A Lower Bound on the Saturation Number and a Strengthening for Triangle-Free Graphs

Calum Buchanan Puck Rombach

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Abstract

The saturation number $\mathrm{sat}(n, H)$ of a graph H and positive integer n is the minimum size of a graph of order n which does not contain a subgraph isomorphic to H but to which the addition of any edge creates such a subgraph. Erdős, Hajnal, and Moon first studied saturation numbers of complete graphs, and Cameron and Puleo introduced a general lower bound on $\mathrm{sat}(n, H)$. In this paper, we present another lower bound on $\mathrm{sat}(n, H)$ with strengthenings for graphs H in several classes, all of which include the class of triangle-free graphs. Demonstrating its effectiveness, we determine the saturation numbers of diameter-3 trees up to an additive constant; these are double stars $S_{s,t}$ of order $s + t$ whose central vertices have degrees s and t. Faudree, Faudree, Gould, and Jacobson determined that $\mathrm{sat}(n, S_{t,t}) = (t - 1)n/2 + O(1)$. We prove that $\mathrm{sat}(n, S_{s,t}) = (st + s)n/(2t + 4) + O(1)$ when $s < t$. We also apply our lower bound to caterpillars and demonstrate an upper bound on the saturation numbers of certain diameter-4 caterpillars.

Mathematics Subject Classifications: 05C35

1 Introduction

Let G and H be finite, simple graphs. If no subgraph of G is isomorphic to H, we say that G is *H-free*. A number of foundational results in extremal graph theory concern global properties of H-free graphs. Turán's theorem, for instance, states that the complete t-partite graph whose partite sets are as balanced as possible uniquely contains the maximum number of edges out of all K_{t+1}-free graphs of a given order n. Erdős, Hajnal, and Moon [3] studied a complementary problem, proving that there is a unique graph of minimum size over all maximally K_{t+1}-free graphs of order n: the complete t-partite graph which is as unbalanced as possible (that is, with $t - 1$ singleton partite sets and one of cardinality $n - t + 1$). The study of maximally K_{t+1}-free graphs was initiated by Zykov, who called these graphs "saturated" [6].

Department of Mathematics and Statistics, University of Vermont, Burlington, VT, U.S.A. (calumjmb@comcast.net, puck.rombach@uvm.edu).

These works sparked interest in *H-saturated* graphs, those whose edge sets are maximal with respect to being H-free, for more general graphs H. The minimum size of an H-saturated graph of order n is called the *saturation number of H*, denoted $\mathrm{sat}(n, H)$.

In 1986, Kászonyi and Tuza introduced a general upper bound on $\mathrm{sat}(n, H)$ [5]. Notably, their bound implies that $\mathrm{sat}(n, H) = O(n)$ for any graph H (with at least one edge). In 2022, Cameron and Puleo proved a general lower bound via a weight function on the edge set of H [2]. For each edge uv in H, assuming $d(u) \leqslant d(v)$, let

$$\mathrm{wt}_{\mathrm{CP}}(uv) = 2|N(u) \cap N(v)| + |N(v) - N(u)| - 1.$$

They proved that there is a constant c depending only on H such that, for any integer n at least the order of H,

$$\mathrm{sat}(n, H) \geqslant \left(\min_{uv \in E(H)} \mathrm{wt}_{\mathrm{CP}}(uv) \right) \frac{n}{2} - c. \tag{1}$$

For a triangle-free graph H or, more generally, a graph in which every edge uv minimizing $\mathrm{wt}_{\mathrm{CP}}$ is contained in no triangles, (1) reduces to

$$\mathrm{sat}(n, H) \geqslant \left(\min_{uv \in E(H)} \max \big\{ d(u), d(v) \big\} - 1 \right) \frac{n}{2} - c. \tag{2}$$

In what follows, we provide a different strengthening of (2) for a general graph H by considering not only the maximum degree of an endpoint of each edge in H, but also the maximum degree of a neighbor of one of its endpoints. To do so, we define two weight functions on the edge set of H.

Definition 1. For each edge uv in a graph H, let

$$\mathrm{wt}_0(uv) = \max \big\{ d(u), d(v) \big\} - 1.$$

If uv is not isolated, that is, if $N(u) \cup N(v) \neq \{u, v\}$, let

$$\mathrm{wt}_1(uv) = \max \big\{ d(w) : w \in (N(u) \cup N(v)) - \{u, v\} \big\}.$$

Let k_0 and k_1 denote the minimum values of wt_0 and wt_1, respectively, over $E(H)$. Further, let

$$k_0' = \min_{\substack{uv \in E(H) \\ \mathrm{wt}_1(uv) = k_1}} \mathrm{wt}_0(uv) \qquad \text{and} \qquad k_1' = \min_{\substack{uv \in E(H) \\ \mathrm{wt}_0(uv) = k_0}} \mathrm{wt}_1(uv).$$

Note that (2) can be rewritten as $\mathrm{sat}(n, H) \geqslant k_0 n/2 - c$. We also note that, if H has an isolated edge, then the bounds (1) and (2) are both trivial. This, however, is the only such case, for $\mathrm{sat}(n, H) = O(1)$ if H contains an isolated edge, and otherwise $\mathrm{sat}(n, H) = \Theta(n)$ [5]. The bounds we provide here concern saturation numbers which grow linearly with n. Thus, we assume throughout that H has no isolated edges, in addition to the obvious assumption that H has at least one edge.

If an edge xy is added to an H-saturated graph G, then this edge is contained in a copy of H in $G+xy$. Suppose xy is the image of the edge uv in H in one such copy. Naturally, the vertices x and y in G must resemble u and v in a number of ways. For instance, one of x or y is the image of the higher-degree endpoint of uv, so $\max\{d_{G+xy}(x), d_{G+xy}(y)\} \geqslant \max\{d_H(u), d_H(v)\} \geqslant \mathrm{wt}_0(uv)+1$. Further, not only must we have $\max\{d_G(x), d_G(y)\} \geqslant \mathrm{wt}_0(uv)$, but at least one of x or y must have a neighbor of degree at least $\mathrm{wt}_1(uv)$. It follows that almost all vertices in G have degree at least k_0 and a neighbor of degree at least k_1 (see Propositions 4 and 5). In Section 2, we use these observations, and similar ones involving k'_0 and k'_1, to obtain a first improvement on (2) for a general graph H.

Noting that there are no restrictions on the degrees of neighbors in an H-saturated graph when $k'_1 \leqslant k_0$, we focus on the case $k'_1 > k_0$. We also note that $k_0 \leqslant k'_0$ and $k_1 \leqslant k'_1$, and the former inequality is strict if and only if the latter is strict as well. We summarize our general lower bound in the following theorem, which combines Lemmas 8 and 10 in Section 2.

Theorem 2. *Let H be a graph with at least one edge and no isolated edges. There is a constant c depending only on H such that, for any $n \geqslant |H|$,*

$$\mathrm{sat}(n,H) \geqslant \left(k_0 + \frac{k'_1 - k_0}{k'_1+1}\right)\frac{n}{2} - c.$$

Further, if $k_1 > k_0$, then $\mathrm{sat}(n,H) \geqslant \big(k_0 + (k'_1-k_0)/k'_1\big)n/2 - c$, and if $k_0 = k_1 < k'_1 < k'_0$, then

$$\mathrm{sat}(n,H) \geqslant \begin{cases} \left(k_0 + \dfrac{k'_0-k_0}{k'_0+1}\right)\dfrac{n}{2} - c: & k'_0 \leqslant k'_1 + \dfrac{k'_0-k_0}{k_0+1}; \\ \left(k_0 + \dfrac{k'_1-k_0}{k'_1}\right)\dfrac{n}{2} - c: & \textit{otherwise.} \end{cases}$$

In Section 3, we strengthen Theorem 2. First, we assume that none of the edges in H which minimize wt_0 are contained in any triangles. In this case, for any pair of nonadjacent vertices x and y with degrees at most k_0 in an H-saturated graph G, at least one must have a neighbor z with $|N_G(z) - \{x,y\}| \geqslant k'_1 - 1$ (see Figure 2a). Second, we assume that at least one degree-(k_0+1) endpoint, say v, of any given edge uv in H which minimizes wt_0 has a neighbor of degree at least k'_1 and is not contained in any triangles. In this case, since one of x or y is the image of such a vertex v in a copy of H in $G+xy$, there exists either $z \in N(y)$ with $|N(z) - (N(y) \cup x)| \geqslant k'_1$ or $z' \in N(x)$ with $|N(z') - (N(x) \cup y)| \geqslant k'_1$ (see Figure 2b). The following theorem summarizes our two main strengthenings of Theorem 2 for such graphs H, phrased in terms of triangle-free graphs for succinctness.

