A Three-Regime Theorem for flow-firing

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Abstract

Graphical chip-firing is a discrete dynamical system where chips are placed on the vertices of a graph and exchanged via simple firing moves. Recent work has sought to generalize chip-firing on graphs to higher dimensions, wherein graphs are replaced by cellular complexes and chip-firing becomes flow-rerouting along the faces of the complex. Given such a system, it is natural to ask (1) whether this firing process terminates and (2) if it terminates uniquely (i.e. is confluent). In the graphical case, these questions were definitively answered by Bjorner–Lovasz–Shor, who developed three regimes which completely determine if a given system will terminate. Building on the work of Duval–Klivans–Martin and Felzenszwalb-Klivans, we answer these questions in a context called flow-firing, where the cellular complexes are 2-dimensional.

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1 Introduction

This paper concerns **flow-firing**, a higher-dimensional analogue of chip-firing. In classical chip-firing chips are placed on the vertices of a graph and move to other vertices via local moves dictated by the graph Laplacian [10]. Flow-firing is a discrete model for cell complexes, where flow along one-dimensional edges is diverted over two-dimensional faces as dictated by the combinatorial Laplacian. Higher dimensional chip-firing was introduced in [3] where the focus was on algebraic considerations. The study of the dynamical properties of flow-firing was initiated in [4], and is continued here. Consider the following

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two questions concerning the fundamental behavior of termination and confluence for flow-firing:

- For which initial configurations does the flow-firing process terminate?
- For which initial configurations does the flow-firing process terminate uniquely?

In the classical case of chip-firing on a graph the answer to the first question is provided by a theorem of Björner, Lovasz and Shor [2]. Their result identifies three regimes for chip-firing behavior. Informally, the theorem states that: (1) If the number of chips is small enough then the process always terminates; (2) If the number of chips is large enough then the process never terminates; and (3) If the number of chips is in a middle range, then one can always find a configuration which terminates and one that does not, see Theorem 6 for a precise statement.

In the graphical case, if a configuration terminates it always does so uniquely, rendering the second question unnecessary. In flow-firing, however, not all initial configurations terminate uniquely. An important setting that terminates uniquely was identified in [4]. Furthermore the authors conjectured a related much larger class of initial configurations would terminate uniquely. Building from that work, our main result reveals a subtler behaviour than originally conjectured, resulting in a generalization of the Three-Regime Theorem for uniqueness of termination for flow-firing.

Denote by $\mathsf{K}(n, r)$ the initial configuration known as the *pulse* of height n and radius r (see Section 2).

Theorem 1 (Three-Regime Theorem for Flow-Firing). *Let $r, n \in \mathbb{Z}_{\geqslant 0}$.*

1. *If $r \leqslant 1$, then firing from $\mathsf{K}(n, r)$ terminates uniquely in the Aztec diamond.*

2. *If $2 \leqslant r \leqslant \lceil \frac{n}{2} \rceil$, then firing from $\mathsf{K}(n, r)$*

    (a) *does not terminate uniquely, but*

    (b) *can terminate in the Aztec diamond.*

3. *If $r > \lceil \frac{n}{2} \rceil$, then*

    (a) *$\mathsf{K}(n, r)$ does not terminate uniquely, and*

    (b) *if $r \geqslant \lceil \frac{n}{\sqrt{3}} \rceil + 1$, then firing from $\mathsf{K}(n, r)$ will never terminate in the Aztec diamond.*

Questions of termination and uniqueness have also been considered in other variations of chip-firing. Two notable examples are labeled chip-firing [7, 8, 9] and root-system chip-firing [5, 6] where the emphasis is on local versus global confluence properties. We prove confluence (unique termination) for subsystems of flow-firing including certain path-firings (Corollary 9).

The remainder of the paper proceeds as follows. Section 2 gives background on flow-firing. We introduce the notion of firing along a path and study its behavior in Section 3. In Sections 4, 5 and 6 we prove Theorem 1, addressing each of the three regimes in turn. We then conclude in Section 6.3 by discussing challenges to improving the bounds of Theorem 1.

Acknowledgements

2   Background

Following [4], we consider flow-firing on the two dimensional grid complex. A **flow configuration** is an assignment of integer values to the edges of the complex. Visually, we depict flow as oriented, with positive values oriented to the right and up, and negative values oriented to the left and down. For example, the flow configuration on the left in Figure 1 has flow value zero on all edges except the center edge, which has a value of 2 and is depicted as a downward facing arrow and labeled with the magnitude 2.

The **degree** of an edge is the number of two dimensional faces in which it is contained. An edge can **fire** if it has at least as many units of flow as its degree. In firing an edge, one unit of flow is diverted around each of the two dimensional cells containing it; an example is shown in Figure 1. In visualizations, an edge with 1 unit of flow will have a directed arrow but we suppress the number 1.
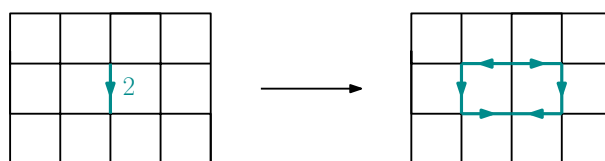

Figure 1: An example of flow-firing on the grid

The flow-firing process starts with a fixed initial flow configuration. From all edges that can fire, one is selected and fired. This continues until no firing moves are possible or continues forever. A flow configuration is **stable** if no flow-firing moves are possible. From a fixed initial configuration, we are interested in whether a stable configuration can be reached or not, and whether that stable configuration is unique. A flow configuration is **conservative** if for each vertex v, the flow into v is equal to the flow out of v.

**Theorem 2** ([4, Proposition 2, Theorem 4]). *Let* K *be a flow configuration.*

- *If there is a vertex v with*

$$|\mathsf{inflow}(v) - \mathsf{outflow}(v)| > 4,$$

  then flow-firing never terminates.

- *If K is conservative, then flow-firing terminates after a finite number of steps.*

Moreover, conservative flow configurations are induced by **face representations**. A face representation of a flow configuration is an assignment of integer values to the 2-cells of the complex. Visually, flow is depicted as oriented, with positive values oriented clockwise and negative values oriented counter-clockwise, see below.
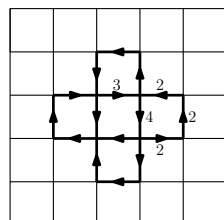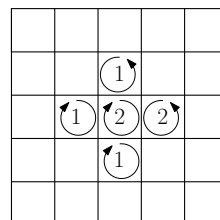
(a) Edge representation (b) Face representation

Henceforth, we assume our configurations are conservative and can therefore be analyzed using the face representation. Given such a configuration K, we denote the weight of a face f by K_f. The **support** of a face configuration is the set of faces f whose weights are nonzero. In the grid, the distance between faces f and g, written $\mathrm{dist}(f,g)$ is the *Manhattan distance*. For instance, $\mathrm{dist}(f,f) = 0$, and if f and g are neighbors, i.e. share an edge, then $\mathrm{dist}(f,g) = 1$.

