Thin Edges in Claw-Free Bricks

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Abstract

A *brick* is a non-bipartite matching covered graph without nontrivial tight cuts. The importance of bricks stems from the fact that they are building blocks of the matching covered graphs. The *bi-contraction* of a vertex u of degree two in a graph G, with precisely two neighbors u_1 and u_2, consists of shrinking the set $\{u, u_1, u_2\}$ to a single vertex. The *retract* of a matching covered graph G is the graph obtained from G by repeatedly bi-contracting vertices of degree two. An edge e of a brick G is *thin* if the retract of $G - e$ is a brick. By showing the existence of thin edge in every brick (other than three basic bricks), Carvalho et al. presented inductive tools for building all the bricks from three basic bricks. However, the lower bound of the number of thin edges in a brick is still unknown.

In this paper, we provide the first nontrivial family of graphs, the numbers of thin edges of which are not a constant: we show that every claw-free brick G with at least 8 vertices has at least $3|V(G)|/8$ thin edges. Consequently, we prove that every claw-free minimal brick G has at least $3|V(G)|/16$ cubic vertices, which shows that Norine and Thomas's conjecture about linear bound of the number of cubic vertices in minimal bricks [J. Combin. Theory Ser. B, 96(4) (2006)] holds for claw-free minimal bricks.

Mathematics Subject Classifications: 05C70,05C75

1 Introduction

Graphs considered in this paper are simple graphs. We follow [1] for undefined notations and terminologies. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For $X, Y \subseteq V(G)$, by $E[X, Y]$ we mean the set of edges of G with one end vertex in X and the other end vertex in Y. Let $\partial(X) = E[X, \overline{X}]$ be an edge cut of G, where $\overline{X} = V(G) \setminus X$. An edge cut $\partial(X)$ is *trivial* if $|X| = 1$ or $|\overline{X}| = 1$. We say that $\partial(X)$ is a k-cut if $|\partial(X)| = k$.

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Let G be a graph with a perfect matching. An edge e in G is *forbidden* if e does not lie in any perfect matchings of G. A nontrivial graph is *matching covered* if it is connected and each of its edges is not forbidden. We denote by $G/X \to x$ the graph obtained from G by contracting X to a single vertex x (and removing any resulting loops, multiple edges). The graphs $G/X \to x$ and $G/\overline{X} \to \overline{x}$ are the two $\partial(X)$*-contractions* of G. An edge cut $\partial(X)$ is *tight* if every perfect matching contains exactly one edge of $\partial(X)$. A matching covered graph without nontrivial tight cuts is called a *brace* if it is bipartite, and a *brick* if it is non-bipartite. Let G be a matching covered graph. We may apply to G a procedure, called a *tight cut decomposition* of G, which produces a list of bricks and braces. Lovász [13] proved that any matching covered graph can be decomposed into a unique list of bricks and braces. Denote by $b(G)$ the number of bricks yield by tight cut decompositions of a matching covered graph G.

We say that an edge e in a matching covered graph G is *removable* if $G-e$ is matching covered. Moreover, a removable edge is *b-invariant* if $b(G-e) = b(G)$. In particular, if e is a b-invariant edge of a brick G, then $b(G-e) = 1$. Carvalho et al. [2, 3] proved a conjecture of Lovász which states that every brick, distinct from K_4, the triangular prism (the complement of a cycle of length 6) and the Petersen graph, has a b-invariant edge. A 2-edge-connected cubic graph is *essentially 4-edge-connected* if it does not contain nontrivial 3-cuts. A brick G is *near-bipartite* if it has a pair of edges $\{e_1, e_2\}$ such that $G - \{e_1, e_2\}$ is bipartite and matching covered. Kothari et al. [9] showed that every essentially 4-edge-connected cubic non-near-bipartite brick G, distinct from the Petersen graph, has at least $|V(G)|$ b-invariant edges. Moreover, they conjectured every essentially 4-edge-connected cubic near-bipartite brick G, distinct from K_4, has at least $|V(G)|/2$ b-invariant edges; Lu et al. [11] confirmed this conjecture. A brick is *solid* if $G-(V(C_1) \cup V(C_2))$ has no perfect matching for any two vertex disjoint odd cycles C_1 and C_2. Carvalho et al. [2] proved that every removable edge of a solid brick, distinct from K_4, is b-invariant; consequently, every solid brick, distinct from K_4, has at least $\frac{|V(G)|}{2}$ b-invariant edges.

The *bi-contraction* of a vertex u of degree two in a graph G, with precisely two neighbors u_1 and u_2, consists of shrinking the set $\{u, u_1, u_2\}$ to a single vertex. The *retract* of a matching covered graph G is the graph obtained from G by repeatedly bi-contracting vertices of degree two. An edge e of a brick G is *thin* if the retract of $G-e$ is a brick. (Thus thin edges of bricks are special types of b-invariant edges.) Carvalho et al. [4] showed that every brick, distinct from K_4, the triangular prism and the Petersen graph, has a thin edge; with the help of thin edges, all the bricks may be generated from three basic bricks: K_4, the triangular prism and the Petersen graph. Similarly, we may define thin edges in braces: an edge e of a brace G is *thin* if the retract of $G-e$ is a brace. Carvalho et al. [6] proved that every brace of order six or more has at least two thin edges. Moreover, they conjectured there exists a positive constant c such that every brace on n vertices has cn thin edges.

For a vertex set $X \subset V(G)$, denote by $G[X]$ the subgraph induced by X, by $N(X)$, or simply $N(u)$ when $X = \{u\}$, the set of all vertices in $\overline{X}$ adjacent to vertices in X. For a vertex x in G, the degree of x, denoted by $d_G(x)$, or simply $d(x)$, is the number of edges incident with x. If $d(u) = 3$, then u is called a cubic vertex. We say that a graph isomorphic to the complete bipartite graph $K_{1,3}$ is a *claw*. Let $G[\{u, u_1, u_2, u_3\}]$ be a claw and $u_i \in N(u)$ for $i \in \{1, 2, 3\}$. We say that u is the *claw-center*. A graph that contains no induced subgraphs isomorphic to the $K_{1,3}$ is *claw-free*. Claw-free graphs have received a lot of attention in connection with the study of various graph properties. Plummer [15] proved that every 3-connected claw-free graph with even number of vertices is a brick. In this paper, we consider the thin edges in claw-free bricks.

Theorem 1. *Let e be an edge of a claw-free brick G. Then e is b-invariant if and only if e is thin.*

Theorem 2. *Let G be a claw-free brick with at least 8 vertices. Then G has at least $3|V(G)|/8$ thin edges.*

A brick G is *minimal* if $G - e$ is not a brick for any $e \in E(G)$. Carvalho et al. [4] proved that every minimal brick has a cubic vertex. Norine and Thomas [14] showed that every minimal brick has at least three cubic vertices. Moreover, they made the following conjecture.

Conjecture 3 ([14])**.** There exists $\alpha > 0$ such that every minimal brick G has at least $\alpha|V(G)|$ cubic vertices.

Lin et al. [10] showed that every minimal brick has at least four cubic vertices. He and Lu [8] proved that every solid minimal brick G has at least $\frac{2}{5}|V(G)|$ cubic vertices. As an application of Theorem 2, we confirm Conjecture 3 to be true for claw-free minimal bricks.

Theorem 4. *Let G be a claw-free minimal brick with at least 8 vertices. Then G has at least $\frac{3|V(G)|}{16}$ cubic vertices.*

In Section 2, we will present some basic properties of removable edges in claw-free bricks. Theorem 1 will be showed in Section 3. Theorems 2 and 4 will be proved in Section 4.

2 Preliminaries

Let G be a graph with a perfect matching. A nonempty vertex set B of G is a *barrier* of G if $o(G - B) = |B|$, where $o(G - B)$ is the number of components with odd number of vertices of $G - B$. A barrier B is *trivial* if $|B| = 1$; otherwise, it is nontrivial. We define a connected subgraph to be an odd (even) component if it contains an odd (even) number of vertices.

Theorem 5. *(Tutte, see [12]) A graph G has a perfect matching if and only if $o(G - X) \leqslant |X|$, for every $X \subseteq V(G)$.*

By Theorem 5, we have the following results directly.

Proposition 6. *If G is matching covered, then, for every barrier B, $G - B$ has no even components and $E(G[B]) = \emptyset$.*

Lemma 7 ([5])**.** *Assume that G is a graph with a perfect matching. An edge xy is forbidden if and only if there exists a barrier containing x and y.*

Let G be a matching covered graph. An edge cut C of G is a *barrier-cut* if there exists a barrier B of G and an odd component O of $G - B$ such that $C = \partial(V(O))$. Let D be a vertex 2-cut in G, that is, $G - D$ is disconnected and $|D| = 2$. Then D is called a *2-separation* if each component of $G - D$ is even. Let $\{u, v\}$ be a 2-separation of G, and let us divide the components of $G - \{u, v\}$ into two nonempty subgraphs G_1 and G_2. Each of the two cuts $\partial(V(G_1) + u)$ and $\partial(V(G_1) + v)$ is a *2-separation cut* associated with $\{u, v\}$. It can be checked that barrier-cuts and 2-separation cuts are tight cuts.

Theorem 8 ([7])**.** *Every matching covered graph that contains a nontrivial tight cut has a nontrivial barrier or a 2-separation.*

Lemma 9 ([16])**.** *Let G be a claw-free brick. Assume that xy is a non-removable edge in G, an edge ab is forbidden in $G - xy$, and B_0 is a barrier of $G - xy$ such that $\{a, b\} \subseteq B_0$. Then $|B_0| \leqslant 3$ and $G - xy - B_0$ contains no even components. Moreover, if $|B_0| = 3$, then the components of $G - xy - B_0$ containing x and y are singletons, respectively.*

Proposition 10. *Let G be a claw-free brick. Assume that $N(u) = \{u_1, u_2, u_3\}$ and $G[\{u, u_2, u_3\}]$ is a triangle. Then uu_1 is not removable.*

Proof. As $\{u_2, u_3\}$ is barrier of $G - uu_1$ and $u_2u_3 \in E(G)$, u_2u_3 is forbidden in $G - uu_1$ by Lemma 7. So the result holds. □

If e is a removable edge and $G - e$ is not a brick, then $G - e$ contains a nontrivial barrier or a 2-separation by Theorem 8. We say a nontrivial barrier (2-separation) in $G - e$ is the barrier (2-separation) associated with e. Now we present some properties of a brick after removing a removable edge.

