On the Order Sequence of a Group

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Abstract

This paper provides a bridge between two active areas of research, the spectrum (set of element orders) and the power graph of a finite group.

The *order sequence* of a finite group G is the list of orders of elements of the group, arranged in non-decreasing order. Order sequences of groups of order n are ordered by elementwise *domination*, forming a partially ordered set. We prove a number of results about this poset, among them the following.

- M. Amiri recently proved that the poset has a unique maximal element, corresponding to the cyclic group. We show that the product of orders in a cyclic group of order n is at least $q^{\phi(n)}$ times as large as the product in any non-cyclic group, where q is the smallest prime divisor of n and ϕ is Euler's function, with a similar result for the sum.
- The poset of order sequences of abelian groups of order p^n is naturally isomorphic to the (well-studied) poset of partitions of n with its natural partial order.
- If there exists a non-nilpotent group of order n, then there exists such a group whose order sequence is dominated by the order sequence of any nilpotent group of order n.
- There is a product operation on finite ordered sequences, defined by forming all products and sorting them into non-decreasing order. The product of order sequences of groups G and H is the order sequence of a group if and only if $|G|$ and $|H|$ are coprime.

The paper concludes with a number of open problems.

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1 Introduction

Let G be a finite group. H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs in [2] defined the following function:

$$\psi(G) = \sum_{g \in G} o(g),$$

where $o(g)$ denotes the order of the element g. They were able to prove the following:

Theorem 1. *For any finite group G of order n, $\psi(G) \leqslant \psi(\mathbb{Z}_n)$ and equality holds if and only if $G \cong \mathbb{Z}_n$, where $\mathbb{Z}_n$ is the cyclic group of order n.*

That is, $\mathbb{Z}_n$ is the unique group of order n with the largest value of $\psi(G)$ for groups of that order. Later S. M. Jafarian Amiri and M. Amiri in [5] and, independently, R. Shen, G. Chen, and C. Wu in [35] investigated the groups with the second largest value of the sum of element orders.

This function ψ has been considered in various works (see [4, 1, 8, 14, 21, 22, 23, 24, 36]). While the goal of some of the papers was to find out the largest, second largest, or least possible values of $\psi(G)$, others aimed to prove new criteria for structural properties (like solvability, nilpotency, etc.) of finite groups.

Later, S. M. Jafarian Amiri and M. Amiri [6] considered the following generalization of the above function defined by $\psi_k(G) = \sum_{g \in G} o(g)^k$ for positive integers $k \geqslant 1$ and they proved that for any positive integer k, $\psi_k(G) < \psi_k(\mathbb{Z}_n)$ for all non-cyclic groups G of order n. Recently, the product of the element orders of a group, which is denoted by $\rho(G)$, has also been considered and several results regarding $\rho(G)$ were proved in [18], including the following:

Theorem 2. *$\rho(G) \leqslant \rho(\mathbb{Z}_n)$ for every finite group G of order n, and $\rho(G) = \rho(\mathbb{Z}_n)$ if and only if $G \cong \mathbb{Z}_n$.*

Inspired by these works and with a goal to give a unified approach to study those functions, we study the *order sequence* of a group G, which is defined as the sequence

$$\mathrm{os}(G) = (o(g_1), o(g_2), \ldots, o(g_n)),$$

where $o(g_i) \leqslant o(g_{i+1})$ for $1 \leqslant i \leqslant n-1$.

For example, one can check that $\mathrm{os}(\mathbb{Z}_6) = (1, 2, 3, 3, 6, 6)$ and $\mathrm{os}(S_3) = (1, 2, 2, 2, 3, 3)$.

The sequence $\mathrm{os}(\mathbb{Z}_n)$ for the cyclic group of order n can be determined explicitly: for each divisor d of n, the entry d occurs $\phi(d)$ times, where ϕ is Euler's function.

Let $E(p^r)$ denote the elementary abelian group of order p^r. Then the order sequence of this group is $(1, p, p, \ldots, p)$. Note that, if p is odd and $r \geqslant 3$ then there are groups with the same order sequence as $E(p^r)$ but not isomorphic to it (non-abelian groups of exponent p). The smallest examples of pairs of groups with the same order sequence have order 16; one such pair is $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times Q_8$, where Q_8 is the quaternion group.

For two groups G and H of order n, we say that $\mathrm{os}(G)$ *dominates* $\mathrm{os}(H)$ if $o(g_i) \geqslant o(h_i)$ for $1 \leqslant i \leqslant n$. This relation is a partial order, but not a total order; there are groups which are incomparable in this order. We shall implicitly think of it as an order on the finite groups of order n; it induces a *partial preorder* on isomorphism classes of groups, that is, a reflexive and transitive relation, since there are groups which have the same order sequence.

However, there is a unique maximal element, namely $\mathbb{Z}_n$ (a recent result of M. Amiri [3]). Theorems 1 and 2 follow immediately from this theorem. In Section 5 we use the maximality of $\mathbb{Z}_n$ under domination to establish bounds on the gaps between the values of ψ and of ρ on cyclic and non-cyclic groups, and characterize groups meeting these bounds (Theorem 25).

In Section 2, we define products of order sequences of two groups G and H, and prove that if $|G|$ and $|H|$ are not coprime, then there exists no group whose order sequence is the same as the product of the order sequences of G and H. We also show that the product of the order sequences of two groups G and H of coprime orders dominates the order sequence of any extension of G by H if G is abelian.

In Section 3, we explicitly determine the order sequence of an abelian p-group; using this, in Theorem 14, we show that the poset of order sequences of abelian groups of order p^n is naturally isomorphic to the (well-studied) poset of partitions of n with its natural partial order.

In Section 4, we find the groups with minimal order sequences among the family of nilpotent groups. In Theorem 22, we show that if there is a non-nilpotent group of order n, a group with minimal order sequence must be non-nilpotent.

Equality of the order sequence defines an equivalence relation on groups of given order, and the equivalence classes are partially ordered by domination. In Section 3, we give a complete description of this order for abelian groups, in terms of the lattice of partitions of an integer. In Section 6, we observe that for all groups the poset can be rather complicated.

Section 7 describes the connection between the order sequence and some well-studied graphs related to groups. We show that the power graph determines the order sequence, which in turn determines the Gruenberg–Kegel graph.

The final Section 8 lists some open problems.

Throughout the paper, most of our notation is standard; for any undefined term, we refer the reader to the books [26, 34].