The flow-firing rules described earlier can be translated to firing rules on faces. Let K be a configuration and suppose f, g are faces of the grid with weights K_f and K_g respectively. If $\mathsf{K}_f \geqslant \mathsf{K}_g + 2$, then a **face fire** of f towards g results in a new configuration K' with weights $\mathsf{K}'_f = \mathsf{K}_f - 1$ and $\mathsf{K}'_g = \mathsf{K}_g + 1$. If the recipient g is understood from context we will often simply refer to firing a face f.

Example 3. Below are three face configurations. The second picture (b) is obtained from the first picture (a) by firing the face with weight 2 to the right. The last picture (c) is obtained from (a) by by firing the face with weight 3 to the left.

(a) (b) (c)

We consider a modification of face-firing called **face-firing with a marked face**. Fix a marked face f_0 and an integer n, the weight of f_0. The marked face acts simultaneously as a source and a sink, while eschewing some of the properties of both. We use the usual face-firing rules away from f_0, but adapt the rules when firing into or out of f_0 in the following way: if f is adjacent to f_0 and has weight $\mathsf{K}_f < n$, we can fire from f_0 to f. If on the other hand $\mathsf{K}_f > n$, we can fire from f to f_0. If $\mathsf{K}_f = n$, then we cannot fire between f_0 and f. Notably, for any configuration K, the value K_{f_0} never changes as a result of firing. Example 4 illustrates face firing moves with a distinguished face.

Example 4. Below are three configurations on the grid complex with a marked face (shaded). To obtain the second configuration (b) from the first configuration (a), we fire the marked face to the left. This firing is only allowed because we are firing the marked face. In the second picture (b), the values of the two faces involved are the same, so we cannot fire the marked face to the left. The third picture (c) is obtained from the second configuration (b) by firing the top middle box to the right.

	3	
1	2	

	3	
2	2	

	2	1
1	2	

$\quad\quad\quad$ (a) $\quad\quad\quad\quad\quad\quad$ (b) $\quad\quad\quad\quad\quad\quad$ (c)

The **(total) weight** of a configuration K is the sum of the non-zero face weights in K:

$$\text{weight}(\mathsf{K}) := \sum_f \mathsf{K}_f$$

and the **support radius** of K is $\text{len}(\mathsf{K}) := \max\{\text{dist}(f, f_0) : \mathsf{K}_f \neq 0\}$.

There are two configurations that will play a significant role in our analysis. The first configuration is the **Aztec diamond**, which is the configuration whose faces f have the following weights

$$\text{Az}(n)_f := \max\{n - \text{dist}(f_0, f) + 1, 0\} \quad \text{for all faces } f \neq f_0,$$

and $\text{Az}(n)_{f_0} = n$.

The second configuration is the **pulse of height n and radius r**, which we denote by $\mathsf{K}(n, r)$. In the edge representation, $\mathsf{K}(n, r)$ can be described as a simple closed curve of radius r around f_0, with n units of flow oriented clockwise. In the face representation, $\mathsf{K}(n, r)$ is the configuration such that $\mathsf{K}(n, r)_f = n$ if $\text{dist}(f_0, f) \leqslant r$ and $\mathsf{K}(n, r)_f = 0$, otherwise. Examples of $\mathsf{K}(4, 2)$ and $\text{Az}(2)$ are shown below. In the edge representation, $\mathsf{K}(4, 2)$ is a closed curve with 4 units of flow along the curve and $\text{Az}(2)$ has 1 unit of flow on every edge in the discrete ℓ_1-ball centered at the distinguished face.

$$\text{Pulse } \mathsf{K}(4, 2) \quad\quad\quad\quad\quad \text{Aztec diamond } \text{Az}(2)$$

The significance of these two configurations originates with the following theorem, which proved the first instance of confluent behavior in the context of flow-firing.

Theorem 5 ([4, Theorem 9]). *Every sequence of flow-firing moves on* $\mathsf{K}(n, 0)$ *terminates in* $\mathrm{Az}(n)$.

The three regimes theorem for flow-firing (Theorem 1) provides a fuller description of the possible behavior of initial configurations around a marked face, in particular showing that the pulse is essentially the only initial configuration that gives a confluent system.

For context and comparison, we give the precise statements of the graphical theorem before proceeding with the proof of the flow-firing theorem.

Theorem 6 ([2, Theorem 3.3]: Three Regimes for Graphical Chip-Firing). *Let G be a finite connected graph with n vertices and m edges. Let $\mathbf{c}$ be a configuration with N chips in total. Then*

1. *If $N < m$ then the chip-firing process terminates after finitely-many firing.*

2. *If $m \leqslant N \leqslant 2m - n$ then there exists an initial configuration which stabilizes and also one which does not.*

3. *If $N > 2m - n$ then the chip-firing process never stabilizes.*

3 Path-Firing

A **path** $\mathcal{P}$ is a collection of successive faces $\mathcal{P} = (f_1, \ldots, f_k)$ such that $\mathrm{dist}(f_{i-1}, f_i) = 1$ for $2 \leqslant i \leqslant k$. A flow-firing process involving only the faces of $\mathcal{P}$ is called **path-firing**. A path is called **standard** if each f_i has value $\max\{0, n - \mathrm{dist}(f_0, f_i) + 1\}$. In other words, a standard path is one in which the weight of f matches $\mathrm{Az}(n)_f$.

Let R be a path configuration on $\mathcal{P}$ such that the face weights of R form a weakly decreasing sequence $\mathsf{R}_1 \geqslant \ldots \geqslant \mathsf{R}_k$. A configuration obtained by path-firing on R along the path $\mathcal{P}$ is called an **intermediate** path configuration; if there are no more path-firing moves left, such a configuration is called **path-stable**. Note that the face weights of any intermediate path configuration always form a weakly decreasing sequence. This means that firing from face f_i can only be directed towards face f_{i+1}—in other words, firing can only happen in one direction along $\mathcal{P}$.

It turns out that path-firing is equivalent to graphical chip-firing along a line, which has been studied in [1]; see also [10, Chapter 5]. We will thus be able to apply standard results from graphical chip-firing to study path-firing. The key ingredient to forge this connection is the notion of **discrete derivatives**, defined below. We thank an anonymous referee for pointing out this connection.

Let R be a path configuration on a path $\mathcal{P}$ as given above. Write $d(\mathsf{R}) = (d_1, \ldots, d_k)$ to denote the **discrete derivative** of R, where $d_k = \mathsf{R}_k$ and $d_i = \mathsf{R}_i - \mathsf{R}_{i+1}$ for $i = 1, \ldots, k$.