Theorem 3. *Let H be a triangle-free graph with at least one edge and no isolated edges. If $k'_1 \geqslant k_0 + \sqrt{2k_0+1}$, or if $k'_1 \geqslant k_0+2$ and at least one degree-(k_0+1) endpoint of every edge in H minimizing wt_0 has a neighbor of degree at least k'_1, then there is a constant c depending only on H such that, for any $n \geqslant |H|$,*

$$\mathrm{sat}(n,H) \geqslant \left(k_0 + \frac{k'_1+1-k_0}{k'_1+2}\right)\frac{n}{2} - c.$$

If, in addition to either of the above conditions, $k_1 > k_0$, then

$$\mathrm{sat}(n, H) \geqslant \left(k_0 + \frac{k_1' + 1 - k_0}{k_1' + 1}\right)\frac{n}{2} - c.$$

Theorem 3 follows from Lemmas 12 and 15 in Section 3. From their proofs, one can also obtain strengthenings of Theorem 2 which do not require the conditions $k_1' \geqslant k_0 + \sqrt{2k_0 + 1}$ or $k_1' \geqslant k_0 + 2$ in Theorem 3. We note one such strengthening (for the case $k_1 > k_0$) in Corollary 16, which we apply in an accompanying discussion of saturation numbers for certain classes of trees. In Theorem 14, we determine the saturation numbers of unbalanced double stars up to an additive constant; in Corollary 17, we determine these numbers precisely for sufficiently large n meeting a divisibility condition (see Sections 3.2 and 3.3). In Theorem 18, we apply Corollary 16 and prove an upper bound on the saturation numbers of certain diameter-4 caterpillars (see Section 3.4). These classes were examined in [4], and the saturation numbers of balanced double stars were determined up to an additive constant.

While Theorem 3 strengthens Cameron and Puleo's lower bound, this is not always the case for Theorem 2. Indeed, Theorem 2 strengthens (1) if and only if there exists an edge minimizing wt_0 which is not contained in any triangle (and $k_1' > k_0$). On the one hand, if $\mathrm{wt}_0(uv) = k_0$ and uv is in no triangle, then $\mathrm{wt}_{\mathrm{CP}}(H) \leqslant \mathrm{wt}_{\mathrm{CP}}(uv) = k_0$; on the other, if every edge minimizing wt_0 is in a triangle, then $\mathrm{wt}_{\mathrm{CP}}(H) > k_0$. For example, suppose that H consists of a triangle with any number ℓ of pendant edges attached to one of its vertices. The minimum value of $\mathrm{wt}_{\mathrm{CP}}$ is 2 in this case, which asymptotically determines the saturation number, while Theorem 2 tells us that the average degree of an H-saturated graph cannot be much less than $1 + (\ell + 1)/(\ell + 2) < 2$. The degrees of second neighbors of an edge, as well as the number of triangles containing it, may be useful in determining saturation numbers of graphs with larger diameters than this one. In this work, we consider only first neighborhoods of edges.

Before proceeding, we note that none of the proofs of our lower bounds use the fact that an H-saturated graph G is H-free. All of these bounds thus hold for the more general *semisaturation number of* H, or the minimum number of edges in a (not necessarily H-free) graph of order n to which the addition of any extra edge creates a new copy of H. Since an H-saturated graph is also H-semisaturated, our upper bounds on the saturation numbers of unbalanced double stars and caterpillars also hold for their semisaturation numbers. Notably, not only do we determine the saturation numbers of unbalanced double stars asymptotically (and, in some cases, exactly), but also their semisaturation numbers.

1.1 Definitions and notations

For the purposes of this paper, a graph G is a pair of sets (V, E), where V, or $V(G)$, is a finite set of vertices, and E, or $E(G)$, is a set of unordered pairs of distinct vertices. The *order* and *size* of G, the numbers of its vertices and edges, respectively, are denoted by $|G|$ and $\|G\|$. The *neighborhood* $N_G(v)$ of a vertex v in G is $\{w \in V : vw \in E\}$, and its *degree* $d_G(v)$ is $|N_G(v)|$. When the graph G is clear from context, we use the notations

$N(v)$ and $d(v)$. The minimum degree of a vertex in G is denoted by $\delta(G)$ and the average degree over V by $d(G)$. For a nonempty subset S of V, $d(S)$ denotes the average degree of a vertex in S; that is, $d(S) = \frac{1}{|S|}\sum_{v\in S} d(v)$. As a convention, we let $d(\varnothing) = 0$. For disjoint subsets S and T of V, $e(S,T)$ denotes the number of edges in G between S and T.

2 Lower bounds for general graphs

We begin with a graph G which is not complete and is (semi)saturated with respect to an arbitrary graph H. If x and y are nonadjacent vertices in G, then xy is an edge in a copy of H contained in $G+xy$. The degrees of x and y must therefore be sufficiently large that xy constitutes such an edge. More precisely, the degree of at least one of x or y must be at least k_0, the minimum value taken by wt_0 over $E(H)$. This implies the following

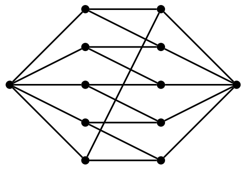
Proposition 4. *Let H be a graph with at least one edge. The vertices in any H-saturated graph of degree strictly less than k_0 form a clique.*

A simple minimum degree bound, along with Proposition 4 and a derivative, shows that the average degree of an H-saturated graph G of order $n \geqslant |H|$ is at least $k_0 - (k_0+1)^2/(4n)$. But, we can say more about pairs of nonadjacent vertices x, y in G. As we assume that H has no isolated edges, some neighbor of x or y must also have sufficiently large degree that xy constitutes an edge in $H \subseteq G + xy$: letting wt_1 and k_1 be as in Definition 1, at least one of x or y has a neighbor of degree at least k_1. Moreover, if xy is to be the image of an edge uv in H minimizing wt_0, then there must be a vertex in $N(x) \cup N(y)$ of degree at least k_1', the minimum value taken by wt_1 over all edges in H which minimize wt_0. We summarize the implications of these statements in the following

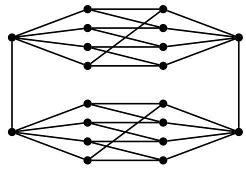
Proposition 5. *Let H be a graph with at least one edge and no isolated edges. The vertices in any H-saturated graph with no neighbor of degree at least k_1 form a clique, and so do the vertices with degree at most k_0 and no neighbor of degree at least k_1'.*

Note that the cliques in Propositions 4 and 5 are of orders at most k_0, k_1 and $k_0 + 1$, respectively, and thus have little effect on the size of an H-saturated graph with large order. In other words, an H-saturated graph cannot have many fewer edges than a graph with minimum degree k_0 in which every minimum-degree vertex has a neighbor of degree at least k_1' and in which every vertex has a neighbor of degree at least k_1. An example of such a graph when $k_0 = 3$, $k_1' = 5$, and $k_1 \leqslant 3$ is shown in Figure 1a. Figure 1b depicts the case $k_1 \in \{4, 5\}$. We preface our lower bounds on $\mathrm{sat}(n, H)$ by proving that these examples have minimum average degree over all such graphs.

Proposition 6. *Let δ and k be positive integers with $\delta < k$. If G is a graph with minimum degree δ in which every vertex of degree δ has a neighbor of degree at least k, then $d(G) \geqslant \delta + (k-\delta)/(k+1)$. If, in addition, every vertex in G of degree at least k has a neighbor of degree strictly larger than δ, then $d(G) \geqslant \delta + (k-\delta)/k$.*



(a) A graph with minimum degree 3 and average degree 10/3 in which every degree-3 vertex has a degree-5 neighbor.



(b) A graph with minimum degree 3 and average degree 17/5 in which every vertex has a degree-5 neighbor.

Figure 1: Graphs as described in Example 7 whose average degrees meet the lower bounds in Proposition 6.

Proof. We partition the vertex set V of G as follows: let $S = \{v \in V : d(v) = \delta\}$, $M = \{v \in V : \delta < d(v) < k\}$, and $L = \{v \in V : d(v) \geqslant k\}$. By assumption, every vertex in S has a neighbor in L, so $e(L, S) \geqslant |S| = |L \cup S| - |L|$. Since $e(L, S) \leqslant \sum_{v \in L} d(v) = d(L)|L|$, we have $|L \cup S| \leqslant (d(L) + 1)|L|$. Let $\ell = d(L)$. Combining the aforementioned inequalities yields

$$|L| \geqslant \frac{1}{\ell + 1}|L \cup S| \quad \text{and} \quad |S| \leqslant \frac{\ell}{\ell + 1}|L \cup S|.$$

Thus,

$$\begin{aligned}\sum_{v \in V} d(v) &\geqslant \ell|L| + \delta|S| + (\delta + 1)|M| \geqslant \frac{\ell(\delta + 1)}{\ell + 1}|L \cup S| + (\delta + 1)|M| \\ &\geqslant \left(\delta + \frac{\ell - \delta}{\ell + 1}\right)|G|.\end{aligned}$$

Since $\ell \geqslant k$ and

$$\frac{\ell - \delta}{\ell + 1} = \frac{k - \delta}{k + 1} + \frac{(\delta + 1)(\ell - k)}{(\ell + 1)(k + 1)} \tag{3}$$

we have $d(G) = \sum d(v)/|G| \geqslant \delta + (k - \delta)/(k + 1)$, as desired.

For the second statement, if every vertex in L has a neighbor in $V - S$, then $|S| \leqslant e(L, S) \leqslant \sum_{v \in L}(d(v) - 1) = (\ell - 1)|L|$. In this case, $|L| \geqslant |L \cup S|/\ell$ and $|S| \leqslant (\ell - 1)|L \cup S|/\ell$. By a similar argument as before, we have

$$\sum_{v \in V} d(v) \geqslant \left(\delta + \frac{\ell - \delta}{\ell}\right)|G|.$$

Since

$$\frac{\ell - \delta}{\ell} = \frac{k - \delta}{k} + \frac{\delta(\ell - k)}{k\ell} \tag{4}$$

we have $d(G) \geqslant \delta + (k - \delta)/k$, as desired. □

The lower bounds in Proposition 6 are tight, as evidenced by the following constructions. Such graphs will be relevant when we discuss saturation upper bounds in Section 3.

