Hindrance from a Wasteful Partial Linkage

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Abstract

Let $D = (V, E)$ be a (possibly infinite) digraph and $A, B \subseteq V$. A hindrance consists of an AB-separator S together with a set of disjoint AS-paths linking a proper subset of A onto S. Hindrances and configurations guaranteeing the existence of hindrances play an essential role in the proof of the infinite version of Menger's theorem and are important in the context of certain open problems as well. This motivates the investigation of circumstances under which hindrances appear. In this paper we show that if there is a "wasteful partial linkage", i.e. a set $\mathcal{P}$ of disjoint AB-paths with fewer unused vertices in B than in A, then there exists a hindrance.

Mathematics Subject Classifications: 05C63, 05C20, 05C40

1 Introduction

Let $G = (A, B, E)$ be a bipartite graph of any cardinality and suppose that our task is to find a matching that covers A. Then a hindrance is a set $X \subseteq A$ for which the neighbourhood $N_G(X)$ of X can be matched into a proper subset of X. We call G hindered if such a set X exists. Note that if X is infinite, this does not preclude the existence of a matching covering A. Indeed, for example if G is a complete bipartite graph with $|A| = |B| = \aleph_0$, then every infinite $X \subseteq A$ is a hindrance although every injection from A to B provides a matching covering A. A necessary and sufficient condition of this manner for the matchability of A was discovered by Aharoni [3, Theorem 1.8]. For more results in infinite matching theory we refer to [1].

Hindrances can be defined in a more general setting. The triple $\mathcal{W} = (D, A, B)$ is a web if $D = (V, E)$ is a digraph of any cardinality and $A, B \subseteq V$ where the vertices in A (in B) have no ingoing (outgoing) edges. Suppose we are tasked with finding a linkage of A to B in D, i.e. a set $\mathcal{P}$ of disjoint AB-paths such that the set $\mathsf{in}(\mathcal{P})$ of the initial vertices of $\mathcal{P}$ is the whole A. If there is an $X \subseteq A$ such that a proper subset of X can be linked onto an XB-separator S; that is, there exists a set $\mathcal{H}$ of disjoint XB-paths with $\mathsf{in}(\mathcal{H}) \subsetneq X$ and with terminal points $\mathsf{ter}(\mathcal{H}) = S$, then we call the web $\mathcal{W}$ hindered. Note

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that adding the vertices in $A \setminus X$ to $\mathcal{H}$ as trivial paths, we obtain a set $\mathcal{H}'$ of disjoint paths linking a proper subset of A onto an AB-separator, namely $S' := S \cup (A \setminus X)$. Such an $\mathcal{H}'$ is called a hindrance. Hindrances play a key role in the proof of the infinite version of Menger's theorem ([3, Theorem 1.6]) and they are also essential in the context of some open problems such as Matroid intersection conjecture [4, Conjecture 1.2]. This motivates us to investigate circumstances under which hindrances appear.

A partial linkage $\mathcal{P}$ in $\mathcal{W}$ is a set of disjoint AB-paths. We call $\mathcal{P}$ wasteful if $|A \setminus \mathsf{in}(\mathcal{P})| > |B \setminus \mathsf{ter}(\mathcal{P})|$. Note that if $|A| > |B|$, then every partial linkage is wasteful. The main result of this paper reads as follows:

Theorem 1. *If a web $\mathcal{W} = (D, A, B)$ admits a wasteful partial linkage $\mathcal{P}$, then it is hindered.*

The proof is based on a natural adaptation of the techniques used in the proof of the infinite version of König's theorem (see [2]). We derive as a consequence the following statement:

Corollary 2. *If a bipartite graph $G = (A, B, E)$ admits a matching M with $|A \setminus V(M)| > |B \setminus V(M)|$, then it is hindered.*

The paper is organized as follows. We introduce the necessary notation in the following section. In Section 3, we prove Theorem 1 and Corollary 2. In the last section (Section 4), we conjecture a matroidal version of Theorem 1 and a possible simplification of the proof of the infinite version of Menger's theorem.

2 Notation

Ordered pairs $\langle u, v\rangle$ in the context of directed edges are written as uv. For a cardinal κ a *κ-set* is a set of cardinality κ. We write ω for the set of natural numbers. A digraph is an ordered pair $D = (V, E)$ where $E \subseteq (V \times V) \setminus \{vv : v \in V\}$ and $uv \in E$ implies $vu \notin E$.[1] Fix a digraph $D = (V, E)$. The *in-neighbourhood* $\boldsymbol{N^-(U)}$ of a set $U \subseteq V$ consists of those $v \in V \setminus U$ for which there is a $w \in U$ with $vw \in E$.