Consider the configuration $\mathsf{R}' = (\mathsf{R}_1, \ldots, \mathsf{R}_i - 1, \mathsf{R}_{i+1} + 1, \ldots, \mathsf{R}_k)$ obtained from R by firing from face f_i towards face f_{i+1}. Then the discrete derivative of R' is

$$d(\mathsf{R}') = \begin{cases} (d_1, \ldots, d_{i-1} + 1, d_i - 2, d_{i+1} + 1, \ldots, d_k) & \text{if } i > 1, \\ (d_1 - 2, d_2 + 1, d_3, \ldots, d_k) & \text{if } i = 1. \end{cases} \tag{1}$$

Notice that R can be recovered from its discrete derivative by taking partial sums: the weight R_i is given by

$$R_i = \sum_{j=i}^{k} d_j.$$

This leads to the following, which follows immediately from Equation (1).

Proposition 7. *Path-firing on R along $\mathcal{P}$ is equivalent to chip-firing on a line with vertices $v_0, v_1, \ldots, v_k$, where for any $0 \leqslant i \leqslant k$, vertex v_i has d_i chips, and vertex v_0 is a sink vertex (with no chips).*

Example 8. Consider a path configuration $R(\ell)$ for a fixed integer ℓ, with $R(\ell)_i = n$ for all $1 \leqslant i \leqslant \ell$. Then the discrete derivative $d(R(\ell)) = (d_0, d_1, \cdots)$ of $R(\ell)$

$$d(R(\ell)) = (0, \cdots, 0, n, 0, \cdots 0)$$

The corresponding graphical configuration has $d_\ell = n$ and $d_i = 0$ for all $i \neq \ell$. This special case of graphical chip-firing is studied in [1] and its final configuration is well-understood (see Corollary 9).

Our ultimate goal is to understand configurations obtained from flow-firing on $K(n, r)$. To do so, we will first analyze path-firing along a single row of $K(n, r)$, which we implicitly introduced in Example 8. Explicitly, for any $\ell \in \mathbb{Z}_{\geqslant 0}$, define

$$\mathcal{P}(\ell) := (f_1 \ldots, f_{\ell + \lfloor \frac{n}{2} \rfloor}),$$

and a configuration $R(\ell)$ on $\mathcal{P}(\ell)$:

$$R(\ell) := \underbrace{(n, n, \ldots, n}_{\ell \text{ times}}, 0, 0, \ldots).$$

Since $\mathcal{P}(\ell)$ does *not* include f_0, the number of chips stays constant while firing along $\mathcal{P}(\ell)$. The discrete derivative of $R(\ell)$ is

$$d(\ell) := \underbrace{(0, 0, \ldots, 0}_{\ell-1 \text{ times}}, n, 0, 0, \ldots).$$

By Proposition 7, flow-firing on a path with initial configuration $R(\ell)$ is the same as graphical chip-firing on a line with $\ell + \lfloor \frac{n}{2} \rfloor + 1$ vertices (equivalently $\ell + \lfloor \frac{n}{2} \rfloor$ edges) where the first vertex is a sink.

Chip-firing in the above context will terminate [10, Theorem 2.5.2], but we can be more precise about its termination behavior using [1, Theorem 1]. Below, we translate this behavior back into the language of flow-firing, so that we can apply it later in Section 5.2.

Corollary 9 ([1, Theorem 1]). *For any $\ell \geqslant 0$, the path that leads to $R(\ell)$ along $\mathcal{P}(\ell)$ terminates uniquely. Let R° be this terminal configuration, and let S_ℓ be the set of faces that fire in the process of obtaining R° from $R(\ell)$. Then*

- If $\ell > \lfloor \frac{n}{2} \rfloor$ then $S_\ell = \{f_{k+1}, f_{k+2}, \ldots, f_{r-1}\}$,

- If $\ell \leqslant \lfloor \frac{n}{2} \rfloor$, then $S_\ell = \{f_1, \ldots, f_{r-1}\}$,

where $k = \ell - \lfloor \frac{n}{2} \rfloor$ and let $r = \mathrm{len}(\mathsf{R}^\circ)$. Moreover, if $\ell \leqslant \lceil \frac{n}{2} \rceil$ then $\mathrm{len}(\mathsf{R}^\circ) = \ell + \lfloor \frac{n}{2} \rfloor$.

As an example, we compute the discrete derivative $d(\mathsf{R}^\circ)$ corresponding to the configuration R° in Corollary 9, when $\ell \leqslant \lceil \frac{n}{2} \rceil$:

$$
d(\mathsf{R}^\circ) = \begin{cases} (\underbrace{1, 1, \ldots, 1}_{\ell + \lfloor \frac{n}{2} \rfloor \text{ times}}, 0, \ldots) & \text{if } n \text{ is odd,} \\[2em] (\underbrace{1, 1, \ldots, 1}_{\ell - 1 \text{ times}}, 0, \underbrace{1, 1, \ldots, 1}_{\frac{n}{2} \text{ times}}, 0, \ldots) & \text{if } n \text{ is even.} \end{cases} \tag{2}
$$

As discussed above, $d(\mathsf{R}^\circ)$ completely determines R°. Determining $d(\mathsf{R}^\circ)$ when $\ell > \lceil \frac{n}{2} \rceil$ is even simpler and is given explicitly in [1].

In addition to describing the termination behavior of $\mathsf{R}(\ell)$, Proposition 7 allows us to better understand intermediate path configurations more generally. Lemma 10 below implies that the entries of discrete derivative cannot be zero "too often" unless it started off that way. We omit the proof because it is similar to the proof of [1, Lemma 1].

Lemma 10. *Let R be a path configuration on a path $\mathcal{P}$ whose face weights form a weakly decreasing sequence. Let R' be an intermediate path configuration obtained from path-firing on R. Let $d(\mathsf{R}') = (d_0, \cdots, d_k, \cdots)$ denote the discrete derivative of R'. Then*

(a) if $d_i = d_{i+1} = 0$, the faces f_i, f_{i+1} and f_{i+2} never fire, and

(b) if $d_i = d_j = 0$ for $i + 1 < j$ and $d_\ell = 1$ for $i < \ell < j$ then the faces f_i, f_{i+1}, f_j, and f_{j+1} never fired.

Lemma 10 significantly limits the types of configurations that can arise from path-firing on along $\mathcal{P}$, and will be useful later in Section 6.1.

4 The first regime

Theorem 1 (Part 1). If $r \leqslant 1$, then firing from $\mathsf{K}(n, r)$ terminates uniquely in the Aztec diamond $\mathrm{Az}(n)$.

