Clustered Colouring of Graph Products

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Abstract

A colouring of a graph G has *clustering* k if the maximum number of vertices in a monochromatic component equals k. Motivated by recent results showing that many natural graph classes are subgraphs of the strong product of a graph with bounded treewidth and a path, this paper studies clustered colouring of strong products of two bounded treewidth graphs, where none, one, or both graphs have bounded degree. For example, in the case of two colours, if n is the number of vertices in the product, then we show that clustering $\Theta(n^{2/3})$ is best possible, even if one of the graphs is a path. However, if both graphs have bounded degree, then clustering $\Theta(n^{1/2})$ is best possible. With three colours, if one of the graphs has bounded degree, then we show that clustering $\Theta(n^{3/7})$ is best possible. However, if neither graph has bounded degree, then clustering $\Omega(n^{1/2})$ is necessary. More general bounds for any given number of colours are also presented.

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1 Introduction

A *colouring* of a graph G is a function that assigns a 'colour' to each vertex of G. A *c-colouring* is a colouring that uses at most c colours. We allow adjacent vertices to be assigned the same colour. A colouring of a graph has *clustering* k if the maximum number of vertices in a monochromatic component equals k. Here, a *monochromatic component* is a component of the subgraph induced by the vertices assigned a single colour. For example, a graph with chromatic number c is c-colourable with clustering 1. See [29] for an extensive survey on this topic.

Most research on clustered colouring gives constant bounds on the clustering (independent of $|V(G)|$). Some other papers focus on the case where the number of colours is so small that dependence on $|V(G)|$ is unavoidable (see [2, 21, 24] for example). We take the latter approach. In particular, we focus on clustered colouring of strong products of two graphs with bounded treewidth (see Section 2 for definitions of these notions). One motivation for considering such graphs is that it has been recently shown that many natural graph classes are subgraphs of the strong product of a graph with bounded treewidth and a path [7, 8, 10, 11, 18–20] or the strong product of two graphs with bounded treewidth [6, 22] (see Section 7 for more on this theme).

We study two variations of these products. In the first, we assume that one of the graphs has bounded maximum degree, generalising the paths in the aforementioned products. We then study the scenario with no restriction on the maximum degree. In each scenario, given an integer $c \geqslant 2$, our goal is to prove tight bounds on the optimal clustering in c-colourings of products $H_1 \boxtimes H_2$, where H_1 and H_2 are graphs of bounded treewidth. Our main results are stated in the next section. They focus on the exponent of $|V(H_1 \boxtimes H_2)|$ in the clustering value. More precise statements accompany their proofs.

1.1 Main results

First, consider 2-colourings of graph products. We show the following asymptotically tight bound on the optimal clustering in this case.

Theorem 1. *For any fixed integer $t \geqslant 1$ and any graphs H_1, H_2 with treewidth at most t, the graph $H_1 \boxtimes H_2$ is 2-colourable with clustering $O(|V(H_1 \boxtimes H_2)|^{2/3})$. Furthermore, there are infinitely many graphs H with treewidth 2 and paths P such that every 2-colouring of $H \boxtimes P$ has clustering $\Omega(|V(H \boxtimes P)|^{2/3})$.*

Theorem 1 shows that in the case of two colours, perhaps surprisingly, the maximum degree condition has no significant impact on the optimal clustering value. For more than two colours, the maximum degree condition has a significant impact on the optimal clustering value of colourings of graph products.

Now consider the case of three colours. When one graph has bounded degree, we prove the following asymptotically tight bound.

Theorem 2. *For any fixed integers $t, \Delta \geqslant 1$, for any graph H_1 with treewidth at most t, and any graph H_2 with treewidth at most t and maximum degree at most Δ, the graph $H_1 \boxtimes H_2$*

is 3-colourable with clustering $O(|V(H_1 \boxtimes H_2)|^{3/7})$. Furthermore, there are infinitely many graphs H with treewidth 3 and paths P such that every 3-colouring of $H \boxtimes P$ has clustering $\Omega(|V(H \boxtimes P)|^{3/7})$.

The following result shows that dropping the maximum degree restriction impacts the exponent in both upper and lower bounds.

Theorem 3. *For any fixed integer $t \geqslant 1$ and any graphs H_1, H_2 with treewidth at most t, the graph $H_1 \boxtimes H_2$ is 3-colourable with clustering $O(|V(H_1 \boxtimes H_2)|^{4/7})$. Furthermore, there exist infinitely many pairs of graphs H_1, H_2 with treewidth 2 such that every 3-colouring of $H_1 \boxtimes H_2$ has clustering $\Omega(|V(H_1 \boxtimes H_2)|^{1/2})$.*

For $c \geqslant 4$ colours, when one graph has bounded degree, we obtain the following general upper bound.

Theorem 4. *For any fixed integers $t, c, \Delta \geqslant 1$, for any graph H_1 with treewidth at most t, and any graph H_2 with treewidth at most t and maximum degree at most Δ, the graph $H_1 \boxtimes H_2$ is c-colourable with clustering $O(|V(H_1 \boxtimes H_2)|^{c/(c^2-c+1)})$.*

We use the lower bound from Theorem 2 to prove the following general lower bound.

Proposition 5. *For any fixed integer $c \geqslant 3$, there are infinitely many graphs H with treewidth c and infinitely many paths P, such that every c-colouring of $H \boxtimes P$ has clustering $\Omega(|V(H \boxtimes P)|^{1/(c-\frac{2}{3})})$.*

There is a sharp change in the asymptotic behaviour of both upper and lower bounds on the clustering in c-colourings of $H_1 \boxtimes H_2$ when we remove the restriction on the maximum degree of H_2. Indeed, while the bounds of Theorem 4 and Proposition 5 are of the form $n^{1/(c-O(1))}$, in the unrestricted setting we prove the following upper and lower bound of the form $n^{\Theta(1/\sqrt{c})}$.

Theorem 6. *For any fixed integers $t \geqslant 1$ and $c \geqslant 4$, for any graphs H_1 and H_2 with treewidth at most t, the graph $H_1 \boxtimes H_2$ is c-colourable with clustering $O(|V(H_1 \boxtimes H_2)|^{1/\lfloor \sqrt{c} \rfloor})$. Furthermore, for any integer $c \geqslant 4$, there are infinitely many pairs of graphs H_1, H_2 with treewidth at most $\sqrt{c}$ such that every c-colouring of $H_1 \boxtimes H_2$ has clustering $\Omega(|V(H_1 \boxtimes H_2)|^{1/2\sqrt{c}})$.*

Of course, Theorems 4 and 6 are only of interest for $c < (t+1)^2$. Indeed, graphs with treewidth t are t-degenerate and thus properly $(t+1)$-colourable; that is, they are $(t+1)$-colourable with clustering 1. A product colouring then shows that $H_1 \boxtimes H_2$ is properly $(t+1)^2$-colourable.

1.2 Summary

The paper is organised as follows. In Section 2 we establish the basic notation and key tools. Our result with two colours, Theorem 1, is proved in Section 3. The results with three colours, Theorems 2 and 3, are proved in Section 4. Our most general results, Theorem 4

together with its associated lower bound given in Proposition 5 as well as Theorem 6, are proved in Section 5. Section 6 studies clustered colouring of products of two graphs, both with bounded treewidth and bounded maximum degree. This setting turns out to be much simpler than the above two scenarios, and the proofs are straightforward. Section 7 concludes with a discussion on planar graphs.

Table 1 summarises all our results. The notation $\Omega(f(n))$ means that for some $\delta = \delta(t, \Delta) > 0$, there are infinitely many graphs H_1, H_2 such that every c-colouring of $H_1 \boxtimes H_2$ has clustering at least $\delta f(|V(H_1 \boxtimes H_2)|)$. The notation $O(f(n))$ means that for some $\delta = \delta(t, \Delta) > 0$ and for any graphs H_1, H_2, there exists a c-colouring of $H_1 \boxtimes H_2$ with clustering at most $\delta f(|V(H_1 \boxtimes H_2)|)$. If $f(n)$ does not depend on n (that is, $c \geqslant 3$ in the first column), the constant δ is absolute (and does not depend on t, Δ).

Table 1: Clustering in c-colourings of $H_1 \boxtimes H_2$, where $n := |V(H_1 \boxtimes H_2)|$.

	$\Delta(H_1) \leqslant \Delta$, $\mathrm{tw}(H_1) \leqslant t$ $\Delta(H_2) \leqslant \Delta$, $\mathrm{tw}(H_2) \leqslant t$	$\mathrm{tw}(H_1) \leqslant t$ $\Delta(H_2) \leqslant \Delta$, $\mathrm{tw}(H_2) \leqslant t$	$\mathrm{tw}(H_1) \leqslant t$ $\mathrm{tw}(H_2) \leqslant t$
$c = 2$	$\Theta(n^{1/2})$	$\Theta(n^{2/3})$	$\Theta(n^{2/3})$
$c = 3$	$O(t^3 \Delta^4)$	$\Theta(n^{3/7})$	$\Omega(n^{1/2})\ O(n^{4/7})$
$c \geqslant 4$	$O(t^2 \Delta^2)$	$\Omega(n^{1/(c-2/3)})\ O(n^{c/(c^2-c+1)})$	$\Omega(n^{1/(2\sqrt{c})})\ O(n^{1/\lfloor\sqrt{c}\rfloor})$

For each case in which upper and lower bounds do not match, we conjecture the recorded upper bound to be asymptotically tight. In particular, we believe the upper bound in Theorem 4 is asymptotically tight, and furthermore conjecture that there exists a construction matching this upper bound where the bounded degree graph is a path.

Conjecture 7. For any fixed integer $c \geqslant 2$, there exists an integer t such that for infinitely many graphs H with treewidth t and paths P, every c-colouring of $H \boxtimes P$ has clustering $\Omega(|V(H \boxtimes P)|^{c/(c^2-c+1)})$.

We also believe that the upper bounds in Theorems 3 and 6 are asymptotically tight.

Conjecture 8. There exists an integer $t \geqslant 1$ such that for infinitely many graphs H_1, H_2 with treewidth t, every 3-colouring of $H_1 \boxtimes H_2$ has clustering $\Omega(|V(H_1 \boxtimes H_2)|^{4/7})$.