A trivial *trail* is a single vertex $v \in V$. A non-trivial trail is a finite, nonempty sequence $T = (v_0v_1, v_1v_2, \dots v_{n-1}v_n)$ where $v_iv_{i+1} \in E$ and $v_iv_{i+1} \neq v_jv_{j+1}$ for $i \neq j$. We write $\boldsymbol{\mathsf{in}(T)}$ for the initial vertex v_0 and $\boldsymbol{\mathsf{ter}(T)}$ for the terminal vertex v_n of the trail (while $\mathsf{in}(T) = \mathsf{ter}(T) = v$ if $T = v$ is a trivial trail). Furthermore, $V(T)$ and $E(T)$ stand for the vertex set and edge set of the subgraph corresponding to T. We say that T is an XY-trail if it is either trivial and an element of $X \cap Y$ or non-trivial with $v_0 \in X$, $v_n \in Y$ and $v_i \notin X \cup Y$ for $0 < i < n$. A *path* P is a trail that is either trivial or non-trivial and has the additional property of not repeating vertices, i.e. $v_i \neq v_j$ for $i \neq j$. For a set $\mathcal{T}$ of trails we let $\boldsymbol{\mathsf{in}(\mathcal{T})} := \{\mathsf{in}(T) : T \in \mathcal{T}\}$ and $\boldsymbol{\mathsf{ter}(\mathcal{T})} := \{\mathsf{ter}(T) : T \in \mathcal{T}\}$,

[1] This restriction of digraphs is just for convenience. Subdividing each edge by a new vertex makes digraphs simple, while the existence of a wasteful linkage and of a hindrance are both invariant under this operation.

moreover $\boldsymbol{V(\mathcal{T})} = \bigcup_{T \in \mathcal{T}} V(T)$ and $\boldsymbol{E(\mathcal{T})} = \bigcup_{T \in \mathcal{T}} E(T)$. We call a set $\mathcal{T}$ of trails *disjoint* if $V(R) \cap V(T) = \emptyset$ for every distinct $R, T \in \mathcal{T}$ and *v-joint* if $\mathsf{ter}(T) = v$ for each $T \in \mathcal{T}$ and $V(R) \cap V(T) = \{v\}$ for every distinct $R, T \in \mathcal{T}$. A set $\mathcal{P}$ of disjoint paths *links X onto Y* if $\mathsf{in}(\mathcal{P}) = X$ and $\mathsf{ter}(\mathcal{P}) = Y$.

A *web* is a triple $\boldsymbol{\mathcal{W}} = (D, A, B)$ where $D = (V, E)$ is a digraph, $A, B \subseteq V$ and the vertices in A (B) have no ingoing (outgoing) edges. An *XY-separator* is an $S \subseteq V$ that meets all XY-paths. A *hindrance* is a set $\mathcal{H}$ of disjoint AS-paths where S is an AB-separator and $\mathcal{H}$ links a proper subset of A onto S. A *partial linkage* of $\mathcal{W}$ is a set $\mathcal{P}$ of disjoint AB-paths.

A *partially linked web* is a quadruple $\boldsymbol{\mathcal{L}} = (D, A, B, \mathcal{P})$ where $\mathcal{W} = (D, A, B)$ is a web and $\mathcal{P}$ is a partial linkage in it. We write $\boldsymbol{\widehat{A}}$ and $\boldsymbol{\widehat{B}}$ for $A \setminus \mathsf{in}(\mathcal{P})$ and $B \setminus \mathsf{ter}(\mathcal{P})$ respectively (we add $\mathcal{L}$ as subscript if it is not clear from the context).

The *alternating trails* in $\mathcal{L}$ consist of the trivial trails corresponding to the elements of $\widehat{A}$ as well as those $T = (v_0v_1, v_1v_2, \dots v_{n-1}v_n)$ for which

1. T is a non-trivial trail in the digraph $\boldsymbol{D^*} = (V, \boldsymbol{E^*})$ we obtain from D by reversing the edges in $E(\mathcal{P})$,
2. $v_0 \in \widehat{A}$,
3. if $v_i = v_j$ for $i < j$, then $v_i \in V(\mathcal{P})$ and $j \neq n$,
4. if $v_i \in V(\mathcal{P})$, $v_{i-1}v_i \in E^* \cap E$ and $i < n$, then $v_iv_{i+1} \in E^* \setminus E$.

(Property (4) states that upon arriving at a vertex $v \in V(\mathcal{P})$ by using a "forward" edge implies that the next edge we use must be the unique "backward" edge from v unless we terminate at v.)

We say that the alternating trails R and T are *strongly disjoint* if they are disjoint and there is no $P \in \mathcal{P}$ such that $V(R) \cap V(P) \neq \emptyset$ and $V(T) \cap V(P) \neq \emptyset$. An alternating trail is *augmenting* if it terminates in $\widehat{B}$. Let $\boldsymbol{O}$ consist of those $v \in \widehat{B}$ for which there exists an $\left|\widehat{A}\right|$-set $\mathcal{T}$ of v-joint augmenting trails and let $\boldsymbol{U} := \widehat{B} \setminus O$. We call these the *popular* and *unpopular* vertices respectively.

3 Proof of Theorem 1

Fix a partially linked web $\mathcal{L} = (D, A, B, \mathcal{P})$ and let $\mathcal{W} := (D, A, B)$.