2 Products and extensions

We begin with a general result on order sequences and domination.

Proposition 3. *Let G and H be groups of the same order. Then* $\mathrm{os}(G)$ *dominates* $\mathrm{os}(H)$ *if and only if there is a bijecton $f : G \to H$ such that $o(g) \geqslant o(f(g))$ for all $g \in G$.*

Proof. The forward implication is clear. For the converse, suppose that the bijection exists, and let $G = \{g_1, \ldots, g_n\}$ where $o(g_1) \leqslant \cdots \leqslant o(g_n)$. Then, for $1 \leqslant k \leqslant n$, there are at least k elements of H whose orders do not exceed $o(g_k)$, namely $f(g_1), \ldots, f(g_k)$; so, if

h_k is the element of H whose order is the kth (in increasing order), then $o(h_k) \leqslant o(g_k)$, as required. □

On the basis of this, we make a stronger definition. For groups G and H of the same order, we say that $\mathrm{os}(G)$ *strongly dominates* $\mathrm{os}(H)$ if there is a bijection $f : G \to H$ such that $o(f(g)) \mid o(g)$ for all $g \in G$.

With this terminology, we can state Amiri's result [3]:

Theorem 4. *The order sequence of $\mathbb{Z}_n$ strongly dominates that of any other group of order n.*

This answers the question asked in [27, Problem 18.1] which was settled by Ladisch [29] for solvable groups.

We use strong domination in the proof of a theorem later in the paper (Theorem 22). But it might warrant further investigation. It is not equivalent to domination as previously defined. The group $\mathbb{Z}_3 : \mathbb{Z}_4$ of order 12, where the generator of $\mathbb{Z}_4$ conjugates the generator of $\mathbb{Z}_3$ to its inverse, dominates the alternating group A_4, but does not strongly dominate it: their order sequences are respectively

$$\begin{aligned} &(1,2,3,3,4,4,4,4,4,4,6,6) \\ \text{and} \quad &(1,2,2,2,3,3,3,3,3,3,3,3) \end{aligned}$$

(these are `SmallGroup(12,i)` for $i = 1$ and $i = 3$ in the GAP library [17].)

We define two operations on non-decreasing sequences x and y. If these sequences have lengths m and n respectively, then xy is the sequence formed by taking all products $x_i y_j$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ and writing them in non-decreasing order. Similarly, $x \vee y$ is obtained by taking all numbers $\mathrm{lcm}\{x_i, y_j\}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ and writing them in non-decreasing order.

Proposition 5. *(a) For two non-decreasing sequences x and y of lengths m and n respectively, $xy = x \vee y$ if and only if x_i and y_j are coprime for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.*

(b) Suppose that $x_i \mid m$ for all i, and $y_j \mid n$ for all j, where $\gcd(m,n) = 1$. Suppose further that each of the sequences x and y contains a unique term equal to 1. Then the product sequence xy and the numbers m and n uniquely determine x and y.

Proof. The proof of (a) is obvious. For (b), we see that x consists of the terms of xy coprime to n, and y consists of the terms coprime to m. □

Theorem 6. *(a) For any two groups G and H, $\mathrm{os}(G \times H) = \mathrm{os}(G) \vee \mathrm{os}(H)$. In particular, $\mathrm{os}(G \times H) = \mathrm{os}(G)\mathrm{os}(H)$ if and only if $|G|$ and $|H|$ are coprime.*

(b) Assume that $\gcd(|G|, |H|) = 1$. Given $\mathrm{os}(G \times H)$ and the numbers $|G|$ and $|H|$, the sequences $\mathrm{os}(G)$ and $\mathrm{os}(H)$ are determined.

(c) If $|G|$ and $|H|$ are not coprime, then there is no group K for which $\mathrm{os}(G)\mathrm{os}(H) = \mathrm{os}(K)$.

Proof. (a) This follows since $o((g,h)) = \mathrm{lcm}\{o(g), o(h)\}$.

(b) This is immediate from the preceding Proposition.

(c) Suppose that $|G|$ and $|H|$ are not coprime, and let p be a prime number dividing both $|G|$ and $|H|$. The number of elements of order p in G is congruent to $-1 \pmod p$. (This follows from the proof of Cauchy's Theorem that G contains elements of order p.) Similarly for H. So the number of terms in $\mathrm{os}(G)\mathrm{os}(H)$ equal to p is congruent to $-2 \pmod p$. Hence $\mathrm{os}(G)\mathrm{os}(H)$ cannot be the order sequence of a group. □

The following corollary is immediate from Theorem 6.

Corollary 7. *Let G, H be two finite groups with $gcd(|G|, |H|) = 1$. Then, $\rho(G \times H) = \rho(G)^{|H|}\rho(H)^{|G|}$.*

Theorem 8. *Let G and H be groups of coprime order, and suppose that G is abelian. Let K be any extension of G by H. Then $\mathrm{os}(G \times H) = \mathrm{os}(G)\mathrm{os}(H)$ strongly dominates $\mathrm{os}(K)$.*

Proof. According to the Schur–Zassenhaus Theorem, G has a complement in K; that is, there is a subgroup of K, which we identify with H, such that $G \cap H = \{1\}$ and $GH = K$. (Since G is abelian, only Schur's part of the proof is required.) Thus every element of K is uniquely written as gh for $g \in G$ and $h \in H$. So the map $f : G \times H \to K$ defined by $f((g,h)) = gh$ is a bijection. By Proposition 3, it suffices to show that $o(g)o(h) \geqslant o(gh)$.

Suppose that $o(g) = m$ and $o(h) = n$. We will prove that $(gh)^{mn} = 1$, from which the result follows.

We have
$$(gh)^n = g.hgh^{-1}.h^2gh^{-2} \cdots h^{n-1}gh^{-(n-1)}h^n,$$
and we have $h^n = 1$. Since G is a normal subgroup of K, the elements g, hgh^{-1}, $\ldots$, $h^{n-1}gh^{-(n-1)}$ all belong to G, and all have order m, since they are conjugate to g. In an abelian group, the product of elements of order m has order dividing m; so $((hg)^n)^m = 1$, and the proof is complete. □

Now we give a couple of results on direct products.

Proposition 9. *Suppose that G_1, G_2 and H are groups such that $|G_1| = |G_2|$ and $\mathrm{os}(G_1)$ dominates $\mathrm{os}(G_2)$. Suppose that either*

(a) $|G_1|$ and $|H|$ are coprime; or

(b) $\mathrm{os}(G_1)$ strongly dominates $\mathrm{os}(G_2)$.

