On Card Guessing Games: Limit Law for One-Time Riffle Shuffle

Markus Kuba[a] Alois Panholzer[b]

Submitted: Nov 4, 2024; Accepted: Apr 25, 2025; Published: Aug 8, 2025

Abstract

We consider a card guessing game with complete feedback. A ordered deck of n cards labeled 1 up to n is riffle-shuffled exactly one time. Then, the goal of the game is to maximize the number of correct guesses of the cards, where one after another a single card is drawn from the top, and shown to the guesser until no cards remain. Improving earlier results, we provide a limit law for the number of correct guesses. As a byproduct, we relate the number of correct guesses in this card guessing game to the number of correct guesses under a two-color card guessing game with complete feedback. Using this connection to two-color card guessing, we can also show a limiting distribution result for the first occurrence of a pure luck guess.

Mathematics Subject Classifications: 05A15, 05A16, 60F05, 60C05

1 Introduction

Different card guessing games have been considered in the literature in many articles [7, 9, 17, 18, 22, 23, 24, 28, 29, 30, 31]. An often discussed setting is the following. A deck of a total of M cards is shuffled, and then the guesser is provided with the total number of cards M, as well as with the individual numbers of say hearts, diamonds, clubs and spades. After each guess of the type of the next card, the person guessing the cards is shown the drawn card, which is then removed from the deck. This process is continued until no more cards are left. Assuming that the guesser tries to maximize the number of correct guesses, one is interested in the total number of correct guesses. Such card guessing games are not only of purely mathematical interest, but there are applications to the analysis of clinical trials [5, 10], fraud detection related to extra-sensory perceptions [7], guessing

[a]Department Applied Mathematics and Physics, University of Applied Sciences - Technikum Wien, Höchstädtplatz 6, 1200 Wien, Austria (kuba@technikum-wien.at).
[b]Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstr. 8-10/104, 1040 Wien, Austria (Alois.Panholzer@tuwien.ac.at).

so-called Zener Cards [28], as well as relations to tea tasting and the design of statistical experiments [11, 29].

The card guessing procedure can be generalized to an arbitrary number $n \geqslant 2$ of different types of cards. In the simplest setting there are two colors, red (hearts and diamonds) and black (clubs and spades), and their numbers are given by non-negative integers m_1, m_2, with $M = m_1+m_2$. One is then interested in the random variable C_{m_1,m_2}, counting the number of correct guesses. Here, not only the distribution and the expected value of the number of correct guesses is known [9, 18, 24, 30, 31], but also multivariate limit laws and additionally interesting relations to combinatorial objects such as Dyck paths and urn models are given [9, 22, 23]. For the general setting of n different types of cards we refer the reader to [9, 17, 28, 29] for recent developments.

Different models of card guessing games involving so-called riffle shuffles are also of importance and are the main topic of this work. The riffle shuffle, also called dovetail shuffle, is a famous card shuffling technique. It consists of splitting a deck of cards in two portions. Then, the two packets are riffled together. The famous *Gilbert-Shannon-Reeds* model (see Subsection 2.1 for details) is the standard mathematical model for such riffle shuffles. We note that experiments reported by Diaconis [8] show that this model is a good description of the way people really shuffle cards; we also refer the reader to the work of Bayer and Diaconis [2] for important applications to mixing up cards.

In this work we consider the following problem. A deck of n cards labeled consecutively from 1 on top to n on bottom is face down on the table. The deck is riffle shuffled once, assuming the Gilbert-Shannon-Reeds model, and placed back on the table, face down. A guesser tries to guess at the cards one at a time, starting from the top. The goal is to maximize the number of correct guesses X_n, assuming that complete feedback is given, i.e., the drawn card is shown to the guessing person, and further assuming that the guesser is using the optimal strategy. Recently, Liu [25] and also Krityakierne and Thanatipanonda [21] made progress on this problem. In [25] an asymptotic expansion of the expected value $\mathbb{E}(X_n)$ is provided for n tending to infinity. An enumerative analysis and a study of higher moments has been carried out in [21]. Therein, precise asymptotics of the first few moments $\mathbb{E}(X_n)$, $\mathbb{E}(X_n^2)$, etc. were provided using both enumerative and symbolic methods. We note in passing that such questions are classical; see for example Ciucu [6], where he studied an optimal strategy under the no feedback game, such that the identities of the card guessed are not revealed, nor is the guesser told whether a particular guess was correct or not. Progress for the no feedback variant was recently obtained in [19, 20]. However, so far the limit law of X_n has proven to be elusive for both variants.