Conjecture 9. For any fixed integer $c \geqslant 4$ there exists an integer k such that for infinitely many graphs H_1, H_2 with treewidth k, every c-colouring of $H_1 \boxtimes H_2$ has clustering $\Omega(|V(H_1 \boxtimes H_2)|^{1/((1+o_c(1))\sqrt{c})})$.

2 Definitions and tools

We use standard graph-theoretic notation. In particular, given a graph G, we write $V(G)$ for its vertex set and $E(G)$ for its edge set.

For a non-empty tree T, a *T-decomposition* of a graph G is a collection $\mathcal{W} = (W_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that:
  (i) for every edge $vw \in E(G)$, there exists a node $x \in V(T)$ with $v, w \in W_x$; and

(ii) for every vertex $v \in V(G)$, the set $\{x \in V(T) \colon v \in W_x\}$ induces a (connected) subtree of T.

Each set W_x in $\mathcal{W}$ is called a *bag*. The *width* of $\mathcal{W}$ is $\max\{|W_x| \colon x \in V(T)\} - 1$. A *tree-decomposition* is a T-decomposition for any tree T. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree-decomposition of G. Intuitively, the treewidth of G measures how 'tree-like' G is. For example, trees have treewidth 1, and outerplanar graphs have treewidth at most 2. Treewidth is of fundamental importance in the graph minor theory of Robertson and Seymour and in algorithmic graph theory; see [3, 15, 27] for surveys on treewidth.

Denote by P_n the path on n vertices. Let F_n be the *fan* graph consisting of a path P_n (called the *base*) and one additional dominant vertex adjacent to every vertex along the path. If v is the dominant vertex and $(w_1, \ldots, w_n)$ is the base path, then the sequence $\{v, w_1, w_2\}, \{v, w_2, w_3\}, \ldots, \{v, w_{n-1}, w_n\}$ defines a P_{n-1}-decomposition of F_n with width 2. This demonstrates that $\mathrm{tw}(F_n) \leqslant 2$. The following is another well-known fact about fans.

Lemma 10. *Every 2-colouring of F_n has clustering at least $\lfloor \sqrt{n} \rfloor$.*

Proof. Consider any red/blue colouring of F_n with clustering k. Without loss of generality, the dominant vertex of F_n is red. So there are at most $k - 1$ red vertices in the base path P_n of F_n. Thus there are at most k blue components in P_n, each with at most k vertices. Hence $n \leqslant (k-1) + k^2 < (k+1)^2$, implying $k > \sqrt{n} - 1$ and $k \geqslant \lfloor \sqrt{n} \rfloor$. $\square$

The *cartesian product* of graphs G and H, denoted by $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (u', v') in $G \,\square\, H$ if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (u', v') in $G \boxtimes H$ if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$, or $uu' \in E(G)$ and $vv' \in E(H)$.

Strong products and clustered colouring are inherently related. In particular, it follows from the definitions that a graph G is c-colourable with clustering at most k if and only if G is contained in $H \boxtimes K_k$ for some c-colourable graph H. Here, a graph G is *contained* in a graph G' if a subgraph of G' is isomorphic to G.

For an integer $m \geqslant 1$ and a graph G, define $\widehat{m\,G}$ to be the graph constructed by taking m pairwise disjoint copies of G and adding one additional vertex adjacent to all vertices in each copy. Note that

$$\mathrm{tw}(\widehat{m\,G}) \leqslant \mathrm{tw}(G) + 1. \tag{1}$$

To see this, take a tree-decomposition of each copy of G, each with width $\mathrm{tw}(G)$. Add the new dominating vertex to each bag of each decomposition, and add some edges to connect the trees together, forming one large tree. This is a tree-decomposition of $\widehat{m\,G}$ with width $\mathrm{tw}(G) + 1$, hence $\mathrm{tw}(\widehat{m\,G}) \leqslant \mathrm{tw}(G) + 1$.

Several proofs in this article use the following two results of Dvořák and Wood [12]. The first allows us to find separators in graphs with bounded treewidth.

Lemma 11 ([12] Lemma 25). *For any integers $n, t \geqslant 1$ and any real number $p > 0$, every graph G on n vertices with treewidth at most t has a set S of at most p vertices such that each component of $G - S$ has at most $\frac{(t+1)n}{p}$ vertices.*

The second lemma is a corollary of the isoperimetric inequality of Bollobás and Leader [4].

Lemma 12 ([12] Lemma 13). *If S is any set of at most $\frac{n^2}{2}$ vertices in $G := P_n \square P_n$ and $k \leqslant \frac{n^2}{e^2}$ and each component of $G - S$ has at most k vertices, then $n^2 \leqslant 4|S|k^{1/2}$.*

In addition, the constructions in this paper typically involve taking the strong products of graphs that contain long paths, producing grid structures. To analyse clustered colourings of these, we use the following result known as the Hex Lemma; see [14, 16] for background, and see [23, Proposition 6.1.4] for the precise version of the statement below.

Lemma 13 (Hex Lemma). *Let G be a plane internal triangulation with outer-cycle (a, b, c, d). In every 2-colouring χ of G with $\chi(a) = \chi(c)$ and $\chi(b) = \chi(d)$, there is a monochromatic path either between a and c or between b and d.*

3 Two colours

We start by studying 2-colourings of graph products, proving Theorem 1. In this case, we demonstrate that the upper bound with no maximum degree restriction and the lower bound with a bounded degree graph are asymptotically tight up to a multiplicative constant. Theorem 1 follows immediately from Lemmas 14 and 15.

Lemma 14. *For any graphs H_1 and H_2 both with treewidth at most t, the graph $G := H_1 \boxtimes H_2$ is 2-colourable with clustering at most $2((t + 1)|V(G)|)^{2/3}$.*

Proof. Let $n_i := |V(H_i)|$ for $i \in \{1, 2\}$. Let $n := |V(G)| = n_1 n_2$. By Lemma 11, for each $i \in \{1, 2\}$, there is a set S_i of at most $(t + 1)^{2/3} n_i / n^{1/3}$ vertices in H_i such that each component of $H_i - S_i$ has at most $((t+1)n)^{1/3}$ vertices. Let $X := (S_1 \times V(H_2)) \cup (V(H_1) \times S_2)$. So

$$|X| \leqslant |S_1| n_2 + |S_2| n_1 \leqslant (t + 1)^{2/3} n_1 n_2 / n^{1/3} + (t + 1)^{2/3} n_2 n_1 / n^{1/3} = 2((t + 1)n)^{2/3}.$$

Colour each vertex in X blue and colour all other vertices red. Each red component Y is contained in $Y_1 \boxtimes Y_2$ for some component Y_1 of $H_1 - S_1$ and some component Y_2 of $H_2 - S_2$. Thus $|V(Y)| \leqslant |V(Y_1 \boxtimes Y_2)| \leqslant ((t+1)n)^{2/3}$. Hence G is 2-coloured with clustering at most $2((t + 1)n)^{2/3}$. $\square$

We now prove that Lemma 14 is asymptotically tight, even when H_2 is a path.

Lemma 15. *There are infinitely many graphs H with treewidth 2 and paths P such that every 2-colouring of $H \boxtimes P$ has clustering at least $\frac{1}{3}|V(H \boxtimes P)|^{2/3}$.*

Proof. Let n be a sufficiently large integer so that $\frac{1}{2}n^2 \geqslant \frac{1}{3}(n^3 + n^2)^{2/3}$, and let $P := P_n$, $H := F_{n^2}$, and $G := H \boxtimes P$ so that $|V(G)| = n^3 + n^2$. With x being the dominating vertex of the fan H, we refer to $\{(x, v) : v \in P\}$, the set of copies of x in G, as the *spine* of G.

Suppose, toward a contradiction, that there exists a 2-colouring of G in red and blue with clustering less than $\frac{1}{2}n^2$. Suppose that two adjacent vertices of the spine are coloured differently. Then there is a red vertex and a blue vertex of G both dominating the same copy of P_{n^2}. Without loss of generality, at least $\frac{1}{2}n^2$ of these vertices are coloured red, and they are all connected into one component by the dominating red vertex, contradicting the fact that the colouring has clustering less than $\frac{1}{2}n^2$.

It follows that the entire spine is monochromatic. Let this colour be red. Since vertices along the spine are dominating in their respective copies of H, this means that all red vertices in the graph G are joined into a single component and therefore there are less than $\frac{1}{2}n^2$ red vertices in G.

Consider the crossed $(n^2 \times n)$-grid, $P_{n^2} \boxtimes P_n$, where the first factor is the base of the fan H. At most $\frac{1}{2}n^2$ columns contain a red vertex. So at least $\frac{1}{2}n^2$ columns are all-blue. Moreover, there is a row R with less than $\frac{1}{2}n$ red vertices. There are at most $\frac{1}{2}n + 1$ blue components in R. At least one of these components intersects at least $(\frac{1}{2}n^2)/(\frac{1}{2}n + 1) \geqslant \frac{1}{2}n$ all-blue columns, which are thus contained in a blue component with at least $(\frac{1}{2}n)n = \frac{1}{2}n^2$ vertices. $\qquad \square$

Note that the bound $\mathrm{tw}(H) \leqslant 2$ in Lemma 15 is best possible. In particular, for any trees T_1 and T_2, by Proposition 36 there exists a 2-colouring of $T_1 \boxtimes T_2$ with clustering at most $|V(T_1 \boxtimes T_2)|^{1/2}$.

We present an alternative proof that the upper bound in Lemma 14 is asymptotically tight, using a symmetric construction.

Lemma 16. *For any integer $n \geqslant 8$, every 2-colouring of $F_n \boxtimes F_n$ has clustering at least $\frac{1}{3}(n+1)^{4/3}$.*

Proof. Let A and B be copies of F_n, with $V(A) = \{a_0, a_1, \ldots, a_n\}$ and $V(B) = \{b_0, b_1, \ldots, b_n\}$, where a_0 dominates the path $a_1, \ldots, a_n$ in A, and b_0 dominates the path $b_1, \ldots, b_n$ in B. The graph $A \boxtimes B$ contains an induced copy of $P_n \boxtimes P_n$ on the vertices (a_i, b_j) with $1 \leqslant i, j \leqslant n$, and consider its subgraph $H \cong P_n \square P_n$.

Suppose, for the sake of contradiction, that there exists a 2-colouring of $A \boxtimes B$ with clustering less than $k := \frac{1}{3}(n+1)^{4/3}$. Without loss of generality, assume that (a_0, b_0) is red. Note that, since $n \geqslant 8$, we have $k \leqslant \frac{n^2}{2}$ and $k \leqslant \frac{n^2}{e^2}$.