3.1 Preparations

Observation 3. *For every infinite cardinal κ, every κ-set of disjoint alternating trails includes a κ-subset of strongly disjoint alternating trails.*

Proof. The desired κ-subset of strongly disjoint alternating trails can be built by a straightforward transfinite recursion. Indeed, let $\mathcal{T}$ be a κ-set of disjoint alternating trails. In step $\alpha < \kappa$ we have already chosen strongly disjoint alternating trails $\mathcal{T}_\alpha = \{T_\beta : \beta < \alpha\}$. The trails in $\mathcal{T}_\alpha$ meet at most $|\alpha| \cdot \aleph_0 < \kappa$ paths in $\mathcal{P}$. Since $|\mathcal{T}| = \kappa$ and the trails in $\mathcal{T}$ are disjoint, we can pick a T_α from $\mathcal{T}$ that is disjoint from all of these paths as well as from all the trails in $\mathcal{T}_\alpha$. □

The proof of Menger's theorem based on augmenting trails (sometimes called augmenting walks) is well known, so we do not provide detailed proofs for the following two lemmas. For detailed proofs in the context of undirected graphs see [6, Lemma 3.3.2 and 3.3.3]. [2]

Lemma 4. *If $\widehat{A} \neq \emptyset$ and there is no augmenting trail, then $\mathcal{W}$ is hindered.*

Proof. For each $P \in \mathcal{P}$, let v_P be the last vertex on P for which there is an alternating trail T with $\mathsf{ter}(T) = v_P$ and let $v_P := \mathsf{in}(P)$ if no alternating trail meets P. One can show (see [6, Lemma 3.3.3]) that $S := \{v_P : P \in \mathcal{P}\}$ is an AB-separator. The initial segment of each $P \in \mathcal{P}$ up to v_P provides a hindrance. □

Lemma 5. *If $\mathcal{T}$ is a set of strongly disjoint augmenting trails, then there is a partial linkage $\mathcal{Q}$ with $\mathsf{in}(\mathcal{Q}) = \mathsf{in}(\mathcal{P}) \cup \mathsf{in}(\mathcal{T})$ and $\mathsf{ter}(\mathcal{Q}) = \mathsf{ter}(\mathcal{P}) \cup \mathsf{ter}(\mathcal{T})$.*

Proof. For $T \in \mathcal{T}$, let $\mathcal{P}_T$ be the set of the finitely many paths in $\mathcal{P}$ meeting T. Then one can find (see [6, Lemma 3.3.2]) a set $\mathcal{Q}_T$ of $|\mathcal{P}_T| + 1$ many AB-paths with $V(\mathcal{Q}_T) \subseteq V(\mathcal{P}_T) \cup V(T)$ and $\mathsf{in}(T), \mathsf{ter}(T) \in V(\mathcal{Q}_T)$. Since the sets $V(\mathcal{P}_T) \cup V(T)$ for $T \in \mathcal{T}$ are pairwise disjoint, the lemma follows. □

The following lemma handles the special case of Theorem 1 where $\left|\widehat{A}\right|$ is infinite and each vertex in $\widehat{B} \setminus \widehat{A}$ is popular.

Lemma 6. *If $\left|\widehat{A}\right| > \left|\widehat{B}\right|$, $\left|\widehat{A}\right| \geqslant \aleph_0$ and $O = \widehat{B} \setminus \widehat{A}$, then a proper subset of A can be linked onto B. In particular, $\mathcal{W}$ is hindered.*

Proof. By Lemma 5, it is enough to find a set $\mathcal{T}$ of strongly disjoint augmenting trails with $\mathsf{ter}(T) = \widehat{B}$. For each $v \in \widehat{B} \cap \widehat{A}$, let the trivial augmenting trail v be in $\mathcal{T}$. By assumption, for each $v \in \widehat{B} \setminus \widehat{A}$ there is an $\left|\widehat{A}\right|$-set $\mathcal{T}_v$ of v-joint augmenting trails. Thus the rest of $\mathcal{T}$ can be constructed by a straightforward transfinite recursion.

Indeed, let $\lambda := \left|\widehat{B} \setminus \widehat{A}\right|$ and let $\widehat{B} \setminus \widehat{A} = \{v_\alpha : \alpha < \lambda\}$ be an enumeration. In step $\alpha < \lambda$ we have already chosen strongly disjoint alternating trails $\mathcal{T}_\alpha = \{T_\beta : \beta < \alpha\}$ where $V(T_\beta) \cap \widehat{A} \cap \widehat{B} = \emptyset$ and $\mathsf{ter}(T_\beta) = v_\beta$ for $\beta < \alpha$. The trails in $\mathcal{T}_\alpha$ meet at most $|\alpha| \cdot \aleph_0 < \left|\widehat{A}\right|$ paths in $\mathcal{P}$ and we know that $\left|\widehat{A} \cap \widehat{B}\right| < \left|\widehat{A}\right|$. Therefore we can pick a $T_\alpha \in \mathcal{T}_{v_\alpha}$ that is disjoint from: the trails in $\mathcal{T}_\alpha$, those paths in $\mathcal{P}$ that meet a trail in $\mathcal{T}_\alpha$, and $\widehat{A} \cap \widehat{B}$. □

[2] For the directed version we could not find a suitable reference. Textbooks are typically reducing vertex-Menger to edge-Menger where the corresponding "augmenting path lemmas" are simpler.

From now on, we assume that $\mathcal{P}$ is wasteful w.r.t. $\mathcal{W}$, i.e. $\left|\widehat{A}\right| > \left|\widehat{B}\right|$. Suppose first that $\left|\widehat{A}\right| \leqslant \aleph_0$. Then $n := \left|\widehat{B}\right| < \aleph_0$. We apply induction on n. If $n = 0$, then $\mathcal{P}$ itself is a hindrance. Assume that $n > 0$. If there is no augmenting trail, then we are done by Lemma 4. If there is an augmenting trail T, then we apply Lemma 5 with $\mathcal{T} = \{T\}$. For the resulting $\mathcal{Q}$, let $\mathcal{L}' := (D, A, B, \mathcal{Q})$. Then $\left|\widehat{A}_{\mathcal{L}'}\right| = \left|\widehat{A}_{\mathcal{L}}\right| - 1$ and $\left|\widehat{B}_{\mathcal{L}'}\right| = \left|\widehat{B}_{\mathcal{L}}\right| - 1 = n - 1$ according to Lemma 5. Since $\mathcal{W}_{\mathcal{L}'} = \mathcal{W}_{\mathcal{L}}$, we are done by applying the induction hypothesis for $n - 1$.