Our approach to showing a configuration *can* terminate in $\mathrm{Az}(n)$ (e.g. Regimes 1 and 2) relies on finding an intermediate configuration of $\mathsf{K}(n, r)$ that does not "violate" the Aztec diamond. A configuration K **violates** the Aztec diamond $\mathrm{Az}(n)$ if some face f has $\mathsf{K}_f > \mathrm{Az}(n)_f$. After finding such a configuration, Lemma 11 allows us to conclude that the initial configuration $\mathsf{K}(n, r)$ *can* terminate in $\mathrm{Az}(n)$.

Lemma 11. *Let K be a configuration with $\mathsf{K}_{f_0} = n$. If $\mathsf{K}_f \leqslant \mathrm{Az}(n)_f$ for all faces f, then there is a sequence of firing moves sending K to $\mathrm{Az}(n)$.*

Proof. The core of this proof comes from two observations.

1. Let f, g be adjacent faces with g strictly further from f_0 than g, so that $\operatorname{dist}(f_0, f) + 1 = \operatorname{dist}(f_0, g)$. If $K_f = \operatorname{Az}(n)_f$ and $K_g < \operatorname{Az}(n)_g$, then $K_g \leqslant K_f - 2$ and hence we can fire from f to g.

2. If g is adjacent to f_0 and $K_g < \operatorname{Az}(n)_g$, then $K_g \leqslant n - 1$ and we can fire from f_0 to g.

Let g be a face such that $K_g < \operatorname{Az}(n)_g$ but all faces strictly closer to f_0 satisfy $K_f = \operatorname{Az}(n)_f$.

If g is adjacent to f_0, then by (2) above, we can fire from f_0 to g. If g is not adjacent to f_0, then there is some face f adjacent to g and strictly closer to f_0; by (1) we can fire from f to g. Repeat this process until $K_g = \operatorname{Az}(n)_g$ for all faces g. $\square$

Proposition 12. *If $r \leqslant 1$, then $\mathsf{K}(n, r)$ terminates uniquely in the Aztec diamond.*

Proof. The $r = 0$ case was proven in [4, Theorem 9]. If $r = 1$, then $\mathsf{K}(n, 1)$ is an intermediate configuration obtained from $\mathsf{K}(n, 0)$ and it terminates uniquely at $\operatorname{Az}(n)$ by [4, Theorem 9]. $\square$

We will see in the subsequent section that although there are many more configurations that *can* terminate in the Aztec diamond, in general this termination will not be unique.

5 The Second Regime

Theorem 1 (Part 2a). If $2 \leqslant r \leqslant \lceil \frac{n}{2} \rceil$, flow-firing on $\mathsf{K}(n, r)$ does not terminate uniquely.

In order to prove this, we will show that there exists (i) a sequence of firings that terminate in $\operatorname{Az}(n)$ and (ii) a sequence of firings that terminate in a configuration that is not $\operatorname{Az}(n)$.

In what follows, we state our results for Regime 2 in terms of $\mathsf{K}(n, r)$. However, both claims in Theorem 1(2) hold more generally for a closed curve with weight n strictly containing $\mathsf{K}(n, 1)$ and contained within $\mathsf{K}(n, r)$ for $2 \leqslant r \leqslant \lceil \frac{n}{2} \rceil$.

5.1 Second regime: termination violating the Aztec diamond.

Our goal in this section is to show that the configuration $\mathsf{K}(n, r)$ need not terminate in the Aztec diamond when $2 \leqslant r \leqslant \lceil \frac{n}{2} \rceil$.

Lemma 13. *Let K be any configuration and let f and g be adjacent faces with*

$$\operatorname{dist}(f_0, f) < \operatorname{dist}(f_0, g) < n\,.$$

If f violates the Aztec diamond but g does not, then one can fire from f to g. In the resulting configuration K', K'_f will violate the Aztec diamond if $\mathsf{K}_f > \operatorname{Az}(n)_f + 1$, and K'_g will violate the Aztec diamond if $\mathsf{K}_g = \operatorname{Az}(n)_g$.

Proof. To see the first part of the lemma, we need to show that $\mathsf{K}_f - \mathsf{K}_g \geqslant 2$. Recall that

$$\mathrm{Az}(n) = \max\{n - \mathrm{dist}(f_0, f) + 1, 0\}.$$

Since $\mathrm{dist}(f_0, f) < \mathrm{dist}(f_0, g) < n$, we have that $\mathrm{Az}(n)_g < \mathrm{Az}(n)_f$. Since they are adjacent,

$$\mathrm{Az}(n)_f - \mathrm{Az}(n)_g = 1.$$

Thus

$$\mathsf{K}_f \geqslant \mathrm{Az}(n)_f + 1 = \mathrm{Az}(n)_g + 2.$$

On the other hand, since K_g does not violate the Aztec diamond, $\mathsf{K}_g \leqslant \mathrm{Az}(n)_g$. Thus

$$\mathsf{K}_f \geqslant \mathsf{K}_g + 2$$

as desired. The second part of the claim is clear from inspection. $\square$

Lemma 14 (Flooding lemma). *Let f be a face such that $\mathsf{K}_f > \mathrm{Az}(n)_f > 0$. Then there is configuration K', reachable from K by flow-firing moves, and a face g such that*

- *$\mathrm{dist}(f_0, g) = 1 + \mathrm{dist}(f_0, f)$ and*

- *$\mathsf{K}'_g > \mathrm{Az}(n)_g$.*

Proof. We explicitly construct K' as in the statement of the lemma. Let V be the set of faces that violate the Aztec diamond in the configuration K. Take $f \in V$ to be a face with maximal distance from f_0. By the maximality of f, every neighbor g of f with $\mathrm{dist}(f_0, g) = 1 + \mathrm{dist}(f_0, f)$ does *not* violate the Aztec diamond. Choose such a neighbor g and a path $\mathcal{P} = (f_0, f_1, \cdots, f_\ell = g)$ that does not pass through f.

We claim that one can fire along $\mathcal{P}$ until g violates the Aztec diamond. There are two cases:

- *Case I: For $1 \leqslant i \leqslant \ell$, no f_i in $\mathcal{P}$ violates the Aztec diamond.*
  First fire along $\mathcal{P}$ to create a standard path. This is always possible because f_i never violates the Aztec diamond. Note that after this process, g has value $\mathrm{Az}(n)_g$. By Lemma 13, we can fire from f into g, which means that g will then violate the Aztec diamond.

- *Case II: There is some $1 \leqslant i \leqslant \ell$ where f_i violates the Aztec diamond.*
  Pick the violating face f_i that is furthest from f_0. Thus f_{i+1} does not violate $\mathrm{Az}(n)$, and so by Lemma 13 we can fire from f_i to f_{i+1}. Repeat this process; it necessarily will terminate because we are in the conservative flow setting. After termination, either we are in *Case I* or the violation has been pushed to $\mathcal{P}_\ell = g$.

