Flag-Transitive Point-Primitive Quasi-Symmetric 2-Designs and Exceptional Groups of Lie Type

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Abstract

Let $\mathcal{D}$ be a non-trivial quasi-symmetric 2-design with two block intersection numbers $x = 0$ and $2 \leqslant y \leqslant 10$, and suppose that G is an automorphism group of $\mathcal{D}$. If G is flag-transitive and point-primitive, then it is known that G is either of affine type or almost simple type. In this paper, we show that the socle of G cannot be a finite simple exceptional group of Lie type.

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1 Introduction

A 2-(v, k, λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a finite incidence structure with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of blocks such that each block contains k points and each two distinct points are contained in λ blocks. It is *non-trivial* if $2 < k < v - 1$. All the 2-designs in this paper are assumed to be non-trivial. The replication number r of $\mathcal{D}$ is the number of blocks containing a given point. The number of blocks is conventionally denoted by b. If $b = v$, we say that $\mathcal{D}$ is *symmetric*. Let $\mathcal{D}$ be a 2-(v, k, λ) design with blocks $B_1, B_2, \cdots, B_b$. The cardinality $|B_i \cap B_j|$ is called a block intersection number of $\mathcal{D}$. If D is symmetric, then D has only one block intersection number, namely λ. Those 2-designs with two block intersection numbers are called *quasi-symmetric*. This concept goes back to [36]. Let x, y denote the two block intersection numbers of a quasi-symmetric design with the standard convention that $x \leqslant y$. We say that quasi-symmetric design is proper if $x < y$ and improper if $x = y$. There are many well known examples of proper quasi-symmetric 2-designs. For example, a linear space with $b > v$ is a quasi-symmetric design with $x = 0$ and $y = 1$. We refer to [30, 37] for more details and additional examples of quasi-symmetric designs.

An *automorphism* of $\mathcal{D}$ is a permutation of the points which preserves the blocks. We write $\mathrm{Aut}(\mathcal{D})$ for the full automorphism group of $\mathcal{D}$, and call its subgroups as automorphism groups. A *flag* of $\mathcal{D}$ is an incident point-block pair. We say that an automorphism

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group G is *flag-transitive* if it acts transitively on the flags of $\mathcal{D}$. If G acts primitively on the points of $\mathcal{D}$, then G is said to be *point-primitive.* There have been extensive works on the classification of flag-transitive point-primitive 2-designs. In 1990, Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [9] classified all flag-transitive linear spaces apart from those with a one-dimensional affine automorphism group. Through a series of papers [31, 32, 33, 34], Regueiro gave the classification of biplanes ($\lambda = 2$). However, moving to the non-symmetric case poses additional challenges, see [1, 4, 14, 20, 21, 22]. More recent and interesting classification results are provided in [19, 28, 29].

This paper is a contribution to the study of quasi-symmetric 2-designs admitting a flag-transitive point-primitive automorphism group. In [26], we proved that for a non-trivial quasi-symmetric 2-design $\mathcal{D}$ with two block intersection numbers $x = 0$ and $2 \leqslant y \leqslant 10$, if $G \leqslant \mathrm{Aut}(\mathcal{D})$ is flag-transitive and point-primitive, then G is either of affine type or almost simple type. Moreover, we proved that the socle of G cannot be an alternating group, and gave the classification results for the case where the socle of G is a sporadic simple group. In this paper, we further investigate the case where the socle of G is a finite simple exceptional group of Lie type. The following are our main results.

Theorem 1. *Let $\mathcal{D}$ be a non-trivial quasi-symmetric 2-design with block intersection numbers $x = 0$ and $2 \leqslant y \leqslant 10$. If $G \leqslant \mathrm{Aut}(\mathcal{D})$ is flag-transitive and point-primitive, then the socle of G cannot be a finite simple exceptional group of Lie type.*

Combining Theorem 1 with [26, Theorem 1.2, 1.4], one obtains the following corollary.

Corollary 2. *Let $\mathcal{D}$ be a non-trivial quasi-symmetric 2-design with block intersection numbers $x = 0$ and $2 \leqslant y \leqslant 10$. Let $G \leqslant \mathrm{Aut}(\mathcal{D})$ be flag-transitive and point-primitive. If the socle of G is not a finite simple classical group, then $\mathcal{D}$ and G are one of the following:*

(1) $\mathcal{D}$ is the unique 2-$(12, 6, 5)$ design with block intersection numbers 0 and 3, $G = \mathrm{M}_{11}$;

(2) $\mathcal{D}$ is the unique 2-$(22, 6, 5)$ design with block intersection numbers 0 and 2, $G = \mathrm{M}_{22}$ or $\mathrm{M}_{22} : 2$.

In [35], Saxl completed the classification of the linear spaces admitting almost simple flag-transitive automorphism groups. Combining Theorem 1 with the result in [35], one obtains the following corollary.