3.2 Elimination of unpopular vertices in countably many steps

Now we can assume that $\aleph_1 \leqslant \left|\widehat{A}\right| =: \kappa$. We can also assume, without loss of generality, that κ is regular. Indeed, otherwise we pick an uncountable regular cardinal λ with $\kappa > \lambda > \left|\widehat{B}\right|$ and delete all but λ many vertices in $\widehat{A}$. Any hindrance $\mathcal{H}$ of the resulting partially linked web can be extended to a hindrance of the original by adding the deleted vertices to $\mathcal{H}$ as trivial paths.

We define recursively partially linked webs $\mathcal{L}_n = (D_n, A, B_n, \mathcal{P}_n)$ for $n < \omega$ (see Figure 1). Let $\mathcal{L}_0 := \mathcal{L}$. Suppose that $\mathcal{L}_n$ is already defined. Let us denote $U_n := U_{\mathcal{L}_n}$ and $O_n := O_{\mathcal{L}_n}$, as well as $\widehat{A}_n := \widehat{A}_{\mathcal{L}_n}$ and $\widehat{B}_n := \widehat{B}_{\mathcal{L}_n}$. We define $B_{n+1} := (B_n \setminus (U_n \setminus A)) \cup N^-_{D_n}(U_n)$. Observe that B_{n+1} separates B_n from A in D_n. To define $D_{n+1} = (V, E_{n+1})$, we define E_{n+1} from E_n by the deletion of the outgoing edges of the vertices in $N^-_{D_n}(U_n)$. Finally, we let $\mathcal{P}_{n+1}$ consist of the initial segments of the paths in $\mathcal{P}_n$ up to their first vertex in B_{n+1}.

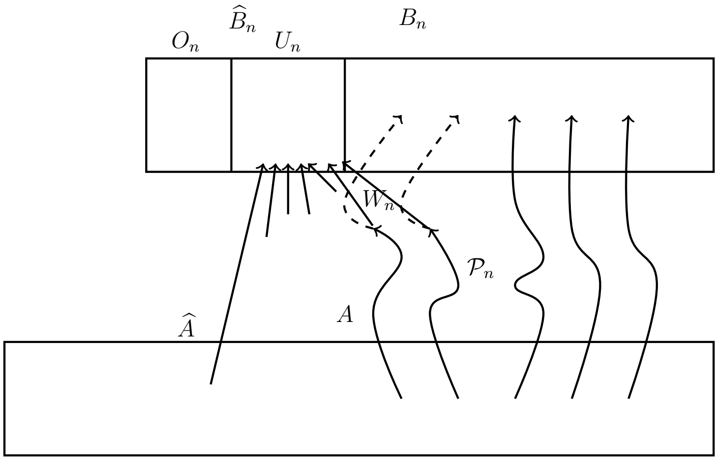

Figure 1: A step of the recursion. (Drawing A and B as disjoint is done purely for clarity.)

Observation 7. *$\widehat{A}_n = \widehat{A}$ for each $n < \omega$.*

Observation 8. *For every $n < \omega$, B_n separates $\bigcup_{k \leqslant n} B_k$ (in particular B) from A in D.*

Proof. For $n = 0$ this is trivial. Suppose we already know for some n that B_n separates $\bigcup_{k\leqslant n} B_k$ from A in D. By construction B_{n+1} separates B_n from A in D_n. Thus it follows by induction and by $E_n \subseteq E$ that B_{n+1} separates $\bigcup_{k\leqslant n+1} B_k$ from A in D_n. Since every edge in $E \setminus E_n$ is an outgoing edge of a vertex in $\bigcup_{k\leqslant n} B_k$, B_{n+1} separates $\bigcup_{k\leqslant n+1} B_k$ from A in D as well. $\square$

Let $\boldsymbol{W_n} := V(\mathcal{P}_n) \setminus V(\mathcal{P}_{n+1})$.

Lemma 9. *For every $n < \omega$, the following holds:*

1. *Every $\mathcal{L}_n$-alternating trail T with $V(T) \cap (W_n \cup N^-_{D_n}(U_n)) \subseteq \{\mathsf{ter}(T)\}$ is an $\mathcal{L}_{n+1}$-alternating trail.*

2. *Every $\mathcal{L}_{n+1}$-alternating trail T with $V(T) \cap W_n \subseteq \{\mathsf{ter}(T)\}$ is an $\mathcal{L}_n$-alternating trail.*

Proof. The lemma follows directly from the definitions by checking formally the properties (1)-(4) in the definition of alternating trails.