In this work we determine the limit law of the number of correct guesses X_n in the full feedback model, starting with n cards labeled one up to n, once riffle shuffled. We translate the enumerative analysis of [21] into a distributional equation. We establish a direct link between the number of correct guesses C_{m_1,m_2} in the two-color card guessing game and the corresponding quantity X_n in the once riffle shuffled model, previously unknown best to the knowledge of the authors. This link allows us to derive the limit law for the number

X_n of correct guesses in the once riffle shuffled case. For the reader's convenience we summarize our main results in the following theorem, collecting the individual results of Theorems 10, 14 and Proposition 13.

Theorem 1. *The normalized random variable $Y_n = (X_n - \frac{n}{2})/\sqrt{n}$ converges in distribution to a generalized gamma distributed random variable G:*

$$\frac{X_n - \frac{n}{2}}{\sqrt{n}} \xrightarrow{\mathcal{L}} G,$$

with density of G given by $f(x) = \sqrt{\frac{2}{\pi}} \cdot 8x^2 e^{-2x^2}$, $x \geqslant 0$. Moreover, the r-th integer moments $\mathbb{E}(Y_n^r)$ converge, for arbitrary but fixed $r \geqslant 1$ and $n \to \infty$, to the moments of the limit law G, expressed in terms of the Gamma function:

$$\mathbb{E}\Big(\frac{X_n - \frac{n}{2}}{\sqrt{n}}\Big)^r \to \frac{\Gamma\big(\frac{r+3}{2}\big)}{2^{\frac{r}{2}-1}\sqrt{\pi}}, \quad r \geqslant 0.$$

If the guesser follows the optimal strategy, the chances of a correct guess are always greater or equal 50 percent. Starting with a deck of n cards, we are also interested in the number of cards P_n (divided by two) remaining in the deck when the first "pure luck guess" with only a 50 percent success chance occurs. We will show in Theorem 23 that P_n, properly normalized, satisfies an arcsine limit law.

1.1 Notation

As a remark concerning notation used throughout this work, we always write $X \stackrel{\mathcal{L}}{=} Y$ to express equality in distribution of two random variables (r.v.) X and Y, and $X_n \xrightarrow{\mathcal{L}} X$ for the weak convergence (i.e., convergence in distribution) of a sequence of random variables X_n to a r.v. X. Furthermore we use $x^{\underline{s}} := x(x-1)\dots(x-(s-1))$ for the falling factorials, and $x^{\overline{s}} := x(x+1)\dots(x+s-1)$ for the rising factorials, $s \in \mathbb{N}_0$. Moreover, $f_n \ll g_n$ denotes that a sequence f_n is asymptotically smaller than a sequence g_n, i.e., $f_n = o(g_n)$, $n \to \infty$.

2 Distributional analysis

2.1 Riffle shuffle model

A riffle shuffle is a certain card shuffling technique. In the mathematical modeling of card shuffling, the *Gilbert-Shannon-Reeds* model [2, 9, 14] describes a probability distribution for the outcome of such a shuffling. We consider a sorted deck of n cards labeled consecutively from 1 up to n. The deck of cards is cut into two packets, assuming that the probability of selecting k cards in the first packet and $n-k$ in the second packet is defined as a binomial distribution with parameters n and $1/2$:

$$\frac{\binom{n}{k}}{2^n}, \quad 0 \leqslant k \leqslant n.$$

Afterwards, the two packets are interleaved back into a single pile: one card at a time is moved from the bottom of one of the packets to the top of the shuffled deck, such that if m_1 cards remain in the first and m_2 cards remain in the second packet, then the probability of choosing a card from the first packet is $m_1/(m_1+m_2)$ and the probability of choosing a card from the second packet is $m_2/(m_1+m_2)$. See Figure 1 for an example of a riffle shuffle of a deck of five cards. For a one-time shuffle, the operation of interleaving