Example 7. It follows from the proof of Proposition 6 that the graphs of minimum size in which every vertex has degree at least δ and a neighbor of degree at least $k > \delta$ are biregular, with all degrees either δ or k, and are such that every vertex has exactly one high-degree neighbor. Note that such a graph contains $k-1$ vertices of degree δ for every vertex of degree k, and thus its order must be divisible by k. Further, the degree-k vertices induce a matching, and thus its order must be divisible by $2k$. It is not difficult to construct such graphs. For $n = 2k\ell$, take ℓ copies of the double star $S_{k,k}$ (obtained from two copies of a star $K_{1,k-1}$ by attaching their central vertices with an edge), and put a $(\delta-1)$-regular graph on the set of leaves (see Figure 1b). If there are no restrictions on the degrees of neighbors of high-degree vertices, then every degree-k vertex corresponds to k degree-δ vertices in a graph of minimum size. For $n = (k+1)\ell$, take ℓ copies of the star $K_{1,k}$ and, if at least one of $\delta-1$, k, or ℓ is even, put a $(\delta-1)$-regular graph on the set of leaves (see Figure 1a). Note that the bound in Proposition 6 is strict whenever n is not divisible by $2k$ in the first case or $k+1$ in the second case, or when $\delta-1$, k and ℓ are all odd in the second case.

We now return to our discussion of saturation numbers. As noted in the introduction, there are no restrictions on the degrees of neighbors in an H-saturated graph when $k_1' \leqslant k_0$, and we simply have $\mathrm{sat}(n,H) \geqslant k_0 n/2 - (k_0+1)^2/8$. Assuming $k_1' > k_0$, we now prove two lower bounds on $\mathrm{sat}(n,H)$, depending on k_1.

Lemma 8. *For any graph H with at least one edge and no isolated edges, and for any $n \geqslant |H|$,*

$$\mathrm{sat}(n,H) \geqslant \left(k_0 + \frac{k_1' - k_0}{k_1' + 1}\right)\frac{n}{2} - c_1.$$

If $k_1 > k_0$, then

$$\mathrm{sat}(n,H) \geqslant \left(k_0 + \frac{k_1' - k_0}{k_1'}\right)\frac{n}{2} - c_2,$$

where $c_1 = \frac{(k_0+1)(k_1'-k_0)}{2k_1'+2} + \frac{(k_0+1)^2}{8}$ and $c_2 = \frac{(k_0+2)(k_1'-k_0)}{2k_1'} + \frac{(k_0+1)^2}{8}$.

Proof. Let G be an H-saturated graph of order n. Partition the vertex set V of G as follows: let $S = \{v \in V : d(v) \leqslant k_0\}$, $M = \{v \in V : k_0 < d(v) < k_1'\}$, and $L = \{v \in V : d(v) \geqslant k_1'\}$. By Propositions 4 and 5, aside from a clique A, every vertex in S has degree k_0, and aside from a clique B, every vertex in S has a neighbor in L. Thus, $e(L,S) \geqslant |S - B| = |L \cup S| - |L| - |B|$, and clearly $e(L,S) \leqslant \sum_{v\in L} d(v) = |L|d(L)$. Letting $\ell = d(L)$, it follows that $|L \cup S| - |B| \leqslant |L|(\ell+1)$, so

$$|L| \geqslant \frac{1}{\ell+1}|L\cup S| - \frac{|B|}{\ell+1} \quad \text{and} \quad |S| \leqslant \frac{\ell}{\ell+1}|L\cup S| + \frac{|B|}{\ell+1}.$$

Thus,

$$\ell|L| + k_0|S| \geqslant \frac{\ell + k_0\ell}{\ell+1}|L\cup S| - \frac{\ell - k_0}{\ell+1}|B| = \left(k_0 + \frac{\ell - k_0}{\ell+1}\right)|L\cup S| - \frac{\ell-k_0}{\ell+1}|B|.$$

Using (3) and noting that $|L \cup S| \geqslant |B|$, it follows that

$$\ell|L| + k_0|S| \geqslant \left(k_0 + \frac{k_1' - k_0}{k_1' + 1}\right)|L \cup S| - \frac{k_1' - k_0}{k_1' + 1}|B|.$$

Since $|B| \leqslant k_0 + 1$, we have

$$\begin{aligned}\sum_{v \in L \cup S} d(v) &= \ell|L| + k_0|S - A| + \sum_{v \in A} d(v) \geqslant \ell|L| + k_0|S| - |A|(k_0 + 1 - |A|) \\ &\geqslant \left(k_0 + \frac{k_1' - k_0}{k_1' + 1}\right)|L \cup S| - \frac{(k_0+1)(k_1' - k_0)}{k_1' + 1} - \frac{(k_0+1)^2}{4}.\end{aligned}$$

Every vertex in M has degree at least $k_0 + 1$ by definition, and S, M, and L partition V, so the degree sum of G is at least $\big(k_0 + (k_1' - k_0)/(k_1' + 1)\big)n - 2c_1$.

For the second statement, suppose that $k_1 > k_0$. By Proposition 5, any vertices in L with no neighbor of degree at least k_1 are adjacent. Since $k_1 \leqslant k_1'$, there is at most one such vertex. Thus, $e(L, S) \leqslant \sum_{v \in L}(d(v) - 1) + 1 = (\ell - 1)|L| + 1$. Since $e(L, S) \geqslant |L \cup S| - |L| - |B|$, we have $|L \cup S| - |B| \leqslant \ell|L| + 1$. If $L = \varnothing$, then $S = B$. Otherwise,

$$|L| \geqslant \frac{1}{\ell}|L \cup S| - \frac{1}{\ell}(|B| + 1) \quad \text{and} \quad |S| \leqslant \frac{\ell - 1}{\ell}|L \cup S| + \frac{1}{\ell}(|B| + 1).$$

Also, in this case, $|L \cup S| \geqslant |B| + 1$, so that using (4) we have

$$\ell|L| + k_0|S| \geqslant \left(k_0 + \frac{k_1' - k_0}{k_1'}\right)|L \cup S| - \frac{(k_0 + 2)(k_1' - k_0)}{k_1'}.$$

Note that the above inequality still holds (and is strict) when $L = \varnothing$, in which case $\ell = 0$. Thus, by the same reasoning with which we concluded the first paragraph of this proof, the degree sum of G is at least $\big(k_0 + (k_1' - k_0)/k_1'\big)n - 2c_2$. The handshake lemma completes the proof. □

We now consider graphs H with $k_1 = k_0 < k_1' < k_0'$. Again, by Proposition 5, almost every vertex of degree at most k_0 in an H-saturated graph has a neighbor of degree at least k_1'. In this case, however, almost every vertex of degree strictly less than k_0' (in particular, almost every vertex of degree k_1') has a neighbor of degree strictly larger than k_1, and $k_1 = k_0$. The constructions in Example 7 give two ideas for such a graph of minimum size, either we have only degree-k_0 and degree-k_1' vertices with two degree-k_1' vertices for every $2(k_1' - 1)$ degree-k_0 vertices (as in Figure 1b), or we have only degree-k_0 and degree-k_0' vertices with one degree-k_0' vertex for every k_0' degree-k_0 vertices (as in Figure 1a).

Example 9. Let us compare, as k_0' varies, the average degree of a graph of the form given in Figure 1a with vertices of degree k_0 and k_0' to the average degree of one as in Figure 1b with vertices of degree k_0 and k_1'. Note that the former graph has average degree $k_0 + (k_0' - k_0)/(k_0' + 1)$ and the latter $k_0 + (k_1' - k_0)/k_1'$. Suppose that $k_1' = 4$ and

$k_0 = k_1 < k_1'$. If $k_0' = 6$, then $(k_0' - k_0)/(k_0' + 1) = 5/7 < (k_1' - k_0)/k_1' = 3/4$. However, if $k_0' = 8$, then $(k_0' - k_0)/(k_0' + 1) = 7/9 > 3/4$. If instead $k_0' = 7$, then the two quantities are equal. In general, we have

$$\frac{k_0' - k_0}{k_0' + 1} \leqslant \frac{k_1' - k_0}{k_1'} \quad \text{if and only if} \quad k_0' - k_1' \leqslant \frac{k_0' - k_0}{k_0 + 1}. \tag{5}$$

We conclude this section, and the proof of Theorem 2, by determining that these constructions are optimal. That is, when $k_0' > k_1'$, these graphs have minimum average degree over all graphs with minimum degree k_0 in which every degree-k_0 vertex has a neighbor of degree at least k_1', and every vertex of degree strictly less than k_0' has a neighbor of degree strictly greater than k_0.

Lemma 10. *For any graph H with $k_0 = k_1 < k_1' < k_0'$, and for any $n \geqslant |H|$,*

$$\mathrm{sat}(n, H) \geqslant \begin{cases} \left(k_0 + \frac{k_0'-k_0}{k_0'+1}\right)\frac{n}{2} - c_1 : & k_0' \leqslant k_1' + \frac{k_0'-k_0}{k_0+1}; \\ \left(k_0 + \frac{k_1'-k_0}{k_1'}\right)\frac{n}{2} - c_2 : & k_0' \geqslant k_1' + \frac{k_0'-k_0}{k_0+1}, \end{cases}$$

where $c_1 = \frac{(k_0+1)(k_0'-k_0)}{2k_0'+2} + \frac{(k_0+1)^2}{8}$ and $c_2 = \frac{(k_0+2)(k_1'-k_0)}{2k_1'} + \frac{(k_0+1)^2}{8}$.