Thus, the resulting configuration K' has $\mathsf{K}'_g > \mathrm{Az}(n)_g$ since g violates the Aztec diamond. $\square$

The Flooding Lemma implies that for a configuration $\mathsf{K}(n, r)$ with r large enough, one can construct a terminal configuration that violates $\mathrm{Az}(n)$.

Theorem 1 (Part 2b). Given a configuration $\mathsf{K}(n,r)$ with $2 \leqslant r \leqslant \lceil \frac{n}{2} \rceil$, there is a stable configuration K^* such that $\mathsf{K}^* \neq \mathrm{Az}(n)$.

Proof. Since $2 \leqslant r \leqslant \lceil \frac{n}{2} \rceil$, there exists at least one face violating $\mathrm{Az}(n)$. Let V be the set of all such faces, and choose $f \in V$ of maximal distance from f_0. Since f is maximal, every face g with $\mathrm{dist}(g, f_0) > \mathrm{dist}(f, f_0)$ has $\mathsf{K}_g \leqslant \mathrm{Az}(n)_g$. Fix such a g with $\mathrm{dist}(g, f_0) = \mathrm{dist}(f, f_0) + 1$. Applying the Flooding Lemma (Lemma 14) to f and g gives a new configuration K' such that some face g with $\mathrm{dist}(g, f_0) > \mathrm{dist}(f, f_0)$ has $\mathsf{K}_g > \mathrm{Az}(n)_g$.

Repeat this process until the violating face h has $\mathrm{dist}(f_0, h) > n$, so that $\mathrm{Az}(n)_h = 0$; note that this is always possible by Lemma 13. Call this configuration K'. Since $\mathsf{K}'_h > 0$, no firing moves on K' can ever decrease the weight of face of h, and so firing on K' will result in a terminal configuration K^* with $\mathsf{K}^*_h > 0$. Hence $\mathsf{K}^* \neq \mathrm{Az}(n)$. $\qquad \square$

Example 15. Below we draw four configurations. The first (a) is $\mathsf{K}(3,2)$, and the other three are obtained from the first by successive firing moves. Using the notation of Lemma 14, for each of the configurations K below, pick f to be the northern-most nonzero face that can fire in K (shaded, in pink), and let g be the neighboring face (shaded, in yellow). Using these choices of f and g, we show a sequence of three iterations of the Flooding Lemma.


| (a) | (b) | (c) | (d) |
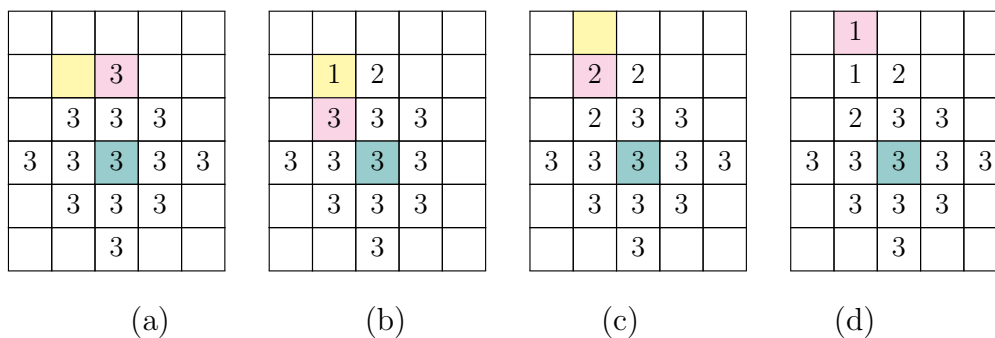
5.2 Second regime: termination in the Aztec diamond

Having constructed a stable configuration from $\mathsf{K}(n,r)$ which is not equal to $\mathrm{Az}(n)$, we now wish to show that $\mathrm{Az}(n)$ *can* appear as a final configuration for $\mathsf{K}(n,r)$ when $2 \leqslant r \leqslant \lceil \frac{n}{2} \rceil$. Our goal is to show:

Theorem 1 (Part 2a). Given a configuration $\mathsf{K}(n,r)$ with $2 \leqslant r \leqslant \lceil \frac{n}{2} \rceil$, there is a stable configuration K^* obtained from firing on $\mathsf{K}(n,r)$ such that $\mathsf{K}^* = \mathrm{Az}(n)$.

Proof. Divide $\mathsf{K}(n,r)$ and $\mathrm{Az}(n)$ into four quadrants, as shown below in the case of $\mathsf{K}(4,2)$. Each quadrant is given in a different color, and $\mathrm{Az}(4)$ is shaded in gray in the background.

Note that the marked face f_0 is not included in any of the quadrants.
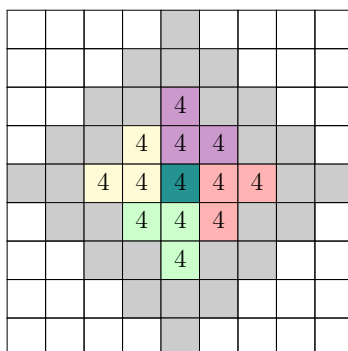
Without loss of generality, we can study a single quadrant and then appeal to symmetry to describe the remaining quadrants. Consider the rows in the first quadrant of $\mathsf{K}(n, r)$ (colored violet in the figure above) and count them from top to bottom. Let $\mathcal{P}(\ell)$ be the path that is obtained from extending the ℓth row of the quadrant to the boundary of $\mathrm{Az}(n)$. The relevant configuration on $\mathcal{P}(\ell)$ is then

$$\mathsf{R}(\ell) := (\underbrace{n, n, \cdots, n}_{\ell \text{ times}}, 0, \dots).$$

whose behavior was studied in Section 3.

Note that $\mathrm{dist}(f_0, f_i) = r - \ell + i$ for each face f_i of $\mathcal{P}(\ell)$ and

$$\mathrm{Az}(n)_{f_i} = n - \mathrm{dist}(f_0, f_i) + 1 = n - r + \ell - i + 1.$$

Let R° be the unique path-stable configuration obtained from path-firing on $\mathsf{R}(\ell)$ given in Corollary 9. Translating the discrete derivative in Corollary 9 into the language of flow-firing, we have that R_i°, the value of face f_i on R°, is

$$\mathsf{R}_i^\circ = \begin{cases} \ell + \lfloor n/2 \rfloor - i & \text{if } n \text{ even and } i < \ell, \\ \ell + \lfloor n/2 \rfloor - i + 1 & \text{else.} \end{cases}$$

Since $r \leqslant \lceil n/2 \rceil$, it follows that

$$\mathrm{Az}(n)_{f_i} \geqslant \lfloor n/2 \rfloor + \ell - i + 1 \geqslant \mathsf{R}_i^\circ .$$

Thus, the value of R_i° cannot exceed $\mathrm{Az}(n)_{f_i}$ along $\mathcal{P}(\ell)$. Firing $\mathsf{K}(n, r)$ along all rows in the analogous way thus yields a configuration which does not violate $\mathrm{Az}(n)$. Thus by Lemma 11, there is a sequence of firing moves that can then be implemented to obtain $\mathrm{Az}(n)$ as a stable configuration. $\qquad \square$

Example 16. The full configurations after path-firing in all four quandrants is shown below. Note that the second configuration does not violate the Aztec diamond $\mathrm{Az}(4)$.