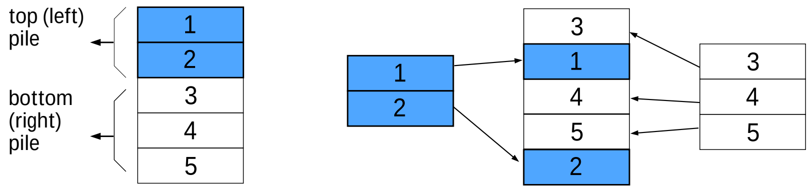

Figure 1: Example of a one-time riffle shuffle: a deck of five cards is split after 2 with probability $\binom{5}{2}/2^5 = 5/16$ and then interleaved.

described above gives rise to an ordered deck (corresponding to the identity permutation) in $n+1$ ways. Each other shuffled deck corresponds to a permutation which contains exactly two proper increasing subsequences and each has multiplicity 1; in total there are $2^n - n - 1$ different such permutations.

In a more combinatorial setting, the outcome of a one-time shuffling in this model might be generated from the 2^n different $\{a, b\}$-sequences of length n, i.e., length-n words over the alphabet $\{a, b\}$, by replacing the a's in such a sequence, let us assume there are $0 \leqslant k \leqslant n$ many, by the increasing sequence $1, 2, \ldots, k$, and the b's in the sequence by the increasing sequence $k+1, k+2, \ldots, n$. Thus, the a's and b's, respectively, correspond to the packet of cards below and above the cut, respectively. Let us denote by $\mathcal{D}_n$ this multiset of permutations on $[n] = \{1, 2, \ldots, n\}$ generated by the family $\mathcal{W}_n = \{a, b\}^n$ of length-n words. Then the $n+1$ words in $\mathcal{W}_n$ of the kind $a^k b^{n-k}$, with $0 \leqslant k \leqslant n$, all generate the identity permutation id_n in $\mathcal{D}_n$, whereas the remaining $2^n - n - 1$ words in $\mathcal{W}_n$ generate pairwise different permutations in $\mathcal{D}_n$.

2.2 First drawn card and the optimal strategy

The optimal strategy for maximizing the number X_n of correctly guessed cards, starting with a deck of n cards, based on the Gilbert-Shannon-Reads model, after a one-time riffle shuffle was discussed before in the literature [21, 25], based on earlier work [9, 14]. This strategy, summarized below, is based on the following Proposition.

Proposition 2 (Guessing the first card [21, 25])**.** *Assume that a deck of n cards has been riffle shuffled once. The probability $p_n(m)$ that the first card being m, $1 \leqslant m \leqslant n$, is given by*

$$p_n(m) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{2^n}, & \text{for } m = 1, \\ \dfrac{\binom{n-1}{m-1}}{2^n}, & \text{for } 2 \leqslant m \leqslant n. \end{cases} \tag{1}$$

For the sake of completeness we include a short proof.

Proof. First, we condition on the cut leading to two decks containing $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$, $0 \leqslant k \leqslant n$, which happens with probability $\frac{\binom{n}{k}}{2^n}$. Each resulting deck has the same probability $1/\binom{n}{k}$. Then, we observe that the probability of a top one is in the case $k > 0$ given by

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n},$$

as there are $\binom{n-1}{k-1}$ different ways of choosing the positions of the other cards. Of course, for $k = 0$ the top card is always one. Thus, we obtain

$$p_n(1) = \frac{1}{2^n} + \sum_{k=1}^{n} \frac{\binom{n}{k}}{2^n} \cdot \frac{k}{n} = \frac{1}{2^n} + \frac{1}{2}.$$

Similarly, for $m > 1$ we observe that only a cut at $m-1$ may lead to a top card labeled m, thus in this situation the subsequences to be interleaved have to be $1, \ldots, m-1$ and $m, \ldots, n$. If m is the top card, there are $\binom{n-1}{n-m}$ different ways of choosing the positions of the other cards, which yields

$$p_n(m) = \frac{\binom{n}{m-1}}{2^n} \cdot \frac{\binom{n-1}{n-m}}{\binom{n}{m-1}} = \frac{\binom{n-1}{m-1}}{2^n}. \qquad \square$$

Now we turn to the *optimal strategy.* The guesser should guess 1 on the first card, as his chance of success is more than 50% by Proposition 2.