Corollary 3. *Let $\mathcal{D}$ be a non-trivial quasi-symmetric 2-design with block intersection numbers $x = 0$ and $y \leqslant 10$. If $G \leqslant \mathrm{Aut}(\mathcal{D})$ is flag-transitive and point-primitive with exceptional socle of Lie type, then $\mathcal{D}$ is a Ree unital and $\mathrm{Soc}(G) = {}^2G_2(q)$ with $q = 3^{2n+1}$.*

This paper is organized as follows. In Section 2, we present some preliminary results on flag-transitive 2-designs and the maximal subgroups of almost simple groups with exceptional socle of Lie type. Then the proof of Theorem 1 is presented in Section 3.

2 Preliminaries

We first present some preliminary results on flag-transitive 2-designs which are used throughout the paper.

Lemma 4. *[10, 1.2 and 1.9] Let $\mathcal{D}$ be a 2-design with the parameters (v, b, r, k, λ). Then the following hold:*

(i) $r(k-1) = \lambda(v-1)$, $r > \lambda$;

(ii) $vr = bk$;

(iii) $b \geqslant v$, $k \leqslant r$. *If $\mathcal{D}$ is non-symmetric, then $b > v$ and $k < r$.*

Lemma 5. *[40, Lemma 2.4] Let $\mathcal{D}$ be a 2-design with the parameters (v, b, r, k, λ), α be a point of $\mathcal{D}$ and G be a flag-transitive automorphism group of $\mathcal{D}$. Then the following hold:*

(i) $|G_\alpha|^3 > \lambda |G|$;

(ii) $r \mid \lambda(v-1, |G_\alpha|)$, *where G_α is the stabilizer of α in G;*

(iii) If d is any non-trivial subdegree of G, then $r \mid \lambda d$. In particular, $\frac{r}{(r,\lambda)} \mid (v-1, d)$.

It is well known that $\mathcal{D}$ is a quasi-symmetric design with intersection numbers 0 and 1 if and only if $\mathcal{D}$ is a finite linear space with $b > v$ (see [37, Chapter 3]). For the linear space $\mathcal{D}$ with a flag-transitive automorphism group G, we know that G is point-primitive [16]. Moreover, G is either almost simple or of affine type [8]. If the socle of G is a finite simple exceptional group of Lie type, then $\mathcal{D}$ is a Ree unital and $\mathrm{Soc}(G) = {}^2G_2(q)$ with $q = 3^{2n+1}$ (see [13, 17, 35]). Thus we suppose that $\mathcal{D}$ is a quasi-symmetric 2-design with block intersection numbers $x = 0$ and $y \geqslant 2$ in the following. Then we have the following lemma.

Lemma 6. *Let $\mathcal{D}$ be a non-trivial quasi-symmetric 2-design with block intersection numbers $x = 0$ and $y \geqslant 2$. Then the following relations hold:*

(i) $(y-1)(r-1) = (k-1)(\lambda-1)$;

(ii) $b \leqslant v(v-1)/k$, $r \leqslant v-1$, $y < \lambda \leqslant k-1$;

(iii) $y \mid k$, $y \mid (r-\lambda)$;

(iv) $v < \frac{k^2-k}{y-1}$;

(v) $(y-1) \cdot \frac{r^2}{\lambda^2} < v-1 < 2(y-1) \cdot \frac{r^2}{\lambda^2}$. *In particular,* $v \leqslant 2(y-1) \cdot \frac{r^2}{(r,\lambda)^2}$.

Proof. It suffices to prove part (v) since other properties are taken from [26, Lemma 2.3]. Note that

$$(y-1)\cdot\frac{r}{\lambda} < k-1 = (y-1)\cdot\frac{r-1}{\lambda-1} < 2(y-1)\cdot\frac{r}{\lambda}$$

and $k-1=(v-1)\cdot\frac{\lambda}{r}$ by Lemma 4(i) and Lemma 6(i). Then we obtain

$$(y-1)\cdot\frac{r}{\lambda} < k-1 = (v-1)\cdot\frac{\lambda}{r} < 2(y-1)\cdot\frac{r}{\lambda},$$

proving part (v). □

To prove Theorem 1, we require some results on maximal subgroups of almost simple groups with exceptional socle of Lie type. The *socle* of a finite group is defined as the subgroup generated by all the minimal normal subgroups. A group G is said to be *almost simple* with socle X if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$, where X is a non-abelian simple group. Throughout this paper, we denote by $[n]$ a group of order n. We also adopt the standard Lie notation for groups of Lie type. Specifically, we write $A_{n-1}(q)$ and $A_{n-1}^{-}(q)$ in place of $\mathrm{PSL}(n,q)$ and $\mathrm{PSU}(n,q)$ respectively, and $B_n(q)$, $C_n(q)$, $D_n^{\pm}(q)$, and $E_6^{-}(q)$ instead of $\mathrm{P\Omega}(2n+1,q)$, $\mathrm{PSp}(2n,q)$, $\mathrm{P\Omega}^{\pm}(2n,q)$, and ${}^2E_6(q)$ respectively. Further notation in group theory can be found in [15, 18]. For a positive integer n and prime number p, let n_p denote the p-part of n, that is, $n_p = p^t$ where $p^t \mid n$ but $p^{t+1} \nmid n$. The first lemma is an elementary result on subgroups of almost simple groups.