Let S be the set of red vertices in H. Since (a_0, b_0) dominates $A \boxtimes B$, we have $|S| \leqslant k$. Each component of $H - S$ is contained in a monochromatic component of $A \boxtimes B$, and thus has at most k vertices. By Lemma 12,

$$n^2 \leqslant 4|S|k^{1/2} \leqslant 4k^{3/2} \leqslant 4(\tfrac{1}{3}(n+1)^{4/3})^{3/2} < n^2,$$

since $n \geqslant 8$, which is the desired contradiction. $\qquad \square$

4 Three colours

We now move to 3-colourings of graph products, proving Theorems 2 and 3.

4.1 Unrestricted maximum degree

We start by proving Theorem 3, where H_1, H_2 both have bounded treewidth. For the upper bound, we build on the 2-colour construction from Lemma 14 to prove the following result with three colours. This approach leads to a bound for all c by induction. However, for $c \geqslant 4$, Lemma 32 yields a better bound than would be achieved with this strategy.

Lemma 17. *For any graphs H_1 and H_2 both with treewidth at most t, the graph $G := H_1 \boxtimes H_2$ is 3-colourable with clustering at most $2(t+1)^{6/7}|V(G)|^{4/7}$.*

Proof. Let $n_i := |V(H_i)|$ for $i \in \{1, 2\}$. Let $n := |V(G)| = n_1 n_2$. By Lemma 11, for each $i \in \{1, 2\}$, there is a set S_i of at most $(t+1)^{6/7}n_i/n^{3/7}$ vertices in H_i such that each component of $H_i - S_i$ has at most $(t+1)^{1/7}n^{3/7}$ vertices. Let $X := (S_1 \times V(H_2)) \cup (V(H_1) \times S_2)$. So

$$|X| \leqslant |S_1|n_2 + |S_2|n_1 \leqslant (t+1)^{6/7}n_1 n_2/n^{3/7} + (t+1)^{6/7}n_2 n_1/n^{3/7} = 2(t+1)^{6/7}n^{4/7}.$$

Each component Y of $G - X$ is contained in $Y_1 \boxtimes Y_2$ for some component Y_1 of $H_1 - S_1$ and some component Y_2 of $H_2 - S_2$, implying that $|V(Y_1 \boxtimes Y_2)| \leqslant (t+1)^{2/7}n^{6/7}$. By Lemma 14, $Y_1 \boxtimes Y_2$ (and hence also Y) is 2-colourable with clustering at most

$$2((t+1)|V(Y_1 \boxtimes Y_2)|)^{2/3} \leqslant 2((t+1)(t+1)^{2/7}n^{6/7})^{2/3} \leqslant 2(t+1)^{6/7}n^{4/7}.$$

By using a third colour for X, we obtain a 3-colouring of G with clustering at most $2(t+1)^{6/7}n^{4/7}$. $\qquad\square$

The next lemma proves the lower bound in Theorem 3, and shows that the upper bound in Theorem 2 for the bounded degree scenario is not optimal in the unrestricted scenario.

Lemma 18. *Every 3-colouring of $F_n \boxtimes F_n$ has clustering at least $\left(1 - \frac{1}{\sqrt{2}}\right)n$.*

Proof. Let A and B be copies of F_n, with $V(A) = \{a_0, a_1, \ldots, a_n\}$ and $V(B) = \{b_0, b_1, \ldots, b_n\}$, where a_0 dominates the path $a_1, \ldots, a_n$ in A, and b_0 dominates the path $b_1, \ldots, b_n$ in B. Suppose there exists a red/blue/green-colouring of $G := A \boxtimes B$ with clustering less than δn, where $\delta := 1 - \frac{1}{\sqrt{2}}$. We may assume without loss of generality that the dominating vertex (a_0, b_0) is red. In particular, there are less than δn red vertices in G.

Let $X = \{(a_0, b_i) : i \in [n]\}$ and $Y = \{(a_i, b_0) : i \in [n]\}$. Note that for any $i, j \in [n]$, the vertices (a_0, b_i) and (a_j, b_0) are adjacent in G. Assume that X contains at least one blue vertex and at least one green vertex. Since each vertex of X dominates Y, it follows that Y contains less than δn blue and less than δn green vertices. This is a contradiction, since

there are less than δn red vertices in G and $\delta < 1/3$ while $|Y| = n$. Therefore, without loss of generality, X contains more than $(1 - \delta)n$ blue vertices and no green vertices.

We can apply similar reasoning to Y. Given that X contains at least one blue vertex, Y has less than δn blue vertices, and therefore Y contains more than $(1 - \delta)n$ green vertices and no blue vertices.

Now focus on the subgraph H of G induced by $\{(a_i, b_j) : i, j \in [n]\}$, which forms a copy of $P_n \boxtimes P_n$. Each vertex of X dominates a row of H, and each vertex of Y dominates a column. Since there are less than δn red vertices in G, there exist more than $(1 - \delta)n$ rows with no red vertices, each dominated by a blue vertex from X. Each such row contains less than δn blue vertices, and hence more than $(1 - \delta)n$ green vertices. Therefore, H contains more than $(1 - \delta)^2 n^2$ green vertices. If we consider columns dominated by green vertices, an identical argument yields that there are more than $(1 - \delta)^2 n^2$ blue vertices in H. Since $2(1 - \delta)^2 = 1$, there are more than n^2 vertices in H coloured blue or green, but this is a contradiction since H has only n^2 vertices. $\qquad\square$

Note that the clustering value in Lemma 18 is best possible (up to the multiplicative constant) for 3-colourings of $F_n \boxtimes F_n$. To see this, colour all the high-degree vertices red. What remains is $P_n \boxtimes P_n$, which can be 2-coloured by alternating rows. So $F_n \boxtimes F_n$ is 3-colourable with clustering n.

4.2 Bounded maximum degree

We now focus on Theorem 2, when H_2 has bounded treewidth and bounded maximum degree. The upper bound follows directly from the general $c \geqslant 2$ result, Theorem 4, presented in Section 5. We prove that this upper bound is asymptotically tight (up to the multiplicative factor) when $c = 3$, which concludes the proof of Theorem 2.

Recall that F_n is the fan graph with base path P_n. Define $H_n := \widehat{n^2\, F_{n^4}}$, and let x be the dominating vertex in H_n (see Figure 1). The *fans of H_n* refer to the n^2 pairwise disjoint copies of F_{n^4} in H_n. Note that H_n is planar, and $\text{tw}(H_n) \leqslant \text{tw}(F_{n^4}) + 1 = 3$ by (1).
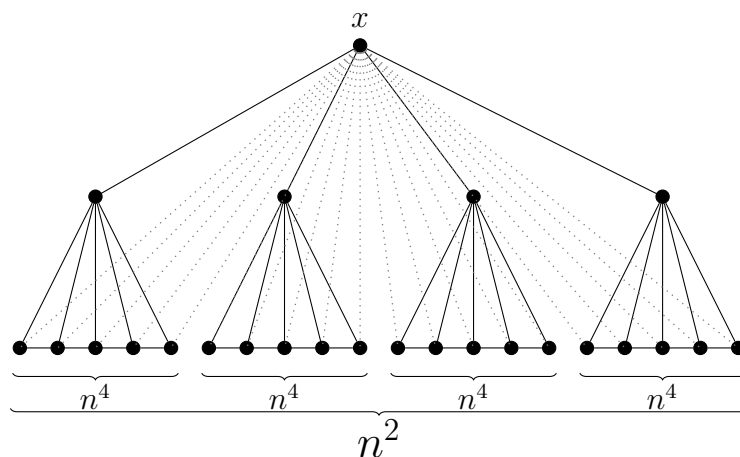

Figure 1: $H_n = \widehat{n^2\, F_{n^4}}$

Lemma 19. *For any $\delta < 1/2$ and any n large enough, every 3-colouring of $G := H_n \boxtimes P_n$ has clustering at least δn^3.*

Since $|V(G)| = n^7 + n^3$ and $\mathrm{tw}(H_n) \leqslant 3$, Lemma 19 proves the lower bound of Theorem 2.

Proof overview: The proof is by contradiction. We first show in Claim 20 that a specific set of vertices of G must be monochromatic, say red. This allows us to bound the total number of red vertices in G. We then restrict our attention to a copy J of $F_{n^4} \boxtimes P_n$ with fewer than δn red vertices. We recognise two subgraphs in J: a *path* induced by the copies of the dominant vertex of F_{n^4} and a *grid* L formed by the strong product of the path of F_{n^4} with P_n. As there are few red vertices, linearly many vertices of the path of J must be of the same non-red colour, say blue. In Claim 21 we argue moreover that these blue vertices all lie in the same blue component S of J. We then consider the intersection K of rows with blue associated dominant vertex and columns disjoint from S in L, and note in Claim 22 that vertices $u, v \in K$ in the same component of $L - S$ remain so when deleting all blue vertices. This allows us to conclude the proof as, loosely speaking, the red vertices split K into few components so there must be a large monochromatic component in the last remaining colour.

Proof. Let $H := H_n$. We refer to $\{(x, v) : v \in P_n\}$, the set of copies of x in G, as the *spine* of G. Suppose, toward a contradiction, that the vertices of G are 3-coloured in red, blue and green, with clustering at most δn^3.

Claim 20. *The spine of G is monochromatic.*

Proof. Suppose that the spine of G is not monochromatic. Let r, b be two adjacent vertices of P_n such that (x, r) and (x, b) have different colours, say red and blue respectively. In G, every vertex $w \in P_n$ has an associated copy $H^{(w)}$ of H on vertex set $\{(z, w) : z \in V(H)\}$ (and edges between two vertices (y, w) and (z, w) precisely when y and z are adjacent in H). Since r and b are adjacent in P_n, then (x, r) and (x, b) are both dominating vertices for $H^{(r)}$. Therefore, as G contains no monochromatic component of size δn^3, there must be less than δn^3 red and less than δn^3 blue vertices in $H^{(r)}$.