If the first guess is incorrect, say the shown card has label $m \geqslant 2$, this implies that the cut was made exactly at $m-1$. The person is left with two increasing subsequences $1, 2, \ldots, m-1$ and $m+1, \ldots, n$. The remaining numbers are then guessed according to the proportions of the lengths of the remaining subsequences until no cards are left.

If the first guess was correct, then the person continues with guessing the number two, etc., i.e., as long as all previous such predictions turned out to be correct, the guesser makes a guess of the number j for the j-th card. This is justified, since by considerations as before one can show easily that the probability that the j-th card has the number j conditioned on the event that the first $j-1$ cards are the sequence of numbers $1, 2, \ldots, j-1$ is for $1 \leqslant j \leqslant n$ given by

$$\frac{2^{n-j} + j}{2^{n-j+1} + j - 1} = \frac{1}{2} + \frac{(1+j)2^{-(n-j+2)}}{1 + (j-1)2^{-(n-j+1)}},$$

and thus exceeds 50%. If such a prediction turns out to be wrong, i.e., gives a number $m > j$ for the j-th card, then again one can determine the two involved remaining subsequences $j, j+1, \ldots, m-1$ and $m+1, \ldots, n$, and all the numbers of the remaining cards are again guessed according to the proportions of the lengths of the remaining subsequences until no cards are left.

2.3 Enumeration and distributional decomposition

Our starting point is the recurrence relation for the generating function

$$D_n(q) := \sum_{\sigma\in\mathcal{D}_n} q^{\#\text{ correct guesses for deck }\sigma} = 2^n \cdot \mathbb{E}(q^{X_n}) = 2^n \sum_{\ell=0}^{n} \mathbb{P}\{X_n = \ell\}\, q^{\ell},$$

counting the number of correct guesses using the optimal strategy when starting with a once-shuffled deck of n different cards, which has been stated in [21] and basically stems from Proposition 2.

Lemma 3 (Recurrence relation for $\boldsymbol{D_n(q)}$ [21])**.** *The generating function $D_n(q)$ satisfies the following recurrence:*

$$D_n(q) = qD_{n-1}(q) + q^n + \sum_{j=0}^{n-2} F_{n-1-j,j}(q), \quad n \geqslant 1, \qquad D_0(q) = 1, \tag{2}$$

where the auxiliary function $F_{m_1,m_2}(q)$ is for $m_1 \geqslant m_2 \geqslant 0$ defined recursively by

$$F_{m_1,m_2}(q) = qF_{m_1-1,m_2}(q) + F_{m_1,m_2-1}(q),$$

with initial values $F_{m_1,0}(q) = q^{m_1}$, and for $m_2 > m_1 \geqslant 0$ by the symmetry relation

$$F_{m_1,m_2}(q) = F_{m_2,m_1}(q).$$

Proof. To keep the work self-contained we give a proof of this recurrence, where we use the before-mentioned combinatorial description of once-shuffled decks of n cards $\sigma = (\sigma_1,\dots,\sigma_n) \in \mathcal{D}_n$ by means of length-n words $w = w_1 \dots w_n \in \mathcal{W}_n$. We count the number of correct guesses, where we distinguish according to the first letter w_1. If $w_1 = a$ then the first drawn card is 1, $\sigma_1 = 1$, and this card will be predicted correctly by the guesser. The guesser keeps his strategy of guessing for the deck of remaining cards, which is order-isomorphic to a deck of $n-1$ cards generated by the length-$(n-1)$ word $w' = w_2 \dots w_n$; to be more precise, if $\sigma = (1, \sigma_2, \dots, \sigma_n) \in \mathcal{D}_n$ and $\sigma' = (\sigma'_1, \dots, \sigma'_{n-1}) \in \mathcal{D}_{n-1}$ are the labels of the cards in the deck generated by the words $w = aw' \in \mathcal{W}_n$ and $w' \in \mathcal{W}_{n-1}$, respectively, then it simply holds $\sigma_i = \sigma'_{i-1} + 1$, $2 \leqslant i \leqslant n$. Since w' is a random word of length $n-1$ if started with a random word w of length n, this yields the summand $qD_{n-1}(q)$ in equation (2).