Lemma 7. *[3, Lemma 3.1] Let G be an almost simple group with socle X, and let H be maximal in G not containing X. Then $G = HX$, and $|H|$ divides $|\mathrm{Out}(X)|\cdot|H\cap X|$.*

A proper subgroup H of G is said to be large if $|H|^3 \geqslant |G|$. In [5], Alavi and Burness determined all the large maximal subgroups of finite simple groups. Furthermore, in [3], Alavi, Bayat and Daneshkhah determined all the large maximal subgroups of almost simple exceptional groups of Lie type.

Lemma 8. *[3, Theorem 1.2] Let G be a finite almost simple group whose socle X is a finite simple exceptional group of Lie type, and let H be a maximal subgroup of G not containing X. If H is a large subgroup of G, then H is either parabolic, or one of the subgroups listed in Table 1.*

Remark 9. (i) For the orders of simple groups of Lie type, one can refer to, for example, [18, Table 5.1.A, B].

(ii) In Table 1, the type of H is an approximate description of the group-theoretic structure of H. For precise structure, one may refer to [7, Table 8.16] for ${}^2B_2(q)$, [7, Table 8.43] for ${}^2G_2(q)$, [7, Table 8.51] for ${}^3D_4(q)$, [27] for ${}^2F_4(q)$, [38, Table 4.1] for $G_2(q)$, [11] for $F_4(q)$ and $E_6^{\epsilon}(q)$, [12] for $E_7(q)$ and [25, Table 5.1] for $E_8(q)$.

Lemma 10. *[24, 3.9] If X is a group of Lie type in characteristic p, acting on the set of cosets of a maximal parabolic subgroup, and X is not $A_{n-1}(q)$, $D_n(q)$ (with n odd) and $E_6(q)$, then there is a unique subdegree which is a power of p.*

Table 1: Large maximal non-parabolic subgroups H of G with socle X in Lemma 8.

X	$H \cap X$ or type of H	Conditions
${}^2B_2(q)\,(q=2^{2n+1} \geqslant 8)$	$(q+\sqrt{2q}+1):4$	$q=8,32$
	${}^2B_2\left(q^{1/3}\right)$	$q>8, 3 \mid 2n+1$
${}^2G_2(q)\,(q=3^{2n+1} \geqslant 27)$	$A_1(q)$	
	${}^2G_2\left(q^{1/3}\right)$	$3 \mid 2n+1$
${}^3D_4(q)$	$A_1\left(q^3\right)A_1(q), \left(q^2+\epsilon q+1\right)A_2^{\epsilon}(q), G_2(q)$	$\epsilon=\pm$
	${}^3D_4\left(q^{1/2}\right)$	q square
	$7^2:\mathrm{SL}(2,3)$	$q=2$
${}^2F_4(q)\,(q=2^{2n+1} \geqslant 8)$	${}^2B_2(q) \wr 2, B_2(q):2, {}^2F_4\left(q^{1/3}\right)$	
	$\mathrm{SU}(3,q):2, \mathrm{PGU}(3,q):2$	$q=8$
	$A_2(3):2, A_1(25)$, $\mathrm{A}_6.2^2$, $5^2:4\mathrm{A}_4$	$q=2$
$G_2(q)$	$A_2^{\pm}(q), A_1(q)^2, G_2\left(q^{1/r}\right)$	$r=2,3$
	${}^2G_2(q)$	$q=3^a, a$ is odd
	$G_2(2)$	$q=5,7$
	$A_1(13), \mathrm{J}_2$	$q=4$
	J_1	$q=11$
	$2^3.A_2(2)$	$q=3,5$
$F_4(q)$	$B_4(q), D_4(q), {}^3D_4(q)$	
	$F_4\left(q^{1/r}\right)$	$r=2,3$
	$A_1(q)C_3(q)$	$p \neq 2$
	$C_4(q), C_2\left(q^2\right), C_2(q)^2$	$p=2$
	${}^2F_4(q)$	$q=2^{2n+1} \geqslant 2$
	${}^3D_4(2)$	$q=3$
	$\mathrm{A}_9, \mathrm{A}_{10}, A_3(3), \mathrm{J}_2, \mathrm{S}_6 \wr \mathrm{S}_2$	$q=2$
	$A_1(q)G_2(q)$	$q>3$ odd
$E_6^{\epsilon}(q)$	$A_1(q)A_5^{\epsilon}(q), F_4(q)$	
	$(q-\epsilon)D_5^{\epsilon}(q)$	$\epsilon=-$
	$C_4(q)$	$p \neq 2$
	$E_6^{\pm}\left(q^{1/2}\right)$	$\epsilon=+$
	$E_6^{\epsilon}\left(q^{1/3}\right)$	
	$(q-\epsilon)^2.D_4(q)$	$(\epsilon,q) \neq (+,2)$
	$\left(q^2+\epsilon q+1\right).{}^3D_4(q)$	$(\epsilon,q) \neq (-,2)$
	$\mathrm{J}_3, \mathrm{A}_{12}, B_3(3), \mathrm{Fi}_{22}$	$(\epsilon,q)=(-,2)$
$E_7(q)$	$(q-\epsilon)E_6^{\epsilon}(q), A_1(q)D_6(q), A_7^{\epsilon}(q), A_1(q)F_4(q), E_7\left(q^{1/r}\right)$	$\epsilon=\pm$ and $r=2,3$
	Fi_{22}	$q=2$
$E_8(q)$	$A_1(q)E_7(q), D_8(q), A_2^{\epsilon}(q)E_6^{\epsilon}(q), E_8\left(q^{1/r}\right)$	$\epsilon=\pm$ and $r=2,3$

3 Proof of Theorem 1

Throughout this section, we assume the following hypothesis.

