $d_2(\lambda')$ are positive. Moreover, $k \geqslant 1$. We conclude that each of the inner-products above is positive, and hence $c(\sigma_{\mu(3)}) \subseteq c(\chi_\lambda^2)$ as required. Notice that $\lambda' = ((x+y)^b, x^{a-b})$. The above argument applies to λ' when $x \geqslant 2$ and $a-b \geqslant 2$, whence we can conclude that $c(\sigma_{\mu(3)}) \subseteq c(\chi_{\lambda'}^2)$. Since $\chi_\lambda^2 = \chi_{\lambda'}^2$, we conclude that $c(\sigma_{\mu(3)}) \subseteq c(\chi_\lambda^2)$. Therefore, we can assume that λ satisfies (i) $b = 1$ or $y = 1$ and (ii) $a - b = 1$ or $x = 1$. It is enough to show that all the inner-products from (1)-(3) are positive in these remaining cases as well. We do this case-by-case.

Case I: Assume that $b = 1$ and $a = b+1 = 2$, whence $\lambda = (2^x, 1^y)$. Since $\lambda \neq (2, 1^{n-2})$, we have $x \geqslant 2$ and $y \geqslant 1$. Moreover, as $n \geqslant 6$ and $\lambda \neq (2^2, 1^2)$, at least one of x or y is greater than two. We conclude that $d_3(\lambda') \geqslant 1$, $d_2(\lambda') \geqslant 1$, and $d_2(\lambda') + l = 2$, whence all the inner-products from (1)-(3) are positive.

Case II: Assume that $b = 1$ and $x = 1$, whence $\lambda = (a, 1^y)$. Since $\lambda \neq (n-1, 1), (2, 1^{n-2})$, we can assume $y \geqslant 2$ and $a \geqslant 3$. Clearly, $d_2(\lambda) = 1$ and $d_2(\lambda') = 1$, whence the inner-products (2) and (3) are positive. Since $n \geqslant 6$, either $y \geqslant 3$ or $a \geqslant 4$, whence one of $d_3(\lambda)$ or $d_3(\lambda')$ is positive. Thus, the inner-product (1) is positive.

Case III: Assume that $y = 1$ and $a = b + 1$, whence $\lambda = ((b+1)^x, b)$. Notice that $l = 1$ in this case. Suppose $x > 2$. Then $d_3(\lambda') \geqslant 1$, which also implies $d_2(\lambda') \geqslant 1$, and we conclude that all inner-products from (1)-(3) are positive. Thus, we may assume that $x \leqslant 2$. If $x = 2$, then $\lambda = (b+1, b+1, b)$. Since $n \geqslant 6$, $b \geqslant 2$. Thus, $d_2(\lambda) = 1$ and $d_2(\lambda') = 1$, whence all inner-products from (1)-(3) are positive once again. If $x = 1$, $\lambda = (b+1, b)$. Since $n \geqslant 6$, $b \geqslant 3$. In this case $d_3(\lambda) = 1$, which implies that $d_2(\lambda) \geqslant 1$. We once again conclude that the inner-products from (1)-(3) are positive.

Case IV: Assume that $y = 1$ and $x = 1$, whence $\lambda = (a, b)$. Clearly $b \geqslant 2$ since $\lambda \neq (n-1, 1)$. If $b \geqslant 3$, then $d_3(\lambda) \geqslant 1$ which implies $d_2(\lambda) \geqslant 1$. Further if $a \geqslant b+2$, note that $d_2(\lambda) = 2$. Once again all the inner-products from (1)-(3) are positive. If $a = b + 1$, then $l = 1$, whence the inner-products from (1)-(3) are positive once again. Finally, we can assume that $b = 2$. Since $n \geqslant 6$ and $\lambda \neq (4, 2)$, we assume $a \geqslant 5$. In this case, $d_3(\lambda) = 1$ and $d_2(\lambda) = 2$, whence all the inner-products from (1)-(3) are positive. □

Lemma 33. *Let $n \geqslant 12$ and λ ($\neq (n), (1^n)$) be a rectangular partition of n. Then $c(\sigma_{\mu(6)}) \subseteq c(\chi_\lambda^4)$.*

Proof. Let $\lambda = (r^s) \vdash n$, where $r, s \geqslant 2$. Note that $d_i(\lambda) = d_i(\lambda') = 1$ for $i = 1, 2$. Also, $d_3(\lambda) = 1$ if $r \geqslant 3$ and $d_3(\lambda') = 1$ if $s \geqslant 3$. Finally, $k = 1$ and $l = 0$. Thus, Lemma 30 yields (i) $\langle \chi_{(n-3,1^3)}, \chi_\lambda^2 \rangle = 1$, (ii) $\langle \chi_\lambda^2, \chi_{(n-3,3)} \rangle$ equals 1 if $r, s \geqslant 3$ and equals 0 otherwise, and (iii) $\langle \chi_\lambda^2, \chi_{(n-3,2,1)} \rangle = 0$. Using Lemma 26, $\langle \chi_\lambda^2, \chi_{(n-2,2)} \rangle = 1$, $\langle \chi_\lambda^2, \chi_{(n-1,1)} \rangle = \langle \chi_\lambda^2, \chi_{(n-1,1^2)} \rangle =$

0. Using Young's rule, we observe that

$$\langle \chi_\lambda^2, \chi_{(n-4,4)} \rangle = \langle \chi_\lambda^2, \sigma_{(n-4,4)} \rangle - \langle \chi_\lambda^2, \sigma_{(n-3,3)} \rangle = \begin{cases} \langle \chi_\lambda^2, \sigma_{(n-4,4)} \rangle - 2 & \text{if } r=2 \text{ or } s=2, \\ \langle \chi_\lambda^2, \sigma_{(n-4,4)} \rangle - 3 & \text{otherwise.} \end{cases}$$

By Lemma 28, $\langle \chi_\lambda^2, \sigma_{(n-4,4)} \rangle = \langle \mathrm{Res}^{S_n}_{S_{n-4}\times S_4} \chi_\lambda, \mathrm{Res}^{S_n}_{S_{n-4}\times S_4} \chi_\lambda \rangle$. To compute this inner-product, we need to compute $c^\lambda_{\mu\nu}$ where $\nu \vdash 4$ (see Equation (1)). Since λ is rectangular, for any $\mu \subseteq \lambda$, the skew diagram $T_{\lambda/\mu}$ must be right justified. Thus, using Theorem 7, we obtain the following:

- there exists $\mu \vdash n-4$ such that $c^\lambda_{\mu,(4)} = 1$ provided $r \geqslant 4$.
- there exists $\mu \vdash n-4$ such that $c^\lambda_{\mu,(1^4)} = 1$ provided $s \geqslant 4$.
- there exists $\mu \vdash n-4$ such that $c^\lambda_{\mu,(3,1)} = 1$ provided $r \geqslant 3$.
- there exists $\mu \vdash n-4$ such that $c^\lambda_{\mu,(2^2)} = 1$.
- there exists $\mu \vdash n-4$ such that $c^\lambda_{\mu,(2,1^2)} = 1$ provided $s \geqslant 3$.

Since $n \geqslant 12$, from the above information, it is easy to conclude that $\langle \chi_\lambda^2, \chi_{(n-4,4)} \rangle > 0$. This yields that $c(\chi_\lambda^4) \supseteq c(\chi_\mu \chi_\nu)$, where μ and ν can be chosen from the set $\{(n-2,2), (n-3,1^3), (n-4,4)\}$. Due to Lemma 25, to get the final result, it is enough to show that $\chi_\eta \in c(\chi_\lambda^4)$ for each $\eta \vdash n$ with the first part equal to $n-6$. By [21, Section 2, Eqn. 24,53,54], we conclude that all such χ_η's occur with the possible exception of $\chi_{(n-6,3^2)}$. On the other hand, using the fact that $\chi^2_{(n-4,4)} = \sigma^2_{(n-4,4)} - 2\sigma_{(n-4,4)}\sigma_{(n-3,3)} + \sigma^2_{(n-3,3)}$ (see Theorem 5), an easy check using Theorem 12 yields that $\chi_{(n-6,3^2)} \in c(\chi^2_{(n-4,4)})$, whence our proof is complete. ☐

For $\lambda, \mu \vdash n$, let $|\lambda \setminus \mu|$ be the number of boxes of T_λ that lie outside of T_μ. For example, if $\lambda = (4,3,2,2)$ and $\mu = (3,3,3,1,1)$, then $\mid \lambda \setminus \mu \mid = 2$. The next lemma is an easy application of Equation (5) and the Pieri rule.

Lemma 34. *[20, Lemma 10] Let $\lambda \vdash n$, $1 \leqslant k \leqslant n-2$, and $r \in \mathbb{N}$. Then $c(\sigma^r_{\mu(k)}\chi_\lambda) = \{\chi_\mu \mid |\lambda \setminus \mu| \leqslant kr\}$.*

Proof. We prove it when $r = 1$. By Equation (5),

$$h_{\mu(k)} * s_\lambda = \sum s_\nu s_1^k,$$

where the sum runs over all sequences $\nu_0 \subseteq \nu_1 \subseteq \cdots \subseteq \nu_k = \lambda$ such that $\nu_0 = \nu \subseteq \lambda$ is a partition of $n-k$, and $|\nu^i/\nu^{i-1}| = 1$ for every $1 \leqslant i \leqslant k$. Using the Pieri rule, we conclude that if $\langle s_\nu s_1^k, s_\mu \rangle > 0$, then $|\lambda \setminus \mu| \leqslant k$. Conversely, if $\mu \vdash n$ is such that $|\lambda \setminus \mu| \leqslant k$, then $\mu \cap \lambda$ is a partition of t where $t \geqslant n-k$. Thus, we can choose $\nu \vdash n-k$ such that $\nu \subseteq \lambda \cap \mu \subseteq \lambda$, and hence $s_\nu s_1^k$ appears in the above summand. Once again the Pieri rule implies that $\langle s_\nu s_1^k, s_\mu \rangle > 0$. This yields the result for $r = 1$. For $r \geqslant 2$, the result follows by induction. ☐

Proof of Theorem 1. If $n \leqslant 11$, the theorem can be verified directly using SageMath ([33]). So we may assume $n \geqslant 12$. Let $\lambda \vdash n$ be non-rectangular. Set $n = 3k + r$, where $0 \leqslant r \leqslant 2$ and $k \geqslant 4$. Using Lemma 32, we conclude that $c(\sigma_{\mu(3)}^k) \subseteq c(\chi_\lambda^{2k})$. If $r \in \{0,1\}$, then by Lemma 18, $c(\sigma_{\mu(3)}^k) = \mathrm{Irr}(S_n)$ and hence $c(\chi_\lambda^{2k}) = \mathrm{Irr}(S_n)$, whence the result follows. If $r = 2$, notice that $c(\sigma_{\mu(3)}^k\chi_\lambda) \subseteq c(\chi_\lambda^{2k+1})$ and $c(\sigma_{\mu(3)}^k\chi_\lambda) = \mathrm{Irr}(S_n)$ by Lemma 34. Hence $c(\chi_\lambda^{2k+1}) = \mathrm{Irr}(S_n)$ and once again the result follows. Now we consider the rectangular partitions. Let $n = 6k + r$ where $0 \leqslant r \leqslant 5$ and $k \geqslant 2$. By Lemma 33, we obtain

$$c(\sigma_{\mu(6)}^k) \subseteq c(\chi_\lambda^{4k}). \tag{6}$$

If $r \in \{0,1\}$, then by Lemma 18, $c(\sigma_{\mu(6)}^k) = \mathrm{Irr}(S_n)$ which implies $c(\chi_\lambda^{4k}) = \mathrm{Irr}(S_n)$, whence the result follows. If $r = 2$, then $\lceil \frac{2(n-1)}{3} \rceil = 4k + 1$. From Equation (6), we get $c(\sigma_{\mu(6)}^k\chi_\lambda) \subseteq c(\chi_\lambda^{4k+1})$. By Lemma 34, it is easy to check that $c(\sigma_{\mu(6)}^k\chi_\lambda) = \mathrm{Irr}(S_n)$, from which the result once again follows. Now we assume that $r \in \{3,4,5\}$. We have seen in the proof of Lemma 33 that $\chi_{(n-3,1^3)} \in c(\chi_\lambda^2)$. Thus, from Equation (6), we get $c(\sigma_{\mu(6)}^k\chi_{(n-3,1^3)}) \subseteq c(\chi_\lambda^{4k+2})$. Once again using Lemma 34 and the fact that $n \geqslant 12$, it is easy to deduce that $c(\sigma_{\mu(6)}^k\chi_{(n-3,1^3)}) = \mathrm{Irr}(S_n)$ when $r \in \{3,4\}$. Since $\lceil \frac{2(n-1)}{3} \rceil = 4k + 2$ when $r \in \{3,4\}$, the result follows in this case as well. When $r = 5$, we get that $c(\chi_\lambda^{4k+2}) \supseteq c(\sigma_{\mu(6)}^k\chi_{(n-3,1^3)}) = \mathrm{Irr}(S_n) \setminus \{\epsilon\}$. Since for any $\nu \vdash n$, $g_{\lambda(1^n)\nu} \leqslant 1$, there exists χ_η $(\neq \epsilon)$ such that $\chi_\eta \in c(\chi_\lambda\chi_\nu)$. This implies that $\chi_\nu \in c(\chi_\eta\chi_\lambda)$ for some χ_η $(\neq \epsilon)$. Thus, $\chi_\nu \in c(\sigma_{\mu(6)}^k\chi_{(n-3,1^3)}\chi_\lambda) \subseteq c(\chi_\lambda^{4k+3})$, whence $c(\chi_\lambda^{4k+3}) = \mathrm{Irr}(S_n)$. Since $\lceil \frac{2(n-1)}{3} \rceil = 4k + 3$ when $r = 5$, our result follows.