Both statements are true for trivial trails because $\widehat{A}_n = \widehat{A}_{n+1}$ by Observation 7. Suppose that $T = (v_0v_1, \dots, v_{n-1}v_n)$ is a non-trivial $\mathcal{L}_n$-alternating trail with $V(T) \cap (W_n \cup N^-_{D_n}(U_n)) \subseteq \{\mathsf{ter}(T)\}$. Since each edge in $E^*_n \setminus E^*_{n+1}$ is an outgoing edge of a vertex in $(W_n \cup N^-_{D_n}(U_n))$, no such edge is used by T. Thus T is a trail in D^*_{n+1}, hence T satisfies (1) w.r.t. $\mathcal{L}_{n+1}$. Clearly, $\mathsf{in}(T) \in \widehat{A}_n = \widehat{A}_{n+1}$, thus (2) remains true. Since $V(\mathcal{P}_{n+1}) \subseteq V(\mathcal{P}_n)$ and T does not repeat vertices that are not in $V(\mathcal{P}_n)$, it does not repeat vertices not in $V(\mathcal{P}_{n+1})$ either. Therefore (3) holds. Since T respects (4) w.r.t. $\mathcal{P}_n$ and T has at most its terminal vertex in W_n, no violation of (4) can occur.

Suppose that $T = (v_0v_1, \dots, v_{n-1}v_n)$ is a non-trivial $\mathcal{L}_{n+1}$-alternating trail with $V(T) \cap W_n \subseteq \{\mathsf{ter}(T)\}$. Note that if an $e \in E^*_{n+1}$ is not in E^*_n, then there is a $P \in \mathcal{P}_n$ with $e \in E(P)$ such that the initial segment P' of P that is in $\mathcal{P}_{n+1}$ does not use e. In other words, e is reversed in D^*_n but not any more in D^*_{n+1}. Hence each edge in $E^*_{n+1} \setminus E^*_n$ is an outgoing edge of a vertex in W_n. Thus no such edge is used by T and therefore T is a trail in D^*_n, hence (1) holds. Clearly, $\mathsf{in}(T) \in \widehat{A}_{n+1} = \widehat{A}_n$, thus (2) holds. We claim that no vertex in $V(\mathcal{P}_n)$ is repeated by T. By definition, $V(\mathcal{P}_n) = V(\mathcal{P}_{n+1}) \cup W_n$ and no vertex in $V(\mathcal{P}_n)$ is repeated. Furthermore, only the terminal vertex of T may be in W_n but terminal vertices are never repeated by the definition of alternating trails. Thus (3) holds. Finally, T respects (4) corresponding to $\mathcal{P}_{n+1}$ and at most its terminal vertex is in W_n, thus (4) holds corresponding to $\mathcal{P}_n$ as well. $\square$

Lemma 10. *For every $n < \omega$:*

(i_n) *there is no κ-set $\mathcal{T}$ of disjoint $\mathcal{L}_n$-augmenting trails;*

(ii_n) *there is no κ-set $\mathcal{T}$ of disjoint $\mathcal{L}_n$-alternating $\widehat{A}W_n$-trails.*

Proof. We apply induction on n. Clearly, (i_0) holds because $\left|\widehat{B}_0\right| < \kappa$ (by $\left|\widehat{B}\right| < \left|\widehat{A}\right| = \kappa$) and every $\mathcal{L}_0$-augmenting trail terminates in $\widehat{B}_0$ by definition. Suppose that we already know (i_n) for some n. Next we prove (ii_n) and (i_{n+1}).

Claim 11. *There is no κ-set of $\mathcal{L}_n$-augmenting trails with* $\mathsf{ter}(\mathcal{T}) \subseteq U_n$ *where any two trails in $\mathcal{T}$ are either disjoint or sharing exactly their terminal vertex.*

Proof. If $|\mathsf{ter}(\mathcal{T})| < \kappa$, then, by the regularity of κ, there is a $v \in \mathsf{ter}(\mathcal{T})$ such that κ many $T \in \mathcal{T}$ terminates at v, contradicting $v \in U_n$. Thus $|\mathsf{ter}(\mathcal{T})| = \kappa$. But then by choosing for each $v \in \mathsf{ter}(\mathcal{T})$ exactly one $T \in \mathcal{T}$ with $\mathsf{ter}(T) = v$, we obtain a κ-set of disjoint $\mathcal{L}_n$-augmenting trails which contradicts (i_n). $\square$

Suppose for a contradiction that (ii_n) fails, i.e. there is a κ-set $\mathcal{T}$ of disjoint $\mathcal{L}_n$-alternating $\widehat{A}W_n$-trails. By Observation 3, we may assume that the trails in $\mathcal{T}$ are strongly disjoint. For $T \in \mathcal{T}$, let P_T be the unique path in $\mathcal{P}_n$ with $\mathsf{ter}(T) \in V(P_T)$ and let v_P be the terminal vertex of the unique path in $\mathcal{P}_{n+1}$ that is a proper initial segment of P_T (exists by the definition of W_n). Let T' be a forward-extension of T that we obtain by going backwards along P_T until v_P and then adding an outgoing edge of v_P in D_n with head in U_n (which exists because v_P is nothing else than the first vertex of P_T that is in $N^-_{D_n}(U_n)$). Then $\mathcal{T}' := \{T' : T \in \mathcal{T}\}$ is a κ-set of $\mathcal{L}_n$-augmenting paths with $\mathsf{ter}(\mathcal{T}') \subseteq U_n$ where any two trails in $\mathcal{T}'$ are either disjoint or sharing exactly their terminal vertex which contradicts Claim 11.