Then $\mathrm{os}(G_1 \times H)$ dominates $\mathrm{os}(G_2 \times H)$.

Proof. Let f be a bijection from G_1 to G_2 satisfying $o(g) \geqslant o(f(g))$ (or, in case (b), $o(f(g)) \mid o(g)$). Then define a bijection $f' : G_1 \times H \to G_2 \times H$ by the rule that $f'((g,h)) = (f(g), h)$.

(a) If $|G_1|$ and $|H|$ are coprime, then

$$o((g,h)) = o(g)o(h) \geqslant o(f(g))o(h) = o((f(g),h)),$$

so by Proposition 3, $\mathrm{os}(G_1 \times H)$ dominates $\mathrm{os}(G_2 \times H)$.

(b) If f satisfies the conditions for strong domination, then

$$o((f(g),h)) = \mathrm{lcm}(o(f(g)), o(h)) \mid \mathrm{lcm}(o(g), o(h)) = o((g,h)),$$

and again Proposition 3 gives the result – indeed we conclude that the domination is strong. □

From this we can prove a two-sided version:

Proposition 10. *Let G_1 and G_2 be groups of the same order, and let H_1 and H_2 be groups of the same order. Suppose that $\mathrm{os}(G_1)$ dominates $\mathrm{os}(G_2)$ and $\mathrm{os}(H_1)$ dominates $\mathrm{os}(H_2)$. Moreover, suppose that one of the following holds:*

(a) $|G_1|$ and $|H_1|$ are coprime;

(b) $\mathrm{os}(G_1)$ strongly dominates $\mathrm{os}(G_2)$ and $\mathrm{os}(H_1)$ strongly dominates $\mathrm{os}(H_2)$.

Then $\mathrm{os}(G_1 \times H_1)$ dominates $\mathrm{os}(G_2 \times H_2)$.

This follows immediately from two applications of the preceding result.

3 Abelian groups and the partition lattice

We start this section by describing the order sequence of a finite abelian p-group. Let $|G| = p^n$ where p is prime and G is abelian. By the Fundamental Theorem of Abelian Groups,

$$G \cong \mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}} \times \cdots \times \mathbb{Z}_{p^{r_k}},$$

where $r_i \geqslant 1$ for $1 \leqslant i \leqslant k$, and $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_k$. Then $r_1 + r_2 + \cdots + r_k = n$, so $(r_1, r_2, \ldots, r_k)$ is a partition of n. (We examine partitions further below.) Define $s_1, s_2, \ldots, s_\ell$, where $\ell = r_1$, by the rule that

$$s_j = |\{i : r_i \geqslant j\}|.$$

Thus, for example, $s_1 = k$.

We note that the numbers s_j in turn determine the numbers r_i: we will see the exact relationship shortly.

Proposition 11. *With the above notation, the number of elements of order dividing p^j in G is $p^{s_1+\cdots+s_j}$.*

Proof. The elements with order dividing p^j form a subgroup A_j of A. Now $A_j \leqslant A_{j+1}$, and A_{j+1}/A_j consists of the elements of order 1 or p in A/A_j; this group is generated by the cosets containing elements of order p^{j+1}, which has rank s_{j+1}, and so its cardinality is $p^{s_{j+1}}$. So $|A_{j+1}| = p^{s_{j+1}}|A_j|$. Since A_0 is the identity group, induction now completes the proof of the Proposition. □

Using this, we immediately have the following corollary which is also proved in [31].

Theorem 12. *Two finite abelian groups have the same order sequence if and only if they are isomorphic.*

Proof. The reverse implication is clear. For the forward implication, it suffices to prove the result for groups of prime power order, by Theorem 6. By Proposition 11, the order sequence determines the numbers $s_1, \ldots, s_\ell$, and hence the numbers r_i, and hence the isomorphism type of the group. □

As we have already seen, we cannot expect such a result for a wider class of groups containing the class of abelian groups; even for p-groups with p an odd prime, it is not true.

We can also count the numbers of elements, or cyclic subgroups, of given order:

Proposition 13. *With the notation introduced before Proposition 11,*

(a) the number of elements of order p^j is $p^{s_1+\cdots+s_{j-1}}(p^{s_j}-1)$;

(b) the number of cyclic subgroups of order p^j is $p^{s_1+\cdots+s_{j-1}-j+1}(p^{s_j}-1)/(p-1)$;

(c) the total number of cyclic subgroups is

$$1 + p^{s_1-1} + \cdots + p^{s_1+\cdots+s_{m-1}-m+1} + (p^{n-m+1}-1)/(p-1)$$

where $|G| = p^n$ and the exponent of G is p^m.

Proof. (a) By Proposition 11, this number is $p^{s_1+\cdots+s_j} - p^{s_1+\cdots+s_{j-1}}$.

(b) Each cyclic subgroup of order p^j has $p^{j-1}(p-1)$ generators.

(c) This is obtained by summing the formulae in (b), noting that the maximum j for which $s_j > 0$ is m and $s_1 + \cdots + s_m = n$. □

Now we relate the preceding analysis of abelian p-groups to the theory of partitions of integers, referring to [30, Section 1.1].

A *partition* of n is a non-increasing sequence $(r_1, \ldots, r_k)$ of positive integers with sum n. Given two partitions $a = (r_1, \ldots, r_k)$ and $b = (s_1, \ldots, s_\ell)$ of n, to compare them we append zeros to the shorter sequence if necessary to make them have the same length; then we write $a \succeq b$ if, for all relevant j, we have

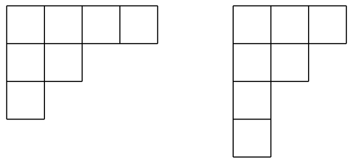
$$r_1 + \cdots + r_j \geqslant s_1 + \cdots + s_j.$$

Thus, from the top, the order begins

$$(n) \succeq (n-1,1) \succeq (n-2,2) \succeq (n-2,1,1).$$

This is called the *natural partial order* on partitions (also called *majorization*). It is not a total order; for example, $(2,2,2)$ and $(3,1,1,1)$ are incomparable.

A partition $(r_1,\ldots,r_k)$ of n can be represented by a *Young diagram* or *Ferrers diagram*, made up of n squares in the plane, in left-aligned rows of lengths $r_1,\ldots,r_k$. The picture shows the partitions $(4,2,1)$ and $(3,2,1,1)$.