Let $\mathcal{F}^{(r)}$ be the set of fans of $H^{(r)}$. As $|\mathcal{F}^{(r)}| = n^2$, by averaging there is some $F \in \mathcal{F}^{(r)}$ which contains less than $2\delta n < n$ vertices coloured red or blue. Hence, the path of length n^4 in F contains a subpath of green vertices of length at least n^3, a contradiction. $\square$

Without loss of generality, we may assume that all vertices of the spine are coloured red. Since the spine is a dominating set for G, there are less than δn^3 red vertices in G. Let $\mathcal{F}$ be the set of fans in H. By averaging, there is an $F \in \mathcal{F}$ such that the subgraph J of G induced by $V(F) \times V(P_n)$ contains less than δn red vertices.

Consider the restriction of the colouring of G to J. Note that J contains a copy of the grid $P_{n^4} \boxtimes P_n$, formed by the strong product of the path of F with P_n (see the graph $G[J]$ on the left of Figure 2). In this grid, rows are copies of the path of F and columns are of the form $\{v\} \boxtimes P_n$, where v is a vertex of the path of F.

Let y be the dominating vertex of the fan F and call a vertex of J of the form (y, v) with $v \in V(P_n)$ a *top vertex*.
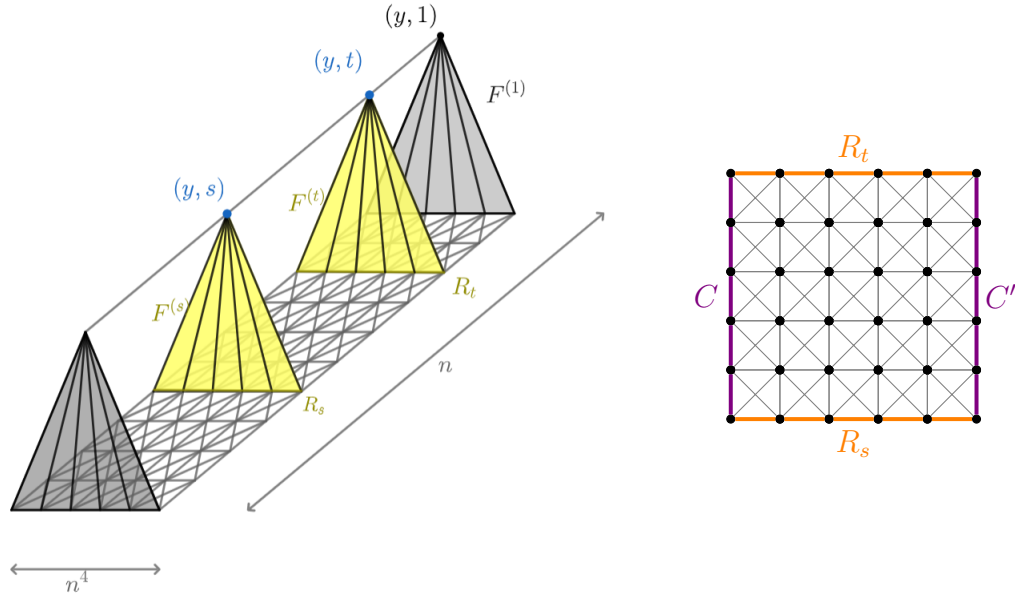
Figure 2: Subgraph $G[J]$ and the subgrid of L bounded by R_s and R_t.

As previously, for $v \in V(P_n)$, denote by $F^{(v)}$ the copy of F associated to v in J. Since there are less than δn red vertices in J, we may assume without loss of generality that there are at least $\frac{1-\delta}{2}n$ blue top vertices.

Claim 21. *All blue top vertices of J are in the same monochromatic blue component in J.*

Proof. Let (y, s), (y, t) be two blue top vertices in J, for some $s, t \in [n]$. We show that there exists a blue path between (y, s) and (y, t) in J. Let R_s and R_t be the vertices in the paths of the fans $F^{(s)}$ and $F^{(t)}$ respectively, both of size n^4. Note that in J, these two rows bound a subgrid of L. Let C denote the first column of this subgrid, and C' the last.

Add two dummy vertices u_s, u_t adjacent to all vertices in R_s and R_t (respectively). Colour them blue. Add two dummy vertices v, v' adjacent to all vertices in C and C' (respectively). Colour them red or green (see Figure 3 for an illustration).

Then by the Hex Lemma (Lemma 13), either there is a blue path from u_s to u_t, or there is a path from v to v' with no blue vertices. Suppose the latter occurs. As previously, this yields a path of length n^4 with no blue vertices. Since there are less than δn red vertices in J, it contains a subpath of n^3 green vertices and we are done. We may therefore assume that there is a blue path from u_s to u_t. This blue path intersects both R_s and R_t. Since (y, s) (respectively (y, t)) is dominating for R_s (respectively R_t), this extends to a blue path from (y, s) to (y, t), as claimed. □

Let S be the vertices of the monochromatic blue component of J containing all blue top vertices. Note that there may be blue vertices in J which are not in S, and as G contains no monochromatic component of size δn^3, we have that $|S| < \delta n^3$.
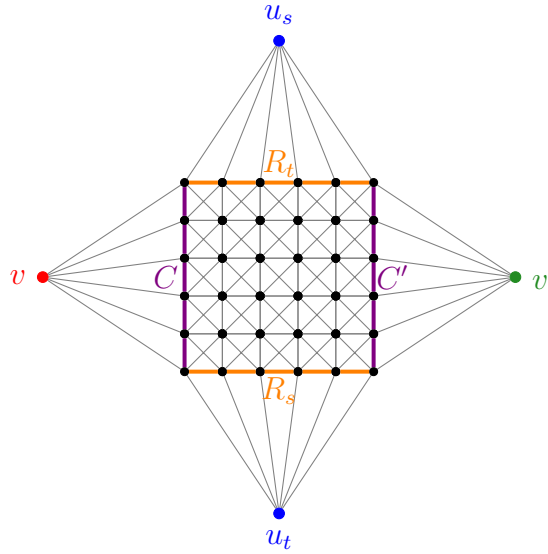
Figure 3: Subgrid of L bounded by R_s and R_t with dummy vertices.

Say that a row of L is *blue dominated* if its associated top vertex is blue, and that a column of L is *S-free* if it does not intersect S. In particular, as there are n^4 columns in L and less than δn^3 elements in S, for any $\varepsilon > 0$, for n large enough, there are $(1 - \varepsilon)n^4$ columns which are S-free. Define the *boundary* of S in J, denoted by ∂S, to be the set of vertices outside of S that are adjacent to a vertex in S.

Claim 22. *Let $u, v \in \partial S$ be vertices simultaneously on blue dominated rows and S-free columns. Suppose u, v are in the same component Q of the grid L after the vertices of S have been removed, and that Q contains an S-free column. Then, there is a path consisting of red and green vertices between u and v in L. In other words, u, v remain in the same component of L after removing all blue vertices.*

Proof. Let Q be a component of $L - S$ that contains an S-free column. Let R_1, R_2, C_1, C_2 be the top row, bottom row, leftmost column, and rightmost column of L, respectively, see Figure 4a.

Let D be the smallest component of L containing Q, bounded up and down by R_1 and R_2, left by a blue path S_1 from R_1 to R_2 (or $S_1 = C_1$ if such a path does not exist), and right by a blue path S_2 from R_1 to R_2 (or $S_2 = C_2$ if such a path does not exist).

We now restrict our attention to D. Let R and R' denote the vertices of its top and bottom rows, and fix a full S-free column K of Q, see Figure 4b. By the Hex Lemma (Lemma 13), either there is a blue path from S_1 to S_2 or a red/green path from R to R'.

Suppose there is a blue path B from S_1 to S_2. If $S_1 = C_1$ and $S_2 = C_2$, then this path has length at least n^4, and is therefore a monochromatic component of size at least δn^3, a contradiction. Otherwise, at least one of S_1, S_2 is a blue monochromatic path crossing every row in D. Without loss of generality, suppose S_1 is such a path. As some rows of L are blue dominated, this path is entirely in S. Hence, as B is blue and touches S_1, $B \subset S$

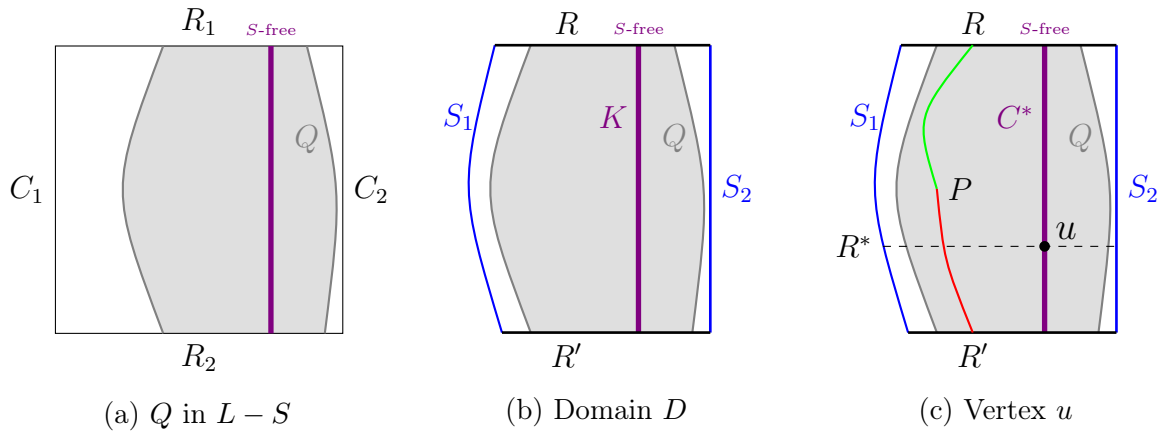(a) Q in $L - S$ (b) Domain D (c) Vertex u

Figure 4: Proof of Claim 22

as well. But B certainly intersects K, an S-free column, which is impossible.

Hence, there must be a path P from R to R' whose vertices are not blue.

It now suffices to show that for each $u \in Q$ which is the intersection of an S-free column with a blue dominated row, there is a path from u to P whose vertices are not blue. Indeed, any $u, v \in Q$ are then connected with non-blue vertices by concatenating the paths from u to P, from v to P, and P itself.

If $u \in P$ we are done, so we may assume that $u \notin P$. Let C^* and R^* be the column and row of u, respectively, see Figure 4c. We let

- P_1 be the shortest path from u to P in D that uses only vertices of C^* above u and possibly vertices of R, and $x \in P$ the other endpoint of P_1, and
- P_2 be the shortest path from u to P in D that uses only vertices of C^* below u and possibly vertices of R', and $y \in P$ the other endpoint of P_2.

