Isoperimetry in Product Graphs

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Abstract

In this short note, we establish an edge-isoperimetric inequality for arbitrary product graphs. Our inequality is sharp for subsets of many different sizes in every product graph. In particular, it implies that the 2^d-element sets with smallest edge-boundary in the hypercube are subcubes and is only marginally weaker than the Bollobás–Leader edge-isoperimetric inequalities for grids and tori. Additionally, it improves two edge-isoperimetric inequalities for products of regular graphs proved by Erde, Kang, Krivelevich, and the first author and answers two questions about edge-isoperimetry in powers of regular graphs raised in their work.

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1 Introduction

Given a graph G with vertex set V, a key part of the (edge-)*isoperimetric problem* is to determine, for every $k \in \mathbb{N}$, the quantity

$$i_k(G) := \min \left\{ \frac{e_G(A, A^c)}{|A|} : A \subseteq V \wedge |A| = k \right\},$$

where $e_G(A, A^c)$ is the number of edges of G with exactly one endpoint in A. For more details about discrete isoperimetric problems, we refer the interested reader to the surveys [2, 3, 9].

In this note, we will consider the isoperimetric problem for *product graphs*. Instances of this problem have been studied in depth for several well-known product graphs, such as hypercubes [1, 8, 10, 11], Hamming graphs [11], grids, and tori [4]. Here, we will investigate the isoperimetric problem for arbitrary product graphs. The motivation for considering this problem in such generality comes partially from the results of (and the questions posed in) the recent work [6], where isoperimetric estimates played a crucial role in studying bond percolation on product graphs.

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Given a positive integer n and an arbitrary sequence of finite graphs $G_1, \ldots, G_n$, the product graph $G_1 \square \cdots \square G_n$ is the graph whose vertex set is $V(G_1) \times \cdots \times V(G_n)$ and whose edges are all pairs $\{u, v\}$ for which there is an index $j \in [\![n]\!]$ such that $u_j v_j \in E(G_j)$ and $u_m = v_m$ for all $m \neq j$.[1] In order to state our main result, we require the following definition. Given an m-vertex graph G, let $\psi_G : [0, \log m] \to [0, \infty)$ be the *convex minorant* of the function $\{\log k : k \in [\![m]\!]\} \ni x \mapsto i_{e^x}(G)$; in other words, ψ_G is the largest convex function satisfying $\psi_G(\log k) \leqslant i_k(G)$ for all $k \in [\![m]\!]$.[2] Observe that ψ_G is piecewise linear and that the only points where its derivative is not continuous are of the form $\log k$ for some integer $k \in [\![m]\!]$. Further, since ψ_G is a convex function defined on the interval $[0, \log m]$ and $\psi_G(x) \geqslant 0 = \psi_G(\log m) = i_m(G)$ for all $x \in [0, \log m]$, we have that ψ_G is decreasing.

Theorem 1. *Let n be a positive integer, let $G_1, \ldots, G_n$ be an arbitrary sequence of finite graphs, and let $\mathbf{G} := G_1 \square \cdots \square G_n$. For every $\emptyset \neq A \subseteq V(\mathbf{G})$,*

$$e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot \min \left\{ \sum_{i=1}^n \psi_{G_i}(h_i) : 0 \leqslant h_i \leqslant \log |V(G_i)| \wedge \sum_{i=1}^n h_i = \log |A| \right\}.$$

In particular, if $G_1 = \cdots = G_n = G$, then

$$e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot n \cdot \psi_G\big((\log |A|)/n\big).$$