If $w_1 = b$ then we first consider the particular case that $w = b^n$, i.e., that the cut of the deck has been at 0. Since in this case the deck of cards corresponds to the identity permutation $\sigma = \mathrm{id}_n$, the guesser will predict all cards correctly using the optimal strategy, which leads to the summand q^n in (2). Apart from this particular case, $w_1 = b$ corresponds to a deck of cards where the first card is $m \geqslant 2$ and thus will cause a wrong prediction by the guesser; however, due to complete feedback, now the guesser knows that the cut is at $m-1$, or in alternative terms, he knows that the remaining deck is generated from a word $w' = w_2 \dots w_n$ that has $j := n-m$ b's and $n-1-j = m-1$ a's, with $0 \leqslant j \leqslant n-2$.

From this point on the guesser changes the strategy, which again could be formulated in alternative terms by saying that the guesser makes a guess for the next letter in the word, in a way that the guess is a if the number of a's exceeds the number of b's in the remaining subword, that the guess is b in the opposite case, and (in order to keep the outcome deterministic) that the guess is a if there is a draw between the number of a's and b's. More generally, let us assume that the word consists of $m_1 \geqslant 0$ a's and $m_2 \geqslant 0$ b's and each of these $\binom{m_1+m_2}{m_1}$ words occur with equal probability, then let us define the r.v. $\hat{C}_{m_1,m_2}$ counting the number of correct guesses by the before-mentioned strategy as well as the generating function $F_{m_1,m_2}(q) = \binom{m_1+m_2}{m_1}\mathbb{E}(q^{\hat{C}_{m_1,m_2}})$. It can be seen immediately that $\hat{C}_{m_1,m_2}$ and so $F_{m_1,m_2}(q)$ is symmetric in m_1 and m_2, and that $F_{m_1,m_2}(q)$ satisfies the recurrence stated in Lemma 3. Moreover, these considerations yield the third summand $\sum_{j=0}^{n-2} F_{n-1-j,j}(q)$ in equation (2). □

When considering the two-color card guessing game (with complete feedback) starting with m_1 cards of type (color) a and m_2 cards of type (color) b it apparently corresponds to the guessing game for the letters of a word over the alphabet $\{a, b\}$ consisting of m_1 a's and m_2 b's as described in the proof of Lemma 3. Thus, the r.v. C_{m_1,m_2} counting the number of correct guesses when the guesser uses the optimal strategy for maximizing correct guesses, i.e., guessing the color corresponding to the larger number of cards present [9, 18, 22, 23], and the r.v. $\hat{C}_{m_1,m_2}$ are equally distributed, $C_{m_1,m_2} \overset{\mathcal{L}}{=} \hat{C}_{m_1,m_2}$. Consequently, the auxiliary function $F_{m_1,m_2}(q)$ stated in Lemma 3 is the generating function of C_{m_1,m_2}:

$$F_{m_1,m_2}(q) = \binom{m_1+m_2}{m_1} \cdot \mathbb{E}(q^{C_{m_1,m_2}}). \tag{3}$$

Remark 4. In most works considering C_{m_1,m_2} it is assumed without loss of generality that $m_1 \geqslant m_2 \geqslant 0$. However, we note that by definition of the two-color card guessing game the order of the parameters is not of relevance under the optimal strategy: $C_{m_1,m_2} = C_{m_2,m_1}$.