This means that we can apply Lemma 11 to obtain Az(4) from the second by firing moves.
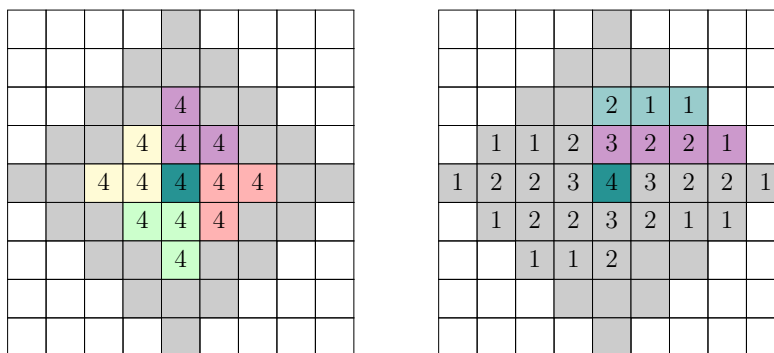
6 The Third Regime

Finally, we turn to the third regime. We first show that when $r > \lceil n/2 \rceil$, $\mathsf{K}(n,r)$ cannot terminate uniquely in Section 6.1. Then, we show that for $r > \lceil \frac{n}{\sqrt{3}} \rceil + 1$, it is impossible for $\mathsf{K}(n,r)$ to terminate in the Aztec diamond in Section 6.2.

6.1 Non-unique termination

We will show that $\mathsf{K}(n,r)$ does not terminate uniquely for $r > \lceil n/2 \rceil$. Our proof relies on decomposing the grid in two distinct ways, shown below:

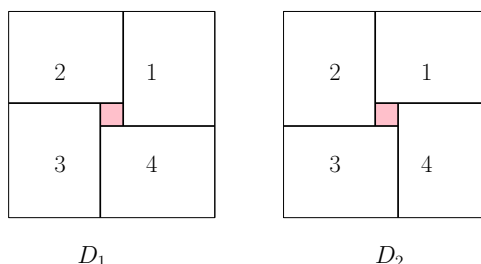

Figure 3: Two distinct ways to decompose the grid into quadrants.

For $i = 1, 2$, we refer to D_i as the decomposition of the grid. Our argument introduces an algorithm that fires identically within each quadrant, but produces two different final cofigurations using the two decompositions in Figure 3.

Theorem 1 (Part 3a). If $r > \lceil n/2 \rceil$ then $\mathsf{K}(n,r)$ does not terminate uniquely.

Proof. It is immediate that $\mathsf{K}(2,r)$ does not terminate uniquely when $r \geqslant 1$. So, suppose $n > 2$. For the sake of contradiction, assume $\mathsf{K}(n,r)$ terminates uniquely.

Consider the following flow-firing algorithm on $\mathsf{K}(n,r)$:

1. Divide $\mathsf{K}(n,r)$ into four quadrants as in either D_1 or D_2 as in Figure 3;

2. While possible, path-fire within each quadrant along all horizontal paths (e.g. rows) away from f_0;

3. While possible, path-fire within each quadrant along all vertical paths (e.g. columns) away from f_0;

4. Return to step (2).

We claim that this process results in a stable configuration, regardless of whether one divides quadrants by D_1 or D_2. By construction, within each quadrant the resulting configuration is stable. Note that any adjacent faces between different quadrants must differ by at most one. This follows from the fact that the quadrants are reflections of one another, with face values shifted exactly one unit up (down) and right (left). Let K_i be the stable configuration obtained from the above process, with the choice of quadrants given by D_i for $i = 1, 2$.

By assumption, $\mathsf{K}(n, r)$ terminates uniquely and so both the supports and the face weights of K_1 and K_2 are the same.

We label the face weights of K_1 and K_2 as follows: first, label the faces of the grid by $(j, k) \in \mathbb{Z}^2$ so that $f_{j,k} \in \mathbb{Z}$ is the face weight at face (j, k). Fix the indexing so that the distinguished face is $f_{0,0}$, and (j, k) increase to the right and up. We will focus on the first quadrant of K_i, denoted by Q_i. Label the face weights in both Q_i as $a_{j,k}^{(i)} \in \mathbb{Z}_{>0}^2$, where the bottom-left face is $a_{1,1}^{(i)}$ for $i = 1, 2$ and again indices increase right and up.
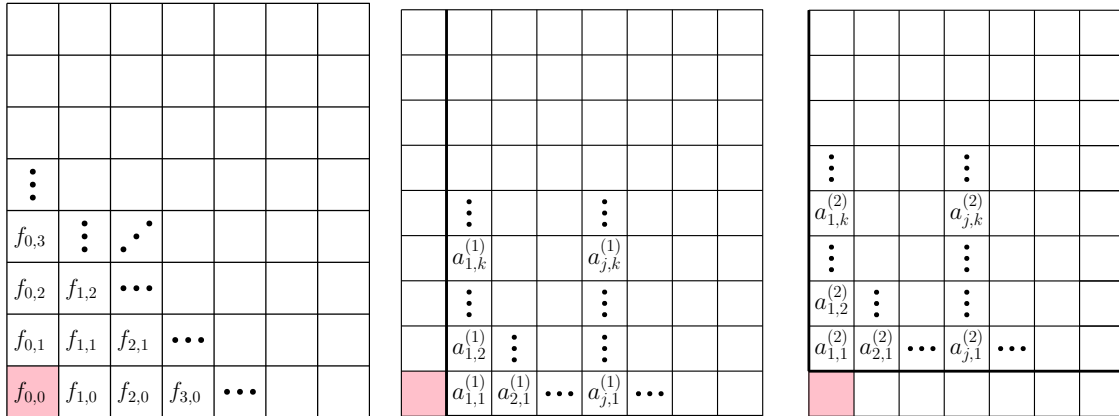

Figure 4: The face labels of the grid, face weights in Q_1 and Q_2 (respectively)

We make three observations:

- because the configurations Q_1 and Q_2 were obtained by the same firing process, $a_{j,k}^{(1)} = a_{j,k}^{(2)}$ for all $j, k \in \mathbb{Z}_{>0}^2$. In other words, Q_2 is obtained by shifting Q_1 up and left by 1;

- by the definitions of D_1 and D_2, we have $a_{1,1}^{(1)}$ is the face weight of $f_{1,0}^{(1)}$ in K_1, while $a_{1,1}^{(2)}$ is the face weight of $f_{0,1}^{(2)}$ in K_2;

- by the assumption that $\mathsf{K}_1 = \mathsf{K}_2$, we have $a_{j,k}^{(1)} = a_{j+1,k-1}^{(2)}$ for all j, k. Therefore $a_{j,k}^{(1)} = a_{j-1,k+1}^{(1)}$ for all j, k.