Hypothesis 11. Let $\mathcal{D}$ be a non-trivial quasi-symmetric 2-design with block intersection numbers $x = 0$ and $2 \leqslant y \leqslant 10$, let $G \leqslant \mathrm{Aut}(\mathcal{D})$ be flag-transitive and point-primitive with socle X a finite simple exceptional group of Lie type. Let α be a point of $\mathcal{D}$ and $H = G_\alpha$.

Before proving the main theorem, we introduce the following Computational methods through two examples.

3.1 Computational methods

Our methods are based on Lemma 6(v) and Lemma 5(ii)(iii). For most of the large maximal non-parabolic subgroups listed in Table 1, we can use Lemma 5(ii) to obtain upper bounds on $\frac{r}{(r,\lambda)}$. Similarly, for parabolic subgroups, we use Lemma 5(iii). Combining these upper bounds with the lower bounds on $\frac{r}{(r,\lambda)}$ derived from Lemma 6(v), we can deduce that only a few small values of q satisfy the conditions.

Example 12. Suppose that $X = F_4(q)$ and $H \cap X = X_\alpha$ is a maximal subgroup of type ${}^3D_4(q)$, where $q = p^f$ for some prime p and positive integer f. According to [11, Table 7], we know that $X_\alpha = {}^3D_4(q).3$. Note that $|X| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$ and $|X_\alpha| = 3q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$ (see [18, Table 5.1.B]). Then $v = |X|/|X_\alpha| = \frac{1}{3}q^{12}(q^8-1)(q^4-1)$. Next, we consider $3(v-1)$ and $|X_\alpha|$ as polynomials in q over the rational field. Using the Magma [6] command **XGCD**, we obtain three rational polynomials $f(q)$, $g(q)$ and $h(q)$ such that

$$f(q) \cdot 3(v-1) + g(q) \cdot |X_\alpha| = h(q).$$

For this example, computation in Magma [6] shows that

$$h(q) = q^8 + q^4 + 1,$$

$$f(q) = -\frac{5}{183}q^{18} + \frac{1}{183}q^{16} + \frac{13}{183}q^{14} - \frac{3}{61}q^{12} - \frac{1}{3}q^8 - \frac{1}{3}q^4 - \frac{1}{3},$$

$$g(q) = \frac{5}{549}q^{14} + \frac{4}{549}q^{12} - \frac{19}{549}q^{10} - \frac{1}{183}q^8 + \frac{22}{549}q^6 + \frac{14}{183}q^4 + \frac{73}{549}q^2 + \frac{34}{549}.$$

Notice that the least common multiple of the denominators of all coefficients of $f(q)$ and $g(q)$ is 549. Then we deduce that $(3(v-1), |X_\alpha|) \leqslant 549(q^8+q^4+1)$. Since $|\mathrm{Out}(X)| = (2,p)f$, we have $\frac{r}{(r,\lambda)} \leqslant (v-1, |G_\alpha|) \leqslant (3(v-1), |X_\alpha| \cdot |\mathrm{Out}(X)|) \leqslant 1098f \cdot (q^8+q^4+1)$ by Lemma 5(ii) and Lemma 7. Furthermore, from Lemma 6(v), we deduce that $v \leqslant 2(y-1) \cdot \frac{r^2}{(r,\lambda)^2} \leqslant 18 \cdot (1098f \cdot (q^8+q^4+1))^2$, i.e.,

$$\frac{1}{3}q^{12}(q^8-1)(q^4-1) \leqslant 18 \cdot \left(1098f \cdot (q^8+q^4+1)\right)^2.$$

This is an inequality on q, and one can show that it holds only for $q \leqslant 9$ by Magma [6]. Then for each prime power $q \leqslant 9$, we can compute the value $a := (v-1, |X_\alpha| \cdot |\mathrm{Out}(X)|)$. Finally, we substitute the values v, a back into the inequality $v \leqslant 18a^2$, and find that the inequality does not hold, leading to a contradiction.

For parabolic subgroups, we use similar processing method.