The final assertion follows since $\mathrm{ccn}(\chi_{(n-2,2)}; S_n) \geqslant \mathrm{ccn}(\sigma_{(n-2,2)}; S_n) = \left\lceil \frac{2(n-1)}{3} \right\rceil$ by Proposition 20. The fact that $\mathrm{ccn}(\chi_{(2^2,1^{n-4})}; S_n) = \left\lceil \frac{2(n-1)}{3} \right\rceil$ follows since $\mathrm{ccn}(\chi_\lambda; S_n) = \mathrm{ccn}(\chi_{\lambda'}; S_n)$. □

As an application of Lemma 32, we prove that every irreducible character of S_n appears as a constituent of the product of all (non-linear) hook characters.

Theorem 35. *Let $n \geqslant 5$. Then* $c\left(\prod_{1\leqslant k\leqslant n-2} \chi_{\mu(k)}\right) = \mathrm{Irr}(S_n)$.

Proof. Let $n \geqslant 6$ be even. Then, using Lemma 24 and Lemma 32, we get the following:

$$\begin{aligned} c\left(\prod_{k=1}^{n-2} \chi_{\mu(k)}\right) &= c\left(\epsilon^{\frac{n}{2}-1}\chi_{(n-1,1)}^2 \prod_{2\leqslant k\leqslant \frac{n}{2}-1} \chi_{\mu(k)}^2\right) \supseteq c(\epsilon^{\frac{n}{2}-1}\sigma_{\mu(2)}\sigma_{\mu(3)}^{\frac{n}{2}-2}) = c(\epsilon^{\frac{n}{2}-1}\sigma_{\mu(1)}^{\frac{3n}{2}-4}) \\ &= \mathrm{Irr}(S_n). \end{aligned}$$

Since $\frac{3n}{2} - 4 \geqslant n - 1$, the last equality follows using Lemma 18 together with Lemma 16(1). When $n = 5, 7$, the result follows by direct computation. Assume now that $n \geqslant 9$ is odd.

Using Lemma 24 and Lemma 32, we get the following:

$$c\left(\prod_{k=1}^{n-2}\chi_{\mu(k)}\right) = c\left(\epsilon^{\frac{n-3}{2}}\chi_{\mu(\frac{n-1}{2})}\chi^2_{(n-1,1)}\prod_{2\leqslant k\leqslant \frac{n-3}{2}}\chi^2_{\mu(k)}\right) \supseteq c(\epsilon^{\frac{n-3}{2}}\chi_{\mu(\frac{n-1}{2})}\sigma_{\mu(2)}\sigma_{\mu(3)}^{\frac{n-5}{2}}).$$

Since $\frac{3n-11}{2} \geqslant n-1$, using Lemma 18 together with Lemma 16(1), we conclude that

$$c(\epsilon^{\frac{n-3}{2}}\chi_{\mu(\frac{n-1}{2})}\sigma_{\mu(1)}^{\frac{3n-11}{2}}) = \mathrm{Irr}(S_n),$$

whence our result follows. □

Now we give a proof of Theorem 1.2.

Proof of Theorem 2. Let $k = \frac{n+1}{2}$ and $\lambda = (k, k-1)$. By [11, Corollary 4.1], we get that $\chi_\lambda^2 = \sum_{l(\mu)\leqslant 4}\chi_\mu$. It follows from Theorem 12 and the Young's rule (Theorem 5) that $c(\sigma_\lambda^2) = c(\chi_\lambda^2)$. By the discussion before Lemma 19, we have $\mathrm{ccn}(\sigma_\lambda; S_n) \geqslant \lceil \log_2 n\rceil$. Let $\alpha_0 = (n)$. For each $r \geqslant 1$, let α_r be the partition obtained from the $\lambda \times \alpha_{r-1}$ matrix A^r (by a decreasing rearrangement of its entries) chosen in the following way: If $\alpha_{r-1} = (a_1, a_2, \ldots, a_s)$, then construct A^r so that its i-th column contains $\lceil \frac{a_i}{2}\rceil$ and $\lfloor \frac{a_i}{2}\rfloor$ and the sum of the entries in its first row is k. It is easy to see that this can always be done. For example: if $\lambda = (11, 10)$, then $\alpha_0 = (21) \to \alpha_1 = (11, 10) \to \alpha_2 = (6, 5^3) \to \alpha_3 = (3^5, 2^3) \to \cdots$. By our construction, α_3 is obtained from the $\lambda\times\alpha_2$ matrix $\left(\begin{smallmatrix}3&3&2&3\\3&2&3&2\end{smallmatrix}\right)$. It is clear that when $r = \lceil \log_2 n\rceil - 1$, the matrix A^{r+1} obtained by the above choice has all its entries equal to 1 or 0, and hence the partition obtained from A^{r+1} is $\alpha^{r+1} = (1^n)$. By the discussion prior to Lemma 18, we conclude that $\mathrm{ccn}(\sigma_\lambda; S_n) \leqslant \lceil \log_2 n\rceil$. Thus, $\mathrm{ccn}(\sigma_\lambda; S_n) = \lceil \log_2 n\rceil$. Since $n \geqslant 5$, we get $\mathrm{ccn}(\sigma_\lambda; S_n) \geqslant 3$, whence Lemma 17 implies the result. □

We end this section with the following conjecture:

Conjecture 36. Let $n \geqslant 5$ and $\lambda = (n-k, k)$ where $1 \leqslant k \leqslant \lfloor \frac{n}{2}\rfloor$. The following hold.

1. If n is odd, then $c(\chi_\lambda^2) = c(\sigma_\lambda^2)$. In particular, $\mathrm{ccn}(\chi_\lambda; S_n) = \mathrm{ccn}(\sigma_\lambda; S_n)$.
2. Let n be even, and $\lambda \neq (\frac{n}{2}, \frac{n}{2})$. If $1 \leqslant k < \lceil \frac{n}{4}\rceil$, then $c(\chi_\lambda^2) = c(\sigma_\lambda^2)$. Otherwise, $c(\chi_\lambda^3) = c(\sigma_\lambda^3)$. In particular, $\mathrm{ccn}(\chi_\lambda; S_n) = \mathrm{ccn}(\sigma_\lambda; S_n)$.
3. If n is even, $\lceil \log_2 n\rceil \leqslant \mathrm{ccn}(\chi_{(\frac{n}{2},\frac{n}{2})}; S_n) \leqslant \lceil \log_2 n\rceil + 1$.

We note that a positive proof of the above conjecture together with Lemma 17, Lemma 18, and Proposition 20 yields that $\mathrm{ccn}(\chi_{(n-k,k)}; S_n) = \lceil \frac{2(n-1)}{k+1}\rceil$ provided $1 \leqslant k \leqslant \sqrt{n}$.

5 Proof of Theorem 3

In this section, we prove Theorem 3. Recall that we denote a hook partition $(n-r, 1^r)$ by $\mu(r)$. The Durfee rank of a partition λ, denoted by $d(\lambda)$, is the size of the largest square contained in it. In particular, partitions with Durfee rank 1 are hook partitions $\mu(r)$, where $0 \leqslant r \leqslant n-1$. Partitions with Durfee rank 2 are called double hooks. They can be written in the form $(n_4, n_3, 2^{d_2}, 1^{d_1})$. Here we assume $n_4 \geqslant n_3 \geqslant 2$. For a proposition P, the notation $((P))$ takes the value 1 if P is true; otherwise, it takes the value 0. The Kronecker coefficients $g_{\mu\nu\lambda}$ when two of the partitions are hooks and the other one is arbitrary were determined by Remmel [27, Theorem 2.1] and reproved by Rosas [30, Theorem 3] (with a slightly different statement). We state this result by mixing both to suit our needs.

Theorem 37. *Let $\mu, \nu, \lambda \vdash n$, where $\mu = (n-e, 1^e)$ and $\nu = (n-f, 1^f)$ are hook-shaped partitions. Assume that $f \geqslant e \geqslant 1$ and $f + e \leqslant n-1$. Then the Kronecker coefficients $g_{\mu\nu\lambda}$ are given by the following:*

1. *If λ is such that $d(\lambda) > 2$, then $g_{\mu\nu\lambda} = 0$.*
2. *If $\lambda = (n)$, then $g_{\mu\nu\lambda} = 1$ if and only if $\mu = \nu$, and if $\lambda = (1^n)$, then $g_{\mu\nu\lambda} = 1$ if and only if $\mu = \nu'$.*
3. *Let $\lambda = (n-r, 1^r)$ be a hook shape where $1 \leqslant r \leqslant n-2$. Then*
$$g_{\mu\nu\lambda} = ((f - e \leqslant r \leqslant e + f)).$$
4. *Let $\lambda = (n_4, n_3, 2^{d_2}, 1^{d_1})$ be a double hook where $n_4 \geqslant n_3 \geqslant 2$ and $x = 2d_2 + d_1$. Then*
$$\begin{aligned} g_{\mu\nu\lambda} &= \left(\left(n_3 - 1 \leqslant \frac{e+f-x}{2} \leqslant n_4\right)\right)((f - e \leqslant d_1)) \\ &\quad + \left(\left(n_3 \leqslant \frac{e+f-x+1}{2} \leqslant n_4\right)\right)((f - e \leqslant d_1 + 1)). \end{aligned}$$

The following result of Blasiak will be required. We give a proof as well since it is a direct application of Theorem 13.

Lemma 38. *[8, Proposition 3.1] Let $\mu(d)$ denote the hook shape $(n-d, 1^d)$ where $1 \leqslant d \leqslant n-1$. Then $g_{\lambda\mu(d)\nu} + g_{\lambda\mu(d-1)\nu} = \displaystyle\sum_{\alpha\vdash d, \beta\vdash n-d} c^{\lambda}_{\alpha\beta} c^{\nu}_{\alpha'\beta}$.*

Proof. By Theorem 13, we have $s_\lambda * (s_{(n-d)} s_{(1^d)}) = \displaystyle\sum_{\alpha\vdash d, \beta\vdash n-d} c^{\lambda}_{\alpha\beta} s_{\alpha'} s_\beta$. On the other hand, using the Pieri rule, it is easy to see that $s_{(n-d)} s_{(1^d)} = s_{\mu(d)} + s_{\mu(d-1)}$. Thus,

$$s_\lambda * s_{\mu(d)} + s_\lambda * s_{\mu(d-1)} = s_\lambda * \left(s_{(n-d)} s_{(1^d)}\right) = \sum_{\alpha\vdash d, \beta\vdash n-d} c^{\lambda}_{\alpha\beta} s_{\alpha'} s_\beta.$$

Taking inner-product with s_ν on both sides yields the result. $\square$

The following lemma will be central to the proof of Theorem 3.