Proof. Let G be an H-saturated graph of order n. Partition the vertex set V of G as follows: let $S = \{v \in V : d(v) \leqslant k_0\}$, $M = \{v \in V : k_0 < d(v) < k_1'\}$, $L = \{v \in V : k_1' \leqslant d(v) < k_0'\}$, and $XL = \{v \in V : d(v) \geqslant k_0'\}$. Aside from a clique B, every vertex in S has a neighbor of degree at least k_1'. We partition $S - B$ into subsets S_L and S_{XL} of vertices with a neighbor in L or XL, respectively (if a vertex has neighbors in both L and XL, assign it to either set arbitrarily). We will show that $d(L \cup S_L)$ is not much less than $k_0 + (k_1' - k_0)/k_1'$ if L is nonempty, and that $d(XL \cup S_{XL})$ is not much less than $k_0 + (k_0' - k_0)/(k_0' + 1)$ if XL is nonempty.

First, suppose $L \neq \varnothing$ and consider $L \cup S_L$. At least one out of any pair of nonadjacent vertices in L has a neighbor in $V - S$, since $d(v) < k_0'$ for all $v \in L$. It follows that at most one vertex in L has all of its neighbors in S, so that $|S_L| \leqslant e(L, S_L) \leqslant \sum_{v \in L}(d(v) - 1) + 1$. That is, $|L \cup S_L| - |L| \leqslant |L|d(L) - |L| + 1$. Letting $\ell = d(L)$, we have

$$|L| \geqslant \frac{1}{\ell}|L \cup S_L| - \frac{1}{\ell} \quad \text{and} \quad |S_L| \leqslant \frac{\ell - 1}{\ell}|L \cup S_L| + \frac{1}{\ell}.$$

Thus,

$$\begin{aligned} \ell|L| + k_0|S_L| &\geqslant \frac{\ell + k_0(\ell - 1)}{\ell}|L \cup S_L| - \frac{\ell - k_0}{\ell} \\ &\geqslant \left(k_0 + \frac{k_1' - k_0}{k_1'}\right)|L \cup S_L| - \frac{k_1' - k_0}{k_1'}. \end{aligned}$$

Note that if $L = \varnothing$, the final inequality above still holds, and is strict.

Now consider $XL \cup S_{XL}$. We have $|S_{XL}| \leqslant e(XL, S_{XL}) \leqslant \sum_{v \in XL} d(v)$. Letting $x = d(XL)$, we have

$$|XL| \geqslant \frac{1}{x+1}|XL \cup S_{XL}| \quad \text{and} \quad |S_{XL}| \leqslant \frac{x}{x+1}|XL \cup S_{XL}|.$$

Thus,

$$x|XL| + k_0|S_{XL}| \geqslant \frac{x(k_0+1)}{x+1}|XL \cup S_{XL}| \geqslant \left(k_0 + \frac{k_0' - k_0}{k_0' + 1}\right)|XL \cup S_{XL}|.$$

We have

$$\sum_{v \in V-M} d(v) = \big(x|XL| + k_0|S_{XL}|\big) + \big(\ell|L| + k_0|S_L|\big) + k_0|B| - \sum_{s \in A}(k_0 - d(s)).$$

If $k_0' - k_1' \geqslant (k_0' - k_0)/(k_0+1)$, then by (5),

$$\sum_{v \in V-M} d(v) \geqslant \frac{k_1' + k_0(k_1' - 1)}{k_1'}|V - M| - \frac{k_1' - k_0}{k_1'}(|B| + 1) - |A|(k_0 + 1 - |A|).$$

It follows that the degree sum of G is at least $\big(k_0 + (k_1' - k_0)/k_1'\big)n - 2c_2$. Otherwise, if $k_0' - k_1' \leqslant (k_0' - k_0)/(k_0+1)$, then

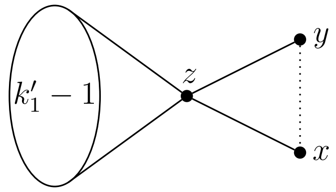
$$\sum_{v \in V-M} d(v) \geqslant \frac{k_0'(k_0+1)}{k_0'+1}|V - M| - \frac{k_0' - k_0}{k_0'+1}|B| - |A|(k_0 + 1 - |A|).$$

In this case, the degree sum of G is at least $\big(k_0 + (k_0' - k_0)/(k_0+1)\big)n - 2c_1$. The handshake lemma completes the proof. □

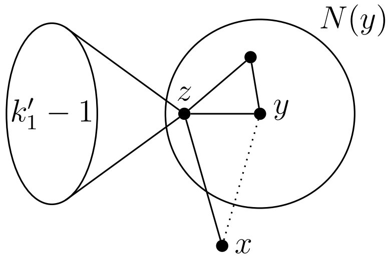
3 Lower bounds for triangle-free graphs and saturation numbers of trees with small diameter

Let G be a graph which is not complete and is (semi)saturated with respect to a triangle-free graph H. In Section 2, we concluded that, since every edge uv in H has a neighbor w of degree at least k_1, at least one out of any pair of nonadjacent vertices in G must have a neighbor of degree at least k_1. Since $\{u, v, w\}$ is not a triangle, exactly one of the edges uw or vw is in H, and thus not only do we have $d(w) \geqslant k_1$, but $|N(w) - \{u, v\}| \geqslant k_1 - 1$. Therefore, at least one out of any pair of nonadjacent vertices x, y in G must have a neighbor z with $|N(z) - \{x, y\}| \geqslant k_1 - 1$. Similarly, if $d(x)$ and $d(y)$ are both at most k_0, then not only must one of x or y have a neighbor of degree at least k_1', but G also has the following property:

for any pair of nonadjacent vertices x, y with degrees at most k_0, there exists $z \in N(x) \cup N(y)$ such that $|N(z) - \{x, y\}| \geqslant k_1' - 1$. (P1)



(a) A high-degree neighbor z of nonadjacent low-degree x, y needs at least $k_1' - 1$ neighbors outside of $\{x, y\}$.



(b) A high-degree neighbor z of low-degree vertex y needs at least $k_1' - 1$ neighbors outside of $N(y) \cup \{x, y\}$.

Figure 2: Illustrations of properties (P1) and (P2).

See Figure 2a.

Further, there is a subset C of k_0 neighbors of either x or y such that $|N(z) - (C \cup \{x, y\})| \geqslant k_1 - 1$ for some $z \in N(x) \cup N(y)$. A number of statements similar to those above can be made about the neighbors of nonadjacent vertices in a graph G which is saturated with respect to a triangle-free graph H.

If every edge uv in H with $\mathrm{wt}_0(uv) = k_0$ has a degree-$(k_0 + 1)$ endpoint with a neighbor w of degree k_1', then we can make a stronger statement. In this case, if x and y are nonadjacent vertices of degrees at most k_0 in an H-saturated graph G, then for at least one of them, having neighborhood N, there exists $z \in N$ with $|N(z) - (N \cup \{x, y\})| \geqslant k_1' - 1$. This is because if, say, y is to play the role of the degree-$(k_0 + 1)$ endpoint of an edge in H minimizing wt_0 with a degree-k_1' neighbor, then every neighbor of y is used, and none can make triangles with z, in this copy of H (see Figure 2b). Thus, in this case, G has the following property:

for any pair of nonadjacent vertices x, y with degrees at most k_0, there exists either $z \in N(y)$ such that $|N(z) - (N(y) \cup x)| \geqslant k_1'$ or $z' \in N(x)$ such that $|N(z') - (N(x) \cup y)| \geqslant k_1'$. (P2)

We will use these properties in a similar manner as we used Proposition 5 in the proofs of Lemmas 8 and 10. Throughout this section, given positive integers $k_0 < k_1'$ and a graph G with property (P1) or (P2), we refer to vertices of degree at least k_1' as *high-degree vertices* and to those of degree at most k_0 as *low-degree*.

Proposition 11. *Let k_0 and k_1' be integers with $k_0 < k_1'$, and let G be a graph with property* (P2). *The set of low-degree vertices v in G such that either*

(i) v has no high-degree neighbor, or

(ii) v has a single high-degree neighbor w, $d(w) = k_1'$, and $N(v) \cap N(w) \neq \varnothing$

is a clique.

Let G be a graph with minimum degree k_0. Suppose that either $k_1' \geqslant k_0+2$ and G has property (P2), or $k_1' \geqslant k_0+\sqrt{2k_0+1}$ and G has property (P1). If G has minimum average degree over all such graphs, then it follows from the proof of Lemma 12 in Section 3.1 that, aside from a small clique, the components of G all resemble Figure 1a (that is, are as described in Example 7), except that the high-degree vertices have degree $k_1'+1$ and low-degree vertices have degree k_0. If, in addition to the above conditions, every high-degree vertex in G has a neighbor of degree strictly larger than k_0, then it follows from the proof of Lemma 15 in Section 3.3 that, aside from a small clique, the components of G all resemble Figure 1b (as in Example 7), but with high-degree vertices of degree $k_1'+1$ and low-degree vertices of degree k_0.