$(4,2,1)$ $(3,2,1,1)$

Figure 1: Young diagrams

The *conjugate* a' of the partition $a=(r_1,\ldots,r_k)$ is the partition $b=(s_1,\ldots,s_\ell)$, where

$$s_i = |\{j : r_j \geqslant i\}|.$$

This corresponds simply to reflecting the Young diagram in the diagonal. So $a''=a$ for any partition a. The two partitions in the figure are conjugates of each other.

An abelian group A of order p^n has a unique expression of the form

$$A \cong \mathbb{Z}_{p^{r_1}} \times \cdots \times \mathbb{Z}_{p^{r_k}},$$

where $a=(r_1,\ldots,r_k)$ is a partition of n, which we will call the *defining partition* of A.

Our arguments in the proof of Theorem 12 show that the partition $(s_1,s_2,\ldots,s_\ell)$ used in the proof of that theorem is conjugate to the defining partition of A. Now the fact that $a''=a$ shows that the construction of $(r_1,\ldots,r_k)$ from $(s_1,\ldots,s_l)$ is formally identical to the construction in the other direction.

Theorem 14. *Let A and C be two abelian groups of order p^n, with defining partitions a and c respectively. Then the following are equivalent:*

(a) $\mathrm{os}(A)$ *dominates* $\mathrm{os}(C)$*;*

(b) $c' \succeq a'$*;*

(c) $a \succeq c$*.*

Proof. $\mathrm{os}(A)$ dominates $\mathrm{os}(C)$ if and only if, for each i, the number of elements of order dividing p^i is at least as large in C as in A. By Proposition 11 and the definition of the natural partial order, the truth of this for all i is equivalent to $c' \succeq a'$. Thus (a) and (b) are equivalent. The equivalence of (b) and (c) is standard; see [30, (1.11)]. □

The poset of partitions of an integer has been the subject of research covering several areas of mathematics. See [10, 19] for some of the results. We can regard the poset of order sequences of groups as a kind of non-commutative generalisation of this. We return briefly to this topic later.

Remark The paper [19] gives estimates for the lengths of maximal chains in the partition lattice. It is very easy to see that the maximum chain length of the lattice of partitions of n tends to infinity with n. Now any finite poset is embeddable in a product of finite chains (for example, take all linear extensions of the given poset). So we conclude:

Proposition 15. *For any finite poset P, there exists n such that P is embeddable in the poset of order sequences of abelian groups of order n.*

We are next interested in the comparison between the number of cyclic subgroups of two abelian p-groups of order p^n. For a group G, let $\mathrm{cyc}(G)$ denote the number of cyclic subgroups of G. The following result is well-known.

Lemma 16. *Let $a = (r_1, r_2, \ldots, r_k)$ and $c = (s_1, s_2, \ldots, s_\ell)$ be two partitions of n. Then $a \succeq c$ if and only if the Young diagram for c can be obtained from that of a by successively moving boxes from a higher row to a lower row (one box at a time), in such a way that each intermediate step is the Young diagram of a partition of n.*

Lemma 17. *Let A, A_1 and A_2 be three abelian groups of order p^n with defining partitions $a = (r_1, r_2, \ldots, r_k), a_1 = (r_1, r_2, \ldots, r_{j_1} - 1, \ldots, r_{j_2} + 1, \ldots, r_k)$, and $a_2 = (r_1, r_2, \ldots, r_{j_1} - 1, \ldots, r_k, 1)$ respectively. Then $\mathrm{cyc}(A) \leqslant \mathrm{cyc}(A_1)$ and $\mathrm{cyc}(A) \leqslant \mathrm{cyc}(A_2)$.*

Proof. Our proof depends on whether the exponent of A_1 and A_2 is the same as the exponent of A, that is whether $j_1 > 1$ or $j_1 = 1$.

Case 1: $j_1 > 1$: In this case, as the exponent is the same for A, A_1 and A_2, the number of parts in the conjugate partitions of A, A_1 and A_2 are the same. Let $(s_1, s_2, \ldots, s_{r_1})$, $(s'_1, s'_2, \ldots, s'_{r_1})$ and $(s''_1, s''_2, \ldots, s''_{r_1})$ denote the conjugate partitions of a, a_1 and a_2 respectively.

By Proposition 13, we have

$$\mathrm{cyc}(A) = 1 + p^{s_1 - 1} + \cdots + p^{s_1 + \cdots + s_{r_1 - 1} - r_1 + 1} + (p^{n - r_1 + 1} - 1)/(p - 1)$$

and

$$\mathrm{cyc}(A_1) = 1 + p^{s'_1 - 1} + \cdots + p^{s'_1 + \cdots + s'_{r_1 - 1} - r_1 + 1} + (p^{n - r_1 + 1} - 1)/(p - 1).$$

As $a \succeq a_1$, by Theorem 14, we have $(s'_1, s'_2, \ldots, s'_{r_1}) \succeq (s_1, s_2, \ldots, s_{r_1})$. Hence, we have $\mathrm{cyc}(A) \leqslant \mathrm{cyc}(A_1)$. The proof of $\mathrm{cyc}(A) \leqslant \mathrm{cyc}(A_2)$ is similar and hence omitted.

Case 2: $j_1 = 1$: In this case, the exponent is not the same for A, A_1 and A_2. The number of parts in the conjugate partitions of a_1 and a_2 is 1 less than the number of parts in the conjugate partition of a. Let $(s_1, s_2, \ldots, s_{r_1})$, $(s'_1, s'_2, \ldots, s'_{r_1 - 1})$ and $(s''_1, s''_2, \ldots, s''_{r_1 - 1})$ denote the conjugate partitions of a, a_1 and a_2 respectively. We show that $\mathrm{cyc}(A) \leqslant \mathrm{cyc}(A_1)$ and $\mathrm{cyc}(A) \leqslant \mathrm{cyc}(A_2)$ will follow by a similar argument.