Suppose for a contradiction that (i_{n+1}) fails, i.e. there is a κ-set $\mathcal{T}$ of disjoint $\mathcal{L}_{n+1}$-augmenting trails. By (i_n) we know that less than κ many $T \in \mathcal{T}$ are $\mathcal{L}_n$-augmenting trails, thus by deleting these we can assume that no $T \in \mathcal{T}$ is an $\mathcal{L}_n$-augmenting trail. then exactly one of the following holds for each $T \in \mathcal{T}$: either T is not even an $\mathcal{L}_n$-alternating trail or it is but $\mathsf{ter}(T) \in B_{n+1} \setminus B_n$. Since one of these two cases occurs κ often, we can assume that one of them holds actually for every $T \in \mathcal{T}$. Suppose first, that no trail in $\mathcal{T}$ is $\mathcal{L}_n$-alternating. Then (2) of Lemma 9 guarantees that each $T \in \mathcal{T}$ meets W_n. Let $\mathcal{T}'$ consists of the initial segments of the trails in $\mathcal{T}$ up to their first vertex in W_n. Then (2) of Lemma 9 ensures that $\mathcal{T}'$ consists of $\mathcal{L}_n$-alternating trails. Since $|\mathcal{T}'| = \kappa$, this contradicts (ii_n). Assume now that every $T \in \mathcal{T}$ is $\mathcal{L}_n$-alternating but terminates in $B_{n+1} \setminus B_n = N^-_{D_n}(U_n)$. Then every $T \in \mathcal{T}$ can be extended forward by a new edge to an $\mathcal{L}_n$-augmenting trail T' that terminates in U_n. But then the set $\mathcal{T}'$ of these extensions contradicts Claim 11. $\square$

Corollary 12. $\left|\widehat{B}_n \cap \widehat{A}\right| < \kappa$ *for each* $n < \omega$.

Proof. Since $\widehat{A} = \widehat{A}_n$, each $v \in \widehat{B}_n \cap \widehat{A}$ counts as a trivial $\mathcal{L}_n$-augmenting trail. Thus the statement follows from (i_n) of Lemma 10. $\square$

Now we show that the popularity of the vertices are preserved during the recursion.

Corollary 13. *Suppose that $v \in O_n$ and let $\mathcal{T}$ be a witness for this, i.e. a κ-set of v-joint $\mathcal{L}_n$-augmenting trails. Then there are only less than κ many $T \in \mathcal{T}$ that are not $\mathcal{L}_{n+1}$-augmenting trails.*

Proof. Let $\mathcal{T}'$ consists of those $T \in \mathcal{T}$ that are not $\mathcal{L}_{n+1}$-augmenting trails. Suppose for a contradiction that $|\mathcal{T}'| = \kappa$. Then (1) of Lemma 9 ensures that each $T \in \mathcal{T}'$ meets

$W_n \cup N^-_{D_n}(U_n)$. Let $\mathcal{T}''$ consist of the initial segments of the trails in $\mathcal{T}'$ until the first vertex in $W_n \cup N^-_{D_n}(U_n)$. If κ many $T \in \mathcal{T}''$ terminates in W_n, then we get a contradiction to (ii_n) of Lemma 10. If κ many $T \in \mathcal{T}''$ terminates in $N^-_{D_n}(U_n)$, then we extend them forward by a new edge to reach U_n and get a contradiction to Claim 11. $\square$

Corollary 14. *$O_n \subseteq O_{n+1}$ for each $n < \omega$.*

Proof. Suppose that $v \in O_n$ and let $\mathcal{T}$ be a witness for this. As earlier, let $\mathcal{T}'$ consists of those $T \in \mathcal{T}$ that are not $\mathcal{L}_{n+1}$-augmenting trails. Then $|\mathcal{T}'| < \kappa$ by Corollary 13. Hence $|\mathcal{T} \setminus \mathcal{T}'| = \kappa$ and therefore $\mathcal{T} \setminus \mathcal{T}'$ witnesses $v \in O_{n+1}$. $\square$

Corollary 15. *$|O_n| < \kappa$ for each $n < \omega$.*

Proof. We apply induction on n. For $n = 0$ the statement holds because $O_0 \subseteq \widehat{B}_0$ and $\left|\widehat{B}_0\right| < \kappa$ by assumption. Assume we already know that $|O_n| < \kappa$ for some $n < \omega$. Suppose for a contradiction that $|O_{n+1}| \geqslant \kappa$. Then, by the induction hypotheses we must have $|O_{n+1} \setminus O_n| \geqslant \kappa$. Let C be a κ-subset of $O_{n+1} \setminus O_n$. By using the definition of O_{n+1}, we can find by a straightforward transfinite recursion a κ-set $\mathcal{T}$ of disjoint $\mathcal{L}_{n+1}$-alternating trails with $\mathsf{ter}(\mathcal{T}) = C$. Note that $O_{n+1} \setminus O_n \subseteq \widehat{B}_{n+1} \setminus \widehat{B}_n \subseteq N^-_{D_n}(U_n)$ by construction. Assume first that κ many $T \in \mathcal{T}$ meets W_n. Then by taking suitable initial segments of these trails, we obtain a κ-set $\mathcal{T}'$ of $\mathcal{L}_{n+1}$-alternating $\widehat{A}W_n$-trails. By (2) of Lemma 9, these are also $\mathcal{L}_n$-alternating trails. But this contradicts (ii_n) of Lemma 10. Thus we may assume that no $T \in \mathcal{T}$ meets W_n. By (2) of Lemma 9, each $T \in \mathcal{T}$ is an $\mathcal{L}_n$-alternating trail as well. Since $\mathsf{ter}(\mathcal{T}) \subseteq N^-_{D_n}(U_n)$, we can extend each $T \in \mathcal{T}$ forward to terminate in U_n. But then the existence of the resulting set of $\mathcal{L}_n$-augmenting trails contradicts Claim 11. $\square$