Label the columns of Q_i by $C_1, C_2, \ldots, C_t$ (from left to right), and let the length of C_j be n_j. Then by the last observation, we must have that $n_{j+1} = n_j - 1$ for $j = 1, \ldots, t - 1$. Thus, without loss of generality, we may describe Q_1 as in Figure 5 below:

Figure 5: The face weights in Q_1

From now on, we focus on Q_1. For ease of notation, let $a_{1,j} := a_{1,j}^{(1)}$ for $1 \leqslant j \leqslant n_1$. Note that $a_{1,n_1} = 1$ and $n_1 > r$.

Write $r = \lceil n/2 \rceil + k$ for some $k \geqslant 1$. Since $n > 2$, we have $r \geqslant k + 2$. Consider the faces $f_{1,0}, \cdots, f_{1,n_1-1}$ in the first column of Q_1 which have face value $a_{1,1}, \cdots, a_{1,n_1}$.

It follows from Corollary 9 and the firing rules to obtain Q_1 that faces $f_{1,k+1}, \ldots, f_{1,n_1-2}$ all fired but faces $f_{1,0}, \ldots, f_{1,k-1}$ may or may not have fired in this process to obtain Q_1. Thus, $a_{1,i} \leqslant n - 1$ for all $k + 1 \leqslant i \leqslant r + 1$. In addition, if $a_{1,\ell} = n$ (in other words, $f_{1,\ell}$ never fired), then $k \geqslant \ell$.

The last observation we make is that the total face weight in Q_1 is

$$\sum_{i,j} a_{i,j} = \frac{nr(r+1)}{2} \tag{3}$$

since face weight is preserved in the firing process to obtain Q_1.

Supposing the above conditions are satisfied, since $a_{1,r+1} \leqslant n - 1$, there are three cases to consider. We will show that each case yields a contradiction, so that $\mathsf{K}(n, r)$ cannot terminate uniquely.

Case 1: Suppose $a_{1,r+1} = n - 1$. Then either $a_{1,r} = n$ or $a_{1,r-1} = n$ by Lemma 10. This means either $k \geqslant r$ or $k \geqslant r - 1$. Since $r \geqslant k + 2$, neither situation is possible. Thus it is impossible that $\mathsf{K}_1 = \mathsf{K}_2$, and so $\mathsf{K}(n, r)$ cannot terminate uniquely in this case.

Case 2: Suppose $a_{1,r+1} = n - 2$. Then, as in the previous case, either $a_{1,r-1} = n$ or $a_{1,r-2} = n$ by Lemma 10. As we have seen in the previous case, $a_{1,r-1} = n$ is not possible. Note that when $a_{1,r-2} = n$, we have either

(i) $(a_{1,r+1}, a_{1,r}, a_{1,r-1}, a_{1,r-2},) = (n - 2, n - 2, n - 1, n)$ or

(ii) $(a_{1,r+1}, a_{1,r}, a_{1,r-1}, a_{1,r-2}) = (n-2, n-1, n-1, n)$.

If $a_{1,r-2} = n$, then $k \geqslant r-2$ which implies that $r = k+2$. This happens only when $n = 3$ or $n = 4$. By counting the total number of chips in Q_1 for $n = 3$ and total face weight count given in (3), we get either

(i) $\frac{3}{2}k(k+1) + 4k + 7 = \frac{3}{2}(k+2)(k+3)$ or

(ii) $\frac{3}{2}k(k+1) + 5k + 9 = \frac{3}{2}(k+2)(k+3)$

neither of which have non-negative integer solutions. Similarly, for $n = 4$, we get

(i) $2k(k+1) + 8k + 17 = 2(k+2)(k+3)$ or

(ii) $2k(k+1) + 9k + 97 = 2(k+2)(k+3)$

neither of which have non-negative integer solutions. Hence, this case is also not possible.

Case 3: Suppose $a_{1,r+1} \leqslant n - 3$.

We will now construct a third terminal configuration K_3 from $\mathsf{K}(n,r)$, and show that it is impossible for $\mathsf{K}_1 = \mathsf{K}_2 = \mathsf{K}_3$.

To construct K_3, consider the intermediate configuration which is obtained from $\mathsf{K}(n,r)$ by applying the algorithm described at the beginning of the proof, but only in the first quarter of D_1. This yields the configuration Q_1 in the first quadrant and $\mathsf{K}(n,r)$ in the other three quadrants.

Recall that $r > \lceil n/2 \rceil$, and consider the $(r+1)^{\text{th}}$ row of Q_1 (counted from south to north). Denote the configuration on this row as R where

$$\mathsf{R} := (a_{1,r+1}, a_{1,r+2}, \cdots, a_{1,n_1}, 0, \cdots).$$

Note that $f_{n_1+1,r}$ is the first face with weight 0 in R.

Let $\tilde{\mathsf{R}}$ be the extension of R outside of Q_1 one unit to the left, i.e. by including the face $f_{0,r}$; this new face has weight n, since no faces in the fourth (i.e. north-west) quadrant have been fired yet:

$$\tilde{\mathsf{R}} := (n, a_{1,r+1}, a_{1,r+2}, \cdots, a_{1,n_1}, 0, \cdots).$$

Now path-fire on $\tilde{\mathsf{R}}$ along this row. Since $a_{1,r+1} \leqslant n-3$, we see that by applying Lemma 10, after path-firing on $\tilde{\mathsf{R}}$ the face weight of $f_{n_1+1,r}$ is non-zero. Call this intermediate configuration $\tilde{\mathsf{K}}$.

Finally, we can obtain a stable configuration K_3 from $\tilde{\mathsf{K}}$ by [4, Theorem 4]. Importantly, in K_3 the face weight of $f_{n_1+1,r}$ is non-zero, because it is non-zero in $\tilde{\mathsf{K}}$. On the other hand, the weight of $f_{n_1+1,r}$ is zero in K_1. Hence $\mathsf{K}_1 \neq \mathsf{K}_3$, and so $\mathsf{K}(n,r)$ cannot terminate uniquely. $\qquad\square$

Example 17. In the case of $\mathsf{K}(3,3)$, the figures below show two distinct final configurations. We decompose the grid as in Figure 3, and perform firings as in the proof of Theorem 1(3a).