Again, using Proposition 13, we have

$$\begin{aligned}\mathrm{cyc}(A) &= 1+p^{s_1-1}+\cdots+p^{s_1+\cdots+s_{r_1-2}-r_1+2}+p^{s_1+\cdots+s_{r_1-1}-r_1+1}+(p^{n-r_1+1}-1)/(p-1)\\ &= 1+p^{s_1-1}+\cdots+p^{s_1+\cdots+s_{r_1-2}-r_1+2}+p^{s_1+\cdots+s_{r_1-1}-r_1+1}+(1+p+\cdots+p^{n-r_1})\end{aligned}$$

and

$$\begin{aligned}\mathrm{cyc}(A_1) &= 1+p^{s'_1-1}+\cdots+p^{s'_1+\cdots+s'_{r_1-2}-r_1+2}+(p^{n-r_1+2}-1)/(p-1)\\ &= 1+p^{s'_1-1}+\cdots+p^{s'_1+\cdots+s'_{r_1-2}-r_1+2}+(1+p+\cdots+p^{n-r_1+1}).\end{aligned}$$

As $a \succeq a_1$, by Theorem 14, we have $(s'_1, s'_2, \ldots, s'_{r_1-1}) \succeq (s_1, s_2, \ldots, s_{r_1})$. Hence, we have $\mathrm{cyc}(A_1) - \mathrm{cyc}(A) \geqslant p^{n-r_1+1} - p^{s_1+\cdots+s_{r_1-1}-r_1+1} \geqslant 0$.

This completes the proof. □

We are now in a position to prove the next result which tells about the connection between the order sequence and the number of cyclic subgroups of p-groups.

Theorem 18. *Let A and C be two abelian groups of order p^n, with defining partitions a and c respectively. If* $\mathrm{os}(A)$ *dominates* $\mathrm{os}(C)$*, then* $\mathrm{cyc}(A) \leqslant \mathrm{cyc}(C)$*.*

Proof. Let

$$A \cong \mathbb{Z}_{p^{r_1}} \times \cdots \times \mathbb{Z}_{p^{r_k}},$$

and

$$C \cong \mathbb{Z}_{p^{t_1}} \times \cdots \times \mathbb{Z}_{p^{t_\ell}},$$

where $a = (r_1, \ldots, r_k)$ and $c = (t_1, \ldots, t_\ell)$ are defining partitions of A and C respectively.

By Theorem 14, we have $a \succeq c$. By Lemma 16, the Young diagram for c can be obtained from that of a by successively moving boxes from a higher row to a lower row (one box at a time), in such a way that each intermediate step is the Young diagram of a partition of n. Let us denote the partitions of the intermediate steps by $b^1, b^2, \ldots, b^r$ and we also denote a by b^0 and c by b^{r+1}. Then we have

$$a(= b^0) \succeq b^1 \succeq b^2 \succeq \cdots \succeq \cdots \succeq b^r \succeq c(= b^{r+1})$$

where the Young diagram for each b^i is obtained from that of b^{i-1} by moving exactly one box from a higher row to a lower row. If B^i denotes the abelian p-group of order p^n with defining partition b^i, by using Lemma 17, we now have

$$\mathrm{cyc}(A) \leqslant \mathrm{cyc}(B^1) \leqslant \mathrm{cyc}(B^2) \leqslant \ldots \leqslant \mathrm{cyc}(B^r) \leqslant \mathrm{cyc}(C).$$

This completes the proof. □

It is clear that the converse of Theorem 18 need not hold, in general. For example, we can consider the groups $\mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_8 \times \mathbb{Z}_8$. Then, $\mathrm{cyc}(\mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 20$ and $\mathrm{cyc}(\mathbb{Z}_8 \times \mathbb{Z}_8) = 16$ but the order sequences of the groups are not comparable.

4 Nilpotent and non-nilpotent groups

In this section, we are primarily interested in studying the minimality of order sequence among finite nilpotent groups.

It is easy to see that the order sequence of any finite p-group of order p^r dominates the order sequence of the elementary abelian p-group $E(p^r)$.

Theorem 19. *Let $G = P_1 \times P_2 \times \cdots \times P_k$ be a finite nilpotent group. If the order sequence of G is minimal among nilpotent groups, then for each $1 \leqslant i \leqslant k$, P_i is a group of prime exponent.*

Proof. We prove by induction on the number of primes k. For $k = 1$ this is of course true. For a nilpotent group G, it can be written as $G = P_1 \times P_2 \times \cdots \times P_k$, where P_i are the Sylow subgroups. Let $G_1 = P_1 \times P_2 \times \cdots \times P_{k-1}$ and $G_2 = P_k$. Then $\mathrm{os}(G_1)$ is minimal among nilpotent groups of its order. For if not, then we could replace it with a nilpotent group G_1^* yielding a smaller sequence, whence $\mathrm{os}(G)$ would dominate $\mathrm{os}(G_1^* \times P_k)$ by Proposition 9. The induction hypothesis now shows that P_i is a group of prime exponent for $1 \leqslant i \leqslant k-1$.

Similarly, the order sequence of P_k is minimal, so P_k is a group of prime exponent. This completes the proof of the Theorem. □

We have already seen that if a group G has the same order sequence as a cyclic group, then G must be cyclic; but it may happen that G is non-abelian but G has the same order sequence as an abelian group. For example, one can take a non-abelian group of order p^3 with every element of order p. We next ask the same question for nilpotent groups. We show first that nilpotency is determined by the order sequence of the group.

Theorem 20. *If G is a nilpotent group, and H is a group of the same order such that $\mathrm{os}(G) = \mathrm{os}(H)$, then H is nilpotent.*

Proof. Let $|G| = |H| = p_1^{a_1} \cdots p_r^{a_r}$. All elements of p_i-power order in G belong to the unique Sylow p_i-subgroup of G, and there are $p_i^{a_i}$ of them. By assumption, H also has exactly $p_i^{a_i}$ elements of p_i-power order, and it has a Sylow p_i-subgroup P_i; thus P_i contains all elements of p_i-power order, and so is the unique Sylow p_i-subgroup, and is normal in H. Since this holds for all i, all the Sylow subgroups of H are normal, and H is nilpotent. □

The proof shows that, if G is nilpotent and $\mathrm{os}(G) = \mathrm{os}(H)$, then for each prime p_i dividing $|G|$, the Sylow p_i-subgroups of G and H have the same order sequence.

Next we show that, if there is a non-nilpotent group of order n, then nilpotent groups cannot realise order sequences which are minimal under domination.

First we require a lemma.

Lemma 21. *For a positive integer n, the following are equivalent:*

- *there exists a non-nilpotent group of order n;*

- *n is divisible by $p^d q$, where p and q are primes and $q \mid p^d - 1$.*

Proof. Suppose first that G is a non-nilpotent group of order n. Then G contains a minimal non-nilpotent subgroup (one all of whose proper subgroups are nilpotent). So it suffices to deal with the case where G is minimal non-nilpotent.