Let A be the subgraph of D induced by the vertices in the region enclosed by P_1, P_2 and the subpath of P between x and y. By the Hex Lemma (Lemma 13), either there is a blue path from $P_1 - u$ to $P_2 - u$, or there is a red/green path from u to P.

Suppose there is a blue path P' from $P_1 - u$ to $P_2 - u$. Note that, as all vertices of P_1 lie above or on row R^* and all vertices of P_2 lie below or on row R^*, any path from P_1 to P_2 must cross row R^*. In particular, P' must be entirely in S as it is blue. Consider the endpoints of P'. If they lie in R and R', then, as P' is blue and in S, both of its endpoints lie to one side of the S-free column $K \subseteq Q$. Thus, P' splits D into two non-trivial parts with one containing Q, contradicting the minimality of D. Hence, by construction, some endpoint of P' must lie on C^*. But C^* is an S-free column, so this is also a contradiction. Therefore, there is a red/green path from u to P, as desired. $\qquad\square$

Let $K \subset \partial S$ be the set of vertices of L simultaneously on blue dominated rows and S-free columns. Recall that L contains at least $(1 - \varepsilon)n^4$ S-free columns and at least $\frac{1-\delta}{2}n$ blue top vertices, therefore $|K| \geqslant \frac{(1-\varepsilon)(1-\delta)}{2}n^5$.

Since $|S| < \delta n^3$, at least one of the n rows of the grid L contains less than δn^2 vertices from S. It follows that the S-free columns of L live in at most δn^2 components of $L - S$.

As K contains only vertices from S-free columns, the vertices of K live in at most δn^2 components of $L - S$. By Claim 22 the vertices of K still live in at most δn^2 components of L after all blue vertices are removed. We now remove the less than δn red vertices. Note that for any $\varepsilon' > 0$, for n large enough, we have at least $(1 - \varepsilon')|K|$ green vertices in K. As each vertex of L has degree at most 8, at each removal we add at most 7 components. After removing all red vertices, for any $\varepsilon'' > 0$, for n large enough, we still have less than $(\delta + \varepsilon'')n^2$ components. Therefore, using $\delta < 1/2$ and choosing ε, ε', and ε'' small enough, there is a component of size at least

$$\frac{(1 - \varepsilon)(1 - \varepsilon')(1 - \delta)}{2(\delta + \varepsilon'')}n^3 > \delta n^3$$

containing only green vertices, a contradiction. $\qquad\square$

5 Arbitrary number of colours

We now present our most general results, with an arbitrary number of colours.

5.1 Bounded maximum degree

5.1.1 The upper bound

In this section, we prove Theorem 4, a general upper bound for products of bounded treewidth graphs, when one graph is additionally assumed to have bounded maximum degree. Specifically, we show the following.

Theorem 23. *For any integers $t, c, \Delta \geqslant 1$, for any graph H_1 with treewidth at most t, and any graph H_2 with treewidth at most t and maximum degree at most Δ, the graph $G := H_1 \boxtimes H_2$ is c-colourable with clustering at most $6^{c+1}(t+1)^2\Delta^c |V(G)|^{c/(c^2-c+1)}$.*

The following lemma is a slightly more precise version of a result by Linial et al. [21, Theorem 1.2].

Lemma 24. *For any integers $t, c \geqslant 1$, every graph G with treewidth at most t has a c-colouring with clustering at most $(t+1)^{(c-1)/c} |V(G)|^{1/c}$.*

As remarked by Linial et al. [21, Theorem 1.2], this bound is asymptotically optimal for fixed c, since there exists a graph G_c with treewidth at most c, on $O(n^{c/2})$ vertices and such that every c-colouring has clustering $\Omega(\sqrt{n})$.

Proof of Lemma 24. We proceed by induction on c. The $c = 1$ case is trivial. Now assume that $c \geqslant 2$. Let G be a graph with n vertices and treewidth at most t. Let $p := (t+1)^{1/c}n^{(c-1)/c}$. If $p \leqslant t + 1$, then $n \leqslant t + 1$ and any colouring of G has clustering at most $n \leqslant (t+1)^{(c-1)/c}n^{1/c}$ as desired. Now assume that $p \geqslant t + 1$. By Lemma 11, there exists a set S of at most p vertices in G such that each component of $G - S$ has at most

$$\frac{(t+1)n}{p} = \frac{(t+1)n}{(t+1)^{1/c}n^{(c-1)/c}} = (t+1)^{(c-1)/c}n^{1/c}$$

vertices. By induction, there exists a $(c-1)$-colouring of $G[S]$ with clustering

$$(t+1)^{(c-2)/(c-1)}p^{1/(c-1)} = (t+1)^{(c-2)/(c-1)}\left((t+1)^{1/c}n^{(c-1)/c}\right)^{1/(c-1)} \leqslant (t+1)^{(c-1)/c}n^{1/c}.$$

Extend this $(c-1)$-colouring of $G[S]$ to a c-colouring of G by giving one new colour to all vertices in $G-S$. This has clustering at most $(t+1)^{(c-1)/c}n^{1/c}$. $\qquad\square$

Lemma 24 is only interesting if $c \leqslant t$, since if $c \geqslant t+1$, then G is $(c-1)$-degenerate and thus properly c-colourable.

We use the following three lemmas from the literature.

Lemma 25 ([13, Lemma 14]). *If a graph G is c-colourable with clustering k, then $G \boxtimes K_\ell$ is c-colourable with clustering $k\ell$.*

Lemma 26 ([13, Lemma 32]). *If a graph G is c-colourable with clustering k and T is a tree with maximum degree at most $\Delta \geqslant 3$, then $G \boxtimes T$ is $(c+1)$-colourable with clustering less than $2k(\Delta-1)^{c-1}$.*

Distel and Wood [9] proved the following extension of a classical result of Ding and Oporowski [5].

Lemma 27 ([9, Theorem 2]). *Every graph H with treewidth at most t and maximum degree at most Δ is contained in $T \boxtimes K_{18(t+1)\Delta}$ for some tree T with maximum degree at most 6Δ and with $|V(T)| \leqslant \max\{|V(H)|/2(t+1), 1\}$.*

We are now ready to prove Theorem 23, which is a more precise version of Theorem 4.

Proof of Theorem 23. The statement is trivially true if $c = 1$, so we will assume $c \geqslant 2$. By Lemma 27, there exists a tree T with $\Delta(T) \leqslant 6\Delta$ and $|V(T)| \leqslant \max\{|V(H_2)|/2(t+1), 1\}$ such that G is contained in $H_1 \boxtimes T \boxtimes K_{18(t+1)\Delta}$.

Define $F := H_1 \boxtimes T$ and let $h := |V(H_1)|$, $\ell := |V(T)|$, $f := |V(F)|$ and $n := |V(G)|$. Note that $f = h\ell \leqslant \frac{n}{2(t+1)}$.

We start by showing that F admits a c-colouring with a bound on its clustering. Assume first that $h \geqslant \ell^{c(c-1)}$. Then

$$\ell^{c(c-1)^2} \leqslant h^{c-1},$$
$$h^{c^2-c+1}\ell^{c(c^2-c+1)} \leqslant (h\ell)^{c^2},$$
$$h^{1/c}\ell \leqslant (h\ell)^{c/(c^2-c+1)}.$$

By Lemma 24, there exists a c-colouring of H_1 with clustering at most $(t+1)^{(c-1)/c}h^{1/c}$. For each $x \in V(H_1)$ and each $y \in V(T)$, colour $(x,y) \in V(F)$ by the colour assigned to x. We obtain a c-colouring of F with clustering at most

$$(t+1)^{(c-1)/c}h^{1/c}\ell \leqslant (t+1)^{(c-1)/c}(h\ell)^{c/(c^2-c+1)}.$$

Now assume that $h \leqslant \ell^{c(c-1)}$. Hence,

$$h^{(c^2-c+1)} \leqslant (h\ell)^{c(c-1)},$$

$$h^{1/(c-1)} \leqslant (h\ell)^{c/(c^2-c+1)}.$$

By Lemma 24, H_1 admits a $(c-1)$-colouring with clustering at most

$$(t+1)^{(c-2)/(c-1)}h^{1/(c-1)} \leqslant (t+1)^{(c-1)/c}h^{1/(c-1)}.$$

Since T has maximum degree at most 6Δ, it follows from Lemma 26 that F has a c-colouring with clustering at most

$$2(t+1)^{(c-1)/c}h^{1/(c-1)}(6\Delta-1)^{c-2} \leqslant 2(t+1)^{(c-1)/c}(6\Delta-1)^{c-1}(h\ell)^{c/(c^2-c+1)}.$$

Finally, let

$$\alpha := 2(t+1)^{(c-1)/c}(6\Delta-1)^{c-2}18(t+1)\Delta\left(\frac{1}{2(t+1)}\right)^{c/(c^2-c+1)} \leqslant 6^{c+1}(t+1)^2\Delta^c.$$

Since $h\ell \leqslant \frac{n}{2(t+1)}$, it follows from Lemma 25 that $G \subseteq H_1 \boxtimes T \boxtimes K_{18(t+1)\Delta}$ has a c-colouring with clustering at most $\alpha n^{c/(c^2-c+1)}$. $\qquad\square$

5.1.2 A lower bound

We now turn our attention to the lower bound for this general case, proving Proposition 5.

Lemma 28. *For any integers $c, k \geqslant 2$, let H be a graph and P be a path such that every $(c-1)$-colouring of H has clustering at least k, and every c-colouring of $H \boxtimes P$ has clustering at least k. Let $J := \widehat{(k-1)\,H}$. Then every $(c+1)$-colouring of $J \boxtimes P$ has clustering at least k.*

Proof. By definition, J is obtained from $k-1$ disjoint copies $H_1, \ldots, H_{k-1}$ of H by adding one dominant vertex. Say $P = (v_1, \ldots, v_p)$. Let J^i be the copy of J in $J \boxtimes P$ corresponding to v_i. For $j \in \{1, \ldots, k-1\}$, let H_j^i be the copy of H_j in J^i, and call $X_j := V(H_j^1 \cup \cdots \cup H_j^p)$ a *column*. Consider any $(c+1)$-colouring of $J \boxtimes P$.