Lemma 11. *Assume that uv is a removable edge in a brick G and D is either a nontrivial barrier or a 2-separation of $G - uv$. Then u and v lie in different components of $G - uv - D$. Moreover, $N(u) \cap N(v) \subset D$.*

Proof. Suppose, to the contrary, that u and v lie in a common component of $G - uv - D$. Then D is either a nontrivial barrier or a 2-separation in G, contradicting the assumption that G is a brick. Thus, u and v lie in different components of $G - uv - D$. Therefore, $N(u) \cap N(v) \subset D$. □

Lemma 12. *Let G be a claw-free brick. Assume that uv is a removable edge in G and B is a nontrivial barrier of $G - uv$. Then $|B| \leqslant 3$.*

Proof. By Lemma 11, u and v lie in different components of $G-uv-B$. Assume that G_1 and G_2 is the components of $G-uv-B$ that contain u and v, respectively. Let K_i consist of vertices in B adjacent to vertices in i components of $G-uv-B$ and let $k_i=|K_i|$. As G is claw-free, we have $i \leqslant 3$. Contracting every component of $G-uv-B$ into a vertex and removing all the resulting multiedges, we get a bipartite graph Q. Assume that the color class of Q containing vertices in B is X, the other one is Y. Assume that u' and v' are the vertices of Y obtained from contracting the components of $G-uv-B$ containing u and v, respectively. Then the degree of every vertex in X is at most 3 as G is claw-free. Specially, if $x \in X$ and $d_Q(x)=3$, then u and v are adjacent to x in G. Therefore, the number of edges in Q with one end in X is $k_1+2k_2+3k_3$. On the other hand, each component in $G-uv-B$ is incident with at least 3 vertices in B, except G_1 and G_2, as G is 3-connected and $uv \in E(G)$. Moreover, G_1 and G_2 are incident with at least two vertices in B, respectively, as G is 3-connected again. Then $d_Q(u')+d_Q(v') \geqslant max\{2k_3,4\}$. Therefore, the number of edges of Q with one end in Y is at least $max\{2k_3,4\}+3(k_1+k_2+k_3-2)$ (By Proposition 6, $G-uv-B$ has no even components). As Q is bipartite, $k_1+2k_2+3k_3 \geqslant max\{2k_3,4\}+3(k_1+k_2+k_3-2)$. We have $6 \geqslant 2k_1+k_2+max\{2k_3,4\}$, and hence $2k_1+k_2 \leqslant 2$. If $k_1 \neq 0$, then $k_1=1$, $k_2=0$ and $k_3 \leqslant 2$. As $|B| \geqslant 1$, we have $k_3 \geqslant 1$. Note that $|B| \geqslant 3$. We have $k_3=2$ and so $|B|=3$. Assume that G_0 is the component of $G-uv-B$ other than G_1 and G_2. As G is 3-connected, every vertex in B has a neighbor in $V(G_0)$. Assume that $x \in B$ and $N(x)$ lies in exactly one component of $G-uv-B$. As $|B \setminus \{x\}|=2$, $G-(B \setminus \{x\})$ is disconnected, contradicting G is 3-connected. Hence $k_1=0$ and $k_2 \leqslant 2$.

Suppose that $k_3>2$. Recall that u is adjacent to every vertex of degree 3 of B. So $d(u) \geqslant 3$. Since $E(G[B])=\emptyset$, we have a claw with u as the claw-center, a contradiction. So $k_3 \leqslant 2$. Since $k_1=0$ and $k_2 \leqslant 2$, we have $|B|=k_1+k_2+k_3 \leqslant 4$. If $k_2+k_3=4$, then $k_2=2=k_3$. So $|V(Q)|=8$. Assume that $x_1,x_2 \in X$, and $d_Q(x_1)=d_Q(x_2)=3$. As Q is a simple graph, $\{x_1y_1,x_2y_2\} \subset E(G)$ or $\{x_1y_2,x_2y_1\} \subset E(G)$ where $\{y_1,y_2\}=Y \setminus \{u',v'\}$. Then $d_Q(y_1) \geqslant 3$ and $d_Q(y_2) \geqslant 3$. Therefore $d_Q(u')=2=d_Q(v')$. Then $E[\{u',v'\},B]=\{x_1u',x_1v',x_2u',x_2v'\}$ in Q. So $N(V(G_1) \cup V(G_2))=\{x_1,x_2\}$ in G. Then $G-\{x_1,x_2\}$ is disconnected, contradicting G is 3-connected. So the result holds. □

Lemma 13. *Assume that uv is a removable edge in a claw-free brick G and B is a nontrivial barrier of $G-uv$. If $|B|=3$, then the components of $G-uv-B$ containing u and v are singletons, respectively. Moreover, $d(u)=d(v)=3$ and $|N(u) \cap N(v)|=1$.*

Proof. Assume that G_i ($i \in \{1,2,3\}$) is the component of $G-uv-B$, where G_1 and G_2 contain u and v, respectively, and K_i and k_i is defined the same as in Lemma 12 for $i \in \{1,2,3\}$. By the proof of Lemma 12, we have $k_1=0$, $k_2 \leqslant 2$ and $k_3 \leqslant 2$. Since $|B|=3$, we have $k_2+k_3=3$. As G is 3-connected, every vertex in B is adjacent to some vertex in G_3.

Assume that $B=\{x_1,x_2,x_3\}$. If $k_2=1$, then $k_3=2$. Let $\{x_1\}=K_2$. So $N(x_1) \cap V(G_1)=\emptyset$ or $N(x_1) \cap V(G_2)=\emptyset$. Note that $N(x_i) \cap V(G_1)=\{u\}$ and $N(x_i) \cap V(G_2)=\{v\}$ for $i \in \{2,3\}$. (Otherwise, suppose, without loss of generality, that $N(x_2) \cap V(G_1) \setminus \{u\} \neq \emptyset$. Then x_2, together with a vertex in $N(x_2) \cap V(G_1) \setminus \{u\}$, a vertex in $N(x_2) \cap V(G_2)$

and a vertex in $N(x_2) \cap V(G_3)$, forms a claw.) If $|V(G_1)| \neq 1$, then $G - \{u, x_1\}$ is disconnected; and if $|V(G_2)| \neq 1$, then $G - \{v, x_1\}$ is disconnected. So $V(G_1) = \{u\}$ and $V(G_2) = \{v\}$. Then $ux_1 \in E(G)$ or $vx_1 \in E(G)$. Assume, without loss of generality, that $ux_1 \in E(G)$. As $\{ux_2, ux_3\} \subset E(G)$ and $E(G[B]) = \emptyset$, we have a claw with u as the claw-center, a contradiction. So $k_2 = 2$ and then $k_3 = 1$. Let $\{x_1, x_2\} = K_2$. Similar to the case when $k_2 = 1$ and $k_3 = 2$, we have $N(x_3) \cap V(G_1) = \{u\}$ and $N(x_3) \cap V(G_2) = \{v\}$. Assume, without loss of generality, that $N(x_1) \cap V(G_2) = \emptyset$. Then $G - \{u, x_1\}$ is disconnected if $|V(G_1)| > 1$. Therefore, G_1 is a singleton. Similarly, G_2 is a singleton.

Then $N(u) \setminus \{v\} \subset B$. As G is claw-free and $E(G[B]) = \emptyset$, $|N(u) \setminus \{v\}| \leqslant 2$. Since G is 3-connected, $|N(u) \setminus \{v\}| = 2$, that is, $d(u) = 3$. Similarly, $d(v) = 3$. So $|N(u) \cap B| = 2$ and $|N(v) \cap B| = 2$. By the inclusion-exclusion principle, $|N(u) \cap N(v)| \geqslant 1$ as $|B| = 3$. Suppose, to the contrary, that $|N(u) \cap N(v)| > 1$. Then $|N(u) \cap N(v)| = 2$ and so $G - \{N(u) \cap N(v)\}$ is disconnected, a contradiction. Therefore, $|N(u) \cap N(v)| = 1$. □

Lemma 14. *Let uv be a removable edge in a claw-free brick G, B be a nontrivial barrier of $G - uv$ and G_0 be a nontrivial component of $G - uv - B$. Assume that $(G - uv)/\overline{V(G_0)} \to \overline{g_0}$ has a nontrivial barrier B'. Then $B' \cup B \setminus \{\overline{g_0}\}$ is a barrier of $G - uv$.*

Proof. Let $H := (G - uv)/\overline{V(G_0)} \to \overline{g_0}$ and $B_0 := B' \cup B \setminus \{\overline{g_0}\}$. Then $\overline{g_0} \in B'$. (Otherwise, B' is a nontrivial barrier of G, contradicting the assumption that G is a brick.)

Note that the component of $G - uv - B$, other than G_0, is a component of $G - uv - B_0$, and the component of $H - B'$ is also the component of $G - uv - B$. So $o(G - uv - B_0) = |B| - 1 + |B'| = |B_0|$, that is, B_0 is a barrier of $G - uv$. □

3 The *b*-invariant edges of claw-free bricks

We first present a lemma, the proof of which will be given in Section 5.

Lemma 15. *Let G be a claw-free brick with at least 8 vertices and $u \in V(G)$. If $d(u) \geqslant 4$, then u is incident with at least $d(u) - 3$ b-invariant edges; if $d(u) = 3$, all the removable edges incident with u are b-invariant.*

Let G be a matching covered graph and let e be a removable edge of G. Let $C := \partial(X)$ be a edge cut of G. We say that C is *peripheral* if C is nontrivial, the cut $C - e$ is tight in $G - e$ and a $(C - e)$-contraction is bipartite. Let C be peripheral. Then $J := (G - e)/\overline{X} \to \overline{x}$ has bipartition $\{B, I\}$, with $\overline{x}$ in I. We then refer to $I \setminus \{\overline{x}\}$ as the inner part of J, whereas B is the outer part of J. We say that $(I \setminus \{\overline{x}\}, B)$ is a pair of vertex sets associated with e. Note that $|I \setminus \{\overline{x}\}| = |B| - 1$.