3.3 The limit partially linked web

Let $E_\omega := \bigcap_{n<\omega} E_n$ and $D_\omega := (V, E_\omega)$. We define $B_\omega := \bigcup_{m<\omega} \bigcap_{m<n<\omega} B_n$ and $\mathcal{P}_\omega := \bigcup_{m<\omega} \bigcap_{m<n<\omega} \mathcal{P}_n$.

In each iteration, each path in the partial linkage was replaced by its initial segment. Roughly speaking, $\mathcal{P}_\omega$ consists of the shortest versions of each path that we get during the process. Similarly, the sets B_n are "getting closer and closer" to A in each iteration. The vertices that arrive at A or become popular at some stage form the set B_ω.

Let $\widehat{A}_\omega := \widehat{A}_{\mathcal{L}_\omega}$, $\widehat{B}_\omega := \widehat{B}_{\mathcal{L}_\omega}$ and $O_\omega := O_{\mathcal{L}_\omega}$.

Observation 16. *$\widehat{A}_\omega = \widehat{A}$.*

Observation 17. *$\mathcal{P}_\omega$ is a partial linkage in $\mathcal{W}_\omega := (D_\omega, A, B_\omega)$.*

Lemma 18. *$\left|\widehat{B}_\omega \cap \widehat{A}\right| < \kappa$.*

Proof. By construction $\widehat{B}_\omega \cap \widehat{A} = \bigcup_{n<\omega}(\widehat{B}_n \cap \widehat{A})$ and we know that $\left|\widehat{B}_n \cap \widehat{A}\right| < \kappa$ for each $n < \omega$ (see Corollary 12). Since κ is a regular uncountable cardinal, we conclude that $\left|\widehat{B}_\omega \cap \widehat{A}\right| = \left|\bigcup_{n<\omega}(\widehat{B}_n \cap \widehat{A})\right| < \kappa$. $\square$

Lemma 19. *B_ω separates B from A in D.*

Proof. Let P be an AB-path in D. By Observation 8, P meets B_n for each n. Let v be the first vertex of P in $\bigcup_{n<\omega} B_n$. Then there is an $m < \omega$ with $v \in B_m$. For every $k \geqslant m$ the set B_k separates B_m from A (see Observation 8). By the choice of v this implies that we must have $v \in B_k$ for every $k \geqslant m$. But then by definition $v \in B_\omega$. $\square$

Lemma 20. $\widehat{B}_\omega \setminus \widehat{A} = \bigcup_{n<\omega} O_n$.

Proof. Suppose that $v \in \widehat{B}_\omega \setminus \widehat{A}$. Then by the definition of $\widehat{B}_\omega$ there is an $m < \omega$ such that $v \in \widehat{B}_n \setminus \widehat{A}$ for $n \geqslant m$. By definition $\widehat{B}_m = U_m \cup O_m$, therefore $v \in U_m \cup O_m$. But $v \notin U_m$ since otherwise $v \in U_m \setminus \widehat{A}$ and therefore $v \notin \widehat{B}_{m+1}$ which contradicts the fact that $v \in \widehat{B}_n \setminus \widehat{A}$ for $n \geqslant m$. Hence $v \in O_m \subseteq \bigcup_{n<\omega} O_n$.

Assume that $v \in \bigcup_{n<\omega} O_n$. Then there is an $m < \omega$ with $v \in O_m$. It follows that $v \notin \widehat{A}$ because $O_m \cap \widehat{A} = \emptyset$ is evident from the definition of popular vertices and the fact that $\left|\widehat{A}\right| = \kappa > 1$. By Corollary 14 we know that $v \in O_n \subseteq \widehat{B}_n$ for each $n \geqslant m$. But then by definition $v \in \widehat{B}_\omega$, thus $v \in \widehat{B}_\omega \setminus \widehat{A}$. $\square$

Corollary 21. $\left|\widehat{B}_\omega \setminus \widehat{A}\right| < \kappa$

Proof. We have just seen that $\widehat{B}_\omega \setminus \widehat{A} = \bigcup_{n<\omega} O_n$ (Lemma 20). Furthermore, $|O_n| < \kappa$ for each $n < \omega$ (see Corollary 15). As κ is a regular uncountable cardinal, we conclude that $\left|\widehat{B}_\omega \setminus \widehat{A}\right| = \left|\bigcup_{n<\omega} O_n\right| < \kappa$. $\square$

Observation 22. *If T is an $\mathcal{L}_n$-augmenting trail for every large enough n, then T is an $\mathcal{L}_\omega$-augmenting trail.*

Proof. From the construction of $\mathcal{L}_\omega$ it is clear that if any of the properties (1)-(4) is violated by T w.r.t. $\mathcal{L}_\omega$, then it is violated w.r.t. $\mathcal{L}_n$ for every large enough n. $\square$

Lemma 23. $\widehat{B}_\omega \setminus \widehat{A} = O_\omega$.

