Colouring Complete Multipartite and Kneser-type Digraphs

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Abstract

The *dichromatic number* of a digraph D is the smallest k such that D can be partitioned into k acyclic subdigraphs, and the dichromatic number of an undirected graph is the maximum dichromatic number over all its orientations. Extending a well-known result of Lovász, we show that the dichromatic number of the Kneser graph $KG(n,k)$ is $\Theta(n-2k+2)$ and that the dichromatic number of the Borsuk graph $BG(n+1,a)$ is $n+2$ if a is large enough. We then study the list version of the dichromatic number. We show that, for any $\varepsilon > 0$ and $2 \leqslant k \leqslant n^{\frac{1}{2}-\varepsilon}$, the list dichromatic number of $KG(n,k)$ is $\Theta(n \ln n)$. This extends a recent result of Bulankina and Kupavskii on the list chromatic number of $KG(n,k)$, where the same behaviour was observed. We also show that for any $\rho > 3$, $r \geqslant 2$ and $m \geqslant \max\{\ln^\rho r, 2\}$, the list dichromatic number of the complete r-partite graph with m vertices in each part is $\Theta(r \ln m)$, extending a classical result of Alon. Finally, we give a directed analogue of Sabidussi's theorem on the chromatic number of graph products.

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1   Introduction

We consider graphs/digraphs without loops or multiple edges/arcs. They are all finite unless otherwise specified. A *proper k-colouring* of an undirected graph $G = (V, E)$ is a mapping $f : V \rightarrow [k] = \{1, \ldots, k\}$ such that $f^{-1}(i)$ is an independent set for every $i \in [k]$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k for which G has a proper k-colouring. A *proper k-colouring* of a digraph $D = (V, A)$ is a mapping $f : V \rightarrow [k]$ such that $f^{-1}(i)$ is acyclic (i.e. the subdigraph induced by $f^{-1}(i)$ has no directed cycles) for every $i \in [k]$, and the *dichromatic number* of D, denoted by $\vec{\chi}(D)$, is

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the minimum k for which D has a proper k-colouring. Note that this definition generalizes the usual colouring, in the sense that the chromatic number of a graph is equal to the dichromatic number of its corresponding bidirected digraph. The notion was introduced by Neumann-Lara in 1982 [24] and it was later rediscovered by Mohar [21]. Since then, it has been shown that many classical results hold also in this setting [3, 14, 15, 16]. However, some fundamental questions remain unanswered. The *dichromatic number* of an undirected graph G, denoted by $\vec{\chi}(G)$, is the maximum dichromatic number over all its orientations. Erdős and Neumann-Lara conjectured the following.

Conjecture 1. [6] For every integer k there exists an integer $r(k)$ such that $\vec{\chi}(G) \geqslant k$ for any undirected graph G satisfying $\chi(G) \geqslant r(k)$.

For instance, $r(1) = 1$ and $r(2) = 3$. But it is already unknown whether $r(3)$ exists. Mohar and Wu [22] managed to prove the fractional analogue of Conjecture 1. Providing further evidence for the conjecture, they showed that Kneser graphs with large chromatic number have large dichromatic number. Improving their bound, we show that the dichromatic number of Kneser graphs is of the order of their chromatic number.

In the 1970s Erdős, Rubin and Taylor [9], and, independently, Vizing [28], introduced the list variant of the colouring problem, which can be carried over to the directed setting as well. A *k-list assignment* to a graph G (or a digraph D) with vertex set V is a mapping $L : V \to \binom{\mathbb{Z}^+}{k} = \{X \subseteq \mathbb{Z}^+ \mid |X| = k\}$. A colouring (a mapping) $f : V \to \mathbb{Z}^+$ is said to be *accepted* by L if $f(v) \in L(v)$ for every $v \in V$ (or just to be *acceptable* if the list assignment is understood). G (resp. D) is *k-list colourable* if every k-list assignment accepts a proper colouring. The *list chromatic number* (or the *choice number*) of G (resp. the *list dichromatic number* of D), denoted by $\chi_\ell(G)$ (resp. $\vec{\chi}_\ell(D)$), is the minimum k such that G (resp. D) is k-list colourable. Similarly, the *list dichromatic number* of G, denoted by $\vec{\chi}_\ell(G)$, is the maximum list dichromatic number over all its orientations. Bensmail, Harutyunyan and Le [2] gave a sample of instances where the list dichromatic number of digraphs behaves as its undirected counterpart. Recently, Bulankina and Kupavskii [4] determined up to a constant factor the list chromatic number of a significant fraction of Kneser graphs. We show that their list dichromatic number is of the same order as the list chromatic number.

The paper is organised as follows. In Section 2, we prove that Kneser graphs have large dichromatic number (at least a multiplicative constant of their chromatic number) and we show that the general lower bound for the chromatic number of Borsuk graphs is also a general lower bound for their dichromatic number. In Section 3, we study the list dichromatic number of the complete multipartite graph K_{m*r}, determining its asymptotics by extending a classical result of Alon [1]. Then we extend Bulankina and Kupavskii's result on the list chromatic number of Kneser graphs to the list dichromatic number. Finally, in Section 4, we prove a directed analogue of Sabidussi's theorem on the chromatic number of Cartesian products of graphs.

2 The dichromatic number of Kneser graphs and Borsuk graphs

The *Kneser graph* with parameters n, k, denoted by $KG(n, k)$, is the undirected graph with vertex set $\binom{[n]}{k}$ where two vertices u, v are adjacent if and only if $u \cap v = \emptyset$. It is well-known [11, 19, 20] that $\chi(KG(n, k)) = n - 2k + 2$ for $1 \leqslant k \leqslant \frac{n}{2}$, as conjectured by Kneser [18, 30].

Mohar and Wu showed that, if k is not too close to $\frac{n}{2}$, the dichromatic number of $KG(n, k)$ is unbounded as well. More precisely, they proved the following.

Theorem 2. [22] *For any positive integers n, k with $1 \leqslant k \leqslant \frac{n}{2}$ we have*

$$\vec{\chi}(KG(n, k)) \geqslant \left\lfloor \frac{n - 2k + 2}{8 \log_2 \frac{n}{k}} \right\rfloor.$$

Note that, since $\vec{\chi}(G) \leqslant \chi(G)$ for any graph G, the estimate in Theorem 2 is sharp up to a constant factor when k is a constant fraction of n. Theorem 6 improves this bound for slower growth rates of k. Note that it cannot be extended to $k = 1$ due to the following result on tournaments.

Theorem 3. [7, 8, 12] *Let T be a tournament of order n. Then $\vec{\chi}(T) \leqslant \frac{n}{\log_2 n}(1 + o(1))$.*

For the proof of Theorem 6, we shall adapt Greene's proof of Kneser's conjecture [11]. The following version of Lusternik–Schnirelmann–Borsuk theorem plays a key role.

Lemma 4. [11] *If the sphere $\mathbb{S}^n$ is covered with $n + 1$ sets, each of which is either open or closed, then one of the sets contains a pair of antipodal points.*

From now on, by $\mathbb{S}^n$ we will denote the embedded unit sphere $S(0, 1) = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} \subseteq \mathbb{R}^{n+1}$ centered at the origin. The following probabilistic lemma will also be of help.

Lemma 5. *Let G be a graph of order $n \geqslant 2$ and D the random orientation of G obtained by orienting every edge independently with probability $\frac{1}{2}$. Let E_ℓ be the event that there exists a subgraph of G isomorphic to $K_{\ell,\ell}$ which is acyclic in D. If $5 \log_2 n \leqslant \ell$ then $\mathbb{P}(E_\ell) < \frac{1}{2}$.*

Proof. Each acyclic orientation of $K_{\ell,\ell}$ can be extended to a transitive tournament on the same vertex set, and different orientations always extend to different tournaments. Therefore, among the 2^{ℓ^2} possible orientations of $K_{\ell,\ell}$, at most $(2\ell)! \leqslant (2\ell)^{2\ell} \leqslant n^{2\ell}$ are acyclic. Since G has at most $\binom{n}{2\ell}(2\ell)! \leqslant n^{2\ell}$ copies of $K_{\ell,\ell}$, we have that $\mathbb{P}(E_\ell) \leqslant n^{4\ell} 2^{-\ell^2} \leqslant 2^{-\ell^2/5} \leqslant 2^{-5}$. $\square$

Theorem 6. *There exist a positive integer n_0 such that, for all $n \geqslant n_0$ and $2 \leqslant k \leqslant \frac{n}{2}$, we have $\vec{\chi}(KG(n, k)) \geqslant \left\lfloor \frac{1}{16} \chi(KG(n, k)) \right\rfloor$.*

Proof. Let $0 < c < \frac{1}{2}$ be a constant and set $t = \frac{-1}{8\log_2 c}$; we will show that $\vec{\chi}(KG(n,k)) \geqslant \lfloor t\chi(KG(n,k)) \rfloor$ if c is smaller than a certain quantity. Picking $c = \frac{1}{4}$ will suffice, although there is some margin to choose larger values. If $cn \leqslant k \leqslant \frac{n}{2}$, then the result is implied by Theorem 2.