Theorem 16 ([5]). *Let G be a brick and let e be a b-invariant edge of G such that $G - e$ is not a brick. Assume that H is the brick of $G - e$, obtained by a tight cut decomposition of $G - e$. Then, one of the following three alternatives holds:*
(i) either G has a peripheral cut $C_1 := \partial(X_1)$ such that $J_1 := (G - e)/\overline{X_1} \to \overline{x_1}$ is bipartite, $H = (G - e)/X_1 \to x_1$ and edge e has one end in the inner part of J_1, the other end in $V(H) - x_1$, or

(ii) G has two peripheral cuts $C_i := \partial(X_i)$, for $i = 1, 2$, such that X_1 and X_2 are disjoint, $J_i := (G-e)/\overline{X_i} \to \overline{x_i}$ is bipartite, $H = ((G-e)/X_1 \to x_1)/X_2 \to x_2$ and edge e has one end in the inner part of J_1, the other end in the inner part of J_2, or
(iii) G has a peripheral cut $C_1 := \partial(X_1)$ such that $J_1 := (G-e)/\overline{X_1} \to \overline{x_1}$ is bipartite, $H = (G-e)/X_1 \to x_1$ and edge e has both ends in the inner part of J_1.

Proof of Theorem 1. By the definition of thin edges, every thin edge is b-invariant. Conversely, assume that uv is b-invariant in G. If $G - uv$ is a brick, then uv is thin. Suppose that $G - uv$ is not a brick. Then one of the three statements of Theorem 16 holds. Let (I_1, B_1) be a pair of vertex sets associated with uv such that $u \in I_1$.

Assume that (ii) of Theorem 16 holds, that is, there exists another pair of vertex sets (I_2, B_2) associated with uv such that $v \in I_2$, and $B_1 \cap B_2 = \emptyset$. Note that $N(u) \setminus \{v\} \subset B_1$ and $N(v) \setminus \{u\} \subset B_2$. Therefore, u, v and two vertices of $N(u) \cap B_1$ form a claw as $E(G[B_1]) = \emptyset$, a contradiction.

Next, we assume that (iii) of Theorem 16 holds. Then $\{u, v\} \subset I_1$. As $|B_1| = |I_1| + 1$, we have $|B_1| \geqslant 3$. Since G is a claw-free brick and uv is b-invariant of G, we have $|B_1| \leqslant 3$ by Lemma 12. Hence $|B_1| = 3$. By Lemma 13, $d(u) = d(v) = 3$. Note that $G - uv$ has exactly two vertices of degree two: u and v. As $N(u) \cup N(v) \setminus \{u, v\} \subset B_1$ and $|N(u) \cap N(v)| = 1$ (by Lemma 13), $N(u) \cup N(v) \setminus \{u, v\} = B_1$. Thus, the retract of $G - uv$ can be obtained from $G - uv$ by contracting $I_1 \cup B_1$ to a singleton. Therefore, the retract of $G - uv$ is a brick by Theorem 16, that is uv is thin in this case.

Now we assume that (i) of Theorem 16 holds. As $u \in I_1$, we have $|B_1| \geqslant 2$. By Lemma 12, $|B_1| \leqslant 3$. Hence $|B_1| = 2$ or $|B_1| = 3$. If $|B_1| = 3$, then $|I_1| = 2$. Let $\{t\} = I_1 \setminus \{u\}$. If $t = v$, then $\{u, v\} \subset I_1$. Similar to last paragraph, the retract of $G - uv$ is a brick. Now we assume that $t \neq v$. As G is 3-connected and $N(t) \subset B_1$, t is adjacent to every vertex in B_1. Since $E(G[B_1]) = \emptyset$, $G[\{t\} \cup B_1]$ is a claw, a contradiction. So we consider the case when $|B_1| = 2$. Let $B_1 = \{u_1, u_2\}$. As $N(u) \setminus \{v\} \subset B_1$, we have $d(u) = 3$. So $N(u) = \{v, u_1, u_2\}$. As G is claw-free and $u_1u_2 \notin E(G)$, at least one of u_1 and u_2 is adjacent to v. Assume that $d(v) = 3$. If $u_1v \in E(G)$ and $u_2v \in E(G)$, then $N(\{u, v\}) \setminus \{u, v\} = \{u_1, u_2\}$. So $G - \{u_1, u_2\}$ is disconnected, a contradiction. Hence either $u_1v \in E(G)$ or $u_2v \in E(G)$. Assume, without loss of generality, that $u_1v \in E(G)$ and $u_2v \notin E(G)$. Let $\{v_1\} = N(v) \setminus \{u, u_1\}$ and $B_1' = \{u_1, u_2, v_1\}$. Denote by Q the component of $G - uv - B_1'$ containing no u and v. Since $|V(Q)| = V(G) \setminus (B_1' \cup \{u, v\})$, $|V(Q)|$ is odd. As $N(u) \cup N(v) \setminus \{u, v\} \subset B_1'$, $o(G - uv - B_1') = 3$. So B_1' is a nontrivial barrier of $G - uv$. Moreover, $E(G[B_1']) = \emptyset$. (Otherwise, B_1' contains a forbidden edge in $G - uv$ by Lemma 7, contradicting the assumption that uv is removable.) Recalling that $d(u) = d(v) = 3$ and $N(u) \cup N(v) \setminus \{u, v\} = B_1'$, the retract of $G - uv$ is isomorphic to $(G - uv)/\overline{Q} \to \overline{q}$. Note that $(G - uv)/Q \to q$ is a bipartite. As uv is a b-invariant edge of G, $(G - uv)/\overline{Q} \to \overline{q}$ contains no nontrivial tight cuts. So $(G - uv)/\overline{Q} \to \overline{q}$ is a brick. Therefore, the retract of $G - uv$ is a brick. Now assume that $d(v) \geqslant 4$. Note that u is the only vertex in $G - uv$ of degree two. So the retract of $G - uv$ can be obtained from $G - uv$ by contracting $I_1 \cup B_1$ to a singleton. Thus, the retract of $G - uv$ is a brick by Theorem 16. Therefore, uv is a thin edge in G. The proof is complete. □

4 The lower bound of thin edges in claw-free bricks

For convenience, we denote by K_4^- the graph obtained from K_4 by removing an edge. We first present several lemmas.

Lemma 17 ([16]). *Let G be a claw-free brick with at least 8 vertices and $u \in V(G)$.*
1. If u is incident with no removable edges, then $d(u) = 3$. Moreover, one of the following statements holds.

a). u lies in a triangle xuv of G, $d(v) = d(x) = 3$ and $N(\{u, v, x\}) \setminus \{u, v, x\}$ is a subset of a clique in G with at least five vertices.

b). $G[\{u\} \cup N(u)] \cong K_4$ or K_4^- and at most one vertex in $N(u)$ is of degree 3.

2. If $d(u) \geqslant 4$, then u is incident with at least $d(u) - 2$ removable edges.

We say that a vertex u of a claw-free brick is *special* if it is incident with no removable edges and $G[\{u\} \cup N(u)]$ is isomorphic to K_4 or K_4^-. By Lemma 17, we have $d(u) = 3$. We call the subgraph induced by a special vertex and its three neighbors is a *special subgraph.* We say that a vertex is a *half* vertex if it lies in exactly two special subgraphs. Denote by $d_r(u)$ and $d_t(u)$ the number of removable edges incident with u and the number of thin edges incident with u, respectively. Therefore, the graph G has at least $(\sum\limits_{u \in V(G)} d_t(u))/2$ thin edges. By Lemma 15 and Theorem 1, for every vertex $x \in V(G)$, if $d(x) = 3$ and x is incident with at least one removable edge, then $d_t(x) \geqslant 1$; if $d(x) \geqslant 4$, then $d_t(x) \geqslant d(u) - 3$.

Lemma 18. *Let H be a special subgraph of a claw-free brick G. Then H contains exactly one special vertex.*

Proof. Let $V(H) = \{u, v, x, y\}$, where u is a special vertex of G. As G is 3-connected and $N(u) = \{v, x, y\}$, every vertex in $N(u)$ has at least one neighbor not in $V(H)$. Suppose that v is also a special vertex of G. Then $d(v) = 3$. By Lemma 17, at most one vertex in $N(u)$ is of degree 3. Therefore, $H \cong K_4^-$, $d(x) \geqslant 4$ and $d(y) \geqslant 4$. Assume that $E(H) = \{uv, ux, uy, vy, xy\}$. As v is a special vertex, vy is not removable in G. Assume that B is a nontrivial barrier of $G - vy$ associated with vy, and G_i is the component of $G - vy - B$, for $i \in \{1, 2, \ldots, |B|\}$, such that $v \in V(G_1)$ and $y \in V(G_2)$. As $\{ux, uy\} \subset E(G)$, $u \in B$. By Lemma 2.5 of [16], we have $|B| \leqslant 3$. If $|B| = 2$, then $x \in B$ as $N(u) = \{v, x, y\}$ and $E(G[B]) \neq \emptyset$. As $d(v) = 3$, $(N(v) \setminus \{u, y\}) \cap V(G_1) \neq \emptyset$. By Lemma 2.6 of [16], $N(u) \cap V(G_1) \setminus \{v\} \neq \emptyset$. Then $d(u) \geqslant 4$, a contradiction. Now we assume that $|B| = 3$. By Lemma 2.7 of [16], $N(v) \cup N(y) \setminus \{v, y\} \subset B$. Then $\{u, x\} \subset B$. As $d(u) = 3$, $N(u) \cap V(G_3) = \emptyset$. Then $G - (B \setminus \{u\})$ is disconnected, a contradiction. So H contains exactly one special vertex. □

Proof of Theorem 2. Assume that L_1 is the set of vertices in G satisfying a). of Lemma 17. If $u \in L_1$, then we assume that the triangle that u lies in is uu_1v and $N(u) \cup N(v) \cup N(u_1) \setminus \{u, v, u_1\} = \{u_2, u_1', v_1\}$. Let $V_u = \{u, v, u_1, u_2, v_1, u_1'\}$ and $V_1 = \{\cup_{u \in L_1} V_u\}$.