Example 13. Suppose that $X = G_2(q)$ and $X_\alpha = [q^5] : \mathrm{GL}(2, q)$ is a parabolic subgroup, where $q = p^f$ for some prime p and positive integer f. Note that $G_2(2)$ is not a simple group. Thus we assume that $q > 2$. Then $v = |X|/|X_\alpha| = (q^6-1)/(q-1)$ and so $(v-1)_p = q$. By Lemma 10, there is a subdegree d which is a power of p. Consequently, $(v-1, d) = q$. Furthermore, by Lemma 5(iii) and Lemma 6(v), we have $(q^6-1)/(q-1) = v \leqslant 18 \cdot \frac{r^2}{(r,\lambda)^2} \leqslant 18q^2$ and so $q = 2$, which is a contradiction.

In [26, Section 3.1], the authors presented a computational method to search for positive integers (v, b, r, k, λ) as potential parameters for the design $\mathcal{D}$. This method is convenient for calculating some small values of v, such as sporadic cases in Table 1. For completeness, we will repeat it in the following paragraph.

Given the value of v, we use Magma [6] to search for positive integers (v, b, r, k, λ) that satisfy the following conditions:

$$2 \leqslant y \leqslant 10, \tag{1}$$

$$k < v < \frac{k^2-k}{y-1}, \tag{2}$$

$$r(k-1) = \lambda(v-1), \tag{3}$$

$$(y-1)(r-1) = (k-1)(\lambda-1), \tag{4}$$

$$y \mid k,\ y \mid (r-\lambda), \tag{5}$$

$$b = \frac{vr}{k}. \tag{6}$$

We provide specific calculation steps:

(1) For a given v and y obtained from (1), substitute v and y into (2) to obtain the range of positive integer k;

(2) Substitute the above possible values of k and v into (3) and (4) to obtain all possible positive integer solutions for (r, λ), where $r > k > \lambda > 1$;

(3) Check whether (y, k, r, λ) satisfies the conditions in (5);

(4) Substitute the positive integer r and corresponding v, k into (6) to check whether b is a positive integer.

Finally, we identify the parameters (v, b, r, k, λ) that satisfy all above conditions as potential parameters for the design $\mathcal{D}$.

3.2 Maximal non-parabolic subgroups in Table 1

According to Lemma 5(i), we have $|H|^3 > \lambda|G|$, which implies that H is a large maximal subgroup of G. Therefore, by Lemma 8, the subgroup H is either a parabolic subgroup or a subgroup such that $H \cap X = X_\alpha$ is listed in Table 1. We shall initially focus on the latter case.

Lemma 14. *Assume Hypothesis 11. The pair (X, X_α) does not belong to any of the cases listed in Table 1, with the exception of the case $(G_2(q), A_2^{\pm}(q))$.*

Proof. For each of the candidate pairs (X, X_α) without specific q value conditions, we can calculate the parameters $v-1$ and $|X_\alpha|$. By using the method in Section 3.1, the computation in Magma [6] gives three polynomials with integer coefficients $f(q)$, $g(q)$ and $h(q)$ such that

$$f(q) \cdot (v-1) + g(q) \cdot |X_\alpha| = h(q).$$

According to Lemma 5(ii) and Lemma 7, we have $\frac{r}{(r,\lambda)} \leqslant (v-1, |G_\alpha|) \leqslant h(q) \cdot |\mathrm{Out}(X)|$. It follows from Lemma 6(v) that

$$v \leqslant 2(y-1) \cdot \frac{r^2}{(r,\lambda)^2} \leqslant 18 \cdot (h(q) \cdot |\mathrm{Out}(X)|)^2 . \tag{7}$$

This inequality (7) is on q, and after computation, we find that only a few small values of q satisfy this inequality. Furthermore, for these q values, and the specific remaining q values listed in Table 1, we directly verify whether inequality $v \leqslant 18 \cdot (v-1, |X_\alpha| \cdot |\mathrm{Out}(X)|)^2$ holds. We provide Example 12 to demonstrate these calculation processes. Finally, for the q values that satisfy above inequality, we search for potential parameters for $\mathcal{D}$. It is worth noting that most cases are excluded in the second step, and no possible parameters are found in the final calculation. The only remaining case is that $(G_2(q), A_2^{\pm}(q))$. In this case, we have $v = \frac{1}{2}q^3(q^3 \pm 1)$ and $h(q) = 12(q^3 \mp 1)$. Then inequality (7) becomes $\frac{1}{2}q^3(q^3 \pm 1) \leqslant 18 \cdot (24f(q^3 \mp 1))^2$ since $|\mathrm{Out}(X)| \leqslant 2f$, where $q = p^f$ for some prime p and positive integer f. This is always true. □

Lemma 15. *Assume Hypothesis 11. If $X = G_2(q)$ with $q \geqslant 3$, then X_α is not a maximal subgroup of type $A_2^{\pm}(q)$.*