Lemma 39. *Let* $1 \leqslant s \leqslant \frac{n-1}{2}$*. Then* $c(\chi_{\mu(s-1)}\chi_{\mu(s)}) \subseteq c(\chi^2_{\mu(s)})$ *and* $c(\chi^2_{\mu(s-1)}) \subseteq c(\chi^2_{\mu(s)})$*. More generally, for any* $k \geqslant 2$*,* $c(\chi_{\mu(s-1)}\chi^{k-1}_{\mu(s)}) \subseteq c(\chi^k_{\mu(s)})$ *and* $c(\chi^k_{\mu(s-1)}) \subseteq c(\chi^k_{\mu(s)})$*.*

Proof. Note that $\mu(1) = (n-1, 1)$, whence the result for $s = 1$ follows from Example 14. So we may assume $s \geqslant 2$. We show that $c(\chi_{\mu(s)}\chi_{\mu(s-1)}) \subseteq c(\chi^2_{\mu(s)})$. Using Theorem 37(1), we note that if the Durfee rank of λ is greater than 2, then χ_λ does not appear as a constituent of both $\chi_{\mu(s)}\chi_{\mu(s-1)}$ and $\chi^2_{\mu(s)}$. Further, by our choice of s, $\chi_{(1^n)}, \chi_{(n)} \notin c(\chi_{\mu(s-1)\mu(s)})$. Thus, we only consider $\lambda \vdash n$ such that $\lambda = \mu(r)$ where $1 \leqslant r \leqslant n-2$ or λ is a double hook shape.

Case I: Suppose $\lambda = (n-r, 1^r)$ where $1 \leqslant r \leqslant n-2$. By Theorem 37(3), $\chi_\lambda \in c(\chi_{\mu(s)}\chi_{\mu(s-1)})$ if and only if $1 \leqslant r \leqslant 2s-1$. The last inequality also implies $0 \leqslant r \leqslant 2s$, whence Theorem 37(3) implies that $\chi_\lambda \in c(\chi^2_{\mu(s)})$.

Case II: Assume that $\lambda = (n_4, n_3, 2^{d_2}, 1^{d_1})$, where $n_4 \geqslant n_3 \geqslant 2$ and $x = 2d_2 + d_1$. Assume that $\chi_\lambda \in c(\chi_{\mu(s)}\chi_{\mu(s-1)})$. By Theorem 37(4), the following expression is positive.

$$((n_3 - 1 \leqslant \frac{2s-1-x}{2} \leqslant n_4))((d_1 \geqslant 1)) + ((n_3 \leqslant \frac{2s-x}{2} \leqslant n_4))((d_1 \geqslant 0)). \tag{7}$$

To show that $\chi_\lambda \in c(\chi^2_{\mu(s)})$, once again using Theorem 37(4), we need to show that the following expression is positive.

$$((n_3 - 1 \leqslant \frac{2s-x}{2} \leqslant n_4))((d_1 \geqslant 0)) + ((n_3 \leqslant \frac{2s-x+1}{2} \leqslant n_4))((d_1 + 1 \geqslant 0)). \tag{8}$$

Notice that if the second summand in expression 7 is positive, then the first summand in expression 8 is positive as well, and we are done. Therefore, we may assume that the second summand in expression 7 is zero. In that case, the first summand in expression 7 is positive. This automatically means that $d_1 \geqslant 1$. We also have $n_3 - \frac{1}{2} \leqslant \frac{2s-x}{2} \leqslant n_4 + \frac{1}{2}$. But since the second summand in expression 7 is zero, it follows that either $\frac{2s-x}{2}$ takes the value $n_3 - \frac{1}{2}$ or $n_4 + \frac{1}{2}$. In the former case, the first summand in expression 8 is positive, and hence we are done. On the other hand, the latter case is not possible. To see this, $\frac{2s-x}{2} = n_4 + \frac{1}{2} \implies s - \frac{1}{2} = n_4 + \frac{x}{2} \geqslant \frac{n_3+n_4+x}{2}$. The last inequality follows since $n_4 \geqslant n_3$. Since $n = n_4 + n_3 + x$, we get $s - \frac{1}{2} \geqslant \frac{n}{2}$, a contradiction to our assumption.

Similar computations yield that $c(\chi^2_{\mu(s-1)}) \subseteq c(\chi^2_{\mu(s)})$. The latter statements follow immediately by using Lemma 15. □

We can now prove the first part of Theorem 3.

Proof of Theorem 3(1). Since $\mathrm{ccn}(\chi_\lambda; S_n) = \mathrm{ccn}(\chi_{\lambda'}; S_n)$, it is enough to prove the result for $\chi_{\mu(2)}$. The result can be easily verified for $5 \leqslant n \leqslant 9$ using SageMath ([33]). So we may assume $n \geqslant 10$.

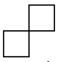
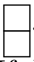
Using Theorem 37, we get that $c(\chi^2_{\mu(2)}) = \{\chi_\lambda \mid \lambda_1 = n-4+i, \lambda_2 \leqslant 2+i; 0 \leqslant i \leqslant 4\}$. We claim the following: for any $2 \leqslant k \leqslant \lfloor \frac{n}{3} \rfloor$,

$$c(\chi^k_{\mu(2)}) \supseteq \{\chi_\lambda \mid \lambda_1 = n-2k+i, \lambda_2 \leqslant k+i; 0 \leqslant i \leqslant 2k\}. \tag{9}$$

We make two observations at this point: (a) $2 \leqslant k \leqslant \lfloor \frac{n}{3} \rfloor$ ensures that $n-2k+i \geqslant k+i$ for every $0 \leqslant i \leqslant 2k$, (b) since $\chi_{\mu(2)} \in c(\chi^2_{\mu(2)})$, we get that $c(\chi^2_{\mu(2)}) \subseteq c(\chi^3_{\mu(2)}) \subseteq c(\chi^4_{\mu(2)}) \subseteq \dots$. We now prove our claim by induction on k. The claim holds for $k = 2$. Thus, we may assume $k \geqslant 3$. Since $c(\chi^{k-1}_{\mu(2)}) \subseteq c(\chi^k_{\mu(2)})$ by observation (b) made above, we conclude by the induction hypothesis that

$$\{\chi_\lambda \mid \lambda_1 = n-2k+2+i, \lambda_2 \leqslant k+i-1; 0 \leqslant i \leqslant 2k-2\} \subseteq c(\chi^k_{\mu(2)}). \tag{10}$$

Thus, it remains to show that $\chi_\lambda \in c(\chi^k_{\mu(2)})$ whenever λ satisfies one of the following conditions: (1) $\lambda_1 = n-2k+i, \lambda_2 \leqslant k+i$ where $i = 0, 1$ and (2) $\lambda_1 = n-2k+i$, $k+i-2 \leqslant \lambda_2 \leqslant k+i$ where $2 \leqslant i \leqslant 2k$. In the latter case, we may assume that $i \leqslant \frac{k+2}{2}$ since $\lambda_2 \leqslant 2k-i$. Now we determine the conditions under which λ is a two-row partition, that is, $\lambda_3 = 0$. In (1), clearly $\lambda_3 > 0$. In (2), if $i < \frac{k}{2}$, then $\lambda_3 > 0$. Thus, $\lambda_3 > 0$ unless (a) $i = \frac{k+1}{2}$ where k is odd, whence $\lambda = (n - \frac{3k-1}{2}, \frac{3k-1}{2})$ and (b) $\frac{k}{2} \leqslant i \leqslant \frac{k}{2}+1$ where k is even, whence $\lambda = (n - \frac{3k}{2}, \frac{3k}{2})$, or $\lambda = (n - \frac{3k}{2}+1, \frac{3k}{2}-1)$.

Case I: Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be as in (1) or (2) with $\lambda_3 > 0$. Let $\tilde{\lambda} = (\lambda_2, \lambda_3, \dots)$ so that $|\tilde{\lambda}| = 2k - i$. Since $\lambda_3 > 0$, there exists $\tilde{\beta} \vdash 2k-2-i$ such that $\tilde{\lambda}/\tilde{\beta}$ is one of the two shapes  or . We make the choice of $\tilde{\beta}$ arbitrary with the above property provided $\lambda_2 \leqslant k+i-1$. If $\lambda_2 = k+i$, notice that $\lambda_3 \leqslant k+i$ and $\lambda_3 = k+i$ only when $i = 0$, in which case $\lambda = (n-2k, k, k)$. Either way, we choose $\tilde{\beta}$ by removing one box from the first row of $\tilde{\lambda}$ and another box from a different admissible row of $\tilde{\lambda}$. Thus, we obtain that $\tilde{\beta}_1 \leqslant k-1+i$ in all cases. We define $\beta = (\beta_1, \beta_2, \dots) \vdash n-2$ as follows:

$$\beta_r = \begin{cases} \lambda_1 & \text{if } r = 1, \\ \tilde{\beta}_{r-1} & \text{if } r \geqslant 2. \end{cases}$$

Let $\alpha = (1,1)$. Clearly $c^\lambda_{\alpha\beta} = 1$. Now we define a partition $\nu = (\nu_1, \nu_2, \dots) \vdash n$ as follows:

$$\nu_r = \begin{cases} n-2k+2+i & \text{if } r = 1, \\ \beta_r & \text{if } r \geqslant 2. \end{cases}$$

Clearly, $\nu_2 \leqslant k-1+i$ since $\beta_2 = \tilde{\beta}_1 \leqslant k-1+i$. By Equation (9), $\chi_\nu \in c(\chi^{k-1}_{\mu(2)})$. Further, ν/β is a horizontal strip with two boxes, which implies that $c^\nu_{\alpha'\beta} = 1$. Thus, we can conclude that $\sum_{\gamma \vdash 2, \delta \vdash n-2} c^\lambda_{\gamma\delta} c^\nu_{\gamma'\delta}$ is positive, whence Lemma 38 implies $g_{\lambda\mu(2)\nu} + g_{\lambda\mu(1)\nu} > 0$. If $g_{\lambda\mu(2)\nu} > 0$, then $\chi_\lambda \in c(\chi_\nu \chi_{\mu(2)}) \subseteq c(\chi^k_{\mu(2)})$ and we are done. If $g_{\lambda\mu(1)\nu} > 0$, then $\chi_\lambda \in c(\chi_\nu \chi_{\mu(1)}) \subseteq c(\chi^{k-1}_{\mu(2)} \chi_{\mu(1)})$. By Lemma 39, $c(\chi_{\mu(1)} \chi^{k-1}_{\mu(2)}) \subseteq c(\chi^k_{\mu(2)})$, whence we conclude that $\chi_\lambda \in c(\chi^k_{\mu(2)})$ and we are done.