Let us note that Theorem 1 gives a sharp bound for every n, every sequence $G_1, \ldots, G_n$, and sets A of many different sizes. To see this, assume first that $G_1 = \cdots = G_n = G$ for some graph G with m vertices. Consider arbitrary integers $k_1, k_2 \in [\![m]\!]$ with $k_1 < k_2$ such that $\psi_G(\log k_i) = i_{k_i}(G)$ for both $i \in [\![2]\!]$ and ψ_G is linear on $[\log k_1, \log k_2]$. Further, let $A_1, A_2 \subseteq V(G)$ be sets witnessing $|A_i| = k_i$ and $e_G(A_i, A_i^c) = i_{k_i}(G) \cdot k_i$ for both $i \in [\![2]\!]$. Then, for all nonnegative integers n_1 and n_2 satisfying $n_1 + n_2 = n$, the set $A := A_1^{n_1} \times A_2^{n_2} \subseteq V(G)^n$ satisfies

$$\frac{e_{G^n}(A, A^c)}{|A|} = n_1 \cdot i_{k_1}(G) + n_2 \cdot i_{k_2}(G) = n_1 \cdot \psi_G(\log k_1) + n_2 \cdot \psi_G(\log k_2)$$

$$= n \cdot \psi_G\left(\frac{n_1}{n} \cdot \log k_1 + \frac{n_2}{n} \cdot \log k_2\right) = n \cdot \psi_G\big((\log |A|)/n\big).$$

The above argument extends to product graphs that are not necessarily powers of a single graph. In this general case, the lower bound on $e_{\mathbf{G}}(A, A^c)$ is achieved by sets A of the form $A_1 \times \cdots \times A_n$, where $A_i \subseteq V(G_i)$ satisfy $e_{G_i}(A_i, A_i^c)/|A_i| = i_{|A_i|}(G_i) = \psi_{G_i}(\log |A_i|)$ and, writing $\partial_- \psi(x)$ and $\partial_+ \psi(x)$ for the left and the right derivatives of ψ at x,

$$\max_{i \in [\![n]\!]} \partial_- \psi_{G_i}(\log |A_i|) \leqslant \min_{i \in [\![n]\!]} \partial_+ \psi_{G_i}(\log |A_i|),$$

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[1] For $n \in \mathbb{N}$, we write $[\![n]\!]$ to denote $\{1, \ldots, n\}$.

[2] Here and throughout the paper, $\log$ denotes the natural logarithm.

where we use the convention that $\partial_{-}\psi_{G_i}(0) = -\infty$ and $\partial_{+}\psi_{G_i}(\log|V(G_i)|) = \infty$. It is not hard to check that the above assumption on left and right derivatives ensures that the minimum in the statement of Theorem 1 is achieved at $(h_1, \ldots, h_n) = (\log|A_1|, \ldots, \log|A_n|)$.

We close this section with an equivalent rephrasing of Theorem 1. Given a graph G, we first let $\phi_G \colon [1, |V(G)|] \to [1, \infty)$ be the function defined by $\phi_G(x) = e^{\psi_G(\log x)}$.

Theorem 2 (Theorem 1 rephrased). *Let n be a positive integer, let $G_1, \ldots, G_n$ be an arbitrary sequence of finite graphs, and let $\mathbf{G} \coloneqq G_1 \square \cdots \square G_n$. For every $\emptyset \neq A \subseteq V(\mathbf{G})$,*

$$e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot \log \left(\min \left\{ \prod_{i=1}^{n} \phi_{G_i}(k_i) : 1 \leqslant k_i \leqslant |V(G_i)| \wedge \prod_{i=1}^{n} k_i = |A| \right\} \right).$$

In particular, if $G_1 = \cdots = G_n = G$, then

$$e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot n \cdot \log \phi_G\big(|A|^{1/n}\big).$$

Organisation

In Section 2, we present the (short) proof of Theorem 1, and in Section 3, we discuss several applications of Theorem 1 and compare them with known results in the literature.