Proof. Suppose to the contrary that X_α is of type $A_2^{\pm}(q)$. According to [38, Table 4.1], we know that $X_\alpha = \mathrm{SL}(3, q) : 2$ or $\mathrm{SU}(3, q) : 2$. Note that $|X| = q^6(q^6-1)(q^2-1)$. Then $v = \frac{1}{2}q^3(q^3+\epsilon)$, where $\epsilon = \pm 1$. Combining Lemma 5(ii)(iii) with [35, Section 4, Case 1 and Section 3, Case 8], we conclude that $\frac{r}{(r,\lambda)}$ divides $\frac{1}{2}(q^3-1)$ for q odd and $\frac{r}{(r,\lambda)}$ divides q^3-1 for q even if $\epsilon = +1$; $\frac{r}{(r,\lambda)}$ divides $\frac{1}{2}(q^3+1)$ for q odd and $\frac{r}{(r,\lambda)}$ divides $3(q^3+1)$ for q even if $\epsilon = -1$. For example, suppose that q is odd. From the factorization $\Omega(7, q) = G_2(q)N_1^{\epsilon}$, it follows that the suborbits of $\Omega(7, q)$ are unions of $G_2(q)$-suborbits. The $\Omega(7, q)$-subdegree are q^6-1, $\frac{1}{2}q^2(q^3-\epsilon)$ and $\frac{1}{2}(q-3)$ times $q^2(q^3-\epsilon)$. Since $\gcd(v-1, q) = 1$, by Lemma 5(iii), we have $\frac{r}{(r,\lambda)}$ divides $\frac{1}{2}(q^3-\epsilon)$.

We give the details only for case where $\epsilon = +1$ and q is odd, since other cases are similar. Since $\frac{r}{(r,\lambda)}$ divides $\frac{1}{2}(q^3-1)$, we have $\frac{r}{\lambda} = \frac{r}{(r,\lambda)}/\frac{\lambda}{(r,\lambda)} = \frac{1}{2u}(q^3-1)$, where u is a positive integer. Suppose that $u \geqslant 3$. Since $y \leqslant 10$, it follows from Lemma 6(v) that $v-1 = \frac{1}{2}(q^6+q^3-2) < 2(y-1)\cdot\frac{(q^3-1)^2}{4u^2} \leqslant \frac{1}{2}(q^6-2q^3+1)$, which is a contradiction. Thus we have $u = 1$ or 2.

If $u = 1$, then by Lemma 6(v), we have $\frac{1}{4}(y-1)(q^3-1)^2 < v-1 = \frac{1}{2}(q^3+2)(q^3-1) < \frac{1}{2}(y-1)(q^3-1)^2$, so $y = 3$. Recall that $k = \frac{\lambda(v-1)}{r}+1$. Then we deduce that $k = q^3+3$. Furthermore, by Lemma 6(i), i.e., $(y-1)(r-1) = (k-1)(\lambda-1)$, we compute that $\lambda = \frac{1}{3}q^3$ and $r = \frac{1}{6}q^3(q^3-1)$. Hence $b = \frac{vr}{k} = \frac{q^6(q^6-1)}{12(q^3+3)}$. Since b is a positive integer, it follows that q^3+3 divides 72, which is impossible.

If $u = 2$, then $k = 2q^3+5$. Furthermore, we can similarly deduce that $6 \leqslant y \leqslant 9$ by Lemma 6(v). We give details only for the case $y = 6$, as other cases are similar. For $y = 6$, by Lemma 6(i), we obtain that $\lambda = \frac{4(2q^3-1)}{3(q^3+7)} < 3$, which contradicts the fact that $y < \lambda$ from Lemma 6(ii). This completes the proof. □

3.3 Maximal parabolic subgroups

In this subsection, we deal with the case $H = G_\alpha$ is a maximal parabolic subgroup.

Lemma 16. *Assume Hypothesis 11. If $X = {}^2B_2(q)$ with $q = 2^{2n+1} \geqslant 8$, then X_α is not a maximal parabolic subgroup $[q^2]:(q-1)$.*

Proof. Since $|X| = q^2(q^2+1)(q-1)$, it follows from the point-transitivity of X that $v = |X|/|X_\alpha| = q^2+1$. Then, by Lemma 5(ii), we have $r \mid \lambda q^2$. Note that $r > \lambda$ and $q = 2^{2n+1}$. Then we deduce that r is even. Recall that $(y-1)(r-1) = (k-1)(\lambda-1)$ by Lemma 6(i). Therefore, $(k-1)_2 \leqslant (y-1)_2$. We write that $(y-1)_2 = a$, where $a = 1, 2, 4$ or 8 since $y \leqslant 10$.

Suppose that $a = 1$. Then we have $(k-1, q^2) = 1$. It follows from Lemma 4(i) that $\frac{r}{\lambda} = \frac{v-1}{k-1} = \frac{q^2}{k-1}$. Therefore, $r \geqslant q^2 = v-1$, which implies that $r = v-1 = q^2$ by Lemma 6(ii). Moreover, we have $\lambda = k-1$ and so $(r,\lambda) = 1$. From [39, Theorem 1] (see also [2, Theorem 1]), we know that $k = q$ and $\lambda = q-1$. It follows from Lemma 6(iii) that $y \mid q$ and $y \mid (q^2-q+1)$, which is impossible. Therefore, we can always assume that $(r,\lambda) > 1$.