3.1 Low-degree vertices with high-degree neighbors

Let H be a triangle-free graph without isolated edges. Aside from a small clique, every low-degree vertex in an H-saturated graph has a high-degree neighbor. Without considering the neighbors of high-degree vertices, Lemma 8 implies that the average degree of an H-saturated graph cannot be much smaller than $k_0 + (k_1' - k_0)/(k_1'+1)$, the average degree of a graph as described in Example 7. Notice, however, that the graph in Figure 1a does not have property (P1) (and thus does not have (P2)) when $k_0 = 3$ and $k_1' = 5$. On the other hand, if we were to have $k_1' = 4$, then this graph would indeed have property (P1) (although it would not necessarily be of minimum size). In the following lemma, we show that, under some extra conditions on H, the average degree of an H-saturated graph cannot be much less than the average degree of a graph with minimum degree k_0 in which every low-degree vertex has a neighbor of degree $k_1'+1$.

Lemma 12. *Let H be a triangle-free graph, and let $n \geqslant |H|$. If $k_1' \geqslant k_0 + \sqrt{2k_0+9/4} - 1/2$, or if at least one degree-(k_0+1) endpoint of every edge in H minimizing* wt_0 *has a neighbor of degree at least k_1' and $k_1' \geqslant k_0+2$, then*

$$\mathrm{sat}(n,H) \geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+2}\right)\frac{n}{2} - c,$$

where $c = \frac{(k_0+1)(k_1'+1-k_0)}{2k_1'+4} + \frac{(k_0+1)^2}{8}$.

Proof. Let G be an H-saturated graph on vertex set V with order n. Partition the high-degree vertices in V into sets $L = \{v \in V : d(v) = k_1'\}$ and $XL = \{v \in V : d(v) > k_1'\}$. Further, let $S = \{v \in V : d(v) \leqslant k_0\}$ and $M = \{v \in V : k_0 < d(v) < k_1'\}$. Let A denote the clique of vertices in G with degree strictly less than k_0, and let B denote the clique of vertices in S with no high-degree neighbor.

We handle the degree sum over XL and the set S_{XL} of vertices in S with a neighbor in XL in a nearly identical manner as we proved the first statement of Lemma 8 or the second statement of Lemma 10. We have $|S_{XL}| \leqslant e(XL, S_{XL}) \leqslant d(XL)|XL|$ and $d(XL) \geqslant k_1'+1$ so that

$$\sum_{v\in XL\cup S_{XL}} d(v) \geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+2}\right)|XL\cup S_{XL}| - |A\cap S_{XL}|(k_0+1-|A|).$$

We now restrict our attention to L and the vertices in $S_L = S - (B \cup S_{XL})$. That is, S_L is the set of vertices in $S - B$ whose only high-degree neighbor(s) lie in L.

Case 1. By the property (P1), if x and y are vertices in S_L which share all of their neighbors in L, then $xy \in E(G)$. It follows that, for any $z \in L$, the set of vertices in $N(z) \cap S_L$ whose only high-degree neighbor is z form a clique (of order at most k_0). Thus, at most $k_0|L|$ vertices in S_L have exactly one neighbor in L, so $2|S_L| - k_0|L| \leqslant e(L, S_L) \leqslant k_1'|L|$. This gives $|L| \geqslant \frac{2}{k_1'+k_0+2}|L \cup S_L|$ and $|S_L| \leqslant \frac{k_1'+k_0}{k_1'+k_0+2}|L \cup S_L|$. Therefore,

$$k_1'|L| + k_0|S_L| \geqslant \frac{2k_1' + k_0k_1' + k_0^2}{k_1' + k_0 + 2}|L \cup S_L|.$$

Note that

$$\frac{2k_1' + k_0k_1' + k_0^2}{k_1' + k_0 + 2} \geqslant \frac{(k_0 + 1)(k_1' + 1)}{k_1' + 2}$$

if and only if $k_1' \geqslant k_0 + \sqrt{2k_0 + 9/4} - 1/2$.

Case 2. Now, we suppose that at least one degree-$(k_0 + 1)$ endpoint of every edge minimizing wt_0 in H has a degree-k_1' neighbor. In this case, G has property (P2). We add to the clique B all those vertices which do not meet condition (ii) of Proposition 11 (those which share neighbors with their unique high-degree neighbor, which lies in L). In doing so, we remove all such vertices from S_L.

We claim that at most one neighbor in S_L of any vertex in L has exactly one neighbor in L. Indeed, suppose for the sake of contradiction that, for some vertex $w \in L$, there exist vertices $u, v \in S_L$ with $N(u) \cap L = N(v) \cap L = \{w\}$. In order for the pair u, v not to contradict property (P1), we must have $uv \in E(G)$. But then u and v both meet condition (ii) of Proposition 11, so $\{u, v\} \subseteq B$. This is a contradiction, for then $u, v \notin S_L$ after all.

It follows from the previous claim and the pigeonhole principle that no more than $|L|$ vertices in S_L have a single neighbor in L. Thus,

$$2|S_L| - |L| \leqslant e(L, S_L) \leqslant k_1'|L|.$$

In other words, $2|L \cup S_L| \leqslant (k_1' + 3)|L|$, so

$$|L| \geqslant \frac{2}{k_1' + 3}|L \cup S_L| \quad \text{and} \quad |S_L| \leqslant \frac{k_1' + 1}{k_1' + 3}|L \cup S_L|.$$

Now,

$$k_1'|L| + k_0|S_L| \geqslant \frac{2k_1' + k_0(k_1' + 1)}{k_1' + 3}|L \cup S_L|.$$

Note that

$$\frac{2k_1' + k_0(k_1' + 1)}{k_1' + 3} \geqslant \frac{(k_0 + 1)(k_1' + 1)}{k_1' + 2}$$

if and only if $k_1' \geqslant k_0 + 2$.

In both of the cases above, it follows that

$$\sum_{v\in L\cup S_L} d(v) \geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+2}\right)|L\cup S_L| - |A\cap S_L|(k_0+1-|A|).$$

Note that the degree sum over $S\cup L\cup XL$ is the degree sum over $L\cup S_L$, $XL\cup S_{XL}$ and B, so

$$\sum_{v\in S\cup L\cup XL} d(v) \geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+2}\right)|L\cup XL\cup S| - \frac{k_1'+1-k_0}{k_1'+2}|B| - |A|(k_0+1-|A|).$$

Noting that $d(v) \geqslant k_0+1$ for all $v\in M$ and that S, M, L, and XL partition V, we have

$$\sum_{v\in V} d(v) \geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+2}\right)n - \frac{(k_0+1)(k_1'+1-k_0)}{k_1'+2} - \frac{(k_0+1)^2}{4},$$

as desired. □

We remark that Lemma 12 applies to a larger class than that of triangle-free graphs. To use property (P1) in Case 1, we require only that none of the edges in H which minimize wt_0 are contained in any triangles. If $k_1' \geqslant k_0 + \sqrt{2k_0+9/4} - 1/2$, then the lower bound in Lemma 12 holds for such a graph H. To use property (P2) in Case 2, we also require that at least one degree-(k_0+1) endpoint of every edge in H minimizing wt_0 has a neighbor of degree at least k_1' and is not contained in any triangles. If $k_1' \geqslant k_0+2$, then the lower bound holds for such a graph H.

3.2 Double stars

We detour here from our regularly scheduled programming to examine the saturation numbers of diameter-3 trees. In particular, we prove that Lemma 12 is tight up to an additive constant for unbalanced double stars. Given positive integers s and t with $s \leqslant t$, let $S_{s,t}$ denote the tree obtained from two stars $K_{1,s-1}$ and $K_{1,t-1}$ by connecting their central vertices with an edge (see Figure 3a). Note that, for $H = S_{s,t}$, we have $k_0 = s-1$, $k_1 = 1$, $k_0' = t-1$, and $k_1' = t$. Further, every edge minimizing wt_0 has a degree-s endpoint with a neighbor of degree t.

Faudree, Faudree, Gould, and Jacobson determined the saturation numbers of balanced double stars up to an additive constant and provided bounds for unbalanced double stars:

Theorem 13 ([4])**.** *For $n \geqslant s^3$,*

$$\frac{s-1}{2}n \leqslant \mathrm{sat}(n, S_{s,s}) \leqslant \frac{s-1}{2}n + \frac{s^2-1}{2}, \quad \textit{and}$$
$$\frac{s-1}{2}n \leqslant \mathrm{sat}(n, S_{s,t}) \leqslant \frac{s}{2}n - \frac{(s-1)^2+8}{8}.$$

(a) The double star $S_{4,5}$.

(b) An $S_{4,5}$-saturated graph.

Figure 3: The double star $S_{4,5}$ on the left and an $S_{4,5}$-saturated graph on the right of order $n = 18$ and size $(12n - 6)/7 = 30$.

In the following theorem, we use Lemma 12 and a construction resembling Figure 3b to determine the saturation numbers of unbalanced double stars up to an additive constant. Later, in Corollary 17 of Section 3.3, we show that the additive constant in the lower bound can be improved to match certain cases of the upper bound construction when n is sufficiently large.

Theorem 14. *For a double star $S_{s,t}$ with $s < t$ and for $n \geqslant q(2t+4) + s$ where $q = \max\{1, \lfloor s/2 \rfloor - 1\}$,*

$$\left(\frac{s(t+1)}{t+2}\right)\frac{n}{2} - c_1 \leqslant \mathrm{sat}(n, S_{s,t}) \leqslant \left(\frac{s(t+1)}{t+2}\right)\frac{n}{2} + c_2,$$

where $c_1 = \frac{s(t-s+2)}{2t+4} + \frac{s^2}{8}$ and $c_2 = \frac{s(s-1)}{2t+4} + \lceil \frac{s}{2} \rceil$.