By the assumption in **Case I** and the discussion prior to it, we are left with the following case:

Case II: (a) $\lambda = (n - \frac{3k-1}{2}, \frac{3k-1}{2})$ when k is odd and (b) $\lambda = (n - \frac{3k}{2}, \frac{3k}{2})$, or $\lambda = (n - \frac{3k}{2} + 1, \frac{3k}{2} - 1)$ when k is even. First, we consider (a). Since $k \geqslant 3$, $\lambda_2 \geqslant 4$. Let $\alpha = (2) \vdash 2$. Let $\beta = (n - \frac{3k-1}{2}, \frac{3k-1}{2} - 2)$. Clearly $c^{\lambda}_{\alpha\beta} = 1$. Let $\nu = (n - \frac{3k-1}{2} + 1, \frac{3k-1}{2} - 2, 1)$. It is immediate that $c^{\nu}_{\alpha'\beta} = 1$ as ν/β is a vertical strip with two boxes. Thus, we obtain $\sum_{\gamma \vdash 2, \delta \vdash n-2} c^{\lambda}_{\gamma\delta} c^{\nu}_{\gamma'\delta} > 0$. Notice that $\chi_\nu \in c(\chi^{k-1}_{\mu(2)})$. By Lemma 38, we get $g_{\lambda\mu(2)\nu} + g_{\lambda\mu(1)\nu} > 0$. Suppose that $g_{\lambda\mu(2)\nu} > 0$. This implies $\chi_\lambda \in c(\chi_\nu \chi_{\mu(2)}) \subseteq c(\chi^{k}_{\mu(2)})$ and we are done. If $g_{\lambda\mu(1)\nu} > 0$, then $\chi_\lambda \in c(\chi_\nu \chi_{\mu(1)}) \subseteq c(\chi^{k-1}_{\mu(2)} \chi_{\mu(1)})$. By Lemma 39, $c(\chi_{\mu(1)} \chi^{k-1}_{\mu(2)}) \subseteq c(\chi^{k}_{\mu(2)})$, whence we conclude that $\chi_\lambda \in c(\chi^{k}_{\mu(2)})$ and we are done. The arguments for the cases in (b) follow along similar lines. We simply mention the choices of α, β, and ν in these cases. If $\lambda = (n - \frac{3k}{2}, \frac{3k}{2})$, then $\lambda_2 \geqslant 6$ as $k \geqslant 3$. We choose $\alpha = (2) \vdash 2$, $\beta = (n - \frac{3k}{2}, \frac{3k}{2} - 2) \vdash n - 2$, and $\nu = (n - \frac{3k}{2} + 1, \frac{3k}{2} - 2, 1) \vdash n$. If $\lambda = (n - \frac{3k}{2} + 1, \frac{3k}{2} - 1)$, then $\lambda_2 \geqslant 5$ as $k \geqslant 3$. We choose $\alpha = (2) \vdash 2$, $\beta = (n - \frac{3k}{2} + 1, \frac{3k}{2} - 3) \vdash n - 2$, and $\nu = (n - \frac{3k}{2} + 2, \frac{3k}{2} - 3, 1) \vdash n$.

By Lemma 34 we get,

$$c(\sigma^{k}_{\mu(2)}) = \{\chi_\lambda \mid \lambda_1 \geqslant n - 2k\} = \{\chi_\lambda \mid \lambda_1 = n - 2k + i; 0 \leqslant i \leqslant 2k\}. \tag{11}$$

Now we compare the sets in the RHS of Equation (9) and Equation (11) by taking $k = \lfloor \frac{n}{3} \rfloor$. It is easy to check that when $n \equiv 0 (\mathrm{mod}\ 3)$, that is, $n = 3k$, then both sets are equal, whence we conclude that $c(\chi^{k}_{\mu(2)}) = c(\sigma^{k}_{\mu(2)})$. Assume now that $n = 3k + 1$. In this case, it is easy to see that $c(\sigma^{k}_{\mu(2)}) \setminus c(\chi^{k}_{\mu(2)}) = \{\chi_\lambda \mid \lambda_1 = \lambda_2 = n - 2k + i; 0 \leqslant i \leqslant \frac{k-1}{2}\}$. We claim that $c(\sigma^{k+1}_{\mu(2)}) = c(\chi^{k+1}_{\mu(2)})$. Since $c(\sigma^{k+1}_{\mu(2)}) \supseteq c(\chi^{k+1}_{\mu(2)})$, using Lemma 34 and the fact that $c(\chi^{k}_{\mu(2)}) \subseteq c(\chi^{k+1}_{\mu(2)})$, we need to show that $\chi_\lambda \in c(\chi^{k+1}_{\mu(2)})$, where λ satisfies one of the following conditions: (1) $\lambda_1 = n - 2k - i$ where $i = 1, 2$, and (2) $\lambda_1 = \lambda_2 = n - 2k + i$ where $0 \leqslant i \leqslant \frac{k-1}{2}$. To show this, we repeat the same process as done previously in **Case I** and **Case II**. More specifically, we note that $\lambda_3 > 0$ in both (1) and (2) except when k is odd and $\lambda = (n - 2k + i, n - 2k + i)$ with $i = \frac{k-1}{2}$. Thus, when $\lambda_3 > 0$, we use the argument in **Case I** above, and for the latter case, the argument in **Case II** is used. When $n = 3k + 2$, it can be proved similarly that $c(\chi^{k+1}_{\mu(2)}) = c(\sigma^{k+1}_{\mu(2)})$. Overall, we conclude that $c(\sigma^{\lceil \frac{n}{3} \rceil}_{\mu(2)}) = c(\chi^{\lceil \frac{n}{3} \rceil}_{\mu(2)})$. The proof of the theorem now follows from Lemma 17 and Lemma 18. $\square$

Remark 40. In the above proof, one can prove that Equation (9) can be strengthened to an equality.

We now move on to the proof of the second statement of Theorem 3. It is enough to consider $\lambda = (\frac{n+1}{2}, 1^{\frac{n-1}{2}})$ when n is odd, and $\lambda = (\frac{n}{2} + 1, 1^{\frac{n}{2} - 1})$ when n is even. In other words, $\lambda = \mu(k)$, where $k = \lfloor \frac{n-1}{2} \rfloor$. We will prove our theorem via a sequence of lemmas.

Lemma 41. *Let $k = \lfloor \frac{n-1}{2} \rfloor$ and $\lambda \vdash n$ be such that $d(\lambda) = 2m$, where $m \geqslant 2$. Then there exists $\alpha \vdash k$ and $\beta \vdash n - k$ such that $d(\alpha) = d(\beta) = m$ and $c^{\lambda}_{\alpha\beta} > 0$.*

Proof. At first, we define the partition $\beta \subseteq \lambda$ by constructing its Young diagram T_β as a sub-diagram of T_λ as follows:

- We choose a box (for T_β) from the first m rows of T_λ if there are at least m boxes below it in its column.

- We say that a box in T_λ satisfy **Property P** if there are at least m boxes right to it (in its row). For each $1 \leqslant j \leqslant m$, we choose those boxes (for T_β) in the j-th column of T_λ that satisfy **Property P** but aren't among the last m boxes in the same column that satisfy **Property P**.

- Note that there is an injection from the set of chosen boxes to the set of non-chosen ones. To see this injection we distinguish two cases:

  1. Consider all those chosen boxes which lie outside the top left $2m \times 2m$ square. Let X be the collection of these boxes. If $(i, j) \in X$ with $j > 2m$, then we map it to $(i + m, j)$ which is a non-chosen box. Similarly, if $(i, j) \in X$ with $i > 2m$, then we map it to $(i, j + m)$ which is once again a non-chosen box.

  2. Consider all those chosen boxes which lie inside the top left $2m \times 2m$ square. Let $A = \{(i, j) \mid 1 \leqslant i, j \leqslant m\}$, $B = \{(i + m, j) \mid 1 \leqslant i, j \leqslant m\}$, $C = \{(i, j + m) \mid 1 \leqslant i, j \leqslant m\}$, $D = \{(i + m, j + m) \mid 1 \leqslant i, j \leqslant m\}$. By our choice, note that all boxes in C are chosen, while none of the boxes of D are chosen, whence we can bijectively map C to D. Note that all the boxes of A are chosen. Moreover, there are m^2 boxes satisfying **Property P** in the first m columns that are not chosen. Let E be the collection of these boxes. Then we can bijectively map A to E. Let B' be the collection of chosen boxes of B, and E' be the collection of those boxes of E which lie below the $2m$-th row. Clearly, B' and E' are in bijection. If $(i, j) \in E'$, then $(i, j+m)$ is a non-chosen box which also lie below the $2m$-th row. Let $F = \{(i, j + m) \mid (i, j) \in E'\}$. Then, we can bijectively map B' to F.

  Therefore, the number of chosen boxes is at most $\frac{n}{2}$, which in turn is strictly less than $n - k$.

- Now if there are more boxes to choose for T_β, we do it by choosing boxes from the first column (column-wise), and then from the second column, and so on until the m-th column.

- If even more boxes are required, we choose boxes from the first row (row-wise), then from the second row, and so on until the m-th row.

- In the above steps, we stop at the point where we have chosen $(n - k)$-many boxes for T_β. Since $d(\lambda) = 2m$, the total number of boxes that are neither in the first m rows nor in the first m columns is clearly less than or equal to k. Thus, by performing the steps mentioned above, we certainly get $(n-k)$-many boxes at some stage.

- Finally, it is also clear that $d(\beta) = m$.

We give an example to illustrate the choice of β (see Figure 6). Let $\lambda = (11, 10^2, 8, 7, 6^2, 4^2, 2^3, 1) \vdash 73$. We have $d(\lambda) = 6$, $m = 3$, and $k = 36$. Thus, $\beta \vdash 37$. The red colored boxes in the first diagram of the given illustration are the ones chosen in steps 1 and 2. The second diagram is obtained from the first by performing the steps after step 2 (indicated by yellow colored boxes) and finally gives $\beta = (8, 7, 6, 3, 2^4, 1^5) \vdash 37$.

(A) (B)

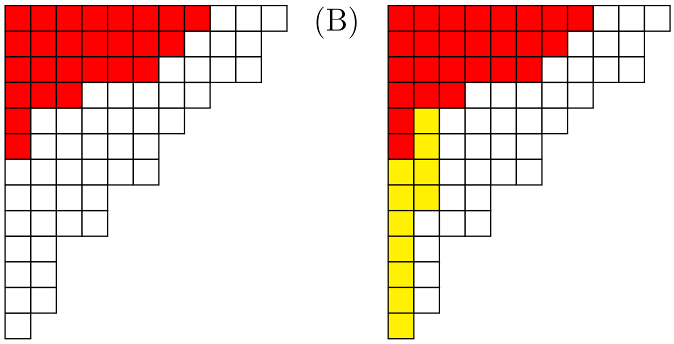


Figure 6: Construction of $\beta \subseteq \lambda$.

Now it is required to fill the skew shape $T_{\lambda/\beta}$ in a way that its type α has Durfee rank equal to m, and it is a LR tableaux. For that, we make some observations on the skew diagram $T_{\lambda/\beta}$. It is easy to see that $T_{\lambda/\beta}$ has no $(m+1) \times (m+1)$ square contained in it. We use the convention that the rows of $T_{\lambda/\beta}$ are labeled with respect to those of T_λ. With this convention, let p be the least positive integer such that the number of boxes (say r_0) in the $(2m+p)$-th row of $T_{\lambda/\beta}$ is less than m, that is, $r_0 < m$. Note that p always exists with the convention that there are zeroes after the last part of the skew partition λ/β. Let $\widetilde{T_{\lambda/\beta}}$ be the sub-diagram of $T_{\lambda/\beta}$ starting from the $(2m+p)$-th row to the last row (say $(2m+p+d)$-th row). We write $\widetilde{T_{\lambda/\beta}} = (r_0, r_1, \ldots, r_d)$, where r_i denotes the number of boxes in the $(2m+p+i)$-th row. Now we make an important observation on $\widetilde{T_{\lambda/\beta}}$. Either $r_0 \geqslant \cdots \geqslant r_d$, or there exists a positive integer c such that $r_0 \geqslant \cdots \geqslant r_{c-1}$, $r_c = r_{c-1} + 1$, and $r_c \geqslant \cdots \geqslant r_d$. To see this, consider the box (if there is any) of T_λ that lies to the immediate left of the first box of the $(2m+p)$-th row of $T_{\lambda/\beta}$. If there is no such box, then $r_0 \geqslant \cdots \geqslant r_d$. But if such a box exists, then (a) it is a part of the diagram T_β and hence it lies in some k-th column of T_λ where $1 \leqslant k \leqslant m$, (b) in the k-th column, our concerned box does not have **Property P**. By our choice of β, all the boxes from the first column to the $(k-1)$-st column are in the diagram of T_β. If all the boxes of the k-th column are also in the diagram of T_β, we once again get the first possibility, otherwise we get the second possibility.

With these observations, we are now in a position to fill $T_{\lambda/\beta}$. We divide this filling into three parts.

- For the first $2m$ rows of $T_{\lambda/\beta}$, we fill each column with numbers 1 to j in increasing order, where j is the number of boxes in that column. Clearly, $j \leqslant m$ by our choice of β. Since $d(\lambda) = 2m$, it is also clear that each of the numbers from 1 to m has been used at least m times. Observe also that the filling until the $2m$-th row is semi-standard and the reverse reading word is a lattice permutation.