Claim 1. ([16]) For every vertex s in V_1, there exists only one vertex t in L_1, such that $s \in V_t$.

Claim 2. $\sum_{u \in V_1} d_t(u) \geqslant |V_1|$.

Proof. By Claim 1, every vertex s in V_1, there exists only one vertex t in L_1, such that $s \in V_t$. Assume that $u \in L_1$. Then every vertex in $\{u_2, u_1', v_1\}$ is of degree at least 5. By Lemma 3.3 of [16], for $s \in \{u_2, u_1', v_1\}$, we have $d_r(s) \geqslant 3$. By Lemma 15 and Theorem 1, for $s \in \{u_2, u_1', v_1\}$, we have $d_t(s) \geqslant 2$. Therefore, $\sum_{u \in V_u} d_t(u) \geqslant |V_u|$. □

Assume that L_2 is the set of vertices in $V(G) \setminus V_1$ incident with no removable edges. By Lemma 17, every vertex in L_2 satisfies b). of Lemma 17. Assume that $u' \in L_2$. Then u' lies in a subgraph $H_{u'}$ isomorphic to K_4 or K_4^-. Let $V_2 = \{\cup_{u' \in L_2} V(H_{u'})\}$ and $V_3 = V(G) \setminus (V_1 \cup V_2)$. By Lemma 17, V_1, V_2 and V_3 are a partition of $V(G)$.

Claim 3. $\sum_{u \in V_2} d_t(u) \geqslant \frac{3|V_2|}{4}$.

Proof. Assume that $u \in V_2$. Note that, for a subgraph that contains u and is isomorphic to K_4 or K_4^-, a special subgraph contains less removable edges than a non-special one. To get the lower bound of thin edges, we may assume that u lies in a special subgraph.

Let $H := G[\{u\} \cup N(u)]$ be a special subgraph. We assume firstly that H is isomorphic to K_4. By Lemma 4.4 of [16], for $s \in V(H)\setminus\{u\}$, we have $d(s) \geqslant 4$ and $d_r(s) \geqslant 2$. By Lemma 15 and Theorem 1, for every vertex $s \in V(H)\setminus\{u\}$, $d_t(s) \geqslant 1$. Therefore, $\sum_{s \in V(H)} d_t(s) \geqslant \frac{3|V(H)|}{4}$. We now assume H is isomorphic to K_4^-. By Lemma 18, H contains exactly one special vertex. Assume that H contains no half vertices. At most one vertex in $V(H)\setminus\{u\}$ is of degree 3, as G is 3-connected and claw-free. By Lemma 4.5 of [16], for $s \in V(H)\setminus\{u\}$, if $d(s) = 3$, then $d_r(s) \geqslant 1$; if $d(s) \geqslant 4$, then $d_r(s) \geqslant 2$. By Lemma 15 and Theorem 1 again, $\sum_{s \in V(H)} d_t(s) \geqslant \frac{3|V(H)|}{4}$. Assume that H contains half vertices. By Lemma 4.2 of [16], this special subgraph contains exactly one half vertex x, and x lies in exactly two special subgraphs, say H_1 and H_2. Let $H_1 := G[\{u\} \cup \{x, y, z\}]$ and $H_2 := G[\{u_1\} \cup \{x, y_1, z_1\}]$, where u and u_1 are special vertices of H_1 and H_2, respectively, $\{x, y, z\} = N(u)$ and $\{x, y_1, z_1\} = N(u_1)$. By Lemma 4.3 of [16], we have $d(z) \geqslant 3$, $d(z_1) \geqslant 3$, $d(y) \geqslant 4$. And if $y_1 z_1 \notin E(G)$, then $d(x) \geqslant 5$ and $d(y_1) \geqslant 4$; if $xz_1 \notin E(G)$, then $d(x) \geqslant 4$ and $d(y_1) \geqslant 5$. So $d_r(z) \geqslant 1$, $d_r(z_1) \geqslant 1$, $d_r(y) \geqslant 2$ and $d_r(x) + d_r(y_1) \geqslant 5$. Then, by Lemma 15 and Theorem 1, $d_t(z) \geqslant 1$, $d_t(z_1) \geqslant 1$, $d_t(y) \geqslant 1$ and $d_t(x) + d_t(y_1) \geqslant 3$. Therefore, $\sum_{s \in V(H_1) \cup V(H_2)} d_t(s) \geqslant \frac{6|V(H_1) \cup V(H_2)|}{7}$. So the claim holds. □

Note that every vertex in V_3 is incident with at least one removable edge. By Lemma 15 and Theorem 1, every vertex in V_3 is incident with at least one thin edge. Therefore, $\sum_{s \in V_3} d_t(s) \geqslant |V_3|$. By Claims 2 and 3, the result follows. □

Lemma 19. *Assume that u is a cubic vertex in a claw-free brick G. Then u is incident with at most two thin edges.*

Proof. We will show that u is incident with at most two removable edges. Suppose, to the contrary, that uu_i ($i \in \{1,2,3\}$) is removable in G. By Proposition 10, $G[\{u, u_2, u_3\}]$ is not a triangle when uu_1 is removable. So $u_2u_3 \notin E(G)$. Similarly, $u_1u_2 \notin E(G)$ and $u_1u_3 \notin E(G)$. Then $G[\{u, u_1, u_2, u_3\}]$ is a claw, a contradiction. So u is incident with at most two removable edges. As every thin edge is removable, the result follows. □

Proof of Theorem 4. If $|V(G)| \leqslant 6$, then it can be checked that G is K_4, the triangular prism and W_5. So we consider $|V(G)| > 6$. By Theorem 2, a claw-free brick G with at least 8 vertices has at least $3|V(G)|/8$ thin edges. If e is a thin edge in a claw-free minimal brick, at least one of the end vertex of e is cubic by Theorems 1 and 16. Therefore, the Theorem 4 holds by Lemma 19. □

5 Proof of the Lemma 15

5.1 The structure of non-b-invariant edges of claw-free bricks

In this subsection, we will consider the structure of non-b-invariant edges in claw-free bricks and we have the following lemmas.

Lemma 20. *Assume that uv is a removable edge in a claw-free brick G such that there exists a barrier B of $G - uv$ with size 3. Then uv is b-invariant.*

Proof. By Lemma 13, $G - uv - B$ has only one nontrivial odd component, say G_0. Let $H := (G-uv)/\overline{V(G_0)} \to \overline{g_0}$. As $(G-uv)/V(G_0) \to g_0$ is bipartite, to complete the proof, we will show that H is a brick by contradiction.

Suppose that B' is a nontrivial barrier of H. Then $\overline{g_0} \in B'$. (Otherwise, B' is a nontrivial barrier of G, contradicting the assumption that G is a brick.) Let $B_0 = B' \cup B \setminus \{\overline{g_0}\}$. By Lemma 14, B_0 is barrier of $G - uv$ and $|B_0| = |B'| - 1 + |B|$. As $|B| = 3$ and $|B'| \geqslant 2$, we have $|B_0| = |B'| + 2 > 3$, which contradicts Lemma 12. Therefore, H contains no nontrivial barriers.

Next, we suppose that S is a 2-separation of H. Then $\overline{g_0} \in S$. (Otherwise, S is a 2-separation of G, contradicting the assumption that G is a brick.) Let $S = \{\overline{g_0}, t\}$. Assume that $\partial(X)$ is a 2-separation cut of H associated with S such that $\overline{g_0} \in X$. Denote by Q_1 and Q_2 the subgraphs of H induced by $X \setminus \{\overline{g_0}\}$ and $\overline{X} \setminus \{t\}$, respectively.

Let $B = \{u_1, u_2, u_3\}$ and $U = B \cup \{t\}$. By Lemma 13, $N(u) \cup N(v) \setminus \{u, v\} \subset \{u_1, u_2, u_3\}$ and the components of $G-uv-B$ that contain u and v are trivial, respectively. So $N(\{u, v\}) \cap (V(Q_1) \cup V(Q_2)) = \emptyset$ and then the components of $G-uv-U$ that contain u and v are trivial, respectively. Hence U is a vertex 4-cut in $G - uv$. By contracting Q_1 and Q_2 into singletons q_1 and q_2, respectively, and removing the edge uv, edges in $G[U]$ and all the resulting multiedges, we obtain a graph Q. As $N_Q(u) \cup N_Q(v) \subset U$ and $q_1q_2 \notin E(Q)$, Q is a bipartite graph. Then U and $W = \{u, v, q_1, q_2\}$ are the color classes of Q. By Lemma 13 again, exactly one of $\{u_1, u_2, u_3\}$, say u_1, is adjacent to both u and v; exactly one of $\{u_2, u_3\}$, say u_2, is adjacent to u; and then u_3 is adjacent to v. As G is claw-free, each vertex of $\{u_1, u_2, u_3\}$ is adjacent to exactly one of q_1 and q_2. Note that $tq_1 \in E(Q)$ and $tq_2 \in E(Q)$. By calculating, $|E_Q[U, W]| = 9$. On the other hand, each component in $G - uv - U$ is incident with at least three vertices in U, except the two components that u and v lie in, respectively, as G is 3-connected and $uv \in E(G)$. By Lemma 13 again, $d(u) = d(v) = 3$. So exactly two vertices in U are incident with both components that u and v lie in. Then $|E_Q[U, W]| \geqslant 10$, a contradiction. Therefore, H is a brick. So the result holds. $\square$

Lemma 21. *Assume that uv is a removable edge in a claw-free brick G. If B is a barrier of $G - uv$ with size 2 and one component of $G - uv - B$ is trivial, then uv is b-invariant.*

Proof. If the components of $G-uv-B$ are both trivial, then $|V(G)| = 4$, that is, $G \cong K_4^-$, which contradicts the assumption that G is matching covered. Assume, without loss of generality, that the component of $G - uv - B$ that u lies in is trivial. Denote by G_0 the only nontrivial odd component of $G - uv - B$. Let $H := (G - uv)/\overline{V(G_0)} \to \overline{g_0}$. As $(G - uv)/V(G_0) \to g_0$ is bipartite, to complete proof we will show that H is a brick by contradiction.