Now suppose that $2 \leqslant k \leqslant cn$. We assume for a contradiction that, for any given orientation of $KG(n,k)$, we can find a partition of its vertex set into $d = \lfloor t(n-2k+2) \rfloor - 1$ acyclic subsets $\mathcal{A}_1, \ldots, \mathcal{A}_d$. Let $X \subseteq \mathbb{S}^d \subseteq \mathbb{R}^{d+1}$ be a set of n points on the unit sphere centered at the origin. We assume that these points together with the origin are in general position. In particular, there are no $d+1$ points of X in a common hyperplane through the origin. The set of vertices of $KG(n,k)$ is assumed to be $\binom{X}{k}$.

Let $s = tk + (1-t)\left(\frac{n}{2}+1\right)$ and $\ell = \left\lceil \frac{1}{d}\binom{\lfloor s \rfloor}{k} \right\rceil$ (note that $d \geqslant 1$; otherwise, the result is immediate). We define U_i as the set of points $x \in \mathbb{S}^d$ for which there exist ℓ different vertices $A_1, \ldots, A_\ell \in \mathcal{A}_i$ such that $x \cdot y > 0$ for every $y \in A_1 \cup \cdots \cup A_\ell$. That is, U_i is the set of poles of the open hemispheres containing all the points of ℓ vertices of $\mathcal{A}_i$. It is clear that U_i is an open set of $\mathbb{S}^d$. Additionally, we define $F = \mathbb{S}^d \setminus U_1 \setminus \cdots \setminus U_d$. By Lemma 4, one of the sets $U_1, \ldots, U_d, F$ contains two antipodal points.

Suppose that U_i contains two antipodal points $x, -x$. Then the hemispheres with pole $x, -x$ each contain the points of ℓ vertices of $\mathcal{A}_i$. Therefore $KG(n,k)[\mathcal{A}_i]$ has a subgraph isomorphic to $K_{\ell,\ell}$. By Lemma 5, $\ell \leqslant 5\log_2\binom{n}{k} \leqslant 5k\log_2 n$. On the other hand,

$$\ell \geqslant \frac{\binom{\lfloor s \rfloor}{k}}{d} \geqslant \frac{1}{n}\frac{\lfloor s \rfloor(\lfloor s \rfloor - 1)\cdots(\lfloor s \rfloor - k + 1)}{k!} \geqslant \frac{1}{n}\left(\frac{\lfloor s \rfloor}{k}\right)^k \geqslant \frac{1}{n}\left(\frac{s-1}{k}\right)^k \geqslant \frac{1}{n}\left(\frac{(1-t)n}{2k}\right)^k.$$

We distinguish two cases.

Case 1. $2 \leqslant k \leqslant n^{\frac{1}{5}}$.

In this case

$$\ell \geqslant \frac{1}{n}\left(\frac{(1-t)n}{2k}\right)^k \geqslant \frac{1}{n}\left(\frac{(1-t)n^{\frac{4}{5}}}{2}\right)^k \geqslant n^{\frac{1}{5}}\left(\frac{(1-t)n^{\frac{1}{5}}}{2}\right)^k,$$

contradicting, when n is large, that $\ell \leqslant 5k\log_2 n \leqslant 5n^{\frac{1}{5}}\log_2 n$.

Case 2. $n^{\frac{1}{5}} \leqslant k \leqslant cn$.

In this case

$$\ell \geqslant \frac{1}{n}\left(\frac{(1-t)n}{2k}\right)^k \geqslant \frac{1}{n}\left(\frac{1-t}{2c}\right)^{n^{\frac{1}{5}}}.$$

Provided that $1 - t - 2c > 0$, this contradicts that $\ell \leqslant 5k\log_2 n \leqslant 5cn\log_2 n$ when n is large. Note that by picking $c = \frac{1}{4}$ we have $1 - t - 2c = 1 - \frac{1}{16} - \frac{1}{2} > 0$.

In conclusion, F must contain two antipodal points $x, -x$. But then the hemispheres with pole $x, -x$ each contain at most $\lfloor s \rfloor - 1$ points of X. Indeed, if one of them contained $\lfloor s \rfloor$ points, it would contain the points of $\binom{\lfloor s \rfloor}{k} > \left(\left\lceil \frac{1}{d}\binom{\lfloor s \rfloor}{k} \right\rceil - 1\right)d = (\ell-1)d$ vertices, so at least ℓ vertices of the same colour would be involved. Hence, there are at least

$n - 2(\lfloor s \rfloor - 1) \geqslant n - 2s + 2 = t(n - 2k + 2) \geqslant d + 1$ points of X on the hyperplane separating the two hemispheres, contradicting the general position of $X \cup \{0\}$. □

Kneser's conjecture remained open for more than two decades [18, 30]. The famous resolution by Lovász [19] was inspired by the analogy between Kneser graphs and Borsuk graphs. Let n be a natural number and $a \in (0, 2)$ a real number. The *Borsuk graph* with parameters $n + 1$ and a, denoted by $BG(n + 1, a)$, is the undirected graph with vertex set $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ where two vertices x, y are adjacent if and only if $\|y - x\| \geqslant a$. The study of the chromatic number of Borsuk graphs can be linked with geometric packing/covering problems. If a is large enough, an $(n + 2)$-colouring of $BG(n + 1, a)$ can be obtained by projecting the faces of an inscribed $(n + 1)$-dimensional simplex. It turns out that this cannot be improved, no matter how close a is to 2. Indeed, it is known that $\chi(BG(n+1, a)) \geqslant n + 2$ for every $a \in (0, 2)$, which is in fact equivalent to the Borsuk–Ulam theorem [20]. The rest of the present section is devoted to prove that the dichromatic number of Borsuk graphs admits the same general lower bound.

Theorem 7. $\vec{\chi}(BG(n + 1, a)) \geqslant n + 2$ *for any* $n \geqslant 1$ *and any* $a \in (0, 2)$.

Proof. Let us denote by $B(x, r)$ the open ball $\{y \in \mathbb{R}^{n+1} \mid \|y - x\| < r\}$. Let $\delta \in (0, 2)$ such that every point in $B(x, \delta) \cap \mathbb{S}^n$ is adjacent to every point in $B(-x, \delta) \cap \mathbb{S}^n$ for any $x \in \mathbb{S}^n$. Let ℓ be an integer that for now remains unspecified, but that is assumed to be as large as desired. We define $m = \left\lceil \sqrt[n+1]{(\ell - 1)(n + 1)} \right\rceil + 1 \leqslant 2 \sqrt[n+1]{(\ell - 1)(n + 1)}$ and $c = \frac{\delta}{m\sqrt{n+1}}$.

An *open hypercube* of $\mathbb{R}^{n+1}$ is the image by a rigid transformation of a product of intervals $(0, \lambda)^{n+1} \subseteq \mathbb{R}^{n+1}$, where $\lambda \in \mathbb{R}^+$. The length of its *side* is λ and the length of its *longest diagonal* is its diameter (i.e. $\lambda\sqrt{n+1}$). Let $\mathcal{Q}_c$ be the set of open hypercubes of side c of the form $(ck_1, ck_1 + c) \times \cdots \times (ck_{n+1}, ck_{n+1} + c)$ with $(k_1, \ldots, k_{n+1}) \in \mathbb{Z}^{n+1}$, i.e. the ones obtained by rescaling the integer lattice by a factor of c. We will make use of the following easy observations about $\mathcal{Q}_c$.