The minimal non-nilpotent groups were determined by Schmidt [33]. A convenient reference is [7], which we use here. Page 3456 of this paper gives a list of five types of group, and Theorem 3 asserts that the Schmidt groups (the minimal non-nilpotent groups) are exactly those of types II, IV and V.

- Type II groups are semidirect products $[P]Q$, where Q is cyclic of order q^r, so that the p-group P is a faithful irreducible Q/Q^q-module with trivial centralizer. Thus the cyclic group Q/Q^q of order q acts fixed-point-freely on $P \setminus \{1\}$, so $q \mid |P| - 1$.
- Type IV groups are semidirect products $[P]Q$, where P is special of rank $2m$ and the cyclic group Q has a faithful irreducible action on $P/\Phi(P)$. Thus q divides $|P/\Phi(P)| = p^{2m}$.
- Type V groups are semidirect products $[P]Q$, where $|P| = p$, Q is cyclic of order q^r, and Q induces an automorphism group of P of order q; so $q \mid p - 1$.

Conversely, suppose that p and q are primes such that $q \mid p^d - 1$. The group $B = \{x \mapsto ax + b\}$ of permutations of the finite field of order p^d, where a runs through the qth roots of unity in the field and b runs through the whole field, is a non-nilpotent group of order $p^d q$. If $p^d q \mid n$, then let K be any abelian group of order $n/(p^d q)$; then $B \times K$ is a non-nilpotent group of order n. □

Theorem 22. *Let n be a positive integer for which there exists a non-nilpotent group of order n. Then there is a non-nilpotent group H of order n such that the order sequence of any nilpotent group G of order n properly dominates* $\mathrm{os}(H)$.

Proof. By Theorem 19, the abelian group G of order n whose Sylow subgroups have prime exponent is dominated by every nilpotent group of order n.

From the proof of Lemma 21, there is a non-nilpotent group B (the group B used here is the one used in the proof of Lemma 21) of order $p^d q$ dividing n. Let A be the direct product of an elementary abelian group of order p^d and a cyclic group of order q. Now B has the same number of elements of order 1 or p as A does; the remaining elements of B all have order q, while A has elements of orders q and pq. Let f be a bijection from A to B mapping the identity to the identity, elements of order p to elements of order p, and the remaining elements arbitrarily. This bijection shows that $\mathrm{os}(A)$ strongly dominates $\mathrm{os}(B)$.

Let K be abelian group of order $n/(p^d q)$, all of whose Sylow subgroups have prime exponent. Then $G \cong A \times K$. By Proposition 9(b), $H = B \times K$ is a non-nilpotent group of order n dominated by G, and hence by every nilpotent group of order n. This completes the proof. □

5 Bounds on several functions

In this section we use the maximality of $\mathbb{Z}_n$ under (strong) domination to establish bounds on the gaps between the values of ψ and of ρ on cyclic and non-cyclic groups, and characterize groups meeting these bounds (Theorem 25). From Theorem 4, we have the following.

Corollary 23. *Let F be a symmetric function of n arguments which is a strictly increasing function of each argument and G be a non-cyclic group of order n. Then $F(\mathrm{os}(G)) < F(\mathrm{os}(\mathbb{Z}_n))$.*

Theorem 1 is an immediate consequence of Corollary 23. We can also establish a gap between the cyclic group and other groups of order n.

Domenico, Monetta and Noce [16] proved the following upper bound for the function $\rho(G)$ where G has a Sylow tower. We recall that G is said to admit a Sylow tower if there exists a normal series

$$\{e\} = G_0 \leqslant G_1 \leqslant G_2 \leqslant \ldots \leqslant G_n = G$$

such that G_{i+1}/G_i is isomorphic to a Sylow subgroup of G for every $0 \leqslant i \leqslant n-1$.

Theorem 24. *Let G be a non-cyclic group of order n admitting a Sylow tower. Then*

$$\rho(G) \leqslant q^{-q}\rho(\mathbb{Z}_n),$$

where q is the smallest prime dividing n. The same inequality holds if $n = p^a q^b$ with $p > q$ or G is a Frobenius group.

The cited paper also includes similar results for other classes such as super-solvable groups.

In this paper, we improve their result in three ways. First, we do not need to assume the existence of a Sylow tower, or any restriction on the group G. Second, our bound is stronger. Third, we can deal with other functions; we give a result for the sum of the orders of group elements as an example, and determine the groups meeting our bound for either sum or product.

Theorem 25. *Let G be a non-cyclic group of order n. Then*

(a) $\rho(G) \leqslant q^{-\phi(n)}\rho(\mathbb{Z}_n)$;

(b) $\psi(G) \leqslant \psi(\mathbb{Z}_n) - n\phi(n)(q-1)/q$;

where q is the smallest prime divisor of n and ϕ is Euler's functon. Equality holds if and only if either $G = \mathbb{Z}_q \times \mathbb{Z}_q$ or G is the quaternion group of order 8.

Proof. We compare the order sequences of G and $\mathbb{Z}_n$. Let

$$\mathrm{os}(G) = (o(g_1), o(g_2), \ldots, o(g_n)) \text{ and } \mathrm{os}(\mathbb{Z}_n) = (o(a_1), o(a_2), \ldots, o(a_n)).$$

By Theorem 4, we have $o(g_i) \leqslant o(a_i)$ for all $1 \leqslant i \leqslant n$. Moreover, $\mathbb{Z}_n$ has $\phi(n)$ elements of order n and the last $\phi(n)$ terms of $\mathrm{os}(G)$ are clearly $\leqslant n/q$. So replacing the $\phi(n)$ elements of $\mathrm{os}(\mathbb{Z}_n)$ equal to n by n/q gives a sequence a which still dominates $\mathrm{os}(G)$. (This is still a non-decreasing sequence since no element of G has order larger than n/q.) Therefore, by Corollary 23,

$$\rho(G) \leqslant q^{-\phi(n)}\rho(\mathbb{Z}_n) \text{ and } \psi(G) \leqslant \psi(\mathbb{Z}_n) - \phi(n)(n - n/q)$$

(the values on the right are the sum and product of elements of a).

It is easy to see that when $G = \mathbb{Z}_q \times \mathbb{Z}_q$ or $G = Q_8$, the equality holds.