Our upper bound is based upon the observation that a graph G_0 obtained from two copies of $K_{1,t+1}$ by joining their sets of leaves with an $(s-2)$-regular bipartite graph, as in the larger component of Figure 3b, is $S_{s,t}$-saturated and has average degree exactly $s(t+1)/(t+2)$. Further, any graph consisting of disjoint copies of G_0 is $S_{s,t}$-saturated. We are able to add a disjoint clique of cardinality s to such a graph to obtain another $S_{s,t}$-saturated graph G with

$$\left(\frac{s(t+1)}{t+2}\right)\frac{n}{2} - \frac{s(t-s+2)}{2t+4}$$

edges, where $n = |G|$. In Corollary 17, we show that this is the precise value of $\mathrm{sat}(n, S_{s,t})$ when n is sufficiently large and equivalent to $s \pmod{2t+4}$. When $n \not\equiv s \pmod{2t+4}$, we add vertices to non-clique components of such a graph G in a manner described below.

Proof of Theorem 14. The lower bound follows from Lemma 12. We provide a construction for the upper bound. Let $S_{s,t}$ be a double star with $s < t$. For $n \geqslant q(2t+4)+s$ where $q = \max\{1, \lfloor (s-2)/2 \rfloor\}$, we construct an n-vertex graph G with the following properties.

(i) We have $V(G) = S \cup L$. For all $v \in S$, $d(v) = s - 1$. For all $v \in L$, $d(v) \geqslant t + 1$.

(ii) For all $v \in L$, $N(v) \subseteq S$, and every $w \in N(v)$ is contained in an independent set of cardinality $t+1$ in $N(v)$.

(iii) Aside from a clique B of order s, every vertex in S has a neighbor in L, and at most one vertex in S has two or more neighbors in L.

We claim that G is $S_{s,t}$-saturated. Since there are no vertices of degree at least t adjacent to any vertices of degree at least s, G is $S_{s,t}$-free. Let x and y be nonadjacent vertices in G. If $x, y \in L$, then both have degree at least $t+1$ by (i), and they have at most one common neighbor by (iii), so x and y are the internal vertices of a copy of $S_{s,t}$ in $G + xy$. If $x \in S - B$, let $z \in N(x) \cap L$. By (ii), there is an independent set I_z of cardinality $t+1$ in $N(z)$ which contains x. There are $t-1$ vertices in $I_z - \{x, y\}$ and $s-1$ neighbors of x which are not in I_z. Therefore, x and z are the internal vertices of a copy of $S_{s,t}$ in $G + xy$. If $x \in B$, we may assume $y \in L$, in which case $B - x$ serves as a set of $s-1$ leaves, and y has a set of $t-1$ neighbors disjoint from B, resulting in a copy of $S_{s,t}$.

We construct G as follows. Let L and S partition the vertex set of G with $|L| = 2\lfloor (n-s)/(2t+4) \rfloor$. Let $r \equiv n - s \pmod{2t+4}$, and let R be a set of r vertices in S. Let B be a clique of order s in S. Let every vertex in L be adjacent to $t+1$ distinct vertices in $S - (B \cup R)$ so that $V(G) - (B \cup R)$ induces a set of at least $2q$ copies of $K_{1,t+1}$. This partitions $S - (B \cup R)$ into classes.

If r is even, make two of these stars into copies of $K_{1,t+1+r/2}$, and put an $(s-2)$-regular bipartite graph on the two sets of $t + 1 + r/2$ vertices in S. Since $|L|$ is even, we can pair up the remaining classes in $S - (B \cup R)$, and put an $(s-2)$-regular bipartite graph on each pair.

If r is odd, let $v \in R$, and repeat the steps in the previous paragraph for $R - v$. If s is even, give v a single neighbor in L, and if s is odd, give v two neighbors in L. If $s > 3$, then take an adjacent pair in $S - B$, delete the edge between them, and give each an edge to v. Repeat this, choosing a different pair of classes at each step for the adjacent pair to ensure condition (ii), until v has degree $s-1$. By our assumption on n, this is always possible, as there are at least $\lfloor s/2 \rfloor - 1$ pairs of classes to choose from.

The resulting graph G meets conditions (i)–(iii). Further, for even r,

$$\begin{aligned} \|G\| &= \left(\frac{s(t+1)}{t+2}\right)\frac{n-r}{2} - \frac{s(t-s+2)}{2t+4} + \frac{sr}{2t+4} \\ &\leqslant \left(\frac{s(t+1)}{t+2}\right)\frac{n}{2} + \frac{s(s+t)}{2t+4}, \end{aligned}$$

and for odd r,

$$\begin{aligned} \|G\| &= \left(\frac{s(t+1)}{t+2}\right)\frac{n-1}{2} - \frac{s(t-s+2)}{2t+4} + \frac{s(r-1)}{2t+4} + \left\lceil \frac{s}{2} \right\rceil \\ &\leqslant \left(\frac{s(t+1)}{t+2}\right)\frac{n}{2} + \frac{s(s-1)}{2t+4} + \left\lceil \frac{s}{2} \right\rceil. \end{aligned}$$

This completes the proof. □

3.3 High-degree neighbors

We now consider triangle-free graphs H with $k_1 > k_0$. In a graph which is saturated with respect to such a graph H, almost every vertex has a neighbor of degree strictly larger than k_0. We now strengthen Lemma 12 in a similar manner to the second statement in Lemma 8. Two corollaries follow from the proof, one which removes the assumption $k_1' \geqslant k_0 + 2$ and the other regarding $S_{s,t}$-saturated graphs of sufficiently large order.

Lemma 15. *Let H be a triangle-free graph with $k_1 > k_0$, and let $n \geqslant |H|$. If $k_1' \geqslant k_0 + \sqrt{2k_0+1}$, or if at least one degree-(k_0+1) endpoint of every edge in H minimizing wt_0 has a neighbor of degree at least k_1' and $k_1' \geqslant k_0 + 2$, then*

$$\mathrm{sat}(n, H) \geqslant \left(k_0 + \frac{k_1' + 1 - k_0}{k_1' + 1}\right)\frac{n}{2} - c,$$

where $c = \frac{(k_0+2)(k_1'+1-k_0)}{2k_1'+2} + \frac{(k_0+1)^2}{8}$.

Proof. Let G be an H-saturated graph with vertex set V of order n, and let S, M, L, and XL partition the vertices $v \in V$ as in the proof of Lemma 12: $S = \{v : d(v) \leqslant k_0\}$, $M = \{v : k_0 < d(v) < k_1'\}$, $L = \{v : d(v) = k_1'\}$, and $XL = \{v : d(v) > k_1'\}$. Again, let A denote the clique $\{v : d(v) < k_0\}$ and let B denote the clique of vertices in S with no high-degree neighbor.

At most one vertex in $L \cup XL$ has all of its neighbors in S, for if there are two then they must be adjacent by Proposition 5. Thus, letting S_{XL} denote the set of vertices in S with a neighbor in XL and $x = d(XL)$, we have $|S_{XL}| \leqslant e(XL, S_{XL}) \leqslant (x-1)|XL| + 1$. By the same arguments used to prove Lemmas 8 and 10, we have

$$x|XL| + k_0|S_{XL}| \geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+1}\right)|XL \cup S_{XL}| - \frac{k_1'+1-k_0}{k_1'+1}.$$

We now consider L and the set S_L of vertices in $S - B$ whose high-degree neighbors have degree exactly k_1'. That is, $S_L = S - (B \cup S_{XL})$.

Case 1. As in the proof of Lemma 12, since G has the property (P1), for any $z \in L$, the set of vertices in $N(z) \cap S_L$ whose only high-degree neighbor is z form a clique (of order at most k_0). It follows that $2|S_L| - k_0|L| \leqslant e(L, S_L) \leqslant (k_1'-1)|L| + 1$.

This gives $2|L \cup S_L| \leqslant (k_1' + k_0 + 1)|L| + 1$, and thus

$$|L| \geqslant \frac{2|L \cup S_L| - 1}{k_1' + k_0 + 1} \quad \text{and} \quad |S| \leqslant \frac{(k_1' + k_0 - 1)|L \cup S_L| + 1}{k_1' + k_0 + 1}.$$

Therefore,

$$k_1'|L| + k_0|S_L| \geqslant \frac{(k_0+2)k_1' + k_0(k_0-1)}{k_1'+k_0+1}|L \cup S_L| - \frac{k_1' - k_0}{k_1'+k_0+1}.$$

Note that

$$\frac{(k_0+2)k_1' + k_0(k_0-1)}{k_1'+k_0+1} \geqslant k_0 + \frac{k_1'+1-k_0}{k_1'+1}$$

if and only if $k_1' \geqslant k_0 + \sqrt{2k_0+1}$. Thus,

$$k_1'|L| + k_0|S_L| \geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+1}\right)|L \cup S_L| - \frac{k_1'-k_0}{k_1'+k_0+1}.$$

Case 2. Now, we suppose that that least one degree-(k_0+1) endpoint of every edge minimizing wt_0 in H has a degree-k_1' neighbor, in which case G has property (P2). As in the proof of Lemma 12, we add to the clique B all those vertices which do not meet condition (ii) of Proposition 11 and, in so doing, remove these vertices from S_L.