- Now we fill the sub-diagram of $T_{\lambda/\beta}$ from $(2m+1)$-st to $(2m+p-1)$-st row (at this step it is assumed $p \geqslant 2$). Let this sub-diagram be written as $(s_1, \ldots, s_{p-1})$, where s_i is the number of boxes in the $(2m+i)$-th row. Then $m \leqslant s_i \leqslant 2m$ (by the choice of p). We fill the last m boxes of the $(2m+i)$-th row with the number $m+i$, and then we fill the remaining boxes of the row from left to right with the least possible numbers in a semi-standard way. Note that these "least possible numbers" will always be less than or equal to m. If not, then it implies that the diagram would contain a $(m+1) \times (m+1)$ square, a contradiction. Thus, we have a filling until the $(2m+p-1)$-st row, which still has the desired properties of a LR tableaux. Further, any number greater than m that has been used occurs exactly m-many times.

- Finally, we fill last remaining sub-diagram $\widetilde{T_{\lambda/\beta}}$, which spans from the $(2m+p)$-th row to the $(2m+p+d)$-th row. Recall the observation we made about this sub-diagram before. If $r_0 \geqslant \cdots \geqslant r_d$, we fill all the boxes of the $(2m+p+i)$-th row with $m+p+i$, where $0 \leqslant i \leqslant d$. If not, then $r_0 \geqslant \cdots \geqslant r_{c-1}$, $r_c = r_{c-1}+1$, and $r_c \geqslant \cdots \geqslant r_d$. Let e be the least positive integer such that $r_{c+e} \leqslant r_0$. If such e does not exist, we take $e = d-c+1$. Then $r_{c-1}+1 = r_c = \cdots = r_{c+e-1}$. We fill all boxes of the $(2m+p+i)$-th row with $m+p+i$, where $0 \leqslant i \leqslant c-1$. For $c \leqslant i \leqslant c+e-1$, we fill the first box of the $(2m+p+i)$-th row with $m+p+i-c$ and the remaining boxes with $m+p+i$. For $c+e \leqslant i \leqslant d$, all boxes of the $(2m+p+i)$-th row are filled with $m+p+i$ once again.

Overall, we get a LR tableaux whose type has Durfee rank m. We end our proof by filling the tableaux in our example according to the method described above (see Figure 7 below). In this case $\alpha = (9, 8^2, 3^2, 2, 1^3)$. □

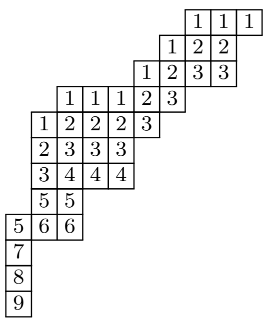


Figure 7: LR tableaux of shape $(11, 10^2, 8, 7, 6^2, 4^2, 2^3, 1)/(8, 7, 6, 3, 2^4, 1^5)$ and type $(9, 8^2, 3^2, 2, 1^3)$.

We prove an analogue of Lemma 41 when $d(\lambda)$ is odd. The proof is similar, but there are subtle changes, so we write it in detail.

Lemma 42. *Let $k = \lfloor \frac{n-1}{2} \rfloor$ and $\lambda \vdash n$ be such that $d(\lambda) = 2m-1$, where $m \geqslant 2$. Then there exists $\alpha \vdash k$ and $\beta \vdash n-k$ such that $m-1 \leqslant d(\alpha) \leqslant m$, $d(\beta) = m$ and $c^{\lambda}_{\alpha\beta} > 0$.*

Proof. As before, we define the partition $\beta \subseteq \lambda$ by constructing its Young diagram T_β as a sub-diagram of T_λ as follows:

- We choose all boxes (for T_β) from the top left $m \times m$ square of T_λ.
- We choose a box (for T_β) from the first $(m-1)$ rows of T_λ if there are at least m boxes below it in its column.
- We say that a box in T_λ satisfy **Property P** if there are at least m boxes right to it (in its row). For each $1 \leqslant j \leqslant m-1$, we choose those boxes (for T_β) in the j-th column of T_λ that satisfy **Property P** but aren't among the last $(m-1)$ boxes in the same column that satisfy **Property P**.
- Note that there is an injection from the set of chosen boxes excluding the box (m, m) to the set of non-chosen ones. To see this injection we distinguish two cases:

  1. Consider all those chosen boxes which lie outside the top left $(2m-1) \times (2m-1)$ square. Let X be the collection of these boxes. If $(i, j) \in X$ with $j > 2m-1$, then we map it to $(i+m, j)$ which is a non-chosen box. Similarly, if $(i, j) \in X$ with $i > 2m-1$, then we map it to $(i, j+m)$ which is once again a non-chosen box.
  2. Consider all those chosen boxes which lie inside the top left $(2m-1) \times (2m-1)$ square except for the box (m, m). Let $A = \{(i, j) \mid 1 \leqslant i, j \leqslant m-1\}$, $B = \{(i+m, j) \mid 1 \leqslant i, j \leqslant m-1\}$, $C = \{(i, j+m) \mid 1 \leqslant i, j \leqslant m-1\}$, $D = \{(i+m, j+m) \mid 1 \leqslant i, j \leqslant m-1\}$, $E = \{(i, m) \mid 1 \leqslant i \leqslant m-1\}$, $F = \{(m, j) \mid 1 \leqslant j \leqslant m-1\}$, $G = \{(i+m, m) \mid 1 \leqslant i \leqslant m-1\}$, $H = \{(m, j+m) \mid 1 \leqslant j \leqslant m-1\}$. By our choice, note that all boxes in C are chosen, while none of the boxes of D are chosen, whence we can bijectively map C to D. Using the same argument, we can bijectively map E to G, and also F to H. Note that all the boxes of A are chosen. Moreover, there are $(m-1)^2$ boxes satisfying **Property P** in the first $m-1$ columns that are not chosen. Let P be the collection of these boxes. Then we can bijectively map A to P. Let B' be the collection of chosen boxes of B, and P' be the collection of those boxes of P which lie below the $(2m-1)$-st row. Clearly, B' and P' are in bijection. If $(i, j) \in P'$, then $(i, j+m)$ is a non-chosen box which also lie below the $(2m-1)$-st row. Let $Q = \{(i, j+m) \mid (i, j) \in P'\}$. Then, we can bijectively map B' to Q.

Therefore, we can conclude that the number of chosen boxes excluding the box (m, m) is at most $\frac{n-1}{2}$. Since $n - k > \frac{n}{2}$, the total number of boxes chosen (which includes the box (m, m)) is at most $n - k$.

- Now, if there are more boxes to choose to form β, we do it in the following sequence:

  1. Choose boxes from the first column (column-wise), then from the second column, and so on until the $(m-1)$-st column.
  2. Choose boxes from the first row (row-wise), then from the second row, and so on until the $(m-1)$-st row.
  3. Choose boxes from the m-th column (column-wise).
  4. Choose boxes from the m-th row (row-wise).

  We stop at that point of the above sequence when the total number of boxes chosen for T_β equals $n - k$.

- Since $d(\lambda) = 2m - 1$, the total number of boxes that are neither in the first m rows nor in the first m columns is clearly less than or equal to k. Thus, by performing the steps mentioned above, we certainly get $(n - k)$-many boxes for T_β at some stage.

- Finally, it is also clear that $d(\beta) = m$.

Now it is required to fill the skew shape $T_{\lambda/\beta}$ in a way that its type α has Durfee rank either $m - 1$ or m and it is a LR tableaux. For that, we make some observations on the skew diagram $T_{\lambda/\beta}$. We use the convention that the rows of $T_{\lambda/\beta}$ are labeled with respect to those of T_λ. With this convention, we have the following observations:

- The sub-diagram of $T_{\lambda/\beta}$ from the first row to the $(2m-1)$-st row has a $m \times (m-1)$ rectangle contained in it. To see this, write $\lambda = (\lambda_1, \ldots, \lambda_{2m-1}, \lambda_{2m}, \ldots, \lambda_l)$ where $\lambda_i \geqslant 2m-1$ if $1 \leqslant i \leqslant 2m-1$, and $\lambda_i \leqslant 2m-1$ if $2m \leqslant i \leqslant l$. Let $\beta = (\beta_1, \beta_2, \ldots)$. Now assume that our assertion is not true. By our choice of β, we have the following inequalities: (a) $\beta_i = \lambda_i$ when $1 \leqslant i \leqslant m-1$, (b) $\beta_m \geqslant m$, $\beta_i = m$ when $m+1 \leqslant i \leqslant 2m-1$, and there exists $j \in \{m, m+1, \ldots, 2m-1\}$ such that $\lambda_j < \beta_m + m - 1$, and (c) $\lambda_i - \beta_i < \beta_i$ for all $2m \leqslant i \leqslant l$. Note that

$$\sum_i (2\beta_i - \lambda_i) = \sum_i \beta_i - \sum_i (\lambda_i - \beta_i) = |T_\beta| - |T_{\lambda/\beta}| \leqslant 2, \tag{12}$$

  since $\beta \vdash n - k$ and $\alpha \vdash k$. Now,

$$\sum_i (2\beta_i - \lambda_i) = \sum_{i=1}^{m-1} \lambda_i + (2\beta_m - \lambda_m) + \sum_{i=m+1}^{2m-1} (2m - \lambda_i) + \sum_{i=2m}^{l} (2\beta_i - \lambda_i).$$

The last summand in the above equation is always non-negative. In (b), if $j = m$, that is, $\lambda_m < \beta_m+m-1$, then $2\beta_m-\lambda_m > \beta_m+1-m \geqslant 1$. Also, $2m+\lambda_i-\lambda_{m+i} \geqslant 2m$ for all $1 \leqslant i \leqslant m-1$. Hence $\sum_i(2\beta_i - \lambda_i) > 2$, a contradiction. Otherwise, in (b), if $j \neq m$, then $2m-\lambda_j > m-\beta_m+1$, whence $(2\beta_m-\lambda_m)+(2m-\lambda_j) > \beta_m+m+1-\lambda_m \geqslant 2m+1-\lambda_m$. But $\lambda_1+2m+1-\lambda_m \geqslant 2m+1$. By pairing i and k, where $2 \leqslant i \leqslant m-1$ and $m+1 \leqslant k \leqslant 2m-1$ with $k \neq j$, we get that $\lambda_i + 2m - \lambda_k \geqslant 2m$ for each such pair. This yields $\sum_i(2\beta_i - \lambda_i) > 2$, once again a contradiction.

- There is no $(m+1) \times (m+1)$ square contained in $T_{\lambda/\beta}$.

- Let p be the least positive integer such that the number of boxes (say r_0) in the $(2m-1+p)$-th row of $T_{\lambda/\beta}$ is less than m, that is, $r_0 < m$. Note that p always exists with the convention that there are zeroes after the last part of the skew partition λ/β. Let $\widetilde{T_{\lambda/\beta}}$ be the sub-diagram of $T_{\lambda/\beta}$ starting from the $(2m-1+p)$-th row to the last row (say $(2m-1+p+d)$-th row). We write $\widetilde{T_{\lambda/\beta}} = (r_0, r_1, \ldots, r_d)$, where r_i denotes the number of boxes in the $(2m-1+p+i)$-th row. Then, either $r_0 \geqslant \cdots \geqslant r_d$, or there exists a positive integer c such that $r_0 \geqslant \cdots \geqslant r_{c-1}$, $r_c = r_{c-1} + 1$, and $r_c \geqslant \cdots \geqslant r_d$. The reason for this is exactly as in the previous lemma.

With these observations, we are now in a position to fill $T_{\lambda/\beta}$. We divide this filling into three parts.

For the first $2m-1$ rows of $T_{\lambda/\beta}$, we fill each column with numbers 1 to j in increasing order, where j is the number of boxes in that column. Clearly, $j \leqslant m$ by our choice of β. Also, since there is a $m \times (m-1)$ rectangle in the first $2m-1$ rows, we conclude that each of the numbers from 1 to m has been used at least $(m-1)$-many times. It is obvious that the filling till the $(2m-1)$-st row is semi-standard, and the reverse reading word is a lattice permutation. To fill the remaining part of the diagram, we distinguish two cases.