Remark 5. As has been pointed out in [22], the two-color guessing procedure for the cards of a deck with m_1 cards of type a (say color red) and m_2 cards of type b (say color black) can be formulated also by means of the so-called sampling without replacement urn model starting with m_1 and m_2 balls of color red and black, respectively, where in each draw a ball is picked at random, the color inspected and then removed, until no more balls are left. Then the urn histories can be described via weighted lattice paths from (m_1, m_2) to the origin with step sets "left" $(-1, 0)$ and "down" $(0, -1)$: at position (k_1, k_2), a left-step and a down-step have weights $\frac{k_1}{k_1+k_2}$ and $\frac{k_2}{k_1+k_2}$, respectively, and reflect the draw of a red ball or a black ball, resp., occurring with the corresponding probabilities. Several quantities of interest for card guessing games can be formulated also via parameters of the sample paths of this urn, such as the first hitting of the diagonal or the first hitting of one of the coordinate axis, which is used in a subsequent section.

Concerning a distributional analysis of X_n, an important intermediate result is the following distributional equation, which we obtain by translating the recurrence relation (2) into a recursion for probability generating functions.

Theorem 6 (One-time riffle and two-color card guessing)**.** *The random variable $X = X_n$ of correctly guessed cards, starting with a deck of n cards, after a one-time riffle satisfies the following decomposition:*

$$X_n \stackrel{\mathcal{L}}{=} I_1\big(X^*_{n-1} + 1\big) + (1 - I_1)\Big(I_2 \cdot n + (1 - I_2) \cdot C_{n-1-J_n,J_n}\Big), \tag{4}$$

where $I_1 \stackrel{\mathcal{L}}{=} \mathrm{Be}(0.5)$, $I_2 \stackrel{\mathcal{L}}{=} \mathrm{Be}(0.5^{n-1})$, and C_{m_1,m_2} denotes the number of correct guesses in a two-color card guessing game, with $\mathrm{Be}(p)$ denoting a Bernoulli distribution with parameter p, such that $P\{I = 1\} = p$ and $P\{I = 0\} = 1 - p$ for $I \stackrel{\mathcal{L}}{=} \mathrm{Be}(p)$. Additionally, X^*_{n-1} is an independent copy of X defined on $n - 1$ cards. Moreover, $J_n \stackrel{\mathcal{L}}{=} \mathrm{B}^*(n - 1, p)$ denotes a truncated binomial distribution:

$$\mathbb{P}\{J_n = j\} = \binom{n-1}{j}/(2^{n-1} - 1), \quad 0 \leqslant j \leqslant n - 2.$$

All random variables I_1, I_2, J_n, as well as C_{m_1,m_2} are mutually independent.

Proof. By definition, the probability generating function of X_n is given as follows:

$$\mathbb{E}(q^{X_n}) = \frac{D_n(q)}{2^n}.$$

Thus, we get from (2) the equation

$$\mathbb{E}(q^{X_n}) = \frac{1}{2} \cdot \mathbb{E}(q^{X_{n-1}+1}) + \frac{1}{2^n} \cdot q^n + \frac{1}{2^n} \sum_{j=0}^{n-2} F_{n-1-j,j}(q). \tag{5}$$

As pointed out above, the probability generating function of C_{m_1,m_2} is given via

$$\mathbb{E}(q^{C_{m_1,m_2}}) = \frac{F_{m_1,m_2}(q)}{\binom{m_1+m_2}{m_1}}.$$

Thus, the last summand in (5) yields the following representation

$$\frac{1}{2^n} \sum_{j=0}^{n-2} F_{n-1-j,j}(q) = \frac{1}{2^n} \sum_{j=0}^{n-2} \mathbb{E}(q^{C_{n-1-j,j}}) \cdot \binom{n-1}{j}$$