Observation 8. *For every* $x \in \mathbb{R}^{n+1}$, $B(x, \delta)$ *contains at least* m^{n+1} *hypercubes of* $\mathcal{Q}_c$.

Proof. Consider an open hypercube Q of longest diagonal δ with $x \in Q$. Clearly $Q \subseteq B(x, \delta)$ and the side of Q is $\frac{\delta}{\sqrt{n+1}}$. This implies the claim. ∎

Observation 9. $B(0, 1 + 2\delta)$ *is contained in any open hypercube* Q *of side* $2c \left\lceil \frac{1+2\delta}{c} \right\rceil$ *centered at the origin. Moreover, one (in fact exactly one) such Q can be obtained as the interior of the closure of the union of* $\left(2 \left\lceil \frac{1+2\delta}{c} \right\rceil\right)^{n+1}$ *hypercubes of* $\mathcal{Q}_c$. ∎

Let $\mathcal{Q}'_c \subseteq \mathcal{Q}_c$ be the set of $\left(2 \left\lceil \frac{1+2\delta}{c} \right\rceil\right)^{n+1}$ hypercubes from Observation 9. For each $Q \in \mathcal{Q}'_c$ choose a point $x_Q \in Q$. Let y_Q be the point where the open ray starting at the origin and passing through x_Q intersects $\mathbb{S}^n$. Since $n \geqslant 1$ we can assume that the points x_Q have been chosen so that $y_Q \neq y_{Q'}$ if $Q \neq Q'$. Let $Y = \{y_Q \mid Q \in \mathcal{Q}'_c\}$. Note that

$$|Y| = |\mathcal{Q}'_c| = \left(2 \left\lceil \frac{(1 + 2\delta)m\sqrt{n+1}}{\delta} \right\rceil\right)^{n+1} \leqslant \left(\frac{8(1 + 2\delta)\sqrt{n+1}}{\delta}\right)^{n+1} (\ell - 1)(n + 1).$$

Observation 10. *For every $x \in \mathbb{S}^n$, $B(x, \delta)$ contains at least m^{n+1} points of Y.*

Proof. Since $B((1+\delta)x, \delta) \subseteq B(0, 1+2\delta)$, all hypercubes of $\mathcal{Q}_c$ intersecting $B((1+\delta)x, \delta)$ are in $\mathcal{Q}_c'$. Hence, by Observation 8, $B((1+\delta)x, \delta)$ contains m^{n+1} hypercubes of $\mathcal{Q}_c'$. The points in Y corresponding to these hypercubes all lie in $B(x, \delta)$. ∎

We now consider the finite induced subgraph $H = BG(n+1, a)[Y]$ of $BG(n+1, a)$. It will be enough to show that $\vec{\chi}(H) \geqslant n + 2$. Let us assume for a contradiction that each orientation of H admits a partition of Y into $n+1$ acyclic subsets $Y_1, \ldots, Y_{n+1}$. For $i \in [n+1]$ let $U_i = \{x \in \mathbb{S}^n \mid |B(x, \delta) \cap Y_i| \geqslant \ell\}$. Clearly, U_i is an open set of $\mathbb{S}^n$. Moreover, $\mathbb{S}^n = U_1 \cup \cdots \cup U_{n+1}$. Indeed, otherwise $B(x, \delta)$ would contain at most $(\ell-1)(n+1) < m^{n+1}$ points of Y for some $x \in \mathbb{S}^n$, contradicting Observation 10. Therefore, by Lemma 4, U_i contains two antipodal points x and $-x$ for some $i \in [n+1]$.

By the choice of δ, we know that in H there is a copy of $K_{\ell,\ell}$ of colour i. Now, $5 \log_2 |Y| \leqslant \ell$ if ℓ is large enough. By Lemma 5, there is an orientation of H such that every copy of $K_{\ell,\ell}$ in H has a directed cycle, a contradiction. □

3 The list dichromatic number of Kneser graphs and complete multipartite graphs

The goal of this section is to study the list dichromatic number of complete multipartite graphs and Kneser graphs. In the first subsection, we study the list dichromatic number of complete multipartite graphs, obtaining tight upper and lower bounds (up to a multiplactive factor). Following this, in the second subsection, we consider Kneser graphs.

3.1 List dichromatic number of complete multipartite graphs

We denote by K_{m*r} the complete r-partite graph with m vertices on each part. Answering a question of Erdős, Rubin and Taylor [9], Alon determined, up to a constant factor, the list chromatic number of K_{m*r}.

Theorem 11. [1] *There exist two positive constants c_1 and c_2 such that for every $m \geqslant 2$ and for every $r \geqslant 2$*

$$c_1 r \ln m \leqslant \chi_\ell(K_{m*r}) \leqslant c_2 r \ln m.$$

More precise results were obtained in [10]. Adapting Alon's proof, we find an analogous bound for the list dichromatic number of K_{m*r} when $r \geqslant 2$ and $m \geqslant \max\{\ln^\rho r, 2\}$, for any $\rho > 3$ (Theorem 14). We remark that it is known that $\vec{\chi}(K_{m*r}) = r$, when m is sufficiently large (see [13]; see also [6]). We will use the following probabilistic result, which is a consequence of the Hoeffding–Azuma inequality.

Theorem 12. (Simple Concentration Bound, [23]) *Let X be a random variable determined by n independent trials, and satisfying the property that changing the outcome of any single trial can affect X by at most c. Then*

$$\mathbb{P}(|X - \mathbb{E}X| > t) \leqslant 2e^{-\frac{t^2}{2c^2 n}}.$$

We will also invoke the following elementary fact.

Remark 13. Let $a \in \mathbb{R}^+$. The function $f : (a, \infty) \to \mathbb{R}$ defined by $f(x) = \left(1 - \frac{a}{x}\right)^x$ is increasing.

Proof. $f'(x) = f(x)\left(\ln\left(1 - \frac{a}{x}\right) + \frac{a}{x-a}\right) \geqslant f(x)\left(\ln\frac{x-a}{x} + \ln\frac{x}{x-a}\right) = 0.$ $\qquad\square$

Theorem 14. *For every $\rho > 3$ there exist constants $c_1, c_2 \in \mathbb{R}^+$ such that if $r \geqslant 2$ and $m \geqslant \max\{\ln^\rho r, 2\}$ then*
$$c_1 r \ln m \leqslant \vec{\chi}_\ell(K_{m*r}) \leqslant c_2 r \ln m.$$

Proof. Let $V_1, \dots, V_r$ be the parts of K_{m*r}. The upper bound is implied by Theorem 11. For the lower bound, we can assume that m is large enough; otherwise, we get the job done by picking a suitable c_1.

Claim 15. *There is a constant c and an orientation D of K_{m*r} such that, if $\ell \geqslant c \ln(rm)$, each subgraph of K_{m*r} isomorphic to K_ℓ or to $K_{\ell,\ell}$ has a directed cycle in D.*

Proof. We orient the edges of K_{m*r} at random, independently and with probability $\frac{1}{2}$. Let E (resp. E') be the event that each subgraph of K_{m*r} isomorphic to K_ℓ (resp. $K_{\ell,\ell}$) has a directed cycle. By Lemma 5, $\mathbb{P}(E), \mathbb{P}(E') > \frac{1}{2}$ if c is sufficently large. Hence $\mathbb{P}(E \cap E') > 0$. $\blacksquare$

Let $k = \lfloor Cr \ln m \rfloor$, where $0 < C \leqslant 1$ is a constant for now unspecified. We start by showing that there exists an assignment of k-lists from a palette $\mathscr{C}$ of $\lfloor r \ln m \rfloor$ colours such that, for any given set $A \subseteq \mathscr{C}$ of at most $\frac{4}{3} \ln m$ colours, each part has at least $\frac{1}{2} m^{1-\delta}$ vertices that avoid the colours from A on their lists, where $\delta = 2C \ln 5$.