Proof. The inclusion "$\supseteq$" holds because $\widehat{B}_\omega = O_\omega \cup U_\omega$ by definition and $O_\omega \cap \widehat{A} = \emptyset$. To show the inclusion "$\subseteq$", let $v \in \widehat{B}_\omega \setminus \widehat{A}$ be given. Since $\widehat{B}_\omega \setminus \widehat{A} = \bigcup_{n<\omega} O_n$ by Lemma 20, there is an $m < \omega$ with $v \in O_m$. Let $\mathcal{T}_m$ be a witness for $v \in O_m$, i.e. a κ-set of v-joint $\mathcal{L}_m$-augmenting trails. We define by recursion $\mathcal{T}_n$ for $n \geqslant m$. If $\mathcal{T}_n$ is already defined, let $\mathcal{T}_{n+1}$ consists of those $T \in \mathcal{T}_n$ that are also $\mathcal{L}_{n+1}$-augmenting trails. Finally, let $\mathcal{T}_\omega := \bigcap_{n \geqslant m} \mathcal{T}_n$. By Observation 22, $\mathcal{T}_\omega$ consists of $\mathcal{L}_\omega$-augmenting trails. We have $|\mathcal{T}_m| = \kappa$ by definition and Corollary 13 ensures that $|\mathcal{T}_n \setminus \mathcal{T}_{n+1}| < \kappa$ for each $n \geqslant m$. Since κ is a regular uncountable cardinal, it follows that $|\mathcal{T}_\omega| = \kappa$. But then $\mathcal{T}_\omega$ witnesses $v \in O_\omega$. $\square$

We intend to apply Lemma 6 to $\mathcal{L}_\omega$. We have $\widehat{A}_\omega = \widehat{A}$ by Observation 16 where $\left|\widehat{A}\right| = \kappa$ by definition. Since $\left|\widehat{B}_\omega \cap \widehat{A}\right| < \kappa$ (Lemma 18) and $\left|\widehat{B}_\omega \setminus \widehat{A}\right| < \kappa$ (Corollary 21), we conclude $\left|\widehat{B}_\omega\right| < \kappa$. Finally, $\widehat{B}_\omega \setminus \widehat{A} = O_\omega$ by Lemma 23. It follows that $\mathcal{L}_\omega$ satisfies the conditions of Lemma 6. Thus there is a hindrance $\mathcal{H}$ in $\mathcal{W}_\omega := (D_\omega, A, B_\omega)$ that links a proper subset of A onto B_ω in D_ω. Since D_ω is a subgraph of D and B_ω separates B from A in D as well (Lemma 19), this $\mathcal{H}$ is a hindrance in $\mathcal{W}$ too. This completes the proof of Theorem 1.

3.4 Proof of Corollary 2

Let D be the orientation of G, in which all the edges are pointing towards B. Note that the vertex covers of G are exactly the AB-separators of D. Then a matching M defines a partial linkage $\mathcal{P}$ in the web $\mathcal{W} := (D, A, B)$. The assumption on M translates to $\mathcal{P}$ being wasteful. Then Theorem 1 ensures that there is a hindrance $\mathcal{H}$ in $\mathcal{W}$. Let $\mathcal{H}'$ consist of the non-trivial paths in $\mathcal{H}$. Then each $P \in \mathcal{H}'$ has exactly one edge and $M' := E(\mathcal{H}')$ is a matching. Moreover, for $X := \mathsf{in}(\mathcal{H}') \cup (A \setminus V(\mathcal{H}))$ we have $N_G(X) = \mathsf{ter}(\mathcal{H}')$ because $\mathsf{ter}(\mathcal{H})$ is a vertex cover in G. Since $\mathcal{H}$ is a hindrance, we know that $A \setminus V(\mathcal{H}) \neq \emptyset$, therefore M' matches $N_G(X)$ to a proper subset of X and thus we conclude that X is a hindrance.

4 Outlook

Hindrances can be defined in the context of further problems such as the previously mentioned Matroid intersection conjecture. For the conjecture and the corresponding matroidal terminology one may refer to authors such as: Aharoni & Ziv [4], Bowler & Carmesin [5] and ourselves [7].

Suppose that M and N are finitary matroids on E and our task is to find an M-independent spanning set of N. An I is an (M, N)-*wave* if $I \in \mathcal{I}(M) \cap \mathcal{I}(N.\mathsf{span}_M(I))$. An I is an (M, N)-*hindrance* if it an (M, N)-*wave* that does not span $N.\mathsf{span}_M(I)$ (see [4, Definition 3.2]). We call (M, N) hindered if there exists an (M, N)-hindrance.

Conjecture 24. If M and N are finitary matroids on a common ground set such that there is an $I \in \mathcal{I}(M) \cap \mathcal{I}(N)$ with $r(M/I) < r(N/I)$, then (M, N) is hindered.

Transforming a so-called κ-hindrance (see [3, p. 35-36]) to a hindrance is an essential part of the proof of the infinite version of Menger's theorem. Together with an auxiliary tool called "Bipartite conversion", it occupies two entire chapters in [3].

Question 25. Is it possible to prove the existence of a hindrance assuming the existence of a κ-hindrance ([3, Theorem 7.30]) directly along the lines of our proof of Theorem 1 avoiding bipartite conversion?

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