By the same reasoning used to prove Lemma 12, at most one neighbor in S_L of any vertex in L has a single high-degree neighbor. Thus,

$$2|S_L| - |L| \leqslant e(L, S_L) \leqslant (k_1'-1)|L| + 1,$$

which can be rewritten as $2|L \cup S_L| \leqslant (k_1'+2)|L| + 1$. It follows that

$$|L| \geqslant \frac{2}{k_1'+2}|L \cup S_L| - \frac{1}{k_1'+2} \quad \text{and} \quad |S| \leqslant \frac{k_1'}{k_1'+2}|L \cup S_L| + \frac{1}{k_1'+2}.$$

Noting that

$$\frac{2k_1' + k_0k_1'}{k_1'+2} = k_0 + \frac{k_1'+1-k_0}{k_1'+1} + \frac{k_1'(k_1'-k_0-1)-2}{(k_1'+1)(k_1'+2)}, \tag{6}$$

whenever $k_1' \geqslant k_0 + 2$, in this case we have

$$\begin{aligned} k_1'|L| + k_0|S_L| &\geqslant \frac{2k_1' + k_0k_1'}{k_1'+2}|L \cup S_L| - \frac{k_1'-k_0}{k_1'+2} \\ &\geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+1}\right)|L \cup S_L| - \frac{k_1'-k_0}{k_1'+2}. \end{aligned}$$

Now, in either Case 1 or Case 2, since $\frac{k_1'+1-k_0}{k_1'+1} > \frac{k_1'-k_0}{k_1'+2} \geqslant \frac{k_1'-k_0}{k_1'+k_0+1}$, and L and XL cannot both have a vertex with all of its neighbors in S, we have

$$x|XL| + k_1'|L| + k_0|S_L \cup S_{XL}| \geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+1}\right)|V - (M \cup B)| - \frac{k_1'+1-k_0}{k_1'+1}.$$

Finally, since $d(v) \geqslant k_0+1$ for all $v \in M$, $d(v) = k_0$ for all $v \in B$, and $|A|(k_0+1-|A|) \geqslant (k_0+1)^2/4$, we have

$$\sum_{v \in V} d(v) \geqslant \left(k_0 + \frac{k_1'+1-k_0}{k_1'+1}\right)n - \frac{(k_0+2)(k_1'+1-k_0)}{k_1'+1} - \frac{(k_0+1)^2}{4},$$

as desired. □

Just like Lemma 12, the bound in Lemma 15 applies to a larger class of graphs H than that of triangle-free graphs. To use property (P1) in Case 1, we require that none of the edges in H which minimize wt_0 are contained in any triangles. To use property (P2) in Case 2, we also require that a degree-(k_0+1) endpoint of any such edge has a neighbor of degree at least k_1' and is not contained in any triangles. Under the assumption $k_1' \geqslant k_0+\sqrt{2k_0+1}$ in Case 1, and under the assumption $k_1' \geqslant k_0+2$ in Case 2, the lower bound on $\operatorname{sat}(n,H)$ in Lemma 15 holds, not only for triangle-free graphs, but any graph H as described above.

To complete the proofs of Lemmas 12 and 15, we needed k_1' to be sufficiently large compared to k_0. However, strengthenings of Theorem 2 can be obtained from these proofs for arbitrary values of k_0 and k_1'. We note one such strengthening below, which we apply to caterpillars in Section 3.4. The interested reader can obtain strengthenings for arbitrary triangle-free graphs in a similar manner.

Corollary 16. *Let H be a triangle-free graph with $k_1 > k_0$, and let $n \geqslant |H|$. If every edge minimizing wt_0 in H has a degree-(k_0+1) endpoint with a neighbor of degree at least k_1', then*

$$\operatorname{sat}(n,H) \geqslant \left(k_0+\frac{2}{k_0+3}\right)\frac{n}{2}-c,$$

where $c=\frac{2k_0+3}{2k_0+6}+\frac{(k_0+1)^2}{8}$.

Proof. Note that the assumption $k_1' \geqslant k_0+2$ was not used in the proof of Lemma 15 until (6). It follows from the proof that

$$k_1'|L|+x|XL|+k_0|S_L\cup S_{XL}| \geqslant \left(k_0+\frac{2(k_1'-k_0)}{k_1'+2}\right)|V-(M\cup B)|-\frac{k_1'-k_0}{k_1'+2}.$$

Since $k_1' \geqslant k_1 \geqslant k_0+1$, and $d(v)=k_0$ for all $v\in B$, we have

$$\sum_{v\in V} d(v) \geqslant \left(k_0+\frac{2}{k_0+3}\right)|G|-\frac{2|B|+1}{k_0+3}-\frac{(k_0+1)^2}{4},$$

and $2|B|+1 \leqslant 2k_0+3$, which completes the proof. □

In addition to providing an improved lower bound when $k_1>k_0$, the techniques used in the proof of Lemma 15 can be used to show that an $S_{s,t}$-saturated graph of sufficiently large order and minimum size cannot have any vertices of degree strictly less than $s-1$. In particular, when $n-s$ is divisible by $2t+4$, the construction provided in the proof of Theorem 14 is optimal. We note that, when $n-s$ is divisible by $t+2$ and t is odd, we can put an $(s-2)$-regular tripartite graph on the leaves of three stars $K_{1,t+1}$ to provide a similar optimal construction when n is sufficiently large.

Corollary 17. *For any $s<t$, and for sufficiently large n,*

$$\operatorname{sat}(n,S_{s,t}) \geqslant \frac{s(t+1)n-s(t-s+2)}{2t+4},$$

and this is tight when $n\equiv s \pmod{2t+4}$.

Proof. Suppose that G is an $S_{s,t}$-saturated graph of order n and that the clique A of vertices in G with degree at most $s-2$ is nonempty. Let $v \in A$. If w is a nonneighbor of v, then w must be the image of either the degree-s or degree-t vertex in the copy of $S_{s,t}$ in $G + vw$, and v must be the image of a leaf. Thus, w has a neighbor of degree at least s.

Let S, L, and XL be as in the proofs of Lemmas 12 and 15; that is, $S = \{v : d(v) < s\}$, $L = \{v : d(v) = t\}$, and $XL = \{v : d(v) > t\}$. Further, let S_L, S_{XL}, and B partition S in the same manner as Case 2 of either proof. The vertex v in A has at most $s-1-|A|$ neighbors in $L \cup XL$. Let C denote this set of high-degree neighbors of v. We have $e(L, S_L) \leqslant (t-1)|L| + |C \cap L|$ and $e(XL, S_{XL}) \leqslant (x-1)|XL| + |C \cap XL|$ where $x = d(XL)$. By similar reasoning to the proof of Lemma 15, we have

$$\sum_{v \in V(G)} d(v) \geqslant \left(s - 1 + \frac{t-s+2}{t+1}\right) n - \frac{|B \cup C|(t-s+2)}{t+1} - \frac{s^2}{4},$$

and the right side of this inequality is strictly larger than

$$\frac{s(t+1)n - s(t-s+2)}{t+2},$$

when n is sufficiently large. Thus, in a minimum $S_{s,t}$-saturated graph G of large order, the set A is empty, and the desired lower bound on $\mathrm{sat}(n, S_{s,t})$ follows from the proof of Lemma 12. Tightness when $n \equiv s \pmod{2t+4}$ follows from the upper bound construction in Theorem 14. □

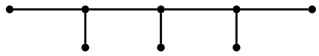
3.4 Shorty the caterpillar and further remarks

Let us continue our discussion of saturation numbers of trees. We would like to apply Lemma 15 to a tree with $k_1 > k_0$. In order that this condition be met, the diameter must be at least 4. Consider the caterpillar P_5^s, obtained from a path of length 4 by attaching s pendant edges to each of the three internal vertices (see Figure 4a). We name this caterpillar Shorty. While the assumption that $k_1' \geqslant k_0 + 2$ in Lemma 12 is met by any unbalanced double star, the same assumption in Lemma 15 is not met by Shorty; every edge uv has $\mathrm{wt}_0(uv) = s+1$ and a degree-$(s+2)$ endpoint with a neighbor of degree k_1', but $k_1' = s+2$. We thus apply Corollary 16 to obtain a lower bound on Shorty's saturation number. The upper bound suggested by Lemma 15 does, however, hold for Shorty (see Figure 4b).

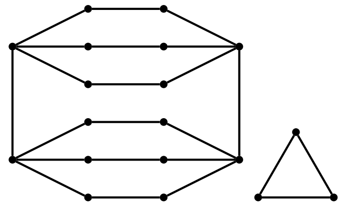
Theorem 18. *For any positive integer s, and for any $n \geqslant q(2s+4)+s+1$ where $q = \max\{2, \lfloor (s-1)/2 \rfloor\}$,*

$$\left(s + \frac{2}{s+3}\right)\frac{n}{2} - c_1 \leqslant \mathrm{sat}(n, P_5^{s-1}) \leqslant \left(s + \frac{2}{s+2}\right)\frac{n}{2} + c_2,$$

where $c_1 = \frac{2s+3}{2s+6} + \frac{(s+1)^2}{8}$ and $c_2 = \frac{s(s+1)}{s+2}$.



(a) The caterpillar P_5^1.



(b) A P_5^1-saturated graph.

Figure 4: The caterpillar P_5^1 on the left and a P_5^1-saturated graph on the right of order $n = 19$ and size $(5n-3)/4 = 23$.

Proof. Let $H = P_5^{s-1}$, so that $k_0 = s$ and $k_1 = k_1' = s+1$. The lower bound follows from Corollary 16. We again provide a construction for the upper bound. In particular, we construct an n-vertex graph G with the following properties:

(i) We have $V(G) = S \cup L$. For all $v \in S$, $d(v) = s$. For all $v \in L$, $d(v) \geqslant s+2$.