We assign to each vertex v of D a random k-list $L(v)$ chosen independently and uniformly among the $\binom{|\mathscr{C}|}{k}$ possible k-lists. Given $i \in [r]$ and $A \subseteq \mathscr{C}$, consider the random variable $X_{i,A} = |\{v \in V_i \mid L(v) \cap A = \emptyset\}|$. Note that there are exactly $\binom{|\mathscr{C}|-|A|}{k}$ k-lists avoiding the colours in A. Devoting ourselves to the case $|A| = \lfloor \frac{4}{3} \ln m \rfloor$, we have that

$$\mathbb{E}X_{i,A} = m\frac{\binom{|\mathscr{C}|-|A|}{k}}{\binom{|\mathscr{C}|}{k}} \geqslant m\left(\frac{|\mathscr{C}| - |A| - k}{|\mathscr{C}| - k}\right)^k = m\left(1 - \frac{|A|}{|\mathscr{C}| - k}\right)^k$$

$$\geqslant m\left(1 - \frac{\frac{4}{3}\ln m}{(1-C)r\ln m - 1}\right)^{Cr\ln m} \geqslant m\left(1 - \frac{4}{5}\right)^{2C\ln m} = m^{1-\delta}$$

if m is large enough and C is not too large, using Remark 13. By the Simple Concentration Bound (Theorem 12),

$$\mathbb{P}(X_{i,A} < \tfrac{1}{2}m^{1-\delta}) \leqslant \mathbb{P}(|X_{i,A} - \mathbb{E}X_{i,A}| > \tfrac{1}{2}m^{1-\delta}) \leqslant 2e^{-\frac{1}{8}m^{1-2\delta}}.$$

Let E be the event that $X_{i,A} < \frac{1}{2}m^{1-\delta}$ for some $i \in [r]$ and $A \subseteq \mathscr{C}$ with $|A| \leqslant \frac{4}{3}\ln m$. We have that

$$\mathbb{P}(E) \leqslant r\binom{|\mathscr{C}|}{\lfloor \frac{4}{3}\ln m \rfloor}2e^{-\frac{1}{8}m^{1-2\delta}} \leqslant (r\ln m)^{\frac{4}{3}\ln m + 1}2e^{-\frac{1}{8}m^{1-2\delta}}$$

$$\leqslant 2e^{\left(m^{\frac{1}{\rho}} + \ln\ln m\right)\left(\frac{4}{3}\ln m + 1\right) - \frac{1}{8}m^{1-2\delta}} \leqslant 2e^{2m^{\frac{1}{\rho}}\ln m - \frac{1}{8}m^{1-2\delta}}$$

if m is large enough. Consequently, if $\delta < \frac{1}{2}(1 - \frac{1}{\rho})$ and m is large enough, there exists a list assignment L' satisfying the desired property. This is the assignment that we are going to use.

Now let f be a proper colouring of D. We claim that there exists a set of indices $I \subseteq [r]$ of size at least $\frac{3r}{4}$ such that $|f(V_i)| \leqslant 4c\ln^2(rm)$ for each $i \in I$. Indeed, if more than $\frac{r}{4}$ parts are coloured with more than $4c\ln^2(rm)$ colours each, then one of the colours appears on more than $\frac{cr\ln^2(rm)}{|\mathscr{C}|} \geqslant c\frac{\ln^2(rm)}{\ln m} \geqslant c\ln(rm)$ parts. By the choice of D, f is not proper, a contradiction.

For each $i \in [r]$ define the set $A_i = \{\gamma \in \mathscr{C} \mid |V_i \cap f^{-1}(\gamma)| \geqslant c\ln(rm)\}$. We claim that if f is acceptable then $|A_i| > \frac{4}{3}\ln m$ for every $i \in I$. Indeed, otherwise, by the choice of the lists, at least $\frac{1}{2}m^{1-\delta}$ vertices of V_i have been coloured with colours not from A_i. Thus one of these colours is used at least

$$\frac{\frac{1}{2}m^{1-\delta}}{4c\ln^2(rm)} \leqslant c\ln(rm)$$

times on V_i. If m is large enough, this implies that

$$m^{1-\delta} \leqslant 8c^2\ln^3(rm) \leqslant 8c^2(m^{\frac{1}{\rho}} + \ln m)^3 \leqslant 9c^2m^{\frac{3}{\rho}}.$$

If we further assume that $\delta < 1 - \frac{3}{\rho}$, we get a contradiction when m is large. Therefore $|A_i| > \frac{4}{3}\ln m$ for every $i \in I$.

Now, by the choice of D, the sets $A_1, \ldots, A_r$ are mutually disjoint. But then

$$|\mathscr{C}| \geqslant \sum_{i=1}^{r} |A_i| \geqslant \sum_{i \in I} |A_i| > \frac{4}{3}|I|\ln m \geqslant r\ln m \geqslant |\mathscr{C}|.$$

This contradiction shows that there is no acceptable proper colouring for the k-list assignment L'. $\qquad\square$

We do not know what happens for other values of m, r. What is clear is that Theorem 14 is not valid in general. Indeed, if $m \leqslant \ln r$ then the following theorem implies that $\vec{\chi}_\ell(K_{m*r}) \leqslant \vec{\chi}_\ell(K_{mr}) \leqslant cr$ for some constant c.

Theorem 16. [2] *Let T be a tournament of order n. Then $\vec{\chi}_\ell(T) \leqslant \frac{n}{\log_2 n}(1 + o(1))$.*

3.2 List dichromatic number of Kneser graphs

Here we investigate the list dichromatic number of Kneser graphs. The list chromatic number of Kneser graphs was recently studied by Bulankina and Kupavskii. They proved the following two results.

Theorem 17. [4] *For any positive integers n, k with $1 \leqslant k \leqslant \frac{n}{2}$ we have $\chi_\ell(KG(n,k)) \leqslant n\ln\frac{n}{k} + n$.*

Theorem 18. [4] *For every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that*

$$\chi_\ell(KG(n,k)) \geqslant c_\varepsilon n \ln n$$

for all n, k with $2 \leqslant k \leqslant n^{\frac{1}{2}-\varepsilon}$.

However, good lower bounds for larger values of k are still unknown. Clearly, the upper bound of Theorem 17 trivially generalises to the dichromatic number. The rest of the subsection is devoted to the proof of the directed analogue of Theorem 18; that is, we show that the lower bound can be strengthened to digraphs. The proof is achieved by a sequence of lemmas, which involve the argument of Bulankina and Kupavskii, as well as ideas from Mohar and Wu [22]. As in Theorem 18, we do not know if the bound on k can be extended to $2 \leqslant k \leqslant n^{1-\varepsilon}$ for an arbitrarily small $\varepsilon > 0$.

Theorem 19. *For every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that $\vec{\chi}_\ell(KG(n,k)) \geqslant c_\varepsilon n \ln n$ for all n, k with $2 \leqslant k \leqslant n^{\frac{1}{2}-\varepsilon}$.*

Let $G = (V, E)$ be a graph, $\mathcal{C}$ a collection of subsets of V and s, t positive integers. We say that $\mathcal{C}$ is an (s,t)-*collection* of V if

(i) $|\mathcal{C}| \leqslant s$;

(ii) $\forall C \in \mathcal{C}$ $|C| \leqslant t$.

Given a list assignment L, we denote by $U = \cup_{v \in V(G)} L(v)$ the total set of colours, referred to as the *palette*, and we set $u = |U|$. The partitions P of V considered in the sequel will always have u (not necessarily non-empty) parts (we always implicitly or explicitly assume that the partitions are *acyclic*, i.e., that each part of P induces an acyclic digraph). It will be convenient to regard as distinct any two partitions arising from different colourings. Thus, partitions will be thought as indexed by U (but for simplicity we will continue calling them just "partitions").

We say that a partition P of V is *covered* by an (s,t)-collection $\mathcal{C}$ (or that $\mathcal{C}$ is an (s,t)-*cover* of P) if each part determined by P is contained in some $C \in \mathcal{C}$. Let $P = (P_1, \ldots, P_u)$ be a partition. We say that a list assignment L of G *accepts* P if, for every $i \in [u]$ and every $v \in P_i$, $i \in L(v)$. Otherwise, we say that L *rejects* P.

In what follows, ℓ_1 and ℓ_2 are integers. Define the function $g(\ell_1, \ell_2, n, s, t, u) := s^u e^{-\frac{n}{2} 2^{-\frac{4\ell_2 tu}{(\ell_1 - \ell_2)n}}}$.