Suppose that B' is a nontrivial barrier of H. Then $\overline{g_0} \in B'$. (Otherwise, B' is a nontrivial barrier of G, contradicting the assumption that G is a brick.) Let $B_0 = B' \cup B \setminus \{\overline{g_0}\}$. By Lemma 14, B_0 is barrier of $G - uv$ and $|B_0| = |B'| - 1 + |B|$. Since $|B| = 2$, we have $|B'| = 2$ by Lemma 12. Hence $|B_0| = 3$. By Lemma 20, uv is b-invariant.

Next, we suppose that S is a 2-separation of H. Then $\overline{g_0} \in S$. (Otherwise, S is a 2-separation of G, contradicting the assumption that G is a brick.) Let $S = \{\overline{g_0}, t\}$. Assume that $\partial(X)$ is a 2-separation cut of H associated with S such that $\overline{g_0} \in X$. Denote by Q_1 and Q_2 the subgraphs of H induced by $X \setminus \{\overline{g_0}\}$ and $\overline{X} \setminus \{t\}$, respectively.

Assume that $B = \{u_1, u_2\}$ and $U = B \cup \{t\}$. Hence U is a vertex 3-cut in $G - uv$. If $t = v$, then $N(v) \cap V(Q_1) \neq \emptyset$ and $N(v) \cap V(Q_2) \neq \emptyset$. So v, u, a vertex in $N(v) \cap V(Q_1)$ and a vertex in $N(v) \cap V(Q_2)$ form a claw, a contradiction. Thus assume, without loss of generality, that v lies in Q_1. As G is claw-free, at least one of u_1v and u_2v belongs to $E(G)$, otherwise, we have a claw with u as the claw-center. As G is 3-connected, $N(u_1) \cap V(Q_2) \neq \emptyset$. If $N(u_1) \cap V(Q_1) \setminus \{v\} \neq \emptyset$, then u_1, u, a vertex in $N(u_1) \cap V(Q_1) \setminus \{v\}$ and a vertex in $N(u_1) \cap V(Q_2)$ form a claw, a contradiction. So $N(u_1) \cap V(Q_1) \setminus \{v\} = \emptyset$. Similarly, $N(u_2) \cap V(Q_1) \setminus \{v\} = \emptyset$. Note that $|V(Q_1)| \geqslant 2$. Then $G - \{v, t_1\}$ is disconnected, a contradiction. Therefore, H is a brick and hence uv is b-invariant. $\square$

Corollary 22. *Let G be a claw-free brick. If uv is a removable but non-b-invariant edge in G, then $G-uv$ contains a vertex 2-cut D and $G-uv-D$ contains exactly two nontrivial components.*

Proof. By Theorem 8, $G - uv$ has a nontrivial barrier or a 2-separation. If $G - uv$ has a nontrivial barrier B, then $|B| \leqslant 2$ by Lemmas 12 and 20. By Lemma 21, we have $|D| = 2$ and $G - uv - D$ contains two nontrivial components. If $G - uv$ has a 2-separation, the result follows by the definition of 2-separation. So the result follows. $\square$

Lemma 23. *Let v_1v_2 be a removable but non-b-invariant edge in a claw-free brick G. Assume that D is a vertex 2-cut of $G - v_1v_2$, where $D = \{s_1, s_2\}$, and G_1 and G_2 are two components of $G - v_1v_2 - D$ such that $v_i \in V(G_i)$ for $i \in \{1, 2\}$. Then $N(s_i) \cap V(G_1) \setminus \{v_1\} \neq \emptyset$ for $i \in \{1, 2\}$.*

Proof. By Corollary 22, we have $|V(G_1)| > 1$. If $N(s_i) \cap V(G_1) \setminus \{v_1\} = \emptyset$, then $G - \{s_{3-i}, v_1\}$ is disconnected, contradicting the fact G is 3-connected. $\square$

By Lemma 11 and Corollary 22, we have the following corollary.

Corollary 24. *Let G be a claw-free brick. If an edge uv is removable but not b-invariant in G, then $|N(u) \cap N(v)| \leqslant 2$.*

In the rest of this paper. Let G be a claw-free brick and uv be a removable but non-b-invariant edge of G. Denote by D a vertex 2-cut of $G - uv$ and by G_1 and G_2 the components of $G - uv - D$ such that $u \in V(G_1)$ and $v \in V(G_2)$. By Corollary 22, we have $|V(G_i)| \geqslant 2$ for $i \in \{1, 2\}$. Let $N(u) = \{v, u_1, u_2, \ldots, u_{d_G(u)-1}\}$.

Lemma 25. *If $u_1 \in N(u) \cap D$, then $N(u_1) \cap V(G_1) \setminus \{u\} \subset N(u) \cap V(G_1)$. Moreover, if $u_1 v \notin E(G)$, then $N(u_1) \cap V(G_1) \setminus \{u\} = N(u) \cap V(G_1)$.*

Proof. By Lemma 23, $N(u_1) \cap V(G_1) \setminus \{u\} \neq \emptyset$ and $N(u_1) \cap V(G_2) \setminus \{v\} \neq \emptyset$. Suppose, to the contrary, that there exists a vertex a in $N(u_1) \cap V(G_1) \setminus \{u\}$ such that $a \notin N(u) \cap V(G_1)$. Then we have a claw with u_1 as the claw-center, a contradiction. Therefore, $N(u_1) \cap V(G_1) \setminus \{u\} \subset N(u) \cap V(G_1)$.

Assume that $u_1 v \notin E(G)$. If there exists a vertex a_1 in $N(u) \cap V(G_1)$ such that $a_1 \notin N(u_1) \cap V(G_1)$, then $G[\{u, a_1, u_1, v\}]$ is a claw, a contradiction. So the result holds. $\square$

Lemma 26. *Let uv be a removable edge of a claw-free brick G. If uv is not b-invariant, then $d(u) \geqslant 4$ and $d(v) \geqslant 4$.*

Proof. Suppose, to the contrary, that $d(u) = 3$. If $N(u) \cap D = \emptyset$, then $G[N(u) \setminus \{v\} \cup \{u\}]$ is a triangle as G is claw-free. By Proposition 10, uv is non-removable, a contradiction. If $|N(u) \cap D| = 2$, then $V(G_1) = \{u\}$, contradicting Corollary 22. So we have $|N(u) \cap D| = 1$. If $N(u) \cap D = \{u_1\}$, then $u_2 \in V(G_1)$. By Lemma 23, $N(u_1) \cap V(G_1) \setminus \{u\} \neq \emptyset$. By Lemma 25, $N(u_1) \cap V(G_1) \setminus \{u\} \subset N(u)$. Then $u_1 u_2 \in E(G)$. So $G[N(u) \setminus \{v\} \cup \{u\}]$ is a triangle. By Proposition 10 again, uv is non-removable, a contradiction. Therefore, $d(u) \geqslant 4$. Similarly, $d(v) \geqslant 4$. $\square$

By Lemma 26, the following corollary can be derived directly.

Corollary 27. *Let uv be a removable edge in a claw-free brick G. If at least one of end vertex of uv is cubic, then it is b-invariant.*

Lemma 28. *Assume that an edge uv is removable but not b-invariant in a claw-free brick G. If $G[N(u) \setminus \{v\}]$ is a complete graph and uu_1 is a removable edge in G, then uu_1 is b-invariant.*

Proof. Suppose, to the contrary, that uu_1 is not b-invariant in G. By Corollary 22, we may assume that D' is a vertex 2-cut of $G - uu_1$, G'_i ($i \in \{1, 2\}$) is the component of $G - uu_1 - D'$, $u \in V(G'_1)$, $u_1 \in V(G'_2)$ and $|V(G'_i)| \geqslant 2$ for $i \in \{1, 2\}$. By Lemma 26, we have $d(u) \geqslant 4$

and $d(v) \geqslant 4$. Since $G[N(u)\backslash\{v\}]$ is a complete graph, $D' \supseteq N(u_1)\cap N(u) \supseteq N(u)\backslash\{v, u_1\}$. By Corollary 24, we have $d(u) \leqslant 4$. So $d(u) = 4$, that is, $N(u_1) \cap N(u) = \{u_2, u_3\}$. Then $D' = \{u_2, u_3\}$ and so $v \in V(G'_1)$. By Lemmas 23 and 25, $N(u_i) \cap V(G'_1) \setminus \{u\} = \{v\}$ for $i \in \{2, 3\}$. If $|V(G'_1)| > 2$, then $G - \{u, v\}$ is disconnected, contradicting the fact G is 3-connected. If $|V(G'_1)| = 2$, then $d(v) = 3$, a contradiction. Therefore, uu_1 is b-invariant in G. □

Lemma 29. *Assume that uv is removable but not b-invariant in a claw-free brick G and $Z = N(u) \cap N(v)$. Then both $G[N(u) \setminus (\{v\} \cup Z)]$ and $G[N(v) \setminus (\{u\} \cup Z)]$ are complete graphs.*

Proof. If $|N(u) \setminus (\{v\} \cup Z)| = 1$, then $G[N(u) \setminus (\{v\} \cup Z)]$ is a complete graph. Now we consider $|N(u) \setminus (\{v\} \cup Z)| \geqslant 2$. Suppose, to the contrary, that $G[N(u) \setminus (\{v\} \cup Z)]$ is not a complete graph. Assume that $\{u_1, u_2\} \subset N(u) \setminus (\{v\} \cup Z)$ and $u_1u_2 \notin E(G)$. Then $G[\{u, u_1, u_2, v\}]$ is a claw, a contradiction. Similarly, $G[N(v) \setminus (\{u\} \cup Z)]$ is a complete graph. □

Lemma 30. *Let G be a brick and uv be a removable but non-b-invariant edge. If $G - uv$ contains at least two vertex-disjoint 2-separations, then $N(u) \cap N(v) = \emptyset$.*

Proof. As G is 3-connected, u and v lie in different components of $G - D - uv$ (and $G - D' - uv$), respectively, where D and D' are two vertex-disjoint 2-separations of $G - uv$. Denote by Q_1 and Q_2 the component of $G - D - uv$ that u lies in and the component of $G - D' - uv$ that v lies in, respectively. Since G is 3-connected, $Q_1 \subset \overline{Q_2}$ or $\overline{Q_2} \subset Q_1$. As D and D' are two vertex-disjoint 2-separations of $G - uv$, every path in $G - uv$ from u to v contains at least one vertex of D and one vertex of D', respectively, thereby the distance between u and v is greater than 2 in $G - uv$. Therefore $N(u) \cap N(v) = \emptyset$. □

5.2 Adjacent non-b-invariant edges in claw-free bricks

To complete the proof of Lemma 15, we will consider a vertex incident with two b-invariant edges in claw-free bricks in this subsection.