2 Proof of Theorem 1

Our argument builds on the beautiful entropy-based proof of an optimal edge-isoperimetric inequality for the hypercube presented by Boucheron, Lugosi, and Massart in [5, Section 4.4]. The *entropy* of a discrete random variable X taking values in a countable set $\mathcal{X}$ is the quantity $H(X)$ defined by

$$H(X) \coloneqq -\sum_{x \in \mathcal{X}} \mathbb{P}(X = x) \log \mathbb{P}(X = x).$$

In particular, if $\mathcal{X}$ is finite and X is uniform on $\mathcal{X}$, then $H(X) = \log|\mathcal{X}|$. Further, given random variables X and Y taking values in countable sets $\mathcal{X}$ and $\mathcal{Y}$, respectively, we define the *conditional entropy* of X given Y, denoted $H(X \mid Y)$, to be the average entropy of the random variable X conditioned on the outcome of Y; in other words,

$$H(X \mid Y) \coloneqq -\sum_{y \in \mathcal{Y}} \mathbb{P}(Y = y) \sum_{x \in \mathcal{X}} \mathbb{P}(X = x \mid Y = y) \log \mathbb{P}(X = x \mid Y = y).$$

We recall four well-known properties of entropy that will be used in the proof.

Fact 3. *Suppose that X, Y, and Z are discrete random variables and that X takes values in a finite set $\mathcal{X}$. We have:*

(i) $H(X) \leqslant \log|\mathcal{X}|$,

(ii) $H(X, Y) = H(X \mid Y) + H(Y),$

(iii) $H(X \mid Y) \leqslant H(X),$

(iv) $H(X \mid Y, Z) \leqslant H(X \mid Y).$

Let n be a positive integer and suppose that $\mathbf{G} = G_1 \square \cdots \square G_n$ for some arbitrary sequence $G_1, \ldots, G_n$ of finite graphs. Consider an arbitrary nonempty set $A \subseteq V(\mathbf{G}) = V(G_1) \times \cdots \times V(G_n)$ and let $X = (X_1, \ldots, X_n)$ be a uniformly chosen random vertex of A. For every $v \in V(\mathbf{G})$ and each $i \in [\![n]\!]$, denote by $v_{(i)}$ the projection of v along the ith coordinate, that is, $v_{(i)} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$. Further, given an $x \in A$, let $A_i(x) \subseteq V(G_i)$ denote the support of X_i conditioned on $X_{(i)} = x_{(i)}$. Our first key observation is that

$$e_{\mathbf{G}}(A, A^c) = \sum_{x \in A} \sum_{i=1}^{n} \frac{e_{G_i}\left(A_i(x), A_i(x)^c\right)}{|A_i(x)|} \geqslant \sum_{x \in A} \sum_{i=1}^{n} i_{|A_i(x)|}(G_i).$$

Denoting by k_i the (random) size of $|A_i(X)|$, we may rewrite the above inequality as

$$e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot \sum_{i=1}^{n} \mathbb{E}[i_{k_i}(G_i)]. \tag{1}$$

By the definition of ψ_{G_i} and by Jensen's inequality, we have, for every $i \in [\![n]\!]$,

$$\mathbb{E}[i_{k_i}(G_i)] \geqslant \mathbb{E}[\psi_{G_i}(\log k_i)] \geqslant \psi_{G_i}\left(\mathbb{E}[\log k_i]\right).$$

Our second key observation is that $\mathbb{E}[\log k_i]$ is the expectation of the logarithm of the size of the support of X_i conditioned on $X_{(i)}$. Since X_i conditioned on $X_{(i)}$ is a uniformly chosen random element of its support, this equals the average entropy of X_i conditioned on $X_{(i)}$, which is precisely the conditional entropy $H(X_i \mid X_{(i)})$. Substituting the above inequality into (1), we conclude that

$$e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot \sum_{i=1}^{n} \psi_{G_i}\left(H(X_i \mid X_{(i)})\right).$$

The main assertion of the theorem now follows as, for each $i \in [\![n]\!]$, the function ψ_{G_i} is decreasing, $0 \leqslant H(X_i \mid X_{(i)}) \leqslant H(X_i) \leqslant \log |V(G_i)|$ for each i, and

$$\sum_{i=1}^{n} H(X_i \mid X_{(i)}) = \sum_{i=1}^{n} H(X_i \mid X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$$
$$\leqslant \sum_{i=1}^{n} H(X_i \mid X_1, \ldots, X_{i-1}) = H(X) = \log |A|.$$