				1				
			1	2	1			
		1	2	2	2	1		
	1	2	2	3	3	2	1	
1	2	3	3	3	3	3	2	1
	1	2	3	3	2	2	1	
		1	2	2	2	1		
			1	2	1			
				1				

				1				
			1	2	1			
		1	2	2	2	1		
	1	2	3	3	2	2	1	
1	2	3	3	3	3	3	2	1
	1	2	2	3	3	2	1	
		1	2	2	2	1		
			1	2	1			
				1				

6.2   Regime 3: Termination outside of the Aztec diamond

Henceforth, let $N = \lceil \frac{n}{\sqrt{3}} \rceil + 1$.

Theorem 1 (Part 3b). If $r \geqslant N$, then $\mathsf{K}(n,r)$ cannot terminate in the Aztec diamond $\mathrm{Az}(n)$.

Remark 18. In fact, Theorem 1(3b) holds for any closed curve of weight n containing $\mathsf{K}(N,r)$.

Proof. It is sufficient to prove that $\mathrm{weight}(\mathsf{K}(n,r)) > \mathrm{weight}(\mathrm{Az}(n))$. Let K denote a single quadrant of the Aztec diamond. The weight of K is

$$\mathrm{weight}(\mathsf{K}) = (n + n - 1 + \cdots + 1) + (n - 1 + n - 2 + \cdots + 1) + \cdots + 1$$
$$= \frac{n(n+1)(n+2)}{6}.$$

Thus summing over all four quadrants *and* f_0, we obtain

$$\mathrm{weight}(\mathrm{Az}(n)) = n + 4\left(\frac{n(n+1)(n+2)}{6}\right) = n + \frac{2}{3}(n)(n+1)(n+2).$$

Let K' denote a single quadrant of $\mathsf{K}(n,r)$. The weight of K' is

$$\mathrm{weight}(\mathsf{K}') = nr + n(r-1) + \cdots 2n + n = n \cdot \frac{r(r+1)}{2}$$

where the first expression comes from adding the total number of chips in each row of the quadrant. Thus by similar logic, summing over all quadrants and f_0 gives

$$\mathrm{weight}(\mathsf{K}(n,r)) = n + 4n \cdot \frac{r(r+1)}{2}.$$

Suppose $r \geqslant \lceil \frac{n}{\sqrt{3}} \rceil + 1$, and note that for n a positive integer, $\lceil n/\sqrt{3} \rceil > n/\sqrt{3}$. Thus

$$\mathrm{weight}(\mathsf{K}(n,r)) > \frac{2}{3}\left(n^3 + 3\sqrt{3}n^2 + 6n\right) + n > \frac{2}{3}\left(n^3 + 3n^2 + 2n\right) + n$$
$$= \mathrm{weight}(\mathrm{Az}(n)). \qquad \square$$

6.3 Bound improvement

Theorem 1(3b) naturally raises the question of whether it is possible for $\mathsf{K}(n,r)$ to terminate in the Aztec diamond when $\lceil \frac{n}{\sqrt{3}} \rceil + 1 > r > \lceil n/2 \rceil$. In the table below, we present the values of $r = \lceil \frac{n}{\sqrt{3}} \rceil + 1$ and $\lceil n/2 \rceil$ for $3 \leqslant n \leqslant 24$, along with the minimum value of r such that $\mathrm{weight}(\mathsf{K}(n,r)) > \mathrm{weight}(\mathrm{Az}(n))$.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$\lceil \frac{n}{2} \rceil$	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12
Minimum r	2	3	4	4	5	6	6	7	7	8	8	9	10	10	11	11	12	12	13	14	14	15
$\lceil \frac{n}{\sqrt{3}} \rceil + 1$	3	4	4	5	6	6	7	7	8	8	9	10	10	11	11	12	12	13	14	14	15	15

The minimum value of r for which $\mathrm{weight}(\mathsf{K}(n,r)) > \mathrm{weight}(\mathrm{Az}(n))$ is precisely $\lceil \frac{n}{\sqrt{3}} \rceil + 1$ approximately one third of the time. Thus, in these cases, the bound for Theorem 1(3b) cannot be improved unless a different approach is taken. On the other hand, for $n > 8$, there is a gap between $\lceil \frac{n}{2} \rceil$ and the minimum value of r. This suggests finding a better upper bound for Regime 2 may be more viable. Below are examples that show improving the bounds for Regime 2 and Regime 3 may be quite subtle.

1. **It is possible to find configurations $\mathsf{K}(n,r)$ with $\lceil \frac{n}{2} \rceil < r < \lceil \frac{n}{\sqrt{3}} \rceil + 1$ that can terminate in $\mathrm{Az}(n)$.** Consider the case $n = 8$; the only ball contained in the region between Regimes 2 and 3 is $\mathsf{K}(8,5)$. First observe that $\mathrm{weight}(\mathsf{K}(8,5)) = \mathrm{weight}(\mathrm{Az}(8))$. For this initial configuration, there is a sequence of firing moves that terminate at $\mathrm{Az}(8)$.

2. **It is possible to find configurations $\mathsf{K}(n,r)$ with $\lceil \frac{n}{2} \rceil < r < \lceil \frac{n}{\sqrt{3}} \rceil + 1$ that cannot terminate in $\mathrm{Az}(n)$.** Consider the case $n = 4$; the only ball contained in the region between Regimes 2 and 3 is $\mathsf{K}(4,3)$. As one can see from the table above, $\mathrm{weight}(\mathsf{K}(4,3)) > \mathrm{weight}(\mathrm{Az}(4))$. So, it is not possible for $\mathsf{K}(4,3)$ to terminate at $\mathrm{Az}(4)$.

3. **It is possible to find configurations within $\mathsf{K}(n,r)$ that can terminate in $\mathrm{Az}(n)$, even if $\mathsf{K}(n,r)$ cannot.** Suppose that $n = 4$. Consider the configuration on the left in the figure which is contained in $\mathsf{K}(4,3)$:

Note that $\mathrm{weight}(\mathsf{K}) = \mathrm{weight}(\mathrm{Az}(4))$. For this initial configuration K, one can find a sequence of firing moves that terminate at the Aztec diamond.

4. **It is possible to find a configuration K with weight(K) = weight(Az(n)) that cannot terminate in Az(n).** Again, take $n = 4$ and consider the configuration on the right in the figure above. This configuration *cannot* terminate in Az(4).

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