First, suppose there exist consecutive vertices v_i and v_{i+1} that are assigned distinct colours, say red and blue respectively. In $J \boxtimes P$, each of v_i and v_{i+1} dominate $H_1^i, \ldots, H_{k-1}^i, H_1^{i+1}, \ldots, H_{k-1}^{i+1}$. If at least $k-1$ of $H_1^i, \ldots, H_{k-1}^i, H_1^{i+1}, \ldots, H_{k-1}^{i+1}$ contain a red vertex, then with v_i we have a red component on at least k vertices, as desired. If at least $k-1$ of $H_1^i, \ldots, H_{k-1}^i, H_1^{i+1}, \ldots, H_{k-1}^{i+1}$ contain a blue vertex, then with v_{i+1} we have a blue component on at least k vertices, as desired. Now assume that at most $k-2$ of $H_1^i, \ldots, H_{k-1}^i, H_1^{i+1}, \ldots, H_{k-1}^{i+1}$ contain a red vertex, and at most $k-2$ of $H_1^i, \ldots, H_{k-1}^i, H_1^{i+1}, \ldots, H_{k-1}^{i+1}$ contain a blue vertex. Thus, at least one of $H_1^i, \ldots, H_{k-1}^i, H_1^{i+1}, \ldots, H_{k-1}^{i+1}$ contains no red vertex and no blue vertex, and is therefore $(c-1)$-coloured. By assumption, this copy of H has a monochromatic component on at least k vertices, as desired.

Now assume that $v_1, \ldots, v_p$ are monochromatic, say red. Since $v_1, \ldots, v_p$ is a connected dominating set in $J \boxtimes P$, the red subgraph is connected. Thus, at most $k-1$ vertices are red. Hence, at most $k-2$ vertices not in $\{v_1, \ldots, v_p\}$ are red. In particular, at most $k-2$

columns contain a red vertex, so some column X_j contains no red vertex. The subgraph of $J \boxtimes P$ induced by X_j, which is isomorphic to $H \boxtimes P$, is c-coloured, and thus has clustering at least k, as desired. $\qquad\square$

The next lemma is folklore, see for instance [25, 26, 28].

Lemma 29. *For any integers $c, k \geqslant 2$, if H is a graph such that every c-colouring of H has clustering at least k, then every $(c+1)$-colouring of $\widehat{(k-1)\,H}$ has clustering at least k.*

Proof. Consider any $(c+1)$-colouring of $\widehat{(k-1)\,H}$. Say the dominant vertex is red. If every copy of H contains a red vertex, then the red component has at least k vertices, as desired. Otherwise, some copy of H is c-coloured, and thus has clustering at least k by assumption. $\qquad\square$

The following is a variant of a result by Linial et al. [21].

Lemma 30. *For any integers $n, k \geqslant 2$ such that $k \leqslant n^3$, let $G_2 = \widehat{n^2\,F_{n^4}}$, and for $c \geqslant 2$, let $G_{c+1} := \widehat{(k-1)\,G_c}$. Then for every $c \geqslant 2$, every c-colouring of G_c has clustering at least k.*

Proof. We proceed by induction on $c \geqslant 2$. For the base case $c = 2$, suppose there exists a red/blue colouring of G_2 with clustering less than k. Say the dominant vertex of G_2 is red. So there exists a copy of F_{n^4} in G_2 with less than k/n^2 red vertices. The path of length n^4 in this copy of F_{n^4} contains a subpath of length at least $n^6/k \geqslant n^3$, whose vertices are all blue, a contradiction. Now assume that $c \geqslant 2$ and every c-colouring of G_c has clustering at least k. By Lemma 29, every $(c+1)$-colouring of G_{c+1} has clustering at least k, as desired. $\qquad\square$

We now prove the following precise version of Proposition 5, which follows immediately since (1) implies that $\mathrm{tw}(G_c) \leqslant \mathrm{tw}(G_{c-1}) + 1 \leqslant c + 1$.

Proposition 31. *For any fixed integers $k, c \geqslant 3$ and any large enough integer n, let $G_2 := \widehat{n^2\,F_{n^4}}$, and $G_d := \widehat{(k-1)\,G_{d-1}}$ for every $d \in [3, c]$. Let $\delta < 1/2$ such that $k = 2\delta n^3$. Then every c-colouring of $G_{c-1} \boxtimes P_n$ has clustering at least $\delta^{\frac{7}{3c-2}} |V(G_{c-1} \boxtimes P_n)|^{1/(c - \frac{2}{3})}$.*

Proof. Let n be large enough such that for every $d \in [3, c]$ we have

$$|V(G_{d-1} \boxtimes P_n)| = (2\delta)^{d-3} n^{3d-2} + o(n^{3d-2}) \leqslant 2^{d-2} \delta^{d-3} n^{3d-2}.$$

Therefore for every $d \in [3, c]$ we obtain

$$\delta^{\frac{7}{3d-2}} |V(G_{d-1} \boxtimes P_n)|^{1/(d-\frac{2}{3})} \leqslant \delta^{\frac{7}{3d-2}} \delta^{\frac{3(d-3)}{3d-2}} 2^{1 - 4/(3d-2)} n^3 \leqslant 2\delta n^3 = k.$$

We reason by induction on d up to c, proving that every d-colouring of $G_{d-1} \boxtimes P_n$ has clustering at least k. The base case $d = 3$ is true by Lemma 19. Assume now that the statement is true for some $d \in [3, c)$; that is, every d-colouring of $G_{d-1} \boxtimes P_n$ has clustering at least k. It follows from Lemma 30 that every $(d-1)$-colouring of G_{d-1} has clustering at least k. Therefore Lemma 28 implies that every $(d+1)$-colouring of $G_d \boxtimes P_n$ has clustering at least k. $\qquad\square$

5.2 Unrestricted maximum degree

By colouring each graph separately with the smallest clustering possible, and then using a product colouring, we prove the following statement that implies the upper bound of Theorem 6.

Lemma 32. *For any integers $c, t \geqslant 1$ and any graphs H_1 and H_2 both with treewidth at most t, the graph $H_1 \boxtimes H_2$ is c-colourable with clustering at most*

$$(t+1)^{2(1-1/\sqrt{c})} |V(H_1 \boxtimes H_2)|^{1/\lfloor\sqrt{c}\rfloor}.$$

Proof. Let $s = \lfloor\sqrt{c}\rfloor$. Given H_1 and H_2 both with treewidth at most t, let $n_i := |V(H_i)|$, $n = n_1 n_2$, apply Lemma 24 to each H_i with s colours, and take the product colouring. So $H_1 \boxtimes H_2$ is s^2-colourable with clustering at most

$$(t+1)^{(s-1)/s} n_1^{1/s} (t+1)^{(s-1)/s} n_2^{1/s} \leqslant (t+1)^{2(\sqrt{c}-1)/\sqrt{c}} n^{1/\lfloor\sqrt{c}\rfloor}. \qquad \square$$

We now prove the lower bound of Theorem 6. Fix an integer $t \geqslant 1$. Consider graphs H_1 and H_2 with treewidth at most t. So both H_1 and H_2 are properly $(t+1)$-colourable. A product colouring shows that $H_1 \boxtimes H_2$ is properly $(t+1)^2$-colourable. That is, $H_1 \boxtimes H_2$ is $(t+1)^2$-colourable with clustering 1. Esperet and Wood [13] showed that in such a result with bounded clustering, $(t+1)^2$ colours is best possible. That is, there exists a family of graphs H_1, H_2 with treewidth at most t such that for any $c < (t+1)^2$, in any c-colouring of $H_1 \boxtimes H_2$ the clustering must increase with $|V(H_1 \boxtimes H_2)|$. We extract the following result from their proof.

Lemma 33. *For any integer $c \geqslant 2$, there are infinitely many graphs H_1, H_2 with treewidth at most $\sqrt{c}$ such that every c-colouring of $G := H_1 \boxtimes H_2$ has clustering $\Omega(|V(G)|^{1/2\sqrt{c}})$.*

Proof. For $t, n \in \mathbb{N}$, define the graph $C_{t,n}$ recursively as follows. Let $C_{1,n} := K_1$, and $C_{t,n} := \widehat{n\, C_{t-1,n}}$ for $t \geqslant 2$. Given c, let t be the integer such that $t^2 \leqslant c < (t+1)^2$. By (1), $\mathrm{tw}(C_{t+1,n}) \leqslant t \leqslant \sqrt{c}$. Let $G := C_{t+1,n} \boxtimes C_{t+1,n}$, where $n \gg t$. Note that $|V(C_{t+1,n})| = n^t + O(n^{t-1})$ and thus $|V(G)| = n^{2t} + O(n^{2t-1})$. Esperet and Wood [13, Theorem 16][1] showed that in any c-colouring of G, there is a monochromatic component with maximum degree at least $\frac{n}{(t+1)^4 12^{t+1}}$. Thus, for fixed t, any c-colouring of G has clustering

$$\Omega(n) = \Omega(|V(G)|^{1/2t}) = \Omega(|V(G)|^{1/2\sqrt{c}}). \qquad \square$$

Consider graphs H_1 and H_2 with bounded treewidth on n_1 and n_2 vertices (respectively). By Lemma 24, each H_i is 2-colourable with clustering $O(\sqrt{n_i})$. Let $n := |V(H_1 \boxtimes H_2)| = n_1 n_2$. By using the product colouring as in the proof of Lemma 32, $H_1 \boxtimes H_2$ is 4-colourable with clustering $O(\sqrt{n})$. Therefore, to prove that the upper bound of Lemma 32 is asymptotically tight when $c = 4$, one must choose H_1 and H_2 to be graphs such that every

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[1] Esperet and Wood [13] actually work in the more general setting of fractional $p : q$-colourings, where a c-colouring is equivalent to a $c : 1$-colouring. They also define $C_{t,n}$ as the closure of a rooted tree, which is equivalent to our recursive definition.

2-colouring of H_i has clustering $\Theta(\sqrt{n_i})$, which is as large as possible. Fans have this property by Lemma 10. This makes fans the natural candidates to show that Lemma 32 is asymptotically tight. However, we now show that $F_n \boxtimes F_n$ is 4-colourable with clustering $O(n^{2/3})$, and this bound is tight.

Proposition 34. *There exists an absolute constant $\delta > 0$ such that for all $n \geqslant 1$, every 4-colouring of $F_n \boxtimes F_n$ has clustering at least $\delta n^{2/3}$.*

Proof. Let $\delta := (48)^{-2/3}$. We make no attempt to optimise this constant.