(This inequality is known as Han's inequality [7], see [5, Theorem 4.1] for a compact statement.) Finally, if $G_1 = \cdots = G_n = G$, then we may use the convexity of ψ_G again to deduce that, for all sequences $(h_i)_{i=1}^n$ that sum to $\log|A|$,

$$\sum_{i=1}^n \psi_G(h_i) \geqslant n \cdot \psi_G\left(\sum_{i=1}^n \frac{h_i}{n}\right) = n \cdot \psi_G\left(\frac{\log|A|}{n}\right),$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3 Applications

3.1 Hamming graphs and the hypercube

Let K_m be the complete graph on m vertices, so that $\mathbf{G} \coloneqq K_m^n$ is the Hamming graph $H(n,m)$. Since $x \mapsto \log_m x$ is concave, we have that $(k-1)/(m-1) \leqslant \log_m k$ for all $k \in [\![m]\!]$; indeed, the secant line $y = (x-1)/(m-1)$ passing through the points $(1,0)$ and $(m,1)$ on the graph of $x \mapsto \log_m x$ lies below the graph on the entire interval $[1,m]$. Equivalently, $i_k(K_m) = m - k \geqslant (m-1) \cdot (1 - \log_m k)$ for all $k \in [\![m]\!]$, and thus

$$\psi_{K_m}(x) \geqslant (m-1) \cdot (1 - x/\log m) \tag{2}$$

for all $x \in [0, \log m]$. Therefore, by Theorem 1, for all nonempty $A \subseteq V(H(n,m))$, we have

$$e_{H(n,m)}(A, A^c) \geqslant |A| \cdot (m-1)\,(n - \log_m |A|). \tag{3}$$

Observe that (3) is sharp whenever A induces a copy of $H(t,m)$ for some $t \in [\![n]\!]$. In this sense, one may view it as a weak version of the edge-isoperimetric inequality for Hamming graphs due to Lindsey [11].[3] In particular, the case $m = 2$, may be viewed as a weak version of the edge-isoperimetric inequality for the hypercube [1, 8, 10, 11].

3.2 The grid

Let P_m be the path with $m \geqslant 3$ vertices, so that $\mathbf{G} \coloneqq P_m^n$ is the n-dimensional $m \times \cdots \times m$ grid. Note that $i_k(P_m) = 1/k$ for every $k \in [\![m-1]\!]$ and that $i_m(P_m) = 0$. For every $z \in [0, \log m)$, let ℓ_z be the line passing through the points (z, e^{-z}) and $(\log m, 0)$, that is, the line $y = e^{-z} \cdot (\log m - x)/(\log m - z)$. Since the points $\{(\log k, 1/k) : k \in [\![m-1]\!]\}$ lie on the graph of the convex function $x \mapsto e^{-x}$ and $\ell_{\log m-1}$ has the largest (that is, least negative) slope among all our lines ℓ_z, we may deduce that

$$\psi_{P_m}(x) \geqslant \begin{cases} e^{-x} & \text{if } 0 \leqslant x \leqslant \log m - 1, \\ e/m \cdot (\log m - x) & \text{if } \log m - 1 \leqslant x \leqslant \log m. \end{cases} \tag{4}$$

In fact, ψ_{P_m} is the piecewise linear function defined by the points $(0,1), \ldots, (\log k^*, 1/k^*)$, and $(\log m, 0)$, where $k^* \in [\![m-1]\!]$ is the index k for which $\ell_{\log k}$ has the largest slope.

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[3]Lindsey's inequality is the stronger statement that each initial interval in the lexicographic ordering of $[\![m]\!]^n$ has the smallest edge-boundary among all sets of the same size.