(ii) For all $v \in L$, $|N(v) \cap L| = 1$. Further, if $u, v \in L$ and $uv \in E(G)$, then $N(u) \cap N(v) = \varnothing$ and there are no edges between $N(u) \cap S$ and $N(v) \cap S$.

(iii) For all $v \in L$, every $w \in N(v) \cap S$ is contained in an independent set of cardinality $s+1$ in $N(v) \cap S$.

(iv) Aside from a clique B of order $s+1$, every vertex in S has a neighbor in L, and at most one vertex has two neighbors in L.

Such a graph G is P_5^{s-1}-free as there is no path of three consecutive vertices that have degree at least $s+1$. We now show that G is P_5^{s-1}-saturated. Let x and y be nonadjacent vertices in G. First, suppose $x, y \in L$. By (i) and (ii), there exists $z \in N(x) \cap L$. Further, $N(x) \cap S$ and $N(z) \cap S$ are disjoint sets of cardinality at least $s+1$. By (iv), $|N(y) \cap (N(x) \cup N(z))| \leqslant 1$, so there exists a set $I_y \subseteq N(y)$ of cardinality s, which is disjoint from $N(x) \cup x$ and $N(z) \cup z$. Let $I_x \subseteq N(x) \cap S$ be of cardinality $(s-1)$, and let $I_z \subseteq N(z) \cap S$ be of cardinality s. We obtain a copy of P_5^{s-1} in $G + xy$ with internal vertices y, x, and z, and leaves $I_y \cup I_x \cup I_z$.

Otherwise, at least one of x or y is in S. We assume, without loss of generality, that this vertex is x. Suppose $x \in B$. Let $I_x = B - x$. By (ii) or (iv), depending on whether y is in L or $S - B$, respectively, there exists $z' \in N(y) \cap L$. If $y \in L$, then $N(y) \cap N(z') = \varnothing$ by (ii), and there exist subsets $I_y \subset N(y)$ and $I_{z'} \subset N(z')$, of cardinalities $s-1$ and s, respectively, such that I_x, I_y, and $I_{z'}$ are pairwise disjoint. On the other hand, if $y \in S$, then by (iii) there is an independent set $I \subseteq N(z')$ of cardinality $(s+1)$ that contains y. Let $I_{z'} = I - y$. Note that there exists a subset $I_y \subseteq N(y)$ of cardinality $(s-1)$ such that $I_{z'}$, I_y, and I_x are pairwise disjoint. We thus obtain a copy of P_5^{s-1} in $G + xy$ with internal vertices x, y, and z', and leaves $I_x \cup I_y \cup I_{z'}$.

Finally, suppose $x \in S - B$. By (iv), there exists $z \in N(x) \cap L$, and by (ii), there is a single vertex z' in $N(z) \cap L$. By (iii), x is contained in an independent set $I \subseteq N(z) \cap S$

of cardinality $s+1$. Let $I_z = I - \{x,y\}$. If $z' \neq y$, let $I_x = N_{G+xy}(x) - z$. Note $|I_x| = s$ by (i), and $I_x \cap I_z = \varnothing$. In this case, we have $N(z') \cap N(z) = \varnothing$, $|N(z') \cap S| \geqslant s+1$, and $I_x \cap N(z') = \varnothing$ by (ii). Thus, $N(z') \cap S$ contains a subset $I_{z'}$ of cardinality s which is disjoint from I_z and I_x. In this case, x, z, and z' make up the internal vertices, and $I_x \cup I_z \cup I_{z'}$ the leaves, of a copy of P_5^{s-1} in $G+xy$. On the other hand, if $y = z'$, then $N(x) \cap N(y) = \{z\}$ by (ii). Let $I_x = N(x) - z$. Note that $|I_z| = s$ in this case. By (i), $|I_x| = s-1$, and there exists a subset $I_y \subset N(y) \cap S$ of cardinality s. By (ii), $I_x \cap I_y$ and $I_y \cap I_z$ are both empty. We have $I_x \cap I_z = \varnothing$ since $I_z \subset I$. Thus, z, x, and y make up the internal vertices, and $I_z \cup I_x \cup I_y$ the leaves, of a copy of P_5^{s-1} in $G+xy$. It follows that G is P_5^{s-1}-saturated.

We construct G as follows. Let S and L partition the vertex set V of G with $|L| = 2\lfloor (n-s-1)/(2s+4) \rfloor$. Let $r \equiv n-s-1 \pmod{2s+4}$, and let R be a set of r vertices in S. Let B be a clique of order $s+1$ in S. Let every vertex in L be adjacent to one other vertex in L and to a distinct set of $s+1$ vertices in $S - (B \cup R)$ so that $V - (B \cup R)$ is a set of at least q double stars $S_{s+2,s+2}$. If r is even, make two of these double stars into copies of $S_{s+2+r/2,s+2}$. Put an $(s-1)$-regular bipartite graph on the two classes of size $s+2+r/2$. If $|L|/2$ is even, put another $(s-1)$-regular bipartite graph on the two remaining classes in this pair. Then, pair up the remaining double stars and similarly put $(s-1)$-regular bipartite graphs between classes which do not correspond to adjacent vertices in L (as in Figure 4b). If $|L|/2$ is odd, then we make a triple of double stars, corresponding to three pairs of classes of vertices in S: (A,B), (C,D), and (E,F). Add three $(s-1)$-regular bipartite graphs with partite sets $(B,C), (D,E), (A,F)$. Now pair up the remaining double stars as in the case where $|L|/2$ is even.

If r is odd, let v be a vertex in R. Repeat the construction in the previous paragraph, replacing R by $R - v$. If s is odd, give v a single neighbor in L, and otherwise give v two unmatched neighbors in L. If $s > 2$, then take an adjacent pair of vertices in $S - B$, delete the edge between them, and give each an edge to v. Repeat this, at each step choosing a different pair of classes for the adjacent pair in $S - B$ to ensure condition (iii), until v has degree s. By our assumption on n, this is always possible, as there are at least $\lfloor (s-1)/2 \rfloor$ pairs of classes to choose from.

The resulting graph G meets conditions (i)–(iv). Further, for even r,

$$\|G\| = \left(s + \frac{2}{s+2}\right)\frac{n}{2} - \frac{s+1}{s+2} + \frac{rs}{2s+4} \leqslant \left(s + \frac{2}{s+2}\right)\frac{n}{2} + s - \frac{2s+1}{s+2}$$

and for odd r,

$$\|G\| = \left(s + \frac{2}{s+2}\right)\frac{n-1}{2} - \frac{s+1}{s+2} + \frac{(r-1)s}{2s+4} + \left\lfloor \frac{s+2}{2} \right\rfloor.$$

One can check that the right side of the above equation is at most $\big(s + 2/(s+2)\big)n/2 + c_2$, which completes the proof. $\square$

We note that the lower bound in Theorem 18 applies to biregular caterpillars of arbitrary diameter. That is, it applies to any caterpillar P_ℓ^{s-1} obtained from a path on $\ell \geqslant 5$

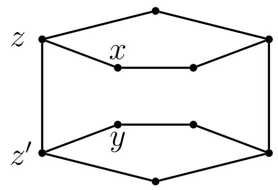


Figure 5: A graph G with property (P2) for $k_0 = 2$ and $k'_1 = 3$. Every vertex has a degree-3 neighbor, but G is not P_5^1-saturated.

vertices by appending $s-1$ leaves to each internal vertex. For $\ell \geqslant 7$, the degrees of second neighbors of the edges of P_ℓ^{s-1} will be relevant in determining their saturation numbers. We also note that any argument which holds for P_ℓ^0 must use the H-free property of saturation, for it is known that the semisaturation number of P_ℓ is asymptotically less than its saturation number for $\ell \geqslant 6$ and $n \geqslant 3\ell - 3$ [1].

We conclude with a remark on our lower bound for P_5^{s-1} and a discussion of potential strengthenings. A graph G with property (P2) ($k_0 = 2$, $k'_1 = 3$) and $6n/5$ edges is depicted in Figure 5. There are at least two reasons why G is not P_5^1-saturated, the former being that the pair of vertices y, z does not meet the following property possessed by nonadjacent pairs in an H-saturated graph for triangle-free H: there should be a subset C of $N(y)$ or $N(z)$, $|C| = k_0$, and a vertex $w \in N(y) \cup N(z)$ such that $|N(w) - (C \cup \{y, z\})| \geqslant k_1 - 1$. It is possible that this property can be used to strengthen our lower bound on $\text{sat}(n, P_\ell^s)$. A stronger lower bound may also follow from a strengthening of Theorem 3 for square-free graphs. For example, consider the nonadjacent pair x, y in G. While this pair does not contradict property (P2), it is the lack of squares in P_5^1 which stops xy from creating a copy in $G + xy$. Indeed, $z \in N(x)$ and $|N(z) - (N(x) \cup y)| = 2$, but one of the vertices in $N(z) - (N(x) \cup y)$ is the high-degree neighbor z' of y. Since y and z' each have only one neighbor outside of $\{x, y, z, z'\}$, neither can play the role of a third high-degree vertex in P_5^1 with x and z as the other high-degree vertices. By symmetry, the same is true if we had used y and z' instead of x and z.

Lending credence to the idea that the upper bound in Theorem 18 may be tight, we note that P_5^0 is simply a path of length 4, and the saturated graphs of minimum size for paths are characterized in [5]. In particular, a disjoint union of copies of $S_{3,3}$ is a minimum P_5-saturated graph, matching our construction.

Acknowledgements

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