Let A and B be copies of F_n, with $V(A) = \{a_0, a_1, \ldots, a_n\}$ and $V(B) = \{b_0, b_1, \ldots, b_n\}$, where a_0 dominates the path $a_1, \ldots, a_n$ in A, and b_0 dominates the path $b_1, \ldots, b_n$ in B. Let $A' := \{(a_1, b_0), \ldots, (a_n, b_0)\}$ and $B' := \{(a_0, b_1), \ldots, (a_0, b_n)\}$.

Suppose, for the sake of contradiction, that $A \boxtimes B$ has a 4-colouring with clustering at most $\delta n^{2/3}$. Let the colours be red, green, blue, and black, where (a_0, b_0) is red. A colour c is *A'-empty* if no vertex in A' is coloured c, otherwise c is *A'-nonempty*. Similarly, a colour c is *B'-empty* if no vertex in B' is coloured c, otherwise c is *B'-nonempty*. A colour c is *A'-small* if at most $\delta n^{2/3}$ vertices in A' are coloured c, otherwise c is *A'-big*. Similarly, a colour c is *B'-small* if at most $\delta n^{2/3}$ vertices in B' are coloured c, otherwise c is *B'-big*.

Since (a_0, b_0) is dominant in G, there are at most $\delta n^{2/3}$ red vertices in total, so red is both A'-small and B'-small. If some colour c is both A'-big and B'-nonempty, then all the vertices in $A' \cup B'$ coloured c are in a single monochromatic component, which is a contradiction. So no colour is both A'-big and B'-nonempty. Similarly, no colour is both A'-nonempty and B'-big. Since $n > 4\delta n^{2/3}$, at least one colour is A'-big, and at least one colour is B'-big.

If at least two colours are A'-big and at least two colours are B'-big, then some colour is both A'-big and B'-big, which is a contradiction. Thus, without loss of generality, exactly one colour is B'-big. We may assume that black is B'-big and A'-empty, blue is B'-small, and green is B'-small.

Let H be the subgraph of G induced by $\{a_1, \ldots, a_n\} \times \{b_1, \ldots, b_n\}$. So $H \cong P_n \boxtimes P_n$. Let S be the union of:

- the set S_b of black vertices in H,
- the set S_r of red vertices in H,
- the set S_{r+} of vertices in H (of any colour) adjacent to a red vertex in $A' \cup B'$,
- the set S_ℓ of blue vertices in H adjacent to a blue vertex in $A' \cup B'$,
- the set S_g of green vertices in H adjacent to a green vertex in $A' \cup B'$.

We claim that $|S_b| \leqslant 4\delta n^{5/3}$. Let $I := \{j \in \{1, \ldots, n\} : (a_0, b_j) \text{ is black}\}$. For each $j \in I$ there are at most $\delta n^{2/3}$ black vertices (a_i, b_j) (since they are dominated by (a_0, b_j)). Thus $|S_b| \leqslant |I| \delta n^{2/3} + (n - |I|)n$. Since each of red, blue and green is B'-small, $n - 3\delta n^{2/3} \leqslant |I| \leqslant n$ implying $|S_b| \leqslant 4\delta n^{5/3}$, as claimed.

We have $|S_r| < \delta n^{2/3}$ directly. Each red vertex in $A' \cup B'$ contributes at most $3n$ vertices to S_{r+}, so $|S_{r+}| \leqslant 3\delta n^{5/3}$.

We claim that $|S_\ell| \leqslant 2\delta n^{5/3}$. Each vertex in S_ℓ is adjacent to a blue vertex in

$A' \cup B'$. Each blue vertex in $A' \cup B'$ contributes at most $\delta n^{2/3}$ vertices to S_ℓ. So $|S_\ell| \leqslant |A' \cup B'|\delta n^{2/3} \leqslant 2\delta n^{5/3}$. By the same argument, $|S_g| \leqslant 2\delta n^{5/3}$.

In total, $|S| < 4\delta n^{5/3} + \delta n^{2/3} + 3\delta n^{5/3} + 2\delta n^{5/3} + 2\delta n^{5/3} \leqslant 12\delta n^{5/3}$.

Every vertex in $H - S$ is blue or green. Suppose, for the sake of contradiction, that vw is an edge of $H - S$ with v blue and w green. Say $v = (a_i, b_j)$ and $w = (a_{i'}, b_{j'})$. Since vw is an edge of $H - S$, we have $|i - i'| \leqslant 1$ (and $|j - j'| \leqslant 1$). Thus v and w are both adjacent to (a_i, b_0) and $(a_{i'}, b_0)$. Since $v \notin S_\ell$, neither (a_i, b_0) nor $(a_{i'}, b_0)$ is blue. Since $w \notin S_g$, neither (a_i, b_0) nor $(a_{i'}, b_0)$ is green. Since black is A'-empty, both (a_i, b_0) and $(a_{i'}, b_0)$ are red, which implies that v and w are in S_{r+}, giving a contradiction. Hence, there is no blue–green edge in $H - S$. This implies that the components of $H - S$ are monochromatic, so each component of $H - S$ has at most $\delta n^{2/3}$ vertices. Now by Lemma 12 with $k = \delta n^{2/3}$,

$$n^2 \leqslant 4|S|k^{1/2} < 4 \cdot 12\delta n^{5/3} (\delta n^{2/3})^{1/2} = 48\,\delta^{3/2}n^2 \leqslant n^2,$$

which is a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The clustering value in Proposition 34 is again best possible (up to the multiplicative constant), as demonstrated by the following result.

Proposition 35. *For any integer $n \geqslant 2$, there exists a 4-colouring of $F_{n^3} \boxtimes F_{n^3}$ with clustering at most $7n^2$.*

Proof. Let $G := F_{n^3} \boxtimes F_{n^3}$. We label the vertices of G as $\{(i,j) \colon 0 \leqslant i, j \leqslant n^3\}$, where $(0,0)$ is dominating for G, and for all $i \in [n^3]$, $(i,0)$ is adjacent with all vertices in $\{(i,j) \colon j \in [n^3]\}$, and $(0,i)$ is adjacent with all vertices in $\{(j,i) \colon j \in [n^3]\}$. Then the vertices $\{(i,j) \colon 1 \leqslant i, j \leqslant n^3\}$ form a copy of $P_{n^3} \boxtimes P_{n^3}$ in G.

Start by colouring $(0,0)$ black. Let $X = \{(0,i) \colon i \in [n^3]\}$ and $Y = \{(i,0) \colon i \in [n^3]\}$, as illustrated in Figure 5. The vertices of X induce a path of length n^3. Colour almost all vertices of X blue, inserting three consecutive black vertices after every n blue vertices. Formally, for every integer $\ell \in \{1, \dots, n^2\}$, colour $(0, \ell n - 1)$, $(0, \ell n)$ and $(0, \ell n + 1)$ black, and the rest of X blue. Similarly, colour Y red, inserting black vertices. Note that there are at most $7n^2$ black vertices.

For every $\ell \in \{1, \dots, n^2\}$ and $i, j \in \{1, \dots, n^3\}$, colour vertices of the form $(\ell n, j)$, $(i, \ell n)$ as follows. Fix $\ell \in \{1, \dots, n^2\}$. For every $j \in \{1, \dots, n^3\}$, colour $(\ell n, j)$ blue, except vertices $(\ell n, \ell' n)$ where $0 < \ell' \leqslant n^2$ and $\ell \neq \ell' \pmod 2$. Colour all other vertices of the form $(\ell n, j)$, $(i, \ell n)$ red.

Finally, colour all remaining vertices green. Note that there is no red vertex in red-dominated columns (or adjacent to one), and no blue vertex in blue-dominated rows (or adjacent to one). The largest blue component has size $2n + 5$, likewise for red, and the largest green component has size $(n + 2)^2 < 7n^2$. $\qquad\qquad\qquad\square$

6   Products of bounded degree graphs

This section considers clustered colouring of the strong product of two graphs, both with bounded treewidth and bounded degree. This setting is much simpler than with none or
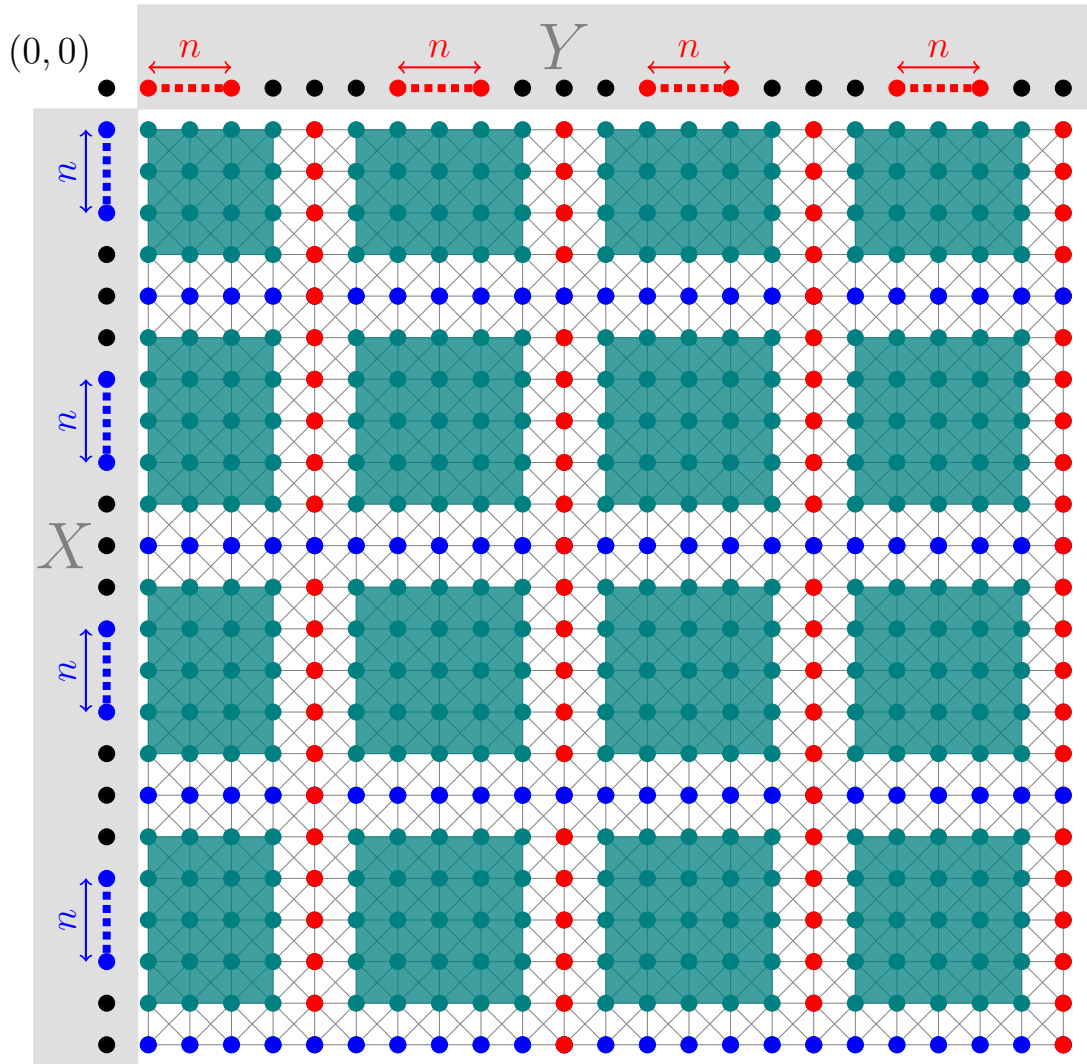
Figure 5: A 4-colouring of $F_{n^3} \boxtimes F_{n^3}$ with clustering at most $7n^2$.

one of the graphs having bounded degree. We only require the following proposition to solve this case.