It is not hard to see that $k^* \in \{\lfloor m/e \rfloor, \lceil m/e \rceil\}$, but whether it is the floor or the ceiling of m/e depends on the value of m. For example, $k^* = \lfloor 3/e \rfloor = 1$ when $m = 3$, whereas $k^* = \lceil 5/e \rceil = 2$ when $m = 5$.

With the lower bound (4) in place, we can now use Theorem 1 to derive edge-isoperimetric inequalities for $\mathbf{G}$. When $|A| \leqslant (m/e)^n$, we have

$$e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot n \cdot e^{-(\log|A|)/n} = n \cdot |A|^{1-1/n}$$

and when $(m/e)^n \leqslant |A| \leqslant m^n$, we have

$$e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot n \cdot \frac{e}{m}\big(\log m - (\log|A|)/n\big) = \frac{|A|}{m} \cdot e \log \frac{m^n}{|A|}.$$

For comparison, Bollobás and Leader [4] showed that, for all $A \subseteq V(\mathbf{G})$ with $|A| \leqslant m^n/2$,

$$e_{\mathbf{G}}(A, A^c) \geqslant \frac{|A|}{m} \cdot \min\left\{ r \cdot \left(\frac{m^n}{|A|}\right)^{1/r} : r \in [\![n]\!] \right\}. \tag{5}$$

Since the minimum above is achieved at $r = n$ whenever $|A| \leqslant (m/e)^n$, our bound matches (5) in this range. In the complementary range $(m/e)^n \leqslant |A| \leqslant m^n/2$, the ratio between the two bounds does not exceed

$$\sup\left\{ \frac{\lceil \log x \rceil \cdot x^{1/\lceil \log x \rceil}}{e \log x} : 2 \leqslant x \leqslant e^n \right\} \leqslant \sup\left\{ \frac{e^{y-1}}{y} : 1/2 \leqslant y \leqslant 1 \right\} = 2e^{-1/2} \leqslant 1.214,$$

where in the first inequality we substituted $y = \log x/\lceil \log x \rceil$ and used that $1/2 \leqslant \log x/\lceil \log x \rceil \leqslant 1$ for all $x \geqslant 2$.

3.3 The torus

Let C_m be the cycle with m vertices, so that $\mathbf{G} := C_m^n$ is the n-dimensional discrete torus with side length m. Since $i_k(C_m) = 2i_k(P_m)$ for all $k \in [\![m]\!]$, we have $\psi_{C_m} = 2\psi_{P_m}$. Thus, Theorem 1 and the estimate (4) yield

$$e_{\mathbf{G}}(A, A^c) \geqslant \begin{cases} 2n \cdot |A|^{1-1/n} & \text{if } |A| \leqslant (m/e)^n, \\ |A|/m \cdot 2e \log(m^n/|A|) & \text{if } |A| \geqslant (m/e)^n. \end{cases}$$

For comparison, Bollobás and Leader [4] showed that, for all $A \subseteq V(\mathbf{G})$ with $|A| \leqslant m^n/2$,

$$e_{\mathbf{G}}(A, A^c) \geqslant \frac{|A|}{m} \cdot \min\left\{ 2r \left(\frac{m^n}{|A|}\right)^{1/r} : r \in [\![n]\!] \right\}, \tag{6}$$

and hence, as in the case of grid graphs, our bound matches (6) whenever $|A| \leqslant (m/e)^n$ and is off by a multiplicative factor of at most $2e^{-1/2}$ in the complementary range.