Lemma 31. *Assume that uv and uu_1 are removable but not b-invariant in a claw-free brick G and D is a vertex 2-cut of $G - uv$. Then $u_1 \notin D$.*

Proof. Suppose, to the contrary, that $u_1 \in D$. Let $D = \{u_1, t\}$. By Corollary 22, we may assume that D' is a vertex 2-cut of $G - uu_1$, G'_i ($i \in \{1, 2\}$) is the component of $G - uu_1 - D'$, $u \in V(G'_1)$, $u_1 \in V(G'_2)$ and $|V(G'_i)| \geqslant 2$ for $i \in \{1, 2\}$. By Lemma 26, we have $d(u) \geqslant 4$.

Assume that $N(u) \cap D = \{u_1\}$. If $u_1v \notin E(G)$, then $G[N(u) \setminus \{v\}]$ is a complete graph by Lemma 29. By Lemma 28, uu_1 is b-invariant, contradicting the assumption that uu_1 is not b-invariant. So $u_1v \in E(G)$. As $v \in N(u) \cap N(u_1)$, $v \in D'$. By Lemma 29, $G[N(v) \setminus \{u, u_1\}]$ is complete. However, by Lemma 23, $N(v) \cap V(G'_1) \setminus \{u\} \neq \emptyset$ and $N(v) \cap V(G'_2) \setminus \{u_1\} \neq \emptyset$, a contradiction.

Now we assume that $N(u)\cap D=\{u_1,u_2\}$. If $u_1v\notin E(G)$, then $N(u_1)\cap V(G_1)\setminus\{u\}=N(u)\cap V(G_1)$ by Lemma 25. By Corollary 24, we have $|N(u)\cap N(u_1)|\leqslant 2$, thereby $d(u)\leqslant 5$. As G is claw-free, $G[N(u)\setminus\{v,u_2\}]$ is complete. If $d(u)=5$, then $u_1u_2\notin E(G)$. Otherwise, since $G[N(u)\setminus\{v,u_2\}]$ is complete, we have $|N(u_1)\cap N(u)|=3$, contradicting Corollary 24. So $N(u_1)\cap N(u)=\{u_3,u_4\}$ by Lemma 25. Then $D'=\{u_3,u_4\}$ and so $\{u_2,v\}\subset V(G'_1)$. Hence $|V(G'_1)|\geqslant 3$. By Lemmas 23 and 25, $\emptyset\neq N(u_i)\cap V(G'_1)\setminus\{u\}\subset N(u)\cap V(G'_1)$ for $i\in\{3,4\}$. Note that $u_3v\notin E(G)$ and $u_4v\notin E(G)$. Then $(N(u_3)\cup N(u_4))\cap V(G'_1)=\{u,u_2\}$. Recalling that $|V(G'_1)|\geqslant 3$, $G-\{u,u_2\}$ is disconnected, a contradiction. So we consider the case when $d(u)=4$. By Lemmas 23 and 25, $\emptyset\neq N(u_i)\cap V(G_1)\setminus\{u\}\subset N(u)\cap V(G_1)$ for $i\in\{1,2\}$. Then $(N(u_1)\cup N(u_2))\cap V(G_1)=\{u,u_3\}$. If $|V(G_1)|\geqslant 3$, then $G-\{u,u_3\}$ is disconnected, a contradiction. So $|V(G_1)|=2$, and then $d(u_3)=3$. As $u_3\in N(u_1)\cap N(u)$, $u_3\in D'$. By Lemma 23, $N(u_3)\cap V(G'_1)\setminus\{u\}\neq\emptyset$ and $N(u_3)\cap V(G'_2)\setminus\{u_1\}\neq\emptyset$. Then $d(u_3)\geqslant 4$ (note that $\{uu_3,u_1u_3\}\subset E(G)$), a contradiction.

Now assume that $u_1v\in E(G)$. By Lemmas 23 and 25, $\emptyset\neq N(u_1)\cap V(G_1)\setminus\{u\}\subset N(u)\cap V(G_1)$. By Corollary 24, $|N(u_1)\cap N(u)|\leqslant 2$. Then $|N(u_1)\cap V(G_1)\setminus\{u\}|=1$ and so $u_1u_2\notin E(G)$. Hence $D'=N(u_1)\cap N(u)=\{u_3,v\}$ and $u_2\in V(G'_1)$. By Lemmas 23 and 25, $\emptyset\neq N(v)\cap V(G'_1)\setminus\{u\}\subset N(u)\cap V(G'_1)$ and $N(v)\cap V(G'_2)\setminus\{u_1\}\neq\emptyset$. On the other hand, as G is claw-free, $G[N(v)\setminus\{u,u_1,u_2\}]$ is complete. So $N(v)\cap V(G'_1)\setminus\{u\}=\{u_2\}$, that is, $u_2v\in E(G)$. Then $N(v)\cap N(u_2)=\{u\}$ (note that $u_3v\notin E(G)$). However, by Lemmas 23 and 25, $N(u_2)\cap N(v)\cap V(G_2)\neq\emptyset$, a contradiction. Therefore, $u_1\notin D$. □

Lemma 32. *Let uv and uu_1 be removable but non-b-invariant edges in a claw-free brick G. Assume that $D=\{x,y\}$ is a vertex 2-cut of $G-uv$.*

1) *If $|N(u)\cap D|<2$, then $d(u)\leqslant 5$.*
2) *If $|N(u)\cap D|=2$ and $xy\notin E(G)$, then $d(u)\leqslant 5$.*
3) *If $|N(u)\cap D|=2$ and $xy\in E(G)$, then $d(u)\geqslant 5$ and $u_1\notin N(D)$.*

Proof. By Corollary 22, we have $|V(G_i)|\geqslant 2$ for $i\in\{1,2\}$. By Lemma 26, we have $d(u)\geqslant 4$. By Lemma 31, $u_1\notin D$ and hence $u_1\in V(G_1)$.

By Lemma 29, $G[N(u)\cap V(G_1)]$ is complete. Then $N(u)\cap V(G_1)\setminus\{u_1\}\subset N(u_1)\cap N(u)$. By Corollary 24, $|N(u)\cap V(G_1)\setminus\{u_1\}|\leqslant 2$. If $|N(u)\cap D|<2$, then $d(u)\leqslant 5$. Then 1) holds.

Now we consider $|N(u)\cap D|=2$ and $xy\notin E(G)$, that is, $D=\{u_2,u_3\}$ and $u_2u_3\notin E(G)$. As G is claw-free, at least one of u_1u_2 and u_1u_3 belongs to $E(G)$, otherwise, we have a claw with u as the claw-center. Recall that $G[N(u)\cap V(G_1)]$ is complete. If $d(u)\geqslant 6$, then $|N(u_1)\cap N(u)|\geqslant 3$, contradicting Corollary 24. So we have $d(u)\leqslant 5$. Then 2) holds.

Next, we consider the case when $|N(u)\cap D|=2$ and $xy\in E(G)$, that is, D is a 2-separation of $G-uv$ and $D=\{u_2,u_3\}$. Suppose, to the contrary, that $d(u)=4$. By Lemmas 23 and 25, $\emptyset\neq N(u_j)\cap V(G_1)\setminus\{u\}\subset N(u)\cap V(G_1)$ for $j\in\{2,3\}$. So $(N(u_2)\cup N(u_3))\cap V(G_1)\setminus\{u\}=\{u_1\}$. Recall that $|V(G_1)|\geqslant 2$. If $|V(G_1)|>2$, then $G-\{u,u_1\}$ is disconnected, a contradiction. Hence $|V(G_1)|=2$. So $d(u_1)=3$, contradicting Lemma 26. Therefore, we have $d(u)\geqslant 5$.

Now we suppose, without loss of generality, that $u_1 \in N(u_2)$. Recall that $G[N(u) \cap V(G_1)]$ is complete. Then $N(u) \cap V(G_1) \setminus \{u_1\} \subset N(u) \cap N(u_1)$. Since $u_1u_2 \in E(G)$, $d(u) = 5$ and $u_1u_3 \notin E(G)$ by Corollary 24. By Corollary 22, we may assume that D' is a vertex 2-cut of $G - uu_1$, G'_i ($i \in \{1,2\}$) is the component of $G - uu_1 - D'$, $u \in V(G'_1)$ and $u_1 \in V(G'_2)$. Note that $N(u_1) \cap N(u) = \{u_2, u_4\}$. Hence $D' = \{u_2, u_4\}$ and $\{u_3, v\} \subset V(G'_1)$. So we have $|V(G'_1)| \geqslant 3$. Assume that $u_2v \notin E(G)$. Note that $u_4v \notin E(G)$. By Lemma 25, $(N(u_2) \cup N(u_4)) \cap V(G'_1) \setminus \{u\} = \{u_3\}$. As $|V(G'_1)| \geqslant 3$, $G - \{u, u_3\}$ is disconnected, a contradiction. So we assume that $u_2v \in E(G)$. By Lemma 25, $N(u_2) \cap V(G_1) \setminus \{u\} \subset N(u) \cap V(G_1)$ and $N(u_2) \cap V(G_2) \setminus \{v\} \subset N(v) \cap V(G_2)$. Hence $N(u_2) \setminus \{v, u_1, u_3, u_4\} \subset N(v) \setminus \{u_2\}$. As $\{u_3, v\} \subset V(G'_1)$ and $N(v) \setminus \{u_2\} \subset V(G'_1)$, $N(u_2) \setminus \{u_1, u_4\} \subset V(G'_1)$. Then $N(u_2) \cap V(G'_2) \setminus \{u_1\} = \emptyset$, contradicting Lemma 23. Therefore, $u_1 \notin N(u_2)$. Similarly, $u_1 \notin N(u_3)$. So 3) holds. □

Lemma 33. *Assume uv is a removable but non-b-invariant edge in a claw-free brick G and D is a vertex 2-cut of $G - uv$. If $u_1 \notin D$, $|N(u) \cap D| = 2$, $E(G[D]) = \emptyset$ and uu_1 is a removable edge in G, then uu_1 is b-invariant.*

Proof. Then $u_1 \in V(G_1)$. Suppose, to the contrary, that uu_1 is not b-invariant. By Corollary 22, we may assume that D' is a vertex 2-cut of $G - uu_1$, G'_i ($i \in \{1,2\}$) is the component of $G - uu_1 - D'$, $u \in V(G'_1)$ and $u_1 \in V(G'_2)$. Let $D = \{u_2, u_3\}$.