We note that $\frac{\binom{n-1}{j}}{2^n}$ cannot be directly translated into a probabilistic setting, as for $q = 1$ the sum $\sum_{j=0}^{n-2} \binom{n-1}{j}$ reduces to $2^{n-1} - 1$, instead of 2^n. However, we observe that the term

$$\mathbb{P}\{J_n = j\} = \frac{\binom{n-1}{j}}{2^{n-1} - 1}$$

corresponds to a truncated binomial distribution J_n with support $\{0, \dots, n-2\}$. Thus we obtain further

$$\begin{aligned}\frac{1}{2^n}\sum_{j=0}^{n-2} F_{n-1-j,j}(q) &= \frac{2^{n-1}-1}{2^n}\sum_{j=0}^{n-2}\mathbb{E}(q^{C_{n-1-j,j}})\cdot\frac{\binom{n-1}{j}}{2^{n-1}-1}\\ &= \frac{1}{2}\Big(1-\frac{1}{2^{n-1}}\Big)\sum_{j=0}^{n-2}\mathbb{E}(q^{C_{n-1-j,j}})\mathbb{P}\{J_n = j\} = \frac{1}{2}\Big(1-\frac{1}{2^{n-1}}\Big)\mathbb{E}(q^{C_{n-1-J_n,J_n}}),\end{aligned}$$

where the two prefactors are translated into the stated Bernoulli distributed random variables. Translating these expressions for the probability generating functions involved into a distributional equation leads to the stated result. Note that the fact that X_{n-1}^* indeed has the same distribution as X defined on a deck of $n-1$ cards follows from equation (2). □

The distributional decomposition together with the properties of the binomial distribution and the limit laws of the two-color card guessing game allow to obtain a limit law for X_n. By the classical de Moivre–Laplace theorem, we can approximate the binomial distribution J_n with mean $\frac{n}{2}$ and standard deviation $\sqrt{n}/2$ by a normal random variable. This suggests that we need to study $C_{n-1-j,j}$ for $j = \frac{n}{2} + x\sqrt{n}$, as n tends to infinity. We recall the limit law for the two-color card guessing game in the required range (see [22, 23] for a complete discussion of all different limit laws of C_{m_1,m_2} depending on the growth behavior of m_1, m_2; additionally, we also refer to [9, 31] for the case $m_1 = m_2$).

Theorem 7 (Limit law for two-color card guessing [22, 23])**.** *Assume that the numbers m_1, m_2 satisfy $m_1 - m_2 \sim \rho\cdot\sqrt{m_1}$, as $m_1 \to \infty$, with $\rho > 0$. Then, the number of correct guesses C_{m_1,m_2} is asymptotically linear exponentially distributed,*

$$\frac{C_{m_1,m_2} - m_1}{\sqrt{m_1}} \xrightarrow{\mathcal{L}} \mathrm{LinExp}(\rho, 2),$$

or equivalently by explicitly stating the cumulative distribution function of LinExp$(\rho, 2)$:

$$\mathbb{P}\{C_{m_1,m_2} \leqslant m_1 + \sqrt{m_1}z\} \to 1 - e^{-z(\rho+z)}, \quad \text{for } z \geqslant 0.$$

In order to derive a limit law for X_n we require first a limit law for C_{n-1-J_n,J_n} as occurring in Theorem 6.

Lemma 8. *The random variable C_{n-1-J_n,J_n}, with J_n denoting a truncated binomial distribution $\mathbb{P}\{J_n = j\} = \binom{n-1}{j}/(2^{n-1}-1)$, $0 \leqslant j \leqslant n-2$, satisfies the following limit law:*

$$\frac{C_{n-1-J_n,J_n} - \frac{n}{2}}{\sqrt{n}} \to G.$$

Here G denotes a generalized gamma distributed random variable with probability density function