3.4 Products of regular graphs

For every $i \in [\![n]\!]$, let G_i be a d_i-regular graph on m_i vertices, let $\mathbf{G} := G_1 \square \cdots \square G_n$, and note that $\mathbf{G}$ is also regular of degree $d := d_1 + \cdots + d_n$. Note that the assumption that G_i is d_i-regular implies that $i_k(G_i) \geqslant d_i - k + 1 = i_k(K_{d_i+1})$ for all $k \in [\![d_i + 1]\!]$; indeed, $\deg_{G_i}(v, A^c) \geqslant d_i - |A| + 1$ for all $v \in A \subseteq V(G_i)$. Consequently, $\psi_{G_i}(x) \geqslant d_i \cdot (1 - x/\log(d_i + 1))$ for all $x \in [0, \log m_i]$, see (2). Thus, by Theorem 1,

$$
e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot \left(d - \max \left\{ \sum_{i=1}^{n} \frac{d_i \cdot h_i}{\log(d_i + 1)} : 0 \leqslant h_i \leqslant \log m_i \wedge \sum_{i=1}^{n} h_i = \log |A| \right\} \right)
$$

$$
\geqslant |A| \cdot \left(d - \max_{i \in [\![n]\!]} \frac{d_i}{\log(d_i + 1)} \cdot \log |A| \right) = |A| \cdot \left(d - D \cdot \log_{D+1} |A| \right),
$$

where $D := \max_{i \in [\![n]\!]} d_i$. This substantially improves [6, Theorem 1].

Assume further that each G_i is connected, so that $i_k(G_i) \geqslant i_k(P_{m_i})$ for all $k \in [\![m_i]\!]$. It follows from (4) that $\psi_{G_i}(x) \geqslant e/m_i \cdot (\log m_i - x)$ for all $x \in [0, \log m_i]$. Therefore, by Theorem 1,

$$
e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot \min \left\{ \sum_{i=1}^{n} \frac{e}{m_i} \cdot (\log m_i - h_i) : 0 \leqslant h_i \leqslant \log m_i \wedge \sum_{i=1}^{n} h_i = \log |A| \right\}
$$

$$
\geqslant |A| \cdot \min \left\{ \sum_{i=1}^{n} \frac{e g_i}{m_i} : g_i \geqslant 0 \wedge \sum_{i=1}^{n} g_i = \log \frac{|V(\mathbf{G})|}{|A|} \right\} = |A| \cdot \frac{e}{M} \cdot \log \frac{|V(\mathbf{G})|}{|A|},
$$

where $M := \max_{i \in [\![n]\!]} m_i$. When $M \geqslant 3$, this improves the respective lower bound on $e_{\mathbf{G}}(A, A^c)$ given by [6, Theorem 2] by a multiplicative factor of $e(1 - 1/M) \log M$.

3.5 Graph powers

Let G be a connected m-vertex graph with minimum degree d, and let $\mathbf{G} := G^n$. For every $k \in [\![m - 1]\!]$, let ℓ_k be the line passing through $(\log k, i_k(G))$ and $(\log m, 0)$, that is, the line $y = i_k(G) \cdot (\log m - x)/(\log m - \log k)$. Let k^* be the smallest index k such that ℓ_k has the least negative slope among all our lines and note that, for all $x \in [0, \log m]$,

$$
\psi_G(x) \geqslant i_{k^*}(G) \cdot \frac{\log m - x}{\log m - \log k^*}. \tag{7}
$$

Let y_G be the y-intercept of ℓ_{k^*}. Note that $y_G \leqslant d$ (as $i_1(G) = d$) and that $y_G = d$ if and only if $k^* = 1$. Further, observe that $y_G = i_{k^*}(G) \cdot \log m/(\log m - \log k^*)$. Hence, by Theorem 1,

$$
e_{\mathbf{G}}(A, A^c) \geqslant |A| \cdot \frac{i_{k^*}(G)}{\log m - \log k^*} \cdot \log \frac{m^n}{|A|} = |A| \cdot y_G \cdot (n - \log_m |A|). \tag{8}
$$

Since (7) holds with equality for all $x \in [\log k^*, \log m]$, inequality (8) is tight for sets A with many different sizes, see the construction described below the statement of Theorem 1.