Lemma 20. *Let $G = (V, E)$ be an undirected graph of order n, $\mathcal{C}$ an (s,t)-collection of V and $\mathcal{P}$ the family of partitions of V covered by $\mathcal{C}$. Let L_1 be an ℓ_1-list assignment for G from a palette of u colours and L_2 a random ℓ_2-list assignment for G where, for every $v \in V$, $L_2(v)$ is chosen independently and equiprobably among $\binom{L_1(v)}{\ell_2}$. If $4tu \leqslant (\ell_1 - \ell_2)n$, then*

$$\mathbb{P}(L_2 \text{ accepts some } P \in \mathcal{P}) < g(\ell_1, \ell_2, n, s, t, u) = s^u e^{-\frac{n}{2} 2^{-\frac{4\ell_2 tu}{(\ell_1 - \ell_2)n}}}.$$

Proof. Let $C = (C_1, \ldots, C_u) \in \mathcal{C}^u$ be any u-tuple of elements of $\mathcal{C}$. For every $v \in V$, let $r_C(v)$ be the number of indices $i \in [u]$ such that $v \in C_i$. Consider the subset of vertices $W_C = \{v \in V \mid r_C(v) \leqslant \frac{2tu}{n}\}$. We claim that $|W_C| > \frac{1}{2}n$. Indeed, otherwise

$$tu \geqslant \sum_{i=1}^{u} |C_i| = \sum_{v \in V} r_C(v) \geqslant \sum_{v \in V \setminus W_C} r_C(v) > tu.$$

Moreover, for any $v \in W_C$ the probability $p_C(v)$ that $v \notin \bigcup_{i \in L_2(v)} C_i$ is at least

$$\frac{\binom{\ell_1 - r_C(v)}{\ell_2}}{\binom{\ell_1}{\ell_2}} = \prod_{k=1}^{\ell_2} \frac{\ell_1 - \ell_2 - r_C(v) + k}{\ell_1 - \ell_2 + k} \geqslant \left(1 - \frac{r_C(v)}{\ell_1 - \ell_2}\right)^{\ell_2}$$

$$\geqslant \left(1 - \frac{2tu}{(\ell_1 - \ell_2)n}\right)^{\ell_2} \geqslant \left(\frac{1}{2}\right)^{\frac{4\ell_2 tu}{(\ell_1 - \ell_2)n}} =: p,$$

using Remark 13 and the inequality $4tu \leqslant (\ell_1 - \ell_2)n$. Therefore, the probability that there is some u-tuple $C = (C_1, \ldots, C_u)$ of elements of $\mathcal{C}$ such that $v \in \bigcup_{i \in L_2(v)} C_i$ for every $v \in V$ is at most

$$\sum_{C \in \mathcal{C}^u} \prod_{v \in W_C} (1 - p_C(v)) < s^u (1 - p)^{\frac{1}{2}n} \leqslant s^u e^{-\frac{1}{2}np}.$$

Since every $P \in \mathcal{P}$ is covered by $\mathcal{C}$, each of the u parts of any such P is contained in some $C \in \mathcal{C}$, so the result follows. $\qquad\square$

Let G, H be graphs. The *tensor product $G \times H$* of G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (v, x) and (w, y) are adjacent if and only if v, w are adjacent in G and x, y are adjacent in H. The tensor product of complete graphs $K_n \times K_n$ is going to play an auxiliary role; we denote it by G_n. Given $S \subseteq V(G_n)$, we call $\pi_1(S)$ and $\pi_2(S)$ the projection of S to the first and second coordinate, respectively. The *rows* (resp. *columns*) of S are the subsets of S of the form $S \cap (\{i\} \times [n])$ (resp. $S \cap ([n] \times \{i\})$), where $i \in [n]$. Now we give some properties of G_n.

Lemma 21. *For any $n \geqslant 2$, there is an orientation D_n of G_n (resp. of $K_2 \times G_n$) such that, for every $S, T \subseteq V(G_n)$ satisfying*

i) $|S|, |T| \geqslant 30 \ln n$ *and*

ii) $\pi_i(S) \cap \pi_i(T) = \emptyset$ *for $i \in \{1, 2\}$,*

the subdigraph of D_n induced by $S \cup T$ (resp. by $(\{1\} \times S) \cup (\{2\} \times T)$) has a directed cycle.

Proof. Since in $G_n[S \cup T]$ (resp. in $(K_2 \times G_n)[(\{1\} \times S) \cup (\{2\} \times T)]$) all edges between S and T (resp. between $\{1\} \times S$ and $\{2\} \times T$) are present, the conclusion follows from Lemma 5. $\qquad\square$

We define an (s_n, t_n)-collection $\mathcal{C}_{G_n}$ of $V(G_n)$ as follows. Let $\mathcal{L}_{G_n} = \{\{i\} \times [n] \mid i \in [n]\} \cup \{[n] \times \{i\} \mid i \in [n]\}$ be the set of rows and columns of $V(G_n)$ and

$$\mathcal{Q}_{G_n} = \begin{cases} \{A \times B \mid A, B \in \binom{[n]}{\lfloor 124 \ln n \rfloor}\} & \text{if } 1 \leqslant \lfloor 124 \ln n \rfloor \leqslant n \\ \{V(G_n)\} & \text{otherwise.} \end{cases}$$

We set $\mathcal{C}_{G_n} = \{L \cup Q \mid L \in \mathcal{L}_{G_n}, Q \in \mathcal{Q}_{G_n}\}$. Note that $|\mathcal{C}_{G_n}| \leqslant s_n := \max\{1, 2n\binom{n}{\lfloor 124 \ln n \rfloor}^2\}$ and $|C| \leqslant t_n := n + \lfloor 124 \ln n \rfloor^2$ for any $C \in \mathcal{C}_{G_n}$.

Lemma 22. *There is an orientation D_n of G_n such that $\mathcal{C}_{G_n}$ covers all acyclic partitions of D_n.*

Proof. It can be assumed that $1 \leqslant \lfloor 124 \ln n \rfloor \leqslant n$. Let D_n be the orientation from Lemma 21, and let S be an acyclic set of D_n. Assume for a contradiction that S is not contained in any $C \in \mathcal{C}_{G_n}$. Let L be the largest row of S, or its largest column if it is larger than its largest row, and let $S' = S \setminus L$. Then S' is not contained in any $Q \in \mathcal{Q}_{G_n}$, so $|\pi_i(S')| > 124 \ln n > 90 \ln n + 2$ for some $i \in \{1, 2\}$. Assume that $i = 1$ (if $i = 2$, the argument below is repeated with rows instead of columns). Let L' be the largest column of S'. We distinguish three cases. We will show that, in each case, we can find two sets in S satisfying the hypotheses of Lemma 21. This will yield a contradiction since S is acyclic.

Case 1. $|L'| > 60 \ln n + 2$.
Recall that $|L| \geqslant |L'|$. Therefore we can find a subset of L and a subset of L' satisfying the hypotheses of Lemma 21.

Case 2. $60 \ln n + 2 \geqslant |L'| \geqslant 30 \ln n$.
Since $|\pi_i(S')| > 90 \ln n + 2$, we can find a subset T of S' such that T and L' satisfy the hypotheses of Lemma 21.

Case 3. $|L'| \leqslant 30 \ln n$.
Let $\{T_1, \ldots, T_k\}$ be a minimal set of columns of S' satisfying $|\pi_i(\bigcup_{j=1}^k T_j)| \geqslant 30 \ln n$. By minimality, $|\pi_i(\bigcup_{j=1}^k T_j)| \leqslant 60 \ln n$. Hence, as in Case 2, we can find a subset T of S' such that T and $\bigcup_{j=1}^k T_j$ satisfy the hypotheses of of Lemma 21.

In any case, Lemma 21 yields a cycle in S, the desired contradiction. $\qquad\square$

We are now in position to determine the order of $\vec{\chi}_\ell(KG(n, k))$ when k is bounded by a constant.