Form now on, we suppose that $a \geqslant 2$. Note that we have $(k-1, q^2) \mid a$. From $\frac{r}{\lambda} = \frac{q^2}{k-1}$, there is an integer m such that $r = \frac{q^2 m}{a}$ and $\lambda = \frac{m(k-1)}{a}$. We claim that m is even and $m < a$. Suppose to the contrary that m is odd. Then $(k-1)_2 = a = (y-1)_2$ and so $\lambda_2 = \left(\frac{m(k-1)}{a}\right)_2 = 1$, i.e., λ is odd. Thus we conclude that $((k-1)(\lambda-1))_2 > ((y-1)(r-1))_2$, which contradicts Lemma 6(ii). If $m = a$, then $r = q^2$ and $\lambda = k-1$. It follows from $(r,\lambda) > 1$ that k is odd. Note that $b = \frac{vr}{k} = \frac{q^2(q^2+1)}{k}$ is an integer. Then we know that $k \mid (q^2+1)$. By solving the following system of equations:

$$r(k-1) = \lambda(v-1),$$

$$(y-1)(r-1) = (k-1)(\lambda-1),$$

we get that $\lambda = \frac{(k-1)(k-y)}{(k-1)^2-(y-1)q^2}$. Since $\lambda = k-1$, we have $k-y = (k-1)^2-(y-1)q^2$, i.e., $k^2-3k+(y+1)-(y-1)q^2 = 0$. It follows from $k \mid (q^2+1)$ that k divides $k^2-3k+(y+1)-(y-1)q^2+(y-1)q^2+(y-1) = k^2-3k+2y$. Then $k \mid 2y$ and so $k \leqslant y$, which contradicts Lemma 6(ii). Therefore, since m is even and $m < a$, we know that r is a power of 2 that is less than q^2. Moreover, from $\lambda = \frac{m(k-1)}{a}$, we know that k is odd. Thus k divides q^2+1. Next, we only need to analyze the possible values of m one by one when a equals 4 or 8.

Suppose that $a = 4$. Then $y = 5$. Let $r = \frac{q^2m}{4}$ and $\lambda = \frac{m(k-1)}{4}$, where m is an even integer. Based on the previous proof, we conclude that $m = 2$. From $\frac{k-1}{2} = \lambda = \frac{(k-1)(k-5)}{(k-1)^2-4q^2}$, we deduce that $k^2-4k+11-4q^2 = 0$. Combining this with the fact that $k \mid (q^2+1)$, we conclude that $k \mid 15$ and so $k = 15$. It follows that $q^2 = 44$, which is impossible. Suppose that $a = 8$. Then $y = 9$. Similarly, we can suppose that $r = \frac{q^2m}{8}$ and $\lambda = \frac{m(k-1)}{8}$, where $m = 2, 4$ or 6. From $\frac{m(k-1)}{8} = \lambda = \frac{(k-1)(k-9)}{(k-1)^2-8q^2}$, we compute that $mk^2-(2m+8)k+m+72-8mq^2 = 0$. It follows from k dividing q^2+1 that $k \mid (72+9m)$. Thus we have $k \mid 45$, $k \mid 27$ and $k \mid 63$ for $m = 2, 4$ and 6 respectively. Then, considering the conditions $y \mid k$ and $y \neq k$, we deduce that $k = 45, 27, 63$ respectively. However, we can directly verify that there are no prime powers q that satisfy the above equations for these k values. This completes the proof. □

Lemma 17. *Assume Hypothesis 11. If $X = {}^2G_2(q)$ with $q = 3^{2n+1} \geqslant 27$, then X_α is not a maximal parabolic subgroup $[q^3]:(q-1)$.*

Proof. Our argument here is similar to that in the proof of Lemma 16. Since $|X| = q^3(q^3+1)(q-1)$, it follows from the point-transitivity of X that $v = |X|/|X_\alpha| = q^3+1$. Then by Lemma 5(ii), we have $r \mid \lambda q^3$. Note that $r > \lambda$ and $q = 3^{2n+1}$. Then we deduce that $r \equiv 0 \pmod 3$. Recall from Lemma 6(i) that $(y-1)(r-1) = (k-1)(\lambda-1)$. Therefore, $(k-1)_3 \leqslant (y-1)_3$. We write that $(y-1)_3 = a$, where $a = 1, 3$ or 9 since $y \leqslant 10$.

Suppose that $a = 1$. Then we have $(k-1, q^3) = 1$. It follows from Lemma 4(i) that $\frac{r}{\lambda} = \frac{v-1}{k-1} = \frac{q^3}{k-1}$. Therefore, $r \geqslant q^3 = v-1$ which implies that $r = v-1 = q^3$ by Lemma 6(ii). Moreover, we have $\lambda = k-1$ and so $(r, \lambda) = 1$. From [39, Theorem 1] (see also [2, Theorem 1]), we know that $k = q, \lambda = q-1$ or $k = q^2, \lambda = q^2-1$. It follows from $y \mid (r-\lambda)$ that $y \mid (q^3-q+1)$ or $y \mid (q^3-q^2+1)$ by Lemma 6(ii), which contradicts that $y \mid k$. Therefore, we can always assume that $(r, \lambda) > 1$ and so $(k, q^3) = 1$.