We now address two questions posed in [6]. First, [6, Question 7.1] asked whether there are constants c_G, C_G such that $i_a(\mathbf{G}) = c_G \cdot \log(m^n/a) + C_G$ for all $a \in [\![m^n]\!]$ (there the assumption is that the graph G is d-regular). In other words, [6, Question 7.1] asks whether $i_a(\mathbf{G})$ is essentially linear in $\log a$. The construction presented below the statement Theorem 1 shows that the lower bound on $i_a(\mathbf{G})$ implied by the theorem is sharp whenever $\log a = (n_1/n) \cdot \log k_1 + (n_2/n) \cdot \log k_2$ for some n_1, n_2 satisfying $n_1 + n_2 = n$ and $k_1, k_2 \in [\![m]\!]^2$ such that $[\log k_1, \log k_2]$ supports one of the linear pieces of ψ_G. This fact implies that $i_a(\mathbf{G})$ is not linear in $\log a$ whenever ψ_G itself is not linear. Since there are regular graphs G for which ψ_G has more than one linear piece (for example, when $G = C_m$ for $m \geqslant 5$), the answer to [6, Question 7.1] is negative.

Further, [6, Question 7.2] asked for a characterisation of m-vertex d-regular graphs G for which sets of the form $B_t := \{u\}^t \times V(G)^{n-t}$ have the smallest edge-boundary among all m^{n-t}-element sets of vertices of $\mathbf{G}$, for all $t \in [\![n]\!]$. We note that this is closely related to the classical problem of finding sufficient conditions for a graph to admit a nested sequence of sets that achieve the smallest edge-boundary (among all sets of a given size), see [3, 9] and references therein. Since

$$e_{\mathbf{G}}(B_t, B_t^c) = |B_t| \cdot t \cdot d = |B_t| \cdot \frac{d}{\log m} \cdot \log \frac{m^n}{|B_t|},$$

it follows from (8) that a sufficient condition is $y_G = d$. We will show below that, for large enough n, this is also a necessary condition.

Suppose that G is an m-vertex, d-regular graph with $y_G < d$, let $k^* \in \{2, \dots, m-1\}$ be the index defined above, and let $S \subseteq V(G)$ be a k^*-element set witnessing $e_G(S, S^c) = |S| \cdot i_{k^*}(G)$. Fix a small positive ε. By Dirichlet's approximation theorem, there exist positive integers s and t such that

$$\left| s \log m - t \log(m/k^*) \right| \leqslant \varepsilon/2, \tag{9}$$

which implies that $(1 - \varepsilon)m^t \leqslant (k^*)^t \cdot m^s \leqslant (1 + \varepsilon)m^t$. Consider the graph $\mathbf{G} := G^{s+t}$ and sets of vertices $A := S^t \times V(G)^s$ and $B := \{u\}^s \times V(G)^t$. Note that, by (9),

$$e_{\mathbf{G}}(A, A^c) = |A| \cdot t \cdot i_{k^*}(G) = |A| \cdot \frac{y_G}{\log m} \cdot t \log(m/k^*) \leqslant |A| \cdot y_G \cdot (s + \varepsilon) \leqslant (1 + 2\varepsilon) \cdot m^t \cdot y_G \cdot s.$$

Let C be a set of size exactly m^t that is obtained by adding to / removing from A at most εm^t vertices in an arbitrary manner. Since $\Delta(\mathbf{G}) = (s + t)d$, we clearly have

$$e_{\mathbf{G}}(C, C^c) - e_{\mathbf{G}}(A, A^c) \leqslant \varepsilon m^t \cdot (s + t)d \leqslant \varepsilon m^t sd \left(1 + \frac{\log m + \varepsilon/2}{\log(m/k^*)} \right),$$

where the second inequality follows from (9). Since we assumed that $y_G < d$, it is clear that choosing ε sufficiently small (as a function of m and $d - y_G$ only) gives $e_{\mathbf{G}}(C, C^c) < m^t sd = e_{\mathbf{G}}(B, B^c)$. This means that the set B does not have the smallest edge boundary among all sets of m^t vertices of $\mathbf{G}$.

Acknowledgements

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