Lemma 23. *There is a constant $c \in \mathbb{R}^+$ such that $\vec{\chi}_\ell(KG(n, k)) \geqslant c \frac{n}{k} \ln \frac{n}{k}$ for every $2 \leqslant k \leqslant \frac{n}{2}$.*

Proof. First note that $G_{\lfloor \frac{n}{k} \rfloor}$ is isomorphic to a subgraph of $KG(n, k)$. Indeed, if we take $\lfloor \frac{n}{k} \rfloor + 1$ pairwise disjoint subsets $I, J_1, \ldots, J_{\lfloor \frac{n}{k} \rfloor} \subseteq [n]$ with $|I| = \lfloor \frac{n}{k} \rfloor$ and $|J_1| = \cdots = |J_{\lfloor \frac{n}{k} \rfloor}| = k - 1$, then the set of vertices $S = \{\{i\} \cup J_j \mid i \in I, 1 \leqslant j \leqslant \lfloor \frac{n}{k} \rfloor\} \subseteq \binom{[n]}{k}$ induces a copy of $G_{\lfloor \frac{n}{k} \rfloor}$.

Thus it suffices to show that $\vec{\chi}_\ell(G_{\tilde{n}}) \geqslant c\tilde{n}\ln\tilde{n}$ for some $c > 0$. Assume that $\tilde{n}$ is large enough. Given $G_{\tilde{n}}$, consider the orientation $D_{\tilde{n}}$ from Lemma 22; we know that $\mathcal{C}_{G_{\tilde{n}}}$ covers all acyclic partitions of $D_{\tilde{n}}$. Let $u_{\tilde{n}} = \ell_{1,\tilde{n}} = \lfloor \tilde{n}\ln\tilde{n} \rfloor$ and $\ell_{2,\tilde{n}} = \lfloor cu_{\tilde{n}} \rfloor$, where $c < 1$ is a positive constant to be defined later. Let $L_{1,\tilde{n}}$ be the canonical $\ell_{1,\tilde{n}}$-list assignment to $D_{\tilde{n}}$ (i.e. $L_{1,\tilde{n}}(v) = [u_{\tilde{n}}]$ for every $v \in V(D_{\tilde{n}})$). It is clear that $4t_{\tilde{n}}u_{\tilde{n}} \leqslant (\ell_{1,\tilde{n}} - \ell_{2,\tilde{n}})\tilde{n}^2$ and

$$\ln g(\ell_{1,\tilde{n}}, \ell_{2,\tilde{n}}, \tilde{n}^2, s_{\tilde{n}}, t_{\tilde{n}}, u_{\tilde{n}}) \leqslant 330\tilde{n}\ln^3\tilde{n} - \frac{1}{2}\tilde{n}^{2-\frac{8c}{(1-c)\log_2 e}} < 0$$

if $\tilde{n}$ is large enough and c has been chosen so that $\frac{8c}{(1-c)\log_2 e} < 1$. Now, by Lemma 20, $\vec{\chi}_\ell(D_{\tilde{n}}) > \ell_{2,\tilde{n}}$. $\qquad\square$

The previous lemma handles the case when k is small. For larger values of k, we need to modify the definition of a cover. Let $H = (V, E)$ be a graph, $\mathcal{C}$ an (s, t)-collection of V and $\lambda \in \mathbb{R}^+$. Consider the graph $K_2 \times H$ and one of its orientations D. We say that an acyclic partition P of $V(D)$ is *semicovered* by the pair $(\mathcal{C}, \lambda)$ (or that $(\mathcal{C}, \lambda)$ is an (s, t)-*semicover* of P) if for every acyclic set $S = (\{1\} \times S_1) \cup (\{2\} \times S_2) \in P$ either $S_1 \subseteq C_1$ and $S_2 \subseteq C_2$ for some $C_1, C_2 \in \mathcal{C}$, or $S_i \subseteq C$ and $|S_i| < \lambda$ for some $i \in \{1, 2\}$ and some $C \in \mathcal{C}$.

Lemma 24. *Let G, H be graphs. Let m_G be the size of G and n_H the order of H. Let D be an orientation of $K_2 \times H$ and $(\mathcal{C}, \lambda)$ an (s, t)-semicover of all acyclic partitions of D. Let ℓ_1, ℓ_2 be positive integers such that $8t\ell_1 \leqslant (\ell_1 - \ell_2)n_H$, $m_G\, g^2(\ell_1, \ell_2, n_H, s, t, 2\ell_1) < 1$ and $\lambda\ell_1 \leqslant n_H$. If $\chi_\ell(G) > \ell_1$, then $\vec{\chi}_\ell(G \times H) > \ell_2$.*

Proof. Suppose that $\vec{\chi}_\ell(G \times H) \leqslant \ell_2$. Let L_1 be any ℓ_1-list assignment for G. Consider a random ℓ_2-list assignment L_2 for $G \times H$, where for each $v \in V(G)$ and each $x \in \{v\} \times V(H)$ $L_2(x)$ is chosen independently and equiprobably among $\binom{L_1(v)}{\ell_2}$. For each edge $\{v, w\}$ of G, let $\mathcal{C}_{\{v,w\}} = \{(\{v\} \times C_1) \cup (\{w\} \times C_2) \mid C_1, C_2 \in \mathcal{C}\}$. We orient the subgraph induced by $\{v, w\} \times V(H)$ according to D (in any of the two possible ways). This results in an orientation of $G \times H$ that we will call $\overrightarrow{G \times H}$. By applying Lemma 20 to $(G \times H)[\{v, w\} \times V(H)]$ with a palette of size $u = 2\ell_1$, we see that the probability that $L_2|_{\{v,w\}\times V(H)}$ accepts some partition covered by $\mathcal{C}_{\{v,w\}}$ is smaller than $g(\ell_1, \ell_2, 2n_H, s^2, 2t, 2\ell_1) = g^2(\ell_1, \ell_2, n_H, s, t, 2\ell_1)$. Therefore, the probability that this happens for some edge $\{v, w\}$ of G is less than $m_G\, g^2(\ell_1, \ell_2, n_H, s, t, 2\ell_1) < 1$. Thus we can find a ℓ_2-list assignment L_2' for $G \times H$ such that, for every $\{v, w\} \in E(G)$, $L_2'|_{\{v,w\}\times V(H)}$ rejects all partitions of $\{v, w\} \times V(H)$ covered by $\mathcal{C}_{\{v,w\}}$.

Since $\vec{\chi}_\ell(G \times H) \leqslant \ell_2$, $\overrightarrow{G \times H}$ has a colouring f_2' which is accepted by L_2' and produces no monochromatic cycles. Let us define a colouring f_1 for G as

$$f_1(v) = \begin{cases} \gamma_v & \text{if } \exists \gamma \; \forall C \in \mathcal{C} \;\; (f_2')^{-1}(\gamma) \cap (\{v\} \times V(H)) \nsubseteq \{v\} \times C, \text{ where } \gamma_v \text{ is any such } \gamma \\ \gamma_v^+ & \text{otherwise, where } \gamma_v^+ \text{ is any } \gamma \text{ maximizing } |(f_2')^{-1}(\gamma) \cap (\{v\} \times V(H))|. \end{cases}$$

Note that $f_1(v) \in L_1(v)$, for each $v \in V(G)$. We will show that $f_1(v)$ is a proper colouring of G, and this contradiction will finish the proof.

Let $\{v, w\}$ be an edge of G, and suppose for a contradiction that $f_1(v) = f_1(w)$. Since $L'_2|_{\{v,w\}\times V(H)}$ rejects all partitions of $\{v, w\} \times V(H)$ covered by $\mathcal{C}_{\{v,w\}}$, either $f_1(v) = \gamma_v$ or $f_1(w) = \gamma_w$. Without loss of generality, we can assume that $f_1(w) = \gamma_w$. Since f'_2 produces no monochromatic cycles, if $f_1(v) = \gamma_v$ then $(\mathcal{C}, \lambda)$ does not semicover all acyclic partitions of D, a contradiction. On the other hand, if $f_1(v) = \gamma_v^+$ then $|(f'_2)^{-1}(\gamma_v^+) \cap (\{v\} \times V(H))| \geqslant \frac{n_H}{\ell_1} \geqslant \lambda$, also contradicting that $(\mathcal{C}, \lambda)$ semicovers all acyclic partitions of D.