By Lemmas 26 and 32, we have $d(u) = 4$ or $d(u) = 5$. Assume that $d(u) = 5$. As G is claw-free, $G[N(u) \cap V(G_1)]$ is complete, that is, $|N(u_1) \cap N(u) \cap V(G_1)| = 1$ when $d(u) = 5$. Since $u_2u_3 \notin E(G)$ and G is claw-free, at least one of u_1u_2 and u_1u_3 belongs to $E(G)$. If $\{u_1u_2, u_1u_3\} \subset E(G)$, then $|N(u) \cap N(u_1)| = 3$, contradicting Corollary 24. Therefore, either $u_1u_2 \in E(G)$ or $u_1u_3 \in E(G)$. Assume, without loss of generality, that $u_1u_2 \in E(G)$ and $u_1u_3 \notin E(G)$. Then $N(u_1) \cap N(u) = \{u_2, u_4\}$ and so $D' = \{u_2, u_4\}$. Hence $\{u_3, v\} \subset V(G'_1)$, thereby $|V(G'_1)| \geqslant 3$. By Lemmas 23 and 25, $\emptyset \neq N(u_2) \cap V(G'_1) \setminus \{u\} \subset N(u) \cap V(G'_1)$. Then $u_2v \in E(G)$ (note that $u_2u_3 \notin E(G)$). On the other hand, $N(u_2) \cap V(G_2) \setminus \{v\} \subset N(v) \cap V(G_2)$ by Lemma 25. Recall that $N(u_2) \cap V(G_1) \setminus \{u\} \subset N(u) \cap V(G_1)$ and $u_2u_3 \notin E(G)$. So $N(u_2) \setminus \{v, u_1, u_4\} \subset N(v) \setminus \{u_2\} \subset V(G'_1)$. Hence $N(u_2) \cap V(G'_2) \setminus \{u_1\} = \emptyset$, which contradicts Lemma 23. Therefore, we have $d(u) = 4$. By Lemmas 23 and 25, $\emptyset \neq N(u_i) \cap V(G_1) \setminus \{u\} \subset N(u) \cap V(G_1)$ for $i \in \{2, 3\}$. Then $(N(u_2) \cup N(u_3)) \cap V(G_1) \setminus \{u\} = \{u_1\}$. By Corollary 22, we have $|V(G_1)| \geqslant 2$. If $|V(G_1)| > 2$, then $G - \{u, u_1\}$ is disconnected, a contradiction. If $|V(G_1)| = 2$, then $d(u_1) = 3$, contradicting Lemma 26. Therefore, uu_1 is b-invariant. □

Lemma 34. *Let uv and uu_1 be two adjacent removable edges of a claw-free brick G. Assume that uv and uu_1 are not b-invariant and $G - uv$ has a nontrivial barrier. Then $d(u) = 4$. Moreover, uu_2 and uu_3 are b-invariant.*

Proof. Let B be a nontrivial barrier of $G - uv$ associated with uv. Assume that G_1 and G_2 are the components of $G - uv - B$, where $u \in V(G_1)$ and $v \in V(G_2)$. By Lemma 21, we have $|B| = 2$ and $|V(G_i)| \geqslant 3$ for $i \in \{1, 2\}$. By Lemmas 26 and 32, we have $d(u) = 4$ or $d(u) = 5$. By Lemma 31, $u_1 \notin B$ and so $u_1 \in V(G_1)$.

If $N(u) \cap B = \emptyset$, then $G[N(u) \setminus \{v\}]$ is complete by Lemma 29. By Lemma 28, uu_1 is b-invariant, contradicting the assumption that uu_1 is not b-invariant. If $|N(u) \cap B| = 2$, then uu_1 is b-invariant by Lemma 33 (note that $E(G[B]) = \emptyset$), a contradiction. So we consider the case when $|N(u) \cap B| = 1$. Let $N(u) \cap B = \{u_2\}$.

Claim A. $d(u) = 4$, $u_2 \in V(G'_1)$ and $u_1u_2 \notin E(G)$.

Proof. We will consider the following two cases.

Case 1. There exists a nontrivial barrier B' of $G - uu_1$.

Assume that G'_1 and G'_2 are the components of $G - uu_1 - B'$, where $u \in V(G'_1)$ and $u_1 \in V(G'_2)$. By Lemma 21, $|B'| = 2$ and $|V(G'_i)| \geqslant 3$ for $i \in \{1, 2\}$.

Assume that $u_2v \notin E(G)$. Then $G[N(u)\setminus\{v\}]$ is complete by Lemma 29. So by Lemma 28, uu_1 is b-invariant, a contradiction. Now we consider the case when $u_2v \in E(G)$. Assume that $d(u) = 5$. As $N(u) \cap N(v) = \{u_2\}$, $G[N(u) \setminus \{u_2, v\}]$ is complete by Lemma 29. So uu_1 lies in a subgraph that is isomorphic to K_4 when $d(u) = 5$. By Lemma 11, $N(u) \cap N(u_1) \subset B'$. So $E(G[B']) \neq \emptyset$, contradicting the fact that uu_1 is removable. Therefore, $d(u) = 4$. Recalling that $G[N(u) \setminus \{u_2, v\}]$ is complete, $u_3 \in N(u_1) \cap N(u)$ and hence $u_3 \in B'$. If $u_2 \in B'$, then $v \in V(G'_1)$. Since $E(G[B']) = \emptyset$, $u_2u_3 \notin E(G)$. Thus, uv is b-invariant by Lemma 33, a contradiction. So $u_2 \notin B'$ and hence $u_2 \in V(G'_1)$ and $u_1u_2 \notin E(G)$.

Case 2. $G - uu_1$ contains no nontrivial barriers.

Then $G - uu_1$ contains only 2-separations. Assume that $G - uu_1$ contains at least two vertex-disjoint 2-separations. Then $N(u_1) \cap N(u) = \emptyset$ by Lemma 30. As G is claw-free, $G[N(u) \setminus \{u_1\}]$ is complete. Thus, uv is b-invariant by Lemma 28, a contradiction. Therefore, all the 2-separations of $G - uu_1$ have a common vertex.

Assume that S' is a 2-separation of $G - uu_1$ and G'_i is the component of $G - uu_1 - S'$ for $i \in \{1, 2\}$, where $u \in V(G'_1)$ and $u_1 \in V(G'_2)$. By Corollary 22, we have $|V(G'_i)| \geqslant 2$ for $i \in \{1, 2\}$.

If $u_2v \notin E(G)$, then $G[N(u) \setminus \{v\}]$ is complete by Lemma 29. By Lemma 28, uu_1 is b-invariant, a contradiction. Now we consider the case when $u_2v \in E(G)$. Assume that $d(u) = 5$. By Lemma 29, $G[N(u)\setminus\{u_2, v\}]$ is complete, that is, $|N(u) \cap N(u_1)| = 2$. Hence $u_1u_2 \notin E(G)$ by Corollary 24. So $N(u_1) \cap N(u) = \{u_3, u_4\}$ and hence $S' = \{u_3, u_4\}$. Then $\{u_2, v\} \subset V(G'_1)$. As $|V(G'_1)|$ is even, $|V(G'_1)| > 3$. By Lemmas 23 and 25, $\emptyset \neq N(u_i) \cap V(G'_1) \setminus \{u\} \subset N(u) \cap V(G'_1)$ for $i \in \{3, 4\}$. Note that $u_3v \notin E(G)$ and $u_4v \notin E(G)$. Therefore, $(N(u_3) \cup N(u_4)) \cap V(G'_1) = \{u, u_2\}$, that is, $\{u, u_2\}$ is a vertex 2-cut in G, a contradiction. So $d(u) = 4$. Since $G[N(u) \setminus \{u_2, v\}]$ is complete, $u_3 \in N(u_1) \cap N(u)$ and hence $u_3 \in S'$. If $u_2 \in S'$, then $v \in V(G'_1)$. By Lemmas 23 and 25, $\emptyset \neq N(u_3) \cap V(G'_1) \setminus \{u\} \subset N(u) \cap V(G'_1)$. Then $N(u_3) \cap V(G'_1) \setminus \{u\} = \{v\}$ as $d(u) = 4$, which contradicts $u_3v \notin E(G)$. Hence $u_2 \notin S'$, thereby $u_2 \in V(G'_1)$ and $u_1u_2 \notin E(G)$. □

Recall that $N(u) \cap B = \{u_2\}$. By the Claim A, we have $d(u) = 4$ and $u_1u_2 \notin E(G)$. By Lemmas 23 and 25, $\emptyset \neq N(u_2) \cap V(G_1) \setminus \{u\} \subset N(u) \cap V(G_1)$. Then $u_2u_3 \in E(G)$. By Lemma 23, $N(u_2) \cap V(G_2) \setminus \{v\} \neq \emptyset$. Therefore, $d(u_2) \geqslant 4$ (note that $\{uu_2, u_2v\} \subset E(G)$).