Conversely, suppose that G is any group attaining either of the bounds. The proof of the inequality shows that $\mathrm{os}(G)$ is obtained from $\mathrm{os}(\mathbb{Z}_n)$ by replacing the $\phi(n)$ orders n by orders n/q, leaving the others as before. This implies that, if m is any divisor of n other than n and n/q, then G contains the same number of elements of order m as $\mathbb{Z}_n$, namely $\phi(m)$ of them; so G has a unique cyclic subgroup of order m, which is normal in G and contains all elements of order m.

We separate into three cases. Suppose first that n has at least three prime divisors, say q (the smallest), r and s. Then G has unique cyclic subgroups of orders n/r and n/s, say H_1 and H_2, respectively. As H_1 is unique subgroup of order n/r, H_1 is normal in G. So is H_2. The product of two normal subgroups is a subgroup. Hence H_1H_2 is a subgroup. Moreover, $|H_1H_2| = |H_1||H_2|/\gcd(|H_1|, |H_2|) = n$, and if $H_1 = \langle a \rangle$ and $H_2 = \langle b \rangle$ then H_1H_2 is generated by ab. Hence H_1H_2 is a cyclic subgroup of order n, neccessarily equal to G, a contradiction.

Next suppose that G is a q-group. Since it is not cyclic, either $G = \mathbb{Z}_q \times \mathbb{Z}_q$, or $|G| > q^3$. In the second case, G contains a unique subgroup of order q. A theorem of Burnside (see [20, Theorem 12.5.2]) shows that G is cyclic or generalized quaternion. Cyclic groups are excluded, and it is easy to see that the only generalized quaternion group satisfying the conditions is Q_8.

Finally suppose that only two primes q and r divide $|G|$. By our earlier remark, the Sylow q-subgroup of G is cyclic and normal. Let R be the Sylow r-subgroup. If $|Q| > q$, then also R is cyclic and normal, whence G is cyclic, a contradiction. So $|Q| = q$. Let R be the Sylow r-subgroup. The number of conjugates of R is 1 or q, and is congruent to 1 mod r; so R is normal in G, and $G = Q \times R$. Now R must be non-cyclic; since r is odd, Burnside's theorem implies that G contains more than $r - 1$ elements of order r, a contradiction. □

It is easy to show that, for any $n \neq q$, we have $\phi(n) \geqslant q$, and therefore the bound in Theorem 25 is stronger than the bound in Theorem 24.

Domenico, Monetta and Noce [16, Proposition 8] also proved the following upper bound for $\rho(G)$ where G is nilpotent.

Theorem 26. *Let G be a non-cyclic nilpotent group of order n. Then*

$$\rho(G) \leqslant q^{-\frac{n(q-1)}{q}} \rho(\mathbb{Z}_n),$$

where q is the smallest prime dividing n.

In the following result, we improve the theorem.

Theorem 27. *Let $G = \mathbb{Z}_m \times G_1$ where G_1 is a nilpotent group and moreover $G_1 = P_1 \times P_2 \times \cdots \times P_r$ where every P_i is a noncyclic p-group (w.r.t the prime p_i), all the p_is are distinct, and $gcd(m, |G_1|) = 1$. Then,*

$$\rho(G) \leqslant \left(\prod_{i=1}^{r} p_i^{-\frac{(p_i-1)}{p_i}} \right)^{|G|} \rho(\mathbb{Z}_{|G|}).$$

Moreover, equality holds when each P_i is $\mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$.

Proof. We prove this by induction on r. When $r = 1$, we have $G = \mathbb{Z}_m \times P_1$. By Corollary 7, we clearly have

$$\rho(G) = \rho(\mathbb{Z}_m \times P_1) = \rho(\mathbb{Z}_m)^{|P_1|} \rho(P_1)^m.$$

As in Theorem 25, we compare the order sequences of P_1 and $\mathbb{Z}_{|P_1|}$. We know that the order sequence of P_1 is dominated by the order sequence of $\mathbb{Z}_{|P_1|}$ and moreover, $\mathbb{Z}_{|P_1|}$ has $\frac{|P_1|(p_1-1)}{p_1}$ elements of order $|P_1|$ and the last $\frac{|P_1|(p_1-1)}{p_1}$ terms of $\mathrm{os}(P_1)$ are clearly $\leqslant \frac{|P_1|}{p_1}$. Therefore, we clearly have

$$\begin{aligned} \rho(\mathbb{Z}_m)^{|P_1|} \rho(P_1)^m &\leqslant \rho(\mathbb{Z}_m)^{|P_1|} \left(p_1^{-\frac{|P_1|(p_1-1)}{p_1}} \rho(\mathbb{Z}_{|P_1|}) \right)^m \\ &= \left(p_1^{-\frac{(p_1-1)}{p_1}} \right)^{m|P_1|} \rho(\mathbb{Z}_m)^{|P_1|} \rho(\mathbb{Z}_{|P_1|})^m \\ &= \left(p_1^{-\frac{(p_1-1)}{p_1}} \right)^{m|P_1|} \rho(\mathbb{Z}_{m|P_1|}). \end{aligned}$$

Thus, when $r = 1$, the statement holds. We assume the statement for $r = \ell - 1$ and prove for $r = \ell$. Let $G = \mathbb{Z}_m \times P_1 \times \cdots \times P_{\ell-1} \times P_\ell$. Moreover, let $H = \mathbb{Z}_m \times P_1 \times \cdots \times P_{\ell-1}$. By Corollary 7, we clearly have

$$\rho(G) = \rho(H \times P_\ell) = \rho(H)^{|P_\ell|} \rho(P_\ell)^{|H|}.$$

By induction and using the fact that P_ℓ is non-cyclic nilpotent, we have

$$\rho(H)^{|P_\ell|}\rho(P_\ell)^{|H|} \leqslant \left(\left(\prod_{i=1}^{\ell-1} p_i^{-\frac{(p_i-1)}{p_i}} \right)^{|H|} \rho(\mathbb{Z}_{|H|}) \right)^{|P_\ell|} \left(p_\ell^{-\frac{|P_\ell|(p_\ell-1)}{p_\ell}} \rho(\mathbb{Z}_{|P_\ell|}) \right)^{|H|}$$
$$= \left(\prod_{i=1}^{\ell} p_i^{-\frac{(p_i-1)}{p_i}} \right)^{|G|} \rho(\mathbb{Z}_{|H|})^{|P_\ell|}\rho(\mathbb{Z}_{|P_\ell|})^{|H|}$$
$$= \left(\prod_{i=1}^{\ell} p_i^{-\frac{(p_i-1)}{p_i}} \right)^{|G|} \rho(\mathbb{Z}_{|G|}).$$

This completes the proof. It is easy to see that when each P_i is $\mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$, the equality holds. □

6 More general groups

Figure 2 shows the the Hasse dagram for the partially ordered set of the 13 isomorphism types of groups of order 60, ordered by domination of the order sequences. The numbers are those in the GAP `SmallGroups` library.