Therefore, we have found a proper colouring f_1 accepted by the ℓ_1-list assignment L_1. Since L_1 was arbitrary we conclude that $\chi_\ell(G) \leqslant \ell_1$, ending the proof. $\qquad\square$

Lemma 25. *For every n there is an orientation D_n of $K_2 \times G_n$ such that all acyclic partitions of D_n are semicovered by $(\mathcal{C}_{G_n}, 2^{13} \ln^2 n)$.*

Proof. The proof is similar to that of Lemma 22. Let D_n be the orientation of $K_2 \times G_n$ from Lemma 21. Assume that $1 \leqslant \lfloor 124 \ln n \rfloor \leqslant n$; otherwise the lemma is trivial. Let $S = (\{1\} \times S_1) \cup (\{2\} \times S_2)$ be an acyclic set in D_n. Assume that S_1 is not contained in any $C \in \mathcal{C}_{G_n}$. As in the proof of Lemma 22 we can find in S_1 two sets $L_1, L'_1 \subseteq S_1$ satisfying the hypotheses of Lemma 21. We can assume that $|L_1| = |L'_1| = \lceil 30 \ln n \rceil$.

We argue by contradiction. Suppose that $|S_2| \geqslant 2^{13} \ln^2 n$ or that S_2 is not contained in any $C \in \mathcal{C}_{G_n}$. Let $T = \{(j_1, j_2) \in S_2 \mid j_1 \notin \pi_1(L_1 \cup L'_1)$ or $j_2 \notin \pi_2(L_1 \cup L'_1)\}$. We claim that $|T| \geqslant 60 \ln n$. Let us consider two cases.

Case 1. $|S_2| \geqslant 2^{13} \ln^2 n$.
We have that $|\pi_1(L_1 \cup L'_1)|, |\pi_2(L_1 \cup L'_1)| \leqslant 2\lceil 30 \ln n \rceil \leqslant 64 \ln n$. Therefore $|T| \geqslant |S_2| - 64^2 \ln^2 n \geqslant 60 \ln n$.

Case 2. $S_2 \nsubseteq C$ for any $C \in \mathcal{C}_{G_n}$.
In this case, $S_2 \nsubseteq Q$ for any $Q \in \mathcal{Q}_{G_n}$. Therefore, $|\pi_i(S_2)| > 124 \ln n$ for some $i \in \{1, 2\}$. Since $|\pi_i(L_1 \cup L'_1)| \leqslant 2\lceil 30 \ln n \rceil$, we have that $|T| \geqslant 124 \ln n - 2\lceil 30 \ln n \rceil \geqslant 60 \ln n$.

Hence $|T| \geqslant 60 \ln n$ in any case. Let $T_1 = \{(j_1, j_2) \in T \mid j_1 \notin \pi_1(L_1), j_2 \notin \pi_2(L_1)\}$ and $T'_1 = T \setminus T_1$. Note that $T'_1 = \{(j_1, j_2) \in T \mid j_1 \notin \pi_1(L'_1), j_2 \notin \pi_2(L'_1)\}$ by the definition of T. Applying Lemma 21 yields the desired contradiction. Indeed, if $|T_1| \geqslant 30 \ln n$, then $(\{1\} \times L_1) \cup (\{2\} \times T_1)$ has a directed cycle, and if otherwise $|T'_1| \geqslant 30 \ln n$, then $(\{1\} \times L'_1) \cup (\{2\} \times T'_1)$ has a directed cycle. $\qquad\square$

Proof of Theorem 19. We fix ε and assume that n is large enough (to deal with the case of n small, we just make sure that c_ε is small enough). If k is bounded by a constant, then Lemma 23 does the job. Therefore we can assume that $k \geqslant 4$. Note that $KG(\lfloor \frac{n}{2} \rfloor, k-2) \times G_{\lfloor \frac{n}{4} \rfloor}$ is a subgraph of $KG(n, k)$. Indeed, consider, for any positive integers n_1, n_2, k_1, k_2 satisfying $2k_1 \leqslant n_1$, $2k_2 \leqslant n_2$, $n_1 + n_2 \leqslant n$ and $k_1 + k_2 = k$, the set of vertices in $KG(n, k)$ of the form

$$S = \left\{ \{i_1, \ldots, i_{k_1}, j_1, \ldots, j_{k_2}\} \; \middle| \; \{i_1, \ldots, i_{k_1}\} \in \binom{[n_1]}{k_1}, \; \{j_1, \ldots, j_{k_2}\} \in \binom{n_1 + [n_2]}{k_2} \right\};$$

the subgraph induced by S is isomorphic to $KG(n_1, k_1) \times KG(n_2, k_2)$. By the proof of Lemma 23, if $k_2 \geqslant 2$ then $G_{\lfloor \frac{n_2}{k_2} \rfloor}$ is isomorphic to a subgraph of $KG(n_2, k_2)$, so we can just take $n_1 = \lfloor \frac{n}{2} \rfloor$, $n_2 = \lceil \frac{n}{2} \rceil$, $k_1 = k - 2$ and $k_2 = 2$.

Note that $KG(\lfloor \frac{n}{2} \rfloor, k-2)$ has $m_n = \frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor}{k-2}\binom{\lfloor \frac{n}{2} \rfloor - k + 2}{k-2}$ edges. Provided that n is sufficiently large, we can find an $\varepsilon' \in (0,1)$ such that $n^{\frac{1}{2}-\varepsilon} \leqslant \lfloor \frac{n}{2} \rfloor^{\frac{1}{2}-\varepsilon'}$. Let $c_{\varepsilon'}$ be the corresponding constant from Theorem 18; let $c'_\varepsilon \in (0,1)$ be a constant for now unspecified, $\ell_{1,n} = \lfloor c_{\varepsilon'} \lfloor \frac{n}{2} \rfloor \ln \lfloor \frac{n}{2} \rfloor \rfloor - 1$ and $\ell_{2,n} = \lfloor c'_\varepsilon \ell_{1,n} \rfloor$. We apply Lemma 24 with the $(s_{\lfloor \frac{n}{4} \rfloor}, t_{\lfloor \frac{n}{4} \rfloor})$-semicover $(\mathcal{C}_{G_{\lfloor \frac{n}{4} \rfloor}}, 2^{13} \ln^2 \lfloor \frac{n}{4} \rfloor)$ of the family of acyclic partitions of $D_{\lfloor \frac{n}{4} \rfloor}$, the orientation of $K_2 \times G_{\lfloor \frac{n}{4} \rfloor}$ from Lemma 25. Clearly

$$8 t_{\lfloor \frac{n}{4} \rfloor} \ell_{1,n} \leqslant (\ell_{1,n} - \ell_{2,n}) \left\lfloor \frac{n}{4} \right\rfloor^2,$$

$$2^{13} \ln^2 \left\lfloor \frac{n}{4} \right\rfloor \ell_{1,n} \leqslant \left\lfloor \frac{n}{4} \right\rfloor^2 \text{ and}$$

$$\ln m_n + 2 \ln g\left(\ell_{1,n}, \ell_{2,n}, \left\lfloor \frac{n}{4} \right\rfloor^2, s_{\lfloor \frac{n}{4} \rfloor}, t_{\lfloor \frac{n}{4} \rfloor}, 2\ell_{1,n}\right) \leqslant 2k \ln n + 660 c_{\varepsilon'} n \ln^3 n - \frac{1}{25} n^{2 - \frac{50 c_{\varepsilon'} c'_\varepsilon}{(1 - c'_\varepsilon) \log_2 e}} < 0$$

if n is large enough and c'_ε has been chosen so that $\frac{50 c_{\varepsilon'} c'_\varepsilon}{(1 - c'_\varepsilon) \log_2 e} < 1$. Thus Lemma 24 and Theorem 18 imply that $\vec{\chi}_\ell(KG(\lfloor \frac{n}{2} \rfloor, k-2) \times G_{\lfloor \frac{n}{4} \rfloor}) > \ell_{2,n}$ if n is large enough. $\qquad \square$

4 Sabidussi's theorem

Tensor products are one of the leitmotifs of Section 3. Let us now take a look at another type of graph product. Let G, H be graphs (resp. digraphs). The *Cartesian product* of G and H is the graph (resp. digraph) $G \square H$ with vertex set $V(G) \times V(H)$ where there is an edge between (u,x) and (v,y) (resp. an arc from (u,x) to (v,y)) if and only if either $u = v$ and $\{x,y\} \in E(H)$ (resp. and $(x,y) \in A(H)$), or $x = y$ and $\{u,v\} \in E(G)$ (resp. and $(u,v) \in A(G)$). A well-known theorem of Sabidussi [26] states that for any two graphs G and H the chromatic number of its Cartesian product is $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. His proof can be adapted to show an analogous result for digraphs.