Now we claim that uu_2 is removable. Suppose, to the contrary, that uu_2 is non-removable. By Lemmas 7 and 9, there exists a nontrivial barrier B_0 of $G - uu_2$ containing a forbidden edge of $G - uu_2$ and $|B_0| \leqslant 3$. As $N(u) \cap N(u_2) = \{u_3, v\} \subset B_0$ and $u_3v \notin E(G)$, $|B_0| \neq 2$. So $|B_0| = 3$. Then the components of $G - uu_2 - B_0$ containing u and u_2 are singletons, respectively, by Lemma 9. Hence $B_0 = N(u) \setminus \{u_2\} = \{u_1, u_3, v\}$. Note that $\{u_2u_3, u_2v\} \subset E(G)$ and $u_1u_2 \notin E(G)$. Then $d(u_2) = 3$, contradicting the fact that $d(u_2) \geqslant 4$. Therefore, uu_2 is removable.

Suppose, to the contrary, that uu_2 is not b-invariant. Since $|N(u) \cap N(u_2)| \neq \emptyset$, all the 2-separations of $G - uu_2$ have a common vertex by Lemma 30. Assume that D' is a vertex 2-cut of $G - uu_2$. As $N(u) \cap N(u_2) = \{u_3, v\}$, $D' = \{u_3, v\}$. By Lemma 31, $v \notin D'$, a contradiction. Therefore, uu_2 is b-invariant. Similarly, uu_3 is b-invariant. □

Lemma 35. *Assume that uv is a removable but non-b-invariant edge of a claw-free brick G and $G - uv$ contains at least two vertex-disjoint 2-separations. If uu_1 is a removable edge in G, then uu_1 is b-invariant.*

Proof. By Lemma 30, $N(u) \cap N(v) = \emptyset$. As G is claw-free, $G[N(u) \setminus \{v\}]$ is complete. Then uu_1 is b-invariant by Lemma 28. □

Lemma 36. *Let uv and uu_1 be two adjacent removable edges of a claw-free brick G. Assume that uv and uu_1 are not b-invariant and $G - uv$ has a 2-separation. Then $d(u) = 4$ or $d(u) = 6$, and each edge of $\partial(\{u\}) \setminus \{uv, uu_1\}$ is b-invariant.*

Proof. Let S be a 2-separation of $G - uv$. Assume that G_1 and G_2 are the components of $G - uv - S$, where $u \in V(G_1)$ and $v \in V(G_2)$. Then $|V(G_i)| \geqslant 2$ by Corollary 22, for $i \in \{1, 2\}$. By Lemma 26, $d(u) \geqslant 4$. By Lemma 31, $u_1 \notin S$ and then $u_1 \in V(G_1)$.

As uu_1 is removable but not b-invariant, assume that S' is a 2-separation of $G - uu_1$ and G'_i ($i \in \{1, 2\}$) is the components of $G - uu_1 - S'$, where $u \in V(G'_1)$ and $u_1 \in V(G'_2)$. By Corollary 22, we have $|V(G'_i)| \geqslant 2$ for $i \in \{1, 2\}$. If $N(u) \cap S = \emptyset$, then $G[N(u) \setminus \{v\}]$ is complete by Lemma 29. By Lemma 28, $N(u) \cap S \neq \emptyset$. Now we consider the following two cases.

Case a. $|N(u) \cap S| = 1$

Let $N(u) \cap S = \{u_2\}$. Assume that $u_2v \notin E(G)$. By Lemma 29, $G[N(u) \setminus \{v\}]$ is complete. Then uu_1 is b-invariant by Lemma 28, a contradiction. So we consider the case when $u_2v \in E(G)$. By Lemma 29, $G[N(u) \setminus \{u_2, v\}]$ is complete. Then $N(u) \cap V(G_1) \setminus \{u_1\} \subset N(u_1) \cap N(u)$. By Corollary 24, $|N(u) \cap V(G_1) \setminus \{u_1\}| \leqslant 2$. As $|N(u) \cap S| = 1$, $d(u) \leqslant 5$. Assume that $d(u) = 5$. Then $u_1u_2 \notin E(G)$. Otherwise, $|N(u_1) \cap N(u)| = 3$, contradicting Corollary 24. So $N(u_1) \cap N(u) = \{u_3, u_4\}$ and hence $S' = \{u_3, u_4\}$. Then $\{u_2, v\} \subset V(G'_1)$. As $|V(G'_1)|$ is even, $|V(G'_1)| > 3$. By Lemma 25, $N(u_i) \cap V(G'_1) \setminus \{u\} \subset N(u) \cap V(G'_1)$ for $i \in \{3, 4\}$. Note that $u_3v \notin E(G)$ and $u_4v \notin E(G)$. So $(N(u_3) \cup N(u_4)) \cap V(G'_1) = \{u, u_2\}$. Hence $G - \{u, u_2\}$ is disconnected, a contradiction. Therefore, $d(u) = 4$. Recall that $N(u) \cap V(G_1) \setminus \{u_1\} \subset N(u_1) \cap N(u)$. Then $u_3 \in N(u_1) \cap N(u)$ and hence $u_3 \in S'$. If $u_2 \in S'$, then $v \in V(G'_1)$. By Lemma 23, $N(u_3) \cap V(G'_1) \setminus \{u\} = \{v\}$ as $d(u) = 4$. However, $u_3v \notin E(G)$. Hence $u_2 \notin S'$, thereby $u_2 \in V(G'_1)$ and $u_1u_2 \notin E(G)$.

Now we claim that uu_2 is removable. Suppose, to the contrary, that uu_2 is non-removable. By Lemma 23, $N(u_2) \cap V(G_1) \setminus \{u\} \neq \emptyset$ and $N(u_2) \cap V(G_2) \setminus \{v\} \neq \emptyset$. So $d(u_2) \geqslant 4$ (note that $\{uu_2, u_2v\} \subset E(G)$). By Lemmas 7 and 9, there exists a nontrivial barrier B_0 of $G - uu_2$ containing a forbidden edge of $G - uu_2$ and $|B_0| \leqslant 3$. As $N(u) \cap N(u_2) = \{u_3, v\} \subset B_0$ and $u_3v \notin E(G)$, $|B_0| \neq 2$. So $|B_0| = 3$. Then the components of $G - uu_2 - B_0$ containing u and u_2 are singletons, respectively, by Lemma 9. Hence $B_0 = N(u) \setminus \{u_2\} = \{u_1, u_3, v\}$. Note that $\{u_2u_3, u_2v\} \subset E(G)$ and $u_1u_2 \notin E(G)$. Then $d(u_2) = 3$, contradicting the fact that $d(u_2) \geqslant 4$. Therefore, uu_2 is removable.

Suppose, to the contrary, that uu_2 is not b-invariant. As $N(u) \cap N(u_2) \neq \emptyset$, all the 2-separations of $G - uu_2$ have a common vertex by Lemma 30. Assume that D' is a vertex 2-cut of $G - uu_2$. As $N(u) \cap N(u_2) = \{u_3, v\}$, $D' = \{u_3, v\}$. By Lemma 31, $v \notin D'$, a contradiction. Thus, uu_2 is b-invariant. Similarly, uu_3 is b-invariant.

Case b. $|N(u) \cap S| = 2$

Let $S = \{u_2, u_3\}$. Assume that $u_2u_3 \notin E(G)$. By Lemma 33, uu_1 is b-invariant, a contradiction. Now we consider the case when $u_2u_3 \in E(G)$. By 3) of Lemma 32, we have $d(u) \geqslant 5$ and $u_1 \notin N(u_i)$ for $i \in \{2, 3\}$.

Suppose, without loss of generality, that $u_2v \notin E(G)$. By Lemmas 23 and 25, $N(u_2) \cap V(G_1) \setminus \{u\} = N(u) \cap V(G_1) \neq \emptyset$. Then $u_1 \in N(u_2)$, a contradiction. Therefore, $u_2v \in E(G)$. Similarly, $u_3v \in E(G)$. By Lemma 25, $N(u_i) \cap V(G_1) \setminus \{u\} \subset N(u) \cap V(G_1)$ for $i \in \{2, 3\}$. Note that $u_1u_2 \notin E(G)$ and $u_1u_3 \notin E(G)$. As G is 3-connected, $|(N(u_2) \cup N(u_3)) \cap V(G_1)| > 2$. So $d(u) \geqslant 6$. By Lemma 29, $G[N(u) \cap V(G_1)]$ is complete. As $N(u) \cap V(G_1) \subset N(u_1) \cap N(u)$, we have $d(u) \leqslant 6$ by Corollary 24. Therefore, $d(u) = 6$.

Now we claim that uu_2 is removable. Suppose, to the contrary, that uu_2 is non-removable. By Lemmas 7 and 9, there exists a nontrivial barrier B_0 of $G - uu_2$ containing a forbidden edge of $G - uu_2$ and $|B_0| \leqslant 3$. By Lemmas 23 and 25, $\emptyset \neq N(u_2) \cap V(G_1) \setminus \{u\} \subset N(u) \cap V(G_1)$, thereby $|(N(u_2) \cap N(u)) \cap V(G_1)| \geqslant 1$. As $\{u_3, v\} \subset N(u_2) \cap N(u)$, $|N(u_2) \cap N(u)| \geqslant 3$. Since $N(u_2) \cap N(u) \subset B_0$, we only need to consider the case when $|B_0| = |N(u_2) \cap N(u)| = 3$. By Lemma 9, $N(u) \setminus \{u_2\} \subset B_0$. Then $d(u) = 4$, a contradiction. Therefore, uu_2 is removable. Recalling that $|N(u_2) \cap N(u)| \geqslant 3$, uu_2 is b-invariant by Corollary 24. Similarly, uu_3, uu_4 and uu_5 are b-invariant. □

Proof of Lemma 15. By Lemmas 34, 35 and 36, u is incident with at least $d(u) - 3$ b-invariant edges when $d(u) \geqslant 4$. If $d(u) = 3$, all the removable edges incident with u are b-invariant by Corollary 27. □

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