Case I: Suppose that in our filling of the first $2m-1$ rows, the number m (hence all the numbers from 1 to $m-1$) has occurred at least m times. In this case, the next two sub-diagrams are to be filled in exactly the same way as we did in the previous lemma. We observe that the type α has Durfee rank m in this case.

Case II: Suppose that in our filling of the first $2m-1$ rows, the number m has occurred exactly $(m-1)$-many times. Now we fill the sub-diagram of $T_{\lambda/\beta}$ from $2m$-th to $(2m-2+p)$-th row (at this step it is assumed that $p \geqslant 2$). Let this sub-diagram be written as $(s_1, \ldots, s_{p-1})$, where s_i is the number of boxes in the $(2m-1+i)$-th row. Then $m \leqslant s_i \leqslant 2m-1$ (by the choice of p). We fill the boxes of the $(2m-1+i)$-th row by the following two steps.

Step 1: We fill the last $m-1$ boxes of the row with the number $m+i$.

Step 2: We fill the row from left to right with least possible numbers in a semi-standard way, but with the restriction that whenever a number greater than m is required it must be the least number that has not been used m-many times (in the filling until then).

The reason we can get this done in a semi-standard way is as follows: For any $1 \leqslant i \leqslant p-1$, consider the $(2m-1+i)$-th row. Notice that this row has at most one number greater than m (ignoring the $(m+i)$'s in the last $(m-1)$ boxes of this row). Indeed, if there are more than two numbers greater than m, then there is a $(m+1)\times(m+1)$ square contained in the diagram $T_{\lambda/\beta}$, a contradiction. Suppose now that this unique number greater than m has appeared in the concerned row. Using induction on i, it easily follows that this number is certainly less than or equal to $m+i$. Thus, we conclude that our filling in step 2 is possible (that is, our filling is semi-standard). Also, the fact that the reverse reading word (until now) is a lattice permutation follows from the choice of our filling.

Now we proceed to fill the last diagram $\widetilde{T_{\lambda/\beta}}$ (at this point, recall the observation we made about this diagram before). Let t be the least positive integer such that for each $t \leqslant j \leqslant p-1$, $m+j$ has so far occurred exactly $(m-1)$-many times. We need to fill $T_{\lambda/\beta}$ from the $(2m-1+p)$-th row to the $(2m-1+p+d)$-th row. If $r_0 \geqslant \cdots \geqslant r_d$, we fill all the boxes of the $(2m-1+p+i)$-th row with $m+p+i$ where $0 \leqslant i \leqslant d$. Otherwise, $r_0 \geqslant \cdots \geqslant r_{c-1}$, $r_c = r_{c-1}+1$, and $r_c \geqslant \cdots \geqslant r_d$. Let e be the least positive integer such that $r_{c+e} \leqslant r_0$. If such e does not exist, we take $e = d-c+1$. Then $r_{c-1}+1 = r_c = \cdots = r_{c+e-1}$. We fill all boxes of the $(2m-1+p+i)$-th row with $m+p+i$ when $0 \leqslant i \leqslant c-1$. For $c \leqslant i \leqslant c+e-1$, we fill the first box of the $(2m-1+p+i)$-th row with the least possible positive number in a semi-standard way which is not between $m+1$ to $m+t-1$ (thus if $t=1$ we have no restriction), and the remaining boxes with $m+p+i$. For $c+e \leqslant i \leqslant d$, all boxes of the $(2m-1+p+i)$-th row are filled with $m+p+i$ once again.

Overall, we get a LR tableaux whose type α has Durfee rank either $m-1$ or m, and we are done. □

A homogeneous symmetric function f of degree n is called *Schur-positive* if it can be written as a non-negative integer linear combination of the Schur functions of degree n. If $f, g \in \Lambda_n$, then we say $f \geqslant g$ if $f-g$ is Schur-positive. Lemma 41 and Lemma 42 implies the following corollary which is interesting in its own right.

Corollary 43. *Let $n \geqslant 5$ and $m \geqslant 2$. Assume that $k = \lfloor \frac{n-1}{2} \rfloor$. Then,*

1. $$\Big(\sum_{\substack{\alpha\vdash k\\ d(\alpha)=m}} s_\alpha\Big)\Big(\sum_{\substack{\beta\vdash n-k\\ d(\beta)=m}} s_\beta\Big) \geqslant \sum_{\substack{\lambda\vdash n\\ d(\lambda)=2m}} s_\lambda,$$

2. $$\Big(\sum_{\substack{\alpha\vdash k\\ m-1\leqslant d(\alpha)\leqslant m}} s_\alpha\Big)\Big(\sum_{\substack{\beta\vdash n-k\\ d(\beta)=m}} s_\beta\Big) \geqslant \sum_{\substack{\lambda\vdash n\\ d(\lambda)=2m-1}} s_\lambda.$$

Lemma 44. *Let $k > 0$ and $\alpha \vdash k, \beta \vdash n-k$ be such that $d(\alpha) = d(\beta) = m$. Then there exists $\eta \vdash n$ such that $d(\eta) = m$ and $c^{\eta}_{\alpha\beta} > 0$. The same conclusion is valid if $d(\alpha) = m-1$ and $d(\beta) = m$.*

Proof. Let $\alpha = (\alpha_1, \alpha_2, \ldots) \vdash k$ and $\beta = (\beta_1, \beta_2, \ldots) \vdash n-k$. We have $\alpha_1 \geqslant \cdots \geqslant \alpha_m \geqslant m$ and $\alpha_{m+1} \leqslant m$. The same holds true for β. Consider the Young diagram of α and fill all the boxes in its i-th row with i. Construct a partition $\eta = (\eta_1, \eta_2, \ldots) \vdash n$ by adjoining the (filled) Young diagram of α to β as follows: For $1 \leqslant i \leqslant m$, the i-th row of T_α is adjoined to the i-th row of T_β. The remaining rows are appended one after the other by putting their boxes one-by-one below the first column, then the second column, and so on until required. Since $\alpha_i \leqslant m$ when $i \geqslant m+1$, it is not required to go beyond the m-th column for appending these rows. By the construction of η, it easily follows that $\eta \vdash n$ has Durfee rank m, $\beta \subseteq \eta$, and η/β is a LR tableaux of type α, whence $c^{\eta}_{\alpha\beta} > 0$ as required. The same construction works if we take $d(\alpha) = m-1$ and $d(\beta) = m$, whence $c^{\eta}_{\alpha\beta} > 0$. We conclude the proof by illustrating the construction with a simple example (see Figure 8 below). Let $\alpha = (6, 6, 3, 3, 3, 2, 1) \vdash 24$ and $\beta = (5, 4, 3, 3, 1) \vdash 16$. Note that $d(\alpha) = d(\beta) = 3$. □

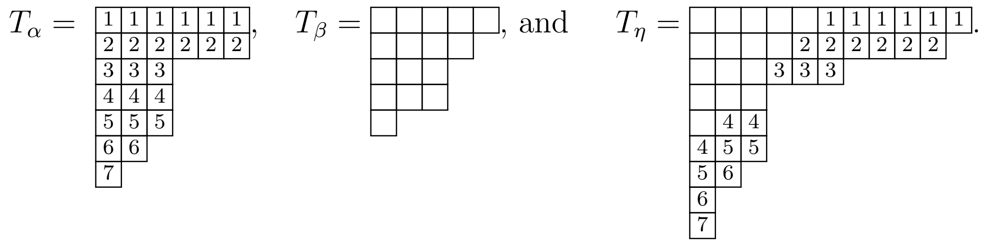


Figure 8: Construction of η.

The following easy observation will be the final ingredient for the proof of Theorem 3(2).

Lemma 45. *Let $\mu \vdash m$, $\nu \vdash n$, and $\lambda \vdash m+n$. If $c^{\lambda}_{\mu\nu} > 0$, then $d(\lambda) \leqslant d(\mu) + d(\nu)$.*

Proof. Consider the skew shape $T_{\lambda/\mu}$. Clearly, it has a square of size $d(\lambda) - d(\mu)$. Assume that $d(\lambda) - d(\mu) > d(\nu)$. By assumption, $c^{\lambda}_{\mu\nu} > 0$, and hence there exists a LR tableaux of shape λ/μ and type ν. Consider such a filling. In this filling, any number greater than $d(\nu)$ must occur at most $d(\nu)$ times. Since the filling is semi-standard, the left bottom corner of the square must have a filling, say a, where $a > d(\nu)$. The last row of the square must be filled with a sequence of numbers that is weakly increasing. Since the reverse reading word of the filling must also be a lattice permutation, if we traverse the row from right to left, we observe that for each entry in this row, we must have filled the diagram with the number a in some box above or in this row. This accounts for at least $(d(\lambda) - d(\mu))$-many occurrences of a in the diagram, a contradiction. □

Proof of Theorem 3(2). Let $k = \lfloor \frac{n-1}{2} \rfloor$. We claim that $c(\chi^2_{\mu(k)})$ consists of all χ_λ's such that $d(\lambda) \leqslant 2$ except possibly the sign character. This can be easily verified using Theorem 37. Indeed, if $\lambda = (n-r, 1^r)$ where $0 \leqslant r \leqslant n-2$, the result follows easily by taking $e = f = k$ in Theorem 37. Let $\lambda = (n_4, n_3, 2^{d_2}, 1^{d_1})$ be a double hook as in Theorem 37. Then $n_4 + n_3 = n - x$ where $x = 2d_2 + d_1$. Notice that the first summand in Theorem 37(4) is always 1. This yields that $g_{\mu(k)\mu(k)\lambda} > 0$, thereby establishing our claim.

Using Theorem 37(2), $\epsilon \in c(\chi^2_{\mu(k)})$ when n is odd and $\epsilon \notin c(\chi^2_{\mu(k)})$ when n is even. Thus, when $n = 7$, $c(\chi^2_{\mu(k)}) = \mathrm{Irr}(S_7)$ and the result follows. Observe that $\chi_{\mu(k)} \in c(\chi^2_{\mu(k)})$ and hence $c(\chi^i_{\mu(k)}) \subseteq c(\chi^{i+1}_{\mu(k)})$ for $i \geqslant 2$. Since the sign character ϵ clearly belongs to $c(\chi^3_{\mu(k)})$, we conclude that $\epsilon \in c(\chi^i_{\mu(k)})$ for $i \geqslant 3$. Thus, when $n = 8$, $c(\chi^3_{\mu(k)}) = \mathrm{Irr}(S_8)$ and the result follows. Therefore, for the rest of the proof we may assume $n \geqslant 9$.