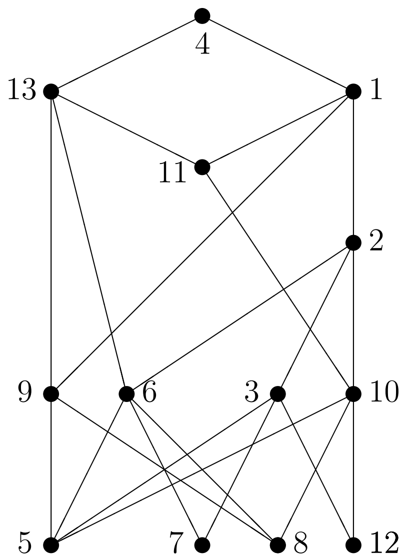


Figure 2: Groups of order 60

Note that `SmallGroup(60,4)` is the cyclic group $\mathbb{Z}_{60}$, `SmallGroup(60,13)` is $\mathbb{Z}_2 \times \mathbb{Z}_{30}$ (the other nilpotent group of order 60) and `Smallgroup(60,1)` is $\mathbb{Z}_5 \times$ `SmallGroup`$(12,1)$, where the second factor is the semidirect product of $\mathbb{Z}_3$ by $\mathbb{Z}_4$, and the element of order 4 inverts the element of order 3. Theorem 22 allows construction of three different non-nilpotent groups whose order sequence is dominated by those of the two nilpotent groups;

these are `SmallGroup(60,`i`)` for $i = 9, 10, 11$. Also, `SmallGroup(60,5)` is the alternating group A_5, the unique non-solvable group of order 60; it is minimal in the domination order but not unique with this property.

This example shows that the domination order will be difficult to understand in general. Notice how abelian groups form only a very small part of the poset, which as noted is a kind of non-abelian generalization of the partition lattice.

7 Connection with graphs

We present here some links between our problem and certain graphs associated with finite groups.

The *power graph* of a finite group G has vertex set G, with an edge $\{g, h\}$ if and only if one of g and h is a power of the other, that is, either $h = g^m$ or $g = h^m$ for some integer m. For a recent survey of properties of this graph, see [28].

Theorem 28. *Suppose that G and H are finite groups with isomorphic power graphs. Then* $\mathrm{os}(G) = \mathrm{os}(H)$.

This is [11, Corollary 3]. It can be seen as follows. The *directed power graph* of G is the directed graph which has an arc from g to h if and only if h is a power of G. It is clear that the order of an element g is the out-degree of g in this graph plus one (one for the element itself). The main theorem of [11] asserts that the power graph determines the directed power graph up to isomorphism.

As a consequence, [12, Theorem 1] (stating that finite abelian groups are determined up to isomorphism by their power graphs) is extended by Theorem 12 of this paper.

Pairs of groups with the same order sequence may or may not have isomorphic power graphs. Examples of order 16 exhibit both behaviours: the 14 groups give rise to 12 different power graphs and 9 order sequences. See [32] for some results on groups with the same power graph.

The *Gruenberg–Kegel graph* of a finite group has vertices the prime divisors of G, wth an edge $\{p, q\}$ if and only if G contains an element of order pq. This small graph contains a surprising amount of information about the group G. Sometimes it is considered as a labelled graph, with each vertex labelled by the corresponding prime. See [13] for a recent survey, and [25, 9] for interesting earlier results.

Proposition 29. *Let G and H be finite groups with* $\mathrm{os}(G) = \mathrm{os}(H)$. *Then the labelled Gruenberg–Kegel graphs of G and H are equal.*

This result is obvious from the definitions.

8 Open questions

Question 30. Investigate the relation of strong domination on groups of given order n.

We note that domination and strong domination coincide for the case when n is a prime power, since in that case the relations of order and divisibility coincide on the divisors of n.

Question 31. Is the condition that G is abelian necessary in Theorem 8?

We have already seen that if a group G has the same order sequence as a cyclic group, then G must be cyclic; whereas a non-abelian group can have the same order sequence as an abelian group. Moreover, nilpotency is characterised by the order sequence, and if there is a non-nilpotent group of order n then a nilpotent group cannot be minimal.

What happens for solvable groups?

Question 32. Is it true that a group having the same order sequence as a solvable group is solvable?

We could ask the same question with "supersolvable" in place of "solvable".

Question 33. If there is a non-solvable group of order n, is it true that at least one group of order n with minimal order sequence is non-solvable?

Note that there is a non-solvable group of order n if and only if n is a multiple of the order of a minimal (non-abelian) simple group; these groups were determined by Thompson [37]. The case $n = 60$ shown in Figure 2 shows that, unlike for nilpotency, there will not be a non-solvable group which is dominated by every solvable group.

Question 34. Let G and H be two non-isomorphic (non-abelian) simple groups of the same order. Is it true that either $\mathrm{os}(G)$ dominates $\mathrm{os}(H)$ or *vice versa*?

From the Classification of Finite Simple Groups, it is known that the only pairs of simple groups of the same order are

- A_8 and $\mathrm{PSL}(3,4)$; and
- $\mathrm{PSp}(2n,q)$ and $\mathrm{P}\Omega(2n+1,q)$, where q is an odd prime power and $n \geqslant 3$.

For the first pair, the $\mathbb{ATLAS}$ of Finite Groups [15] shows that $\mathrm{os}(A_8)$ dominates $\mathrm{os}(\mathrm{PSL}(3,4))$. The answer is not known in the other cases.

Question 35. Given a sequence of n natural numbers, is it the order sequence of a group? If it is, then how many groups, and can we construct them? (For computational complexity reasons, it is better to take the input to be the collected order sequence, the set of pairs $(m, s(m))$ where m is the order of an element and $s(m)$ the number of elements of this order, since this only requires a polylogarithmic amount of data.)

Question 36. Investigate further the poset of order sequences of finite groups.

Question 37. Given a finite poset P, find the minimum value of n such that P is embeddable in the order sequence poset of groups of order n.

We can ask the same question restricting to abelian groups.

For example, the 2-element antichain is represented by two groups of order 12, or two abelian groups of order 36; these are the smallest possible orders.

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