Theorem 26. *Let G and H be digraphs. Then $\vec{\chi}(G \square H) = \max\{\vec{\chi}(G), \vec{\chi}(H)\}$.*

Proof. Let $N = \max\{\vec{\chi}(G), \vec{\chi}(H)\}$. Note that both G and H are isomorphic to a subdigraph of $G \square H$. Therefore, $\vec{\chi}(G \square H) \geqslant N$.

Now, let f_G, f_H be N-colourings of G, H. Let $f : V(G \square H) \to [N]$ be the N-colouring of $G \square H$ defined by $f(g,h) \equiv f_G(g) + f_H(h) \mod N$. We claim that f is a proper colouring of $G \square H$. We argue by contradiction. Let

$$(g_1, h_1), \ldots, (g_1, h_{s_1}), (g_2, h_{s_1+1}), \ldots, (g_2, h_{s_2}), \ldots, (g_r, h_{s_{r-1}+1}), \ldots, (g_r, h_{s_r})$$

be the successive vertices of a monochromatic cycle, where $g_i \neq g_{i+1}$ and $h_{s_i} = h_{s_i+1}$ for $i \in [r-1]$, and $g_r \neq g_1$ and $h_{s_r} = h_1$. The fact that f is constant on these vertices implies that

$$f_H(h_1) = \cdots = f_H(h_{s_1}) = f_H(h_{s_1+1}) = \cdots = f_H(h_{s_r}).$$

If not all of $h_1, \ldots, h_{s_r}$ are identical, we have found a monochromatic cycle in H. If they are all identical, then $r \geqslant 2$, and we can similarly see that $f_G(g_1) = \cdots = f_G(g_r)$, yielding a monochromatic cycle in G. Therefore, $\vec{\chi}(G \square H) \leqslant N$. $\qquad \square$

Hedetniemi's conjecture proposes a similar statement for tensor products: given any two graphs G and H, the chromatic number of its tensor product is (conjectured to be) $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. That was refuted by Shitov [27]; at the time, the conjecture had been standing for more than five decades. Before the first counterexamples were known, its directed version was formulated. Given two digraphs G and H, its tensor product $G \times H$ is defined to be the digraph with vertex set $V(G) \times V(H)$ where there is an arc from (g_1, h_1) to (g_2, h_2) if and only if (g_1, g_2) is an arc of G and (h_1, h_2) is an arc of H.

Conjecture 27. [12] Let G and H be digraphs. Then $\vec{\chi}(G \times H) = \min\{\vec{\chi}(G), \vec{\chi}(H)\}$.

Note that counterexamples to Conjecture 27 can be obtained by taking any counterexample to Hedetniemi's conjecture and replacing all its edges with bidirected arcs. But what happens if G, H are oriented graphs? In [12] it was proved that Conjecture 27 holds when $\min\{\vec{\chi}(G), \vec{\chi}(H)\} \leqslant 2$.

On the positive side, Zhu proved the fractional version of Hedetniemi's conjecture [29]. We wonder if this result can be generalized to the dichromatic number of digraphs, if the fractional dichromatic number of a digraph is defined in the natural way.

Update note. After the completion of the first version of this paper, it came to our attention that Theorem 26 has already been proved by Costa and Silva [5]. The question of whether Conjecture 27 holds for oriented graphs has been recently answered in the negative by Picasarri-Arrieta [25].

Acknowledgements

References

[1] N. Alon. Choice numbers of graphs: a probabilistic approach. *Combinatorics, Probability and Computing*, 1(2):107–114, 1992.

[2] J. Bensmail, A. Harutyunyan, and N. K. Le. List coloring digraphs. *Journal of Graph Theory*, 87(4):492–508, 2018.

[3] D. Bokal, G. Fijavž, M. Juvan, P. Kayll, and B. Mohar. The circular chromatic number of a digraph. *Journal of Graph Theory*, 46:227–240, 2004.

[4] V. Bulankina and A. Kupavskii. Choice number of Kneser graphs. *Discrete Mathematics*, 345(11), 2022.

[5] I. L. Costa and A. S. Silva. Acyclic coloring of products of digraphs. *Discrete Applied Mathematics*, 349:59–69, 2024.

[6] P. Erdős. Problems and results in number theory and graph theory. In *Proceedings of the 9th Manitoba Conference on Numerical Mathematics and Computing*, pages 3–21, 1979.

[7] P. Erdős, J. Gimbel, and D. Kratsch. Some extremal results in cochromatic and dichromatic theory. *Journal of Graph Theory*, 15(6):579–585, 1991.

[8] P. Erdős and L. Moser. On the representation of directed graphs as unions of orderings. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences, Series A*, 9:125–132, 1964.

[9] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium 26*, pages 125–157, 1980.

[10] N. Gazit and M. Krivelevich. On the asymptotic value of the choice number of complete multi-partite graphs. *Journal of Graph Theory*, 52(2):123–134, 2006.

[11] J. E. Greene. A new short proof of Kneser's conjecture. *The American Mathematical Monthly*, 109(10):918–920, 2002.

[12] A. Harutyunyan. *Brooks-type results for coloring of digraphs*. PhD thesis, Simon Fraser University, 2011.

[13] A. Harutyunyan, L. Harutyunyan, and N. Hovhannisyan. Coloring k-partite sparse digraphs. *Discrete Applied Mathematics*, 353:1–3, 2024.

[14] A. Harutyunyan and B. Mohar. Gallai's theorem for list coloring of digraphs. *SIAM Journal on Discrete Mathematics*, 25(1):170–180, 2011.

[15] A. Harutyunyan and B. Mohar. Strengthened Brooks' theorem for digraphs of girth at least three. *Electronic Journal of Combinatorics*, 18:#P195, 2011.

[16] A. Harutyunyan and B. Mohar. Two results on the digraph chromatic number. *Discrete Mathematics*, 312:1823–1826, 2012.

[17] A. Harutyunyan and G. Puig i Surroca. Colouring complete multipartite and Kneser-type digraphs. In D. Král' and J. Nešetřil, editors, *Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications*, pages 545–551. Masaryk University Press, 2023.

[18] M. Kneser. Aufgabe 360. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 58(27):107–116, 1955.

[19] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A*, 25(3):319–324, 1978.

[20] J. Matoušek, A. Björner, and G. M. Ziegler. *Using the Borsuk-Ulam Theorem*. Berlin: Springer, 2003.

[21] B. Mohar. Circular colorings of edge-weighted graphs. *Journal of Graph Theory*, 43:107–116, 2003.

[22] B. Mohar and H. Wu. Dichromatic number and fractional chromatic number. *Forum of Mathematics, Sigma*, 4:e32, 2016.

[23] M. Molloy and B. Reed. *Graph Colouring and the Probabilistic Method.* Springer, 2002.

[24] V. Neumann-Lara. The dichromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, 33:265–270, 1982.

[25] L. Picasarri-Arrieta. *Digraph colouring.* PhD thesis, Université Côte d'Azur, 2024.

[26] G. Sabidussi. Graphs with given group and given graph-theoretical properties. *Canadian Journal of Mathematics*, 9:515–525, 1957.

[27] Y. Shitov. Counterexamples to Hedetniemi's conjecture. *Annals of Mathematics*, 190(2):663–667, 2019.

[28] V. G. Vizing. Vertex colorings with given colors (in Russian). *Metody Diskretnogo Analiza*, 29:3–10, 1976.

[29] X. Zhu. The fractional version of Hedetniemi's conjecture is true. *European Journal of Combinatorics*, 32(7):1168–1175, 2011.

[30] G. M. Ziegler. Aufgabe 360: The Kneser Conjecture. In U. Rehmann and Y. Tschinkel, editors, *Martin Kneser Collected Works*, pages 743–749. Springer International Publishing, 2021.