Now we claim that $c(\chi^i_{\mu(k)}) \setminus \{\epsilon\} = \{\chi_\lambda \mid d(\lambda) \leqslant 2^{i-1}\} \setminus \{\epsilon\}$ for every $2 \leqslant i \leqslant l$, where l is the least integer such that $2^{l-1} \geqslant \lfloor \sqrt{n} \rfloor$. Setting $i = l$ in the claim, we obtain that $c(\chi^l_{\mu(k)}) = \mathrm{Irr}(S_n)$. Since $c(\chi^{l-1}_{\mu(k)}) \subsetneq \mathrm{Irr}(S_n)$, our theorem is proved once we establish our claim. We prove it using induction on i. For $i = 2$, we have the claim from the first paragraph of this proof. Let us assume it is true for all i such that $2 \leqslant i < r \leqslant l$. We prove the claim for $i = r$. We first show that $\{\chi_\lambda \mid d(\lambda) \leqslant 2^{r-1}\} \setminus \{\epsilon\} \subseteq c(\chi^r_{\mu(k)})$. Let $(1^n) \neq \lambda \vdash n$ be such that $2 < d(\lambda) \leqslant 2^{r-1}$. By using Lemma 41, Lemma 42, and Lemma 44, we conclude that there exist $\alpha \vdash k$, $\beta \vdash n-k$, and $\eta \vdash n$ such that $d(\eta) = \lceil \frac{d(\lambda)}{2} \rceil$ and $c^\lambda_{\alpha\beta} c^\eta_{\alpha'\beta} > 0$, whence Lemma 38 yields that $g_{\lambda\mu(k)\eta} + g_{\lambda\mu(k-1)\eta} > 0$. Since $d(\eta) \leqslant 2^{r-2}$, by induction hypothesis, $\chi_\eta \in c(\chi^{r-1}_{\mu(k)})$. Thus, if $g_{\lambda\mu(k)\eta} > 0$, then $\chi_\lambda \in c(\chi_{\mu(k)}\chi_\eta) \subseteq c(\chi^r_{\mu(k)})$ as desired. Otherwise, $g_{\lambda\mu(k-1)\eta} > 0$, whence $\chi_\lambda \in c(\chi_{\mu(k-1)}\chi_\eta) \subseteq c(\chi_{\mu(k-1)}\chi^{r-1}_{\mu(k)}) \subseteq c(\chi^r_{\mu(k)})$ by Lemma 39 as desired. Now we show that $c(\chi^r_\lambda) \setminus \{\epsilon\} \subseteq \{\chi_\lambda \mid d(\lambda) \leqslant 2^{r-1}\}$. Let $\chi_\nu \in c(\chi^r_\lambda) \setminus \{\epsilon\}$. By induction hypothesis, we can conclude that $\chi_\nu \in c(\chi_\lambda \chi_{\mu(k)})$ for some λ with $d(\lambda) \leqslant 2^{r-2}$. This implies that $g_{\lambda\mu(k)\nu} > 0$, whence by Lemma 38, we conclude that there exist $\alpha \vdash k, \beta \vdash n-k$ such that $c^\lambda_{\alpha\beta} c^\nu_{\alpha'\beta} > 0$. Since $d(\lambda) \leqslant 2^{r-2}$, we obtain that $d(\alpha), d(\beta) \leqslant 2^{r-2}$. Using Lemma 45, we get $d(\nu) \leqslant 2^{r-1}$ which yields the desired inclusion. □

Remark 46. If $\lambda = (\frac{n+1}{2}, 1^{\frac{n-1}{2}})$ where n is odd, then $\mathrm{ccn}(\sigma_\lambda; S_n) = 2$. If n is even, $\mathrm{ccn}(\sigma_\lambda; S_n) = 2$ when $\lambda = (\frac{n}{2}, 1^{\frac{n}{2}})$, and $\mathrm{ccn}(\sigma_\lambda; S_n) = 3$ when $\lambda = (\frac{n}{2}+1, 1^{\frac{n}{2}-1})$. In these cases, $\mathrm{ccn}(\sigma_\lambda; S_n)$ is much less than $\mathrm{ccn}(\chi_\lambda; S_n)$. In contrast, when λ is $(n-2, 2)$ or $(\frac{n+1}{2}, \frac{n-1}{2})$ (when n is odd), we have seen that $\mathrm{ccn}(\sigma_\lambda; S_n) = \mathrm{ccn}(\chi_\lambda; S_n)$. Moreover, the same is conjectured for all the irreducible characters indexed by two-row partitions (Conjecture 36).

6 Proof of Theorem 4

We briefly discuss the irreducible characters of A_n to set down the notations. We denote $\mathrm{Res}^{S_n}_{A_n} \chi_\lambda$ by $\chi_\lambda \downarrow$. Recall that $m(\pi)$ denotes the cycle-type of $\pi \in S_n$ and is a partition of n. Let $\mathrm{DOP}(n)$ denote the set of all partitions of n with distinct and odd parts. Further, let $\mathrm{SP}(n)$ denote the set of all self-conjugate partitions of n. The folding algorithm defines a bijection $\varphi : \mathrm{DOP}(n) \to \mathrm{SP}(n)$ (see [26, Lemma 4.6.16]). For $\mu \in \mathrm{DOP}(n)$, the conjugacy class of S_n parameterized by μ (say C_μ) splits into two conjugacy classes of A_n of equal size, that is, $C_\mu = C^+_\mu \sqcup C^-_\mu$. As convention, we assume that $w_\mu \in C^+_\mu$. Set $w^+_\mu := w_\mu$ and w^-_μ to be a fixed element of C^-_μ. If $\mu \in \mathrm{DOP}(n)$, we write $\mu = (2m_1+1, 2m_2+1, \dots)$. The following theorem describes the irreducible characters of A_n and their character values.

Theorem 47. *[26, Theorem 5.12.5] Let $\lambda \vdash n$. Then:*

1. *If $\lambda \neq \lambda'$, then $\chi_\lambda \downarrow$ is an irreducible character of A_n. Further, $\chi_\lambda \downarrow = \chi_{\lambda'} \downarrow$.*

2. *If $\lambda = \lambda'$, then $\chi_\lambda \downarrow$ decomposes into two irreducible characters of A_n, say χ_λ^+ and χ_λ^-, that is, $\chi_\lambda \downarrow = \chi_\lambda^+ + \chi_\lambda^-$. Moreover, for any odd permutation π, $\chi_\lambda^-(w) = \chi_\lambda^+(\pi w \pi^{-1})$ for all $w \in A_n$.*

3. *We have $\chi_\lambda^+(w) = \chi_\lambda^-(w) = \chi_\lambda(w)/2$ unless $m(w)$ is the partition $\mu := \varphi^{-1}(\lambda)$ having distinct and odd parts, in which case*

$$\chi_\lambda^\pm(w_\mu^+) = \frac{1}{2}\left(\epsilon_\mu \pm \sqrt{\epsilon_\mu z_\mu}\right),$$

and $\chi_\lambda^\pm(w_\mu^-) = \chi_\lambda^\mp(w_\mu^+)$. Here, $\epsilon_\mu = (-1)^{m_1+m_2+\cdots}$ and z_μ is the size of the centralizer of w_μ in S_n.

We begin with an important lemma whose proof is easy.

Lemma 48. *Let $\lambda, \mu \vdash n$ be such that $\lambda \neq \lambda'$ and $\mu = \mu'$. Then $c(\chi_\lambda \downarrow \chi_\mu^+) \setminus \{\chi_\mu^\pm\} = c(\chi_\lambda \downarrow \chi_\mu^-) \setminus \{\chi_\mu^\pm\}$.*

Proof. Let $\nu \vdash n$ be such that $\nu \neq \nu'$. Let $\theta = \varphi^{-1}(\mu)$. Using Theorem 47, we get

$$\langle \chi_\lambda \downarrow \chi_\mu^+, \chi_\nu \downarrow \rangle = \frac{2}{n!}\left[\sum_{\substack{w \in A_n \\ m(w) \neq \theta}} \frac{\chi_\lambda(w)\chi_\mu(w)\chi_\nu(w)}{2} + \frac{|C_\theta|}{2}\chi_\lambda(w_\theta)\chi_\nu(w_\theta)\left(\chi_\mu^+(w_\theta^+) + \chi_\mu^+(w_\theta^-)\right)\right]. \quad (13)$$

Now expanding the inner-product $\langle \chi_\lambda \downarrow \chi_\mu^-, \chi_\nu \downarrow \rangle$ as above and using the fact $\chi_\mu^\pm(w_\theta^-) = \chi_\mu^\mp(w_\theta^+)$, we conclude that $\langle \chi_\lambda \downarrow \chi_\mu^+, \chi_\nu \downarrow \rangle = \langle \chi_\lambda \downarrow \chi_\mu^-, \chi_\nu \downarrow \rangle$. If $\nu \vdash n$ is self-conjugate and $\nu \neq \mu$, a similar computation yields $\langle \chi_\lambda \downarrow \chi_\mu^+, \chi_\nu^\pm \rangle = \langle \chi_\lambda \downarrow \chi_\mu^-, \chi_\nu^\pm \rangle$. ☐

The following theorem of Bessenrodt and Behns will be required.

Theorem 49. *[6, Theorem 5.1] Let $n \geqslant 5$ and $\lambda, \mu \vdash n$. Let $d(\chi_\lambda \chi_\mu) = \max\{d(\nu) \mid \chi_\nu \in c(\chi_\lambda \chi_\mu)\}$. Then $d(\chi_\lambda \chi_\mu) = 1$ if and only if one of them is $\chi_{(n)}$ or $\chi_{(1^n)}$, and the other one is $\chi_{(n-r,1^r)}$, where $0 \leqslant r \leqslant n-1$.*

Before moving further, we make the following observation: Suppose $\nu \vdash n$ with $d(\nu) = 2$. Then $\nu = \nu'$ implies that n is even. Indeed, if $\nu = \nu'$, then the unfolding $\varphi^{-1}(\nu)$ is a partition of n with two distinct and odd parts, whence n is even. As a result, if n is odd, then $\chi_\nu \downarrow$ is an irreducible character of A_n.

Lemma 50. *Let $n \geqslant 5$ be odd and $k = \frac{n-1}{2}$. Then, for every $1 \leqslant r \leqslant \frac{n-3}{2}$, there exists $\nu \vdash n$ with $d(\nu) = 2$ and $\chi_\nu \downarrow \in c(\chi_{\mu(r)} \downarrow \chi_{\mu(k)}^+) \cap c(\chi_{\mu(r)} \downarrow \chi_{\mu(k)}^-)$.*

Proof. Using Theorem 37 and Theorem 49, we conclude that there exists $\nu \vdash n$ with $d(\nu) = 2$ and $\chi_\nu \in c(\chi_{\mu(r)}\chi_{\mu(k)})$. This yields that $\chi_\nu \downarrow \in c(\chi_{\mu(r)} \downarrow \chi_{\mu(k)} \downarrow)$, whence our assertion holds by Lemma 48. ☐

Next, we determine the irreducible constituents of $\chi^{\pm 2}_{\mu(k)}$ and $\chi^{+}_{\mu(k)}\chi^{-}_{\mu(k)}$, where n is odd and $k = \frac{n-1}{2}$.

Lemma 51. *Let $n \geqslant 5$ be odd and $k = \frac{n-1}{2}$. We have the following:*

1. *If $n \equiv 3 (\text{mod } 4)$, then* $\chi^{\pm 2}_{\mu(k)} = \sum\limits_{\substack{0 \leqslant i \leqslant \frac{n-3}{2} \\ k \equiv i (\text{mod } 2)}} \chi_{\mu(i)} \downarrow + \sum\limits_{\substack{\{\nu,\nu'\} \\ d(\nu)=2}} \chi_\nu \downarrow + \chi^{\mp}_{\mu(k)}$.

2. *If $n \equiv 1 (\text{mod } 4)$, then* $\chi^{\pm 2}_{\mu(k)} = \sum\limits_{\substack{0 \leqslant i \leqslant \frac{n-3}{2} \\ k \equiv i (\text{mod } 2)}} \chi_{\mu(i)} \downarrow + \sum\limits_{\substack{\{\nu,\nu'\} \\ d(\nu)=2}} \chi_\nu \downarrow + \chi^{\pm}_{\mu(k)}$.