Proposition 36. *For any integer $k \geqslant 1$ and any graphs H_1, H_2 that are c-colourable with clustering k, the graph $H_1 \boxtimes H_2$ is c-colourable with clustering $k |V(H_1 \boxtimes H_2)|^{1/2}$.*

Proof. Let $G := H_1 \boxtimes H_2$ and $n := |V(G)|$. Without loss of generality, $|V(H_2)| \leqslant |V(H_1)|$, so $|V(H_2)| \leqslant n^{1/2}$. Fix a c-colouring of H_1 with clustering k. Colour each vertex (x, y) of G by the colour assigned to x. Now G is c-coloured. If $(x_1, y_1)(x_2, y_2)$ is a monochromatic edge in G, then x_1 and x_2 had the same colour in H_1, and $x_1 = x_2$ or $x_1 x_2 \in E(H_1)$. Hence, each monochromatic component M in G projects to a monochromatic component in H_1, implying that $|V(M)| \leqslant k |V(H_2)| \leqslant k n^{1/2}$, as claimed. $\square$

Now consider graphs H_1, H_2 with treewidth at most t and maximum degree at most Δ. By Lemma 27, each H_i is contained in $T_i \boxtimes K_{O(t\Delta)}$ for some tree T_i. As noted by Alon, Ding, Oporowski, and Vertigan [1, Theorem 2.2], any proper 2-colouring of T_i determines a 2-colouring of H_i with clustering $O(t\Delta)$. Therefore:

- For $c = 2$, Proposition 36 implies that $H_1 \boxtimes H_2$ is 2-colourable with clustering $O(t\Delta |V(H_1 \boxtimes H_2)|^{1/2})$. This bound is tight for fixed t and Δ, since the Hex Lemma (Lemma 13) implies that any 2-colouring of $P_n \boxtimes P_n$ has clustering at least $n = |V(P_n \boxtimes P_n)|^{1/2}$.

- For $c = 3$, Esperet and Wood [13, Theorems 4 and 33] proved that $H_1 \boxtimes H_2$ is 3-colourable with bounded clustering, at most $O(t^3\Delta^4)$.

- For $c = 4$, using the product of the 2-colourings of each H_i, it is easy to see that $H_1 \boxtimes H_2$ is 4-colourable with bounded clustering, at most $O(t^2\Delta^2)$.

7 Planar graphs

Now consider clustered colouring of planar graphs. Linial et al. [21] introduced the following definition. Let $f_c(n)$ be the minimum integer such that every n-vertex planar graph is c-colourable with clustering $f_c(n)$. The 4-Colour Theorem says that $f_c(n) = 1$ for every $c \geqslant 4$. Linial et al. [21] showed that $f_2(n) = \Theta(n^{2/3})$ and $\Omega(n^{1/3}) \leqslant f_3(n) \leqslant O(n^{1/2})$.

One approach for closing the gap in these bounds on $f_3(n)$ is to apply the Planar Graph Product Structure Theorem of Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [10], which says that every planar graph is contained in $H \boxtimes P$ for some graph H of treewidth at most 8 and some path P. The idea is to 3-colour $H \boxtimes P$ with small clustering, thus determining a 3-colouring of G. Two issues arise, however. First, it may be that $|V(H \boxtimes P)|$ is significantly larger than $|V(G)|$ (the best bounds are $|V(H)| \leqslant |V(G)|$ and $|V(P)| \leqslant O(|V(G)|^{(1+\epsilon)/2})$ by Hendrey and Wood [17]). So clustering $O(|V(H \boxtimes P)|^\beta)$ does not immediately imply clustering $O(|V(G)|^\beta)$. Second, by Lemma 19, there are graphs H with treewidth 3 and there are paths P such that every 3-colouring of $H \boxtimes P$ has clustering $\Omega(|V(H \boxtimes P)|^{3/7})$. So the best upper bound on $f_3(n)$ that one could hope for using this method is $f_3(n) \leqslant O(n^{3/7})$, which is between the known bounds mentioned above. Determining $f_3(n)$ is a tantalising open problem.

Acknowledgements

References

[1] Noga Alon, Guoli Ding, Bogdan Oporowski, and Dirk Vertigan. Partitioning into graphs with only small components. *J. Combin. Theory Ser. B*, 87(2):231–243, 2003.

[2] Michael A Bekos, Carla Binucci, Michael Kaufmann, Chrysanthi Raftopoulou, Antonios Symvonis, and Alessandra Tappini. Coloring outerplanar graphs and planar 3-trees with small monochromatic components. 2019, arXiv:1911.10863.

[3] Hans L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998.

[4] Béla Bollobás and Imre Leader. Edge-isoperimetric inequalities in the grid. *Combinatorica*, 11(4):299–314, 1991.

[5] Guoli Ding and Bogdan Oporowski. Some results on tree decomposition of graphs. *J. Graph Theory*, 20(4):481–499, 1995.

[6] Marc Distel, Kevin Hendrey, Nikolai Karol, David R. Wood, and Jung Hon Yip. Treewidth 2 in the planar graph product structure theorem. *Discrete Math. & Theoretic. Comput. Sci.*, 27:2:#8, 2025.

[7] Marc Distel, Robert Hickingbotham, Tony Huynh, and David R. Wood. Improved product structure for graphs on surfaces. *Discrete Math. & Theoretic. Comput. Sci.*, 24(2):#6, 2022.

[8] Marc Distel, Robert Hickingbotham, Michał T. Seweryn, and David R. Wood. Powers of planar graphs, product structure, and blocking partitions. *Innovations in Graph Theory*, 1:39–86, 2024.

[9] Marc Distel and David R. Wood. Tree-partitions with small bounded degree trees. 2022, arXiv:2210.12577.

[10] Vida Dujmović, Gwenaël Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David R. Wood. Planar graphs have bounded queue-number. *J. ACM*, 67(4):#22, 2020.

[11] Vida Dujmović, Pat Morin, and David R. Wood. Graph product structure for non-minor-closed classes. *J. Combin. Theory Ser. B*, 162:34–67, 2023.

[12] Zdeněk Dvořák and David R. Wood. Product structure of graph classes with strongly sublinear separators. 2022, arXiv:2208.10074.

[13] Louis Esperet and David R. Wood. Colouring strong products. *European J. Combin.*, 121:103847, 2024.

[14] David Gale. The game of Hex and the Brouwer fixed-point theorem. *Amer. Math. Monthly*, 86(10):818–827, 1979.

[15] Daniel J. Harvey and David R. Wood. Parameters tied to treewidth. *J. Graph Theory*, 84(4):364–385, 2017.

[16] Ryan B. Hayward and Bjarne Toft. Hex, inside and out—the full story. CRC Press, 2019.

[17] Kevin Hendrey and David R. Wood. Short paths in the planar graph product structure theorem. 2025, arXiv:2502.01927.

[18] Robert Hickingbotham and David R. Wood. Shallow minors, graph products and beyond-planar graphs. *SIAM J. Discrete Math.*, 38(1):1057–1089, 2024.

[19] Petr Hliněný and Jan Jedelský. Hereditary graph product structure theory and induced subgraphs of strong products. 2024, arXiv:2403.16789.

[20] Freddie Illingworth, Alex Scott, and David R. Wood. Product structure of graphs with an excluded minor. *Trans. Amer. Math. Soc. Ser. B*, 11:1233–1248, 2024.

[21] Nathan Linial, Jiří Matoušek, Or Sheffet, and Gábor Tardos. Graph colouring with no large monochromatic components. *Combin. Probab. Comput.*, 17(4):577–589, 2008.

[22] Chun-Hung Liu, Sergey Norin, and David R. Wood. Product structure and tree decompositions. 2024, arXiv:2410.20333.

[23] Jiří Matoušek and Jaroslav Nešetřil. Invitation to discrete mathematics. Clarendon Press, 1998.

[24] Jiří Matoušek and Aleš Přívětivý. Large monochromatic components in two-colored grids. *SIAM J. Discrete Math.*, 22(1):295–311, 2008.

[25] Sergey Norin, Alex Scott, Paul Seymour, and David R. Wood. Clustered colouring in minor-closed classes. *Combinatorica*, 39(6):1387–1412, 2019.

[26] Patrice Ossona de Mendez, Sang-il Oum, and David R. Wood. Defective colouring of graphs excluding a subgraph or minor. *Combinatorica*, 39(2):377–410, 2019.

[27] Bruce A. Reed. Algorithmic aspects of tree width. In *Recent advances in algorithms and combinatorics*, vol. 11, pp. 85–107. Springer, 2003.

[28] Jan van den Heuvel and David R. Wood. Improper colourings inspired by Hadwiger's conjecture. *J. London Math. Soc.*, 98:129–148, 2018. `arXiv:1704.06536`.

[29] David R. Wood. Defective and clustered graph colouring. *Electron. J. Combin.*, #DS23, 2018. Version 1.