From now on, we suppose that $a \geqslant 3$. Let $r = \frac{q^3m}{a}$ and $\lambda = \frac{m(k-1)}{a}$ for some integer m. Suppose that $m = a$. Then $r = q^3$ and $\lambda = k-1$. Similar to the proof of Lemma 16, we can deduce that $k \mid 2y$. Note that $k \equiv 1 \pmod 3$ since $(r, \lambda) > 1$. Then we get $k = y$ since $y \equiv 1 \pmod 3$, which contradicts Lemma 6(ii).

Suppose that $a = 3$. Then $y = 4$ or 7. Let $r = \frac{q^3m}{3}$ and $\lambda = \frac{m(k-1)}{3}$, where $m = 1$ or 2. We give the details only for the case where $y = 4$ and $m = 1$, as other cases follow similarly. From $\frac{k-1}{3} = \lambda = \frac{(k-1)(k-4)}{(k-1)^2-3q^3}$, we deduce that $k^2-5k+13-3q^3 = 0$. Note that $b = \frac{vr}{k} = \frac{q^3(q^3+1)}{k}$ is an integer. Then $k \mid (q^3+1)$, and so we conclude that $k \mid 16$. Recall that $y = 4$ divides k and $\lambda = \frac{k-1}{3}$ is an integer. Thus we must have $k = 16$. It follows that $q^3 = 16^2 - 5 \cdot 16 + 13 = 189$, which is impossible. Suppose that $a = 9$. Then $y = 10$. Similarly, we can suppose that $r = \frac{q^2 m}{9}$ and $\lambda = \frac{m(k-1)}{9}$, where $m \leqslant 8$. From $\frac{m(k-1)}{9} = \lambda = \frac{(k-1)(k-10)}{(k-1)^2 - 9q^3}$, we compute that $mk^2 - (2m+9)k + m + 90 - 9mq^3 = 0$. Note that $b = \frac{vr}{k} = \frac{mq^3(q^3+1)}{k}$ is an integer. Then $k \mid m(q^3+1)$, and so we conclude that $k \mid (90 + 10m)$. It can be verified that for the cases $m \leqslant 8$, there are no integers k that satisfy all the above conditions simultaneously. For example, if $m = 1$, then $k \mid 100$. By the fact that $y = 10$ divides k and $\lambda = \frac{k-1}{9}$ is an integer, we have $k = 100$. It follows that $9q^3 = 100^2 - 11 \cdot 100 + 91 = 8991$, which is impossible. This completes the proof. □

Lemma 18. *Assume Hypothesis 11. The point stabilizer $H = G_\alpha$ is not a parabolic subgroup of G.*

Proof. Firstly, by Lemmas 16 and 17, we know that $(X, X_\alpha) \neq ({}^2B_2(q), [q^2] : (q-1))$ or $({}^2G_2(q), [q^3] : (q-1))$. We then assume that $X \neq E_6(q)$. By Lemma 10, there is a unique subdegree which is a power of p. Moreover, we have $(v-1)_p \leqslant 2q$ for all parabolic subgroups, with the equality holding only when $q = 2^f$. Then by Lemma 5(iii) and Lemma 6(v), we have

$$v \leqslant 2(y-1) \cdot \frac{r^2}{(r,\lambda)^2} \leqslant 18(v-1,d)^2 \leqslant 18 \cdot 4q^2. \tag{8}$$

Similar to the final computation process in Lemma 14, we can eliminate this case. Specifically, one can refer to the calculation in Example 13 for details.

Finally, we suppose that $X = E_6(q)$. If G contains a graph automorphism and $H = P_i$ with $i = 2$ or 4, then there is a unique subdegree which is a power of p (see [35, p.345]). Therefore, we can deal with these two cases in the same way as the proof in the previous paragraph. If $H \cap X$ is P_1 with type $D_5(q)$, then $v = (q^8 + q^4 + 1)(q^9 - 1)/(q^8 - 1)$. Note that the right coset action of G on H is rank 3 by [23]. Two non-trivial subdegrees are $d_1 := q(q^8-1)(q^3+1)/(q-1)$ and $d_2 := q^8(q^5-1)(q^4+1)/(q-1)$, respectively. By Lemma 6(iii) and Lemma 6(v), we have $v \leqslant 18 \cdot (v-1, d_1)^2$, which is impossible by Magma [6]. If $H \cap X$ is P_3 with type $A_1(q)A_4(q)$, then $v = (q^3+1)(q^4+1)(q^9-1)(q^{12}-1)/(q-1)(q^2-1)$. This case can be ruled out similar to the proof of Example 12 and we omit the details. This completes the proof. □

Proof of Theorem 1. By Lemma 5(i), the point stabilizer G_α is maximal in G. Thus we can apply Lemma 8 and analyze each possible case. In Lemmas 14 and 15, we excluded the case where G_α is a maximal non-parabolic subgroup of G. In Lemmas 16, 17 and 18, we handled the case where G_α is a parabolic subgroup of G. Putting together, this completes the proof of Theorem 1.

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