3. $\chi^{+}_{\mu(k)}\chi^{-}_{\mu(k)} = \sum\limits_{\substack{0 \leqslant i \leqslant \frac{n-3}{2} \\ k \equiv i+1 (\text{mod } 2)}} \chi_{\mu(i)} \downarrow + \sum\limits_{\substack{\{\nu,\nu'\} \\ d(\nu)=2}} \chi_\mu \downarrow.$

Proof. Since we have assumed that n is odd, we conclude that $\nu \neq \nu'$ if $d(\nu) = 2$. Using Theorem 37, we have the following decomposition of $\chi_{\mu(k)} \downarrow^2$ into irreducible characters of A_n.

$$\chi_{\mu(k)} \downarrow^2 = 2 \sum_{0 \leqslant i \leqslant \frac{n-3}{2}} \chi_{\mu(i)} \downarrow + 4 \sum_{\substack{\{\nu,\nu'\} \\ d(\nu)=2}} \chi_\nu \downarrow + \chi^{+}_{\mu(k)} + \chi^{-}_{\mu(k)}. \tag{14}$$

Notice that $\varphi^{-1}(\mu(k)) = (n)$. Hence, $\chi^{\pm}_{\lambda}(w^{+}_{(n)}) = \frac{1}{2}((-1)^k \pm \sqrt{(-1)^k n})$. Let $0 \leqslant i \leqslant \frac{n-3}{2}$. Note that $4\chi^{+2}_{\mu(k)}(\pi) - \chi_{\mu(k)} \downarrow^2 (\pi) = 0$ if $\pi \in A_n$ and $m(\pi) \neq (n)$. Further,

$$\begin{aligned} & 4\chi^{+2}_{\mu(k)}(w^{+}_{(n)}) - \chi_{\mu(k)} \downarrow^2 (w^{+}_{(n)}) + 4\chi^{+2}_{\mu(k)}(w^{-}_{(n)}) - \chi_{\mu(k)} \downarrow^2 (w^{-}_{(n)}) \\ = \; & ((-1)^k + \sqrt{(-1)^k n})^2 + ((-1)^k - \sqrt{(-1)^k n})^2 - 2 = 2((-1)^{2k} + (-1)^k n) - 2 \\ = \; & 2(-1)^k n. \end{aligned}$$

Since $\chi_{\mu(i)} \downarrow (w^{+}_{(n)}) = \chi_{\mu(i)} \downarrow (w^{-}_{(n)}) = (-1)^i$, we get

$$\langle 4\chi^{+2}_{\mu(k)} - \chi_{\mu(k)} \downarrow^2, \chi_{\mu(i)} \downarrow \rangle = 2(-1)^{k+i}.$$

This yields

$$\langle \chi^{+2}_{\mu(k)}, \chi_{\mu(i)} \downarrow \rangle = \frac{1}{4} \langle \chi_{\mu(k)} \downarrow^2, \chi_{\mu(i)} \downarrow \rangle + \frac{1}{2}(-1)^{k+i} = \frac{1}{2}(1 + (-1)^{k+i}).$$

The last equality follows from Equation (14). Similar computations yield

(a) $\langle \chi^{-2}_{\mu(k)}, \chi_{\mu(i)} \downarrow \rangle = \frac{1}{4} \langle \chi_{\mu(k)} \downarrow^2, \chi_{\mu(i)} \downarrow \rangle + \frac{1}{2}(-1)^{k+i} = \frac{1}{2}(1 + (-1)^{k+i})$.

(b) $\langle \chi^{+}_{\mu(k)}\chi^{-}_{\mu(k)}, \chi_{\mu(i)} \downarrow \rangle = \frac{1}{4} \langle \chi_{\mu(k)} \downarrow^2, \chi_{\mu(i)} \downarrow \rangle + \frac{1}{2}(-1)^{k+i+1} = \frac{1}{2}(1 + (-1)^{k+i+1})$.

Thus,

$$\langle \chi^{\pm 2}_{\mu(k)}, \chi_{\mu(i)}\downarrow\rangle = \begin{cases} 1 & \text{if } k \equiv i (\text{mod } 2),\\ 0 & \text{otherwise,}\end{cases} \text{ and } \langle \chi^{+}_{\mu(k)}\chi^{-}_{\mu(k)}, \chi_{\mu(i)}\downarrow\rangle = \begin{cases} 1 & \text{if } k \equiv i+1 (\text{mod } 2),\\ 0 & \text{otherwise.}\end{cases}$$

Let $\nu \vdash n$ be such that $d(\nu) = 2$. Using Equation (14) and the fact that $\chi_\nu(w_{(n)}) = 0$, we obtain

$$\langle \chi^{\pm 2}_{\mu(k)}, \chi_\nu\downarrow\rangle = \langle \chi^{+}_{\mu(k)}\chi^{-}_{\mu(k)}, \chi_\nu\downarrow\rangle = \frac{1}{4}\langle \chi_{\mu(k)}\downarrow^2, \chi_\nu\downarrow\rangle = 1.$$

It remains to determine the memberships of both $\chi^{+}_{\mu(k)}$ and $\chi^{-}_{\mu(k)}$ in $\chi^{\pm 2}_{\mu(k)}$ and $\chi^{+}_{\mu(k)}\chi^{-}_{\mu(k)}$. Since $\chi_{\mu(k)}\downarrow^2 = \chi^{+2}_{\mu(k)} + \chi^{-2}_{\mu(k)} + 2\chi^{+}_{\mu(k)}\chi^{-}_{\mu(k)}$, using Equation (14) we can conclude that $\chi^{\pm}_{\mu(k)} \notin c(\chi^{+}_{\mu(k)}\chi^{-}_{\mu(k)})$. Assume that $n \equiv 3 (\text{mod } 4)$. In this case, we have $\chi^{-}_{\mu(k)} = \overline{\chi^{+}_{\mu(k)}}$ and hence it is easily seen that $\langle \chi^{+2}_{\mu(k)}, \chi^{-}_{\mu(k)}\rangle = \langle \chi^{-2}_{\mu(k)}, \chi^{+}_{\mu(k)}\rangle = 1$. When $n \equiv 1 (\text{mod } 4)$, a direct computation yields the desired result. □

We are now ready to prove the theorem.

Proof of Theorem 4. Let $k = \lceil \frac{n-1}{2} \rceil$. Let $n \in \{6, 8\}$. We have seen in the proof of Theorem 3(2) that $c(\chi^2_{\mu(k)}) = \mathrm{Irr}(S_n) \setminus \{\epsilon\}$. Thus, $c(\chi_{\mu(k)}\downarrow^2) = \mathrm{Irr}(A_n)$ and the result follows in this case.

Since $d(\lambda) = d(\lambda')$, using the proof of Theorem 3(2), we conclude that for $n \geqslant 9$ we have $\mathrm{ccn}(\chi_{\mu(k)}\downarrow; A_n) = \mathrm{ccn}(\chi_{\mu(k)}; S_n)$. Thus, when $n \geqslant 10$ is even, the result once again follows by Theorem 3(2). Assume that n is odd. Note that $\chi_{\mu(k)}\downarrow^3 = \chi^{+3}_{\mu(k)} + \chi^{-3}_{\mu(k)} + 3\chi^{+2}_{\mu}\chi^{-}_{\mu(k)} + 3\chi^{+}_{\mu(k)}\chi^{-2}_{\mu(k)}$. We claim that $c(\chi^{+3}_{\mu}) = c(\chi^{-3}_{\mu(k)}) \supseteq c(\chi^{+2}_{\mu}\chi^{-}_{\mu(k)}) = c(\chi^{+}_{\mu(k)}\chi^{-2}_{\mu(k)})$. We first show that $c(\chi^{+3}_{\mu(k)}) \supseteq c(\chi^{+}_{\mu(k)}\chi^{-2}_{\mu(k)})$. Using the previous lemma, we have the following:

$$\chi^{+3}_{\mu(k)} = \sum_{\substack{0\leqslant i\leqslant \frac{n-3}{2}\\ k\equiv i(\text{mod } 2)}} (\chi_{\mu(i)}\downarrow \chi^{+}_{\mu(k)}) + \sum_{\substack{\{\nu,\nu'\}\\ d(\nu)=2}} (\chi_\nu\downarrow \chi^{+}_{\mu(k)}) + \chi\chi^{+}_{\mu(k)}, \tag{15}$$

where χ is $\chi^{-}_{\mu(k)}$ (resp. $\chi^{+}_{\mu(k)}$) when $n \equiv 3 (\text{mod } 4)$ (resp. $n \equiv 1 (\text{mod } 4)$). Also,

$$\chi^{+}_{\mu(k)}\chi^{-2}_{\mu(k)} = \sum_{\substack{0\leqslant i\leqslant \frac{n-3}{2}\\ k\equiv i(\text{mod } 2)}} (\chi_{\mu(i)}\downarrow \chi^{+}_{\mu(k)}) + \sum_{\substack{\{\nu,\nu'\}\\ d(\nu)=2}} (\chi_\nu\downarrow \chi^{+}_{\mu(k)}) + \chi'\chi^{+}_{\mu(k)}, \tag{16}$$

where χ' is $\chi^{+}_{\mu(k)}$ (resp. $\chi^{-}_{\mu(k)}$) when $n \equiv 3 (\text{mod } 4)$ (resp. $n \equiv 1 (\text{mod } 4)$). The first two summands of both equations are the same. The last summands in the above two equations differ only by the irreducible characters $\chi_{\mu(i)}\downarrow$ where $0 \leqslant i \leqslant \frac{n-3}{2}$, and possibly one of $\chi^{+}_{\mu(k)}$ or $\chi^{-}_{\mu(k)}$. By Lemma 50, for $1 \leqslant i \leqslant \frac{n-3}{2}$, we conclude that all $\chi_{\mu(i)}\downarrow$ appear as constituents of the second summand in both the equations. Further, both $\chi^{+}_{\mu(k)}$ and $\chi^{-}_{\mu(k)}$ appear as constituents of the second summand once again, by Lemma 51. Finally, note that $\chi\chi^{+}_{\mu(k)}$ contains the trivial character as well, whence the result follows. Now,

$$\chi^{-}_{\mu(k)}\chi^{+2}_{\mu(k)} = \sum_{\substack{0\leqslant i\leqslant \frac{n-3}{2}\\ k\equiv i(\text{mod } 2)}} (\chi_{\mu(i)}\downarrow \chi^{-}_{\mu(k)}) + \sum_{\substack{\{\nu,\nu'\}\\ d(\nu)=2}} (\chi_\nu\downarrow \chi^{-}_{\mu(k)}) + \chi\chi^{-}_{\mu(k)}. \tag{17}$$

Comparing Equation (16), Equation (17), and using Lemma 48, the first two summands of both the equations have the same irreducible constituents except possibly $\chi^{\pm}_{\mu(k)}$. The third summands of both equations are the same when $n \equiv 1 (\mathrm{mod}\ 4)$, and differ only by the irreducible characters $\chi^{\pm}_{\mu(k)}$ when $n \equiv 3 (\mathrm{mod}\ 4)$, by Lemma 51. Using Lemma 51 once again, when $d(\nu) = 2$, $\{\chi^{+}_{\mu(k)}, \chi^{-}_{\mu(k)}\} \subseteq c(\chi_\nu \downarrow \chi^{\pm}_{\mu(k)})$. This implies $\{\chi^{+}_{\mu(k)}, \chi^{-}_{\mu(k)}\} \subseteq c(\chi^{+}_{\mu(k)}\chi^{-2}_{\mu(k)}) \cap c(\chi^{-}_{\mu(k)}\chi^{+2}_{\mu(k)})$. We conclude that $c(\chi^{+2}_{\mu}\chi^{-}_{\mu(k)}) = c(\chi^{+}_{\mu(k)}\chi^{-2}_{\mu(k)})$, as required. Finally,

$$\chi^{-3}_{\mu(k)} = \sum_{\substack{0 \leqslant i \leqslant \frac{n-3}{2} \\ k \equiv i (\mathrm{mod}\ 2)}} (\chi_{\mu(i)} \downarrow \chi^{-}_{\mu(k)}) + \sum_{\substack{\{\nu,\nu'\} \\ d(\nu)=2}} (\chi_\nu \downarrow \chi^{-}_{\mu(k)}) + \chi' \chi^{-}_{\mu(k)}. \tag{18}$$

Comparing the above equation with Equation (15) and using similar arguments as above, it follows that $c(\chi^{+3}_{\mu(k)}) = c(\chi^{-3}_{\mu(k)})$. Thus, the claim is established and we conclude that $c(\chi^{+3}_{\mu(k)}) = c(\chi^{-3}_{\mu(k)}) = c(\chi_{\mu(k)} \downarrow^3)$. From Lemma 51, it is clear that $\mathrm{ccn}(\chi^{\pm}_{\mu(k)}; A_n) \geqslant 3$. Let $n = 5, 7$. Since $\mathrm{ccn}(\chi_{\mu(k)} \downarrow; A_n) = 2$, using Lemma 16(1), we get $c(\chi_{\mu(k)} \downarrow^3) = \mathrm{Irr}(A_n)$, whence the result follows for $n = 5, 7$. For $n \geqslant 9$, using Theorem 3(2) and Lemma 17, we get the desired result. □

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