$$f(x) = \sqrt{\frac{2}{\pi}}\cdot 8x^2 e^{-2x^2}, \quad x \geqslant 0.$$

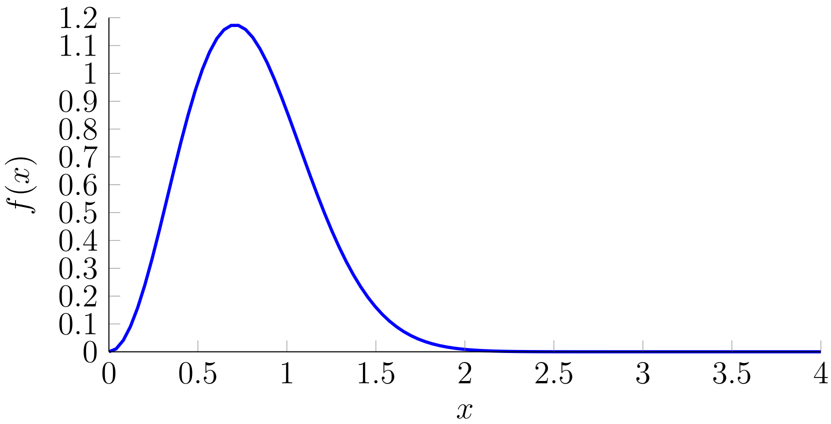

Figure 2: Plot of the density function $f(x)$ of the generalized Gamma distribution occurring in Theorem 10 and Lemma 8.

Remark 9. This special instance of a generalized Gamma distribution is also known as a Maxwell-Boltzmann distribution with parameter $a = 1/2$, which is of importance for describing particle speeds in idealized gases.

The first three raw integer moments of G are

$$\mathbb{E}(G) = \mu_G = \sqrt{\frac{2}{\pi}} \approx 0.7979, \quad \mathbb{E}(G^2) = \frac{3}{4}, \quad \mathbb{E}(G^3) = \sqrt{\frac{2}{\pi}}.$$

Consequently, the standard deviation σ_G and the skewness γ_G are given by

$$\sigma_G = \sqrt{\mathbb{E}(G^2) - \mu_G^2} \approx 0.3367, \quad \gamma_G = \frac{\mathbb{E}(G^3) - 3\mu_G \mathbb{E}(G^2) + 2\mu_G^3}{\sigma_G^3} \approx 0.4857,$$

leading to a right-skewed distribution, in agreement with the numerical observations of the limit law of X_n (which turns out to be G as well) in [21]. See Figure 2 for a plot of the density function of G.

Proof of Lemma 8. We consider the distribution function

$$F_n(x) = \mathbb{P}\big\{C_{n-1-J_n,J_n} \leqslant \frac{n}{2} + x\sqrt{n}\big\}$$

for fixed positive real x. Conditioning on the truncated binomial distribution gives

$$F_n(x) = \sum_{j=0}^{n-2} \mathbb{P}\big\{C_{n-1-j,j} \leqslant \frac{n}{2} + x\sqrt{n}\big\}\mathbb{P}\{J_n = j\}.$$

We can use the symmetry of the binomial distribution, as well as of the random variable C_{m_1,m_2}, to get

$$F_n(x) \sim 2 \cdot \sum_{j=\lfloor n/2 \rfloor}^{n-2} \mathbb{P}\big\{C_{j,n-1-j} \leqslant \frac{n}{2} + x\sqrt{n}\big\} \cdot \mathbb{P}\{J_n = j\}.$$

Our task is to evaluate this sum for n tending to infinity. We proceed using standard methods, namely the local limit theorem for the binomial distribution $\mathbb{P}\{J_n = j\}$, as well as approximating the sum by an integral. We also require the asymptotics of the two-color card-guessing game in the required range, as stated in Theorem 7. Combining all these three steps yields the desired asymptotic expansion of $F_n(x)$. We start by replacing the binomial distribution with the Gaussian law. By the local limit theorem for the binomial distribution and also approximating the sum by an integral, we get for large n the asymptotics

$$F_n(x) \sim 2\int_{n/2}^{n-2} \mathbb{P}\big\{C_{j,n-1-j} \leqslant \frac{n}{2} + x\sqrt{n}\big\} \cdot \frac{e^{-\frac{(j-\mu_n)^2}{2\sigma_n^2}}}{\sigma_n\sqrt{2\pi}} dj,$$

where $\mu_n = n/2$ and $\sigma_n = \sqrt{n}/2$. Substituting $j = \mu_n + t\sigma_n$, we obtain further

$$F_n(x) \sim 2\int_{0}^{\infty} \mathbb{P}\big\{C_{n/2+t\sqrt{n}/2,n-1-n/2-t\sqrt{n}/2} \leqslant \frac{n}{2} + x\sqrt{n}\big\} \cdot \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt.$$

Next, we use the limit law from Theorem 7 for the two-color card guessing game with

$$m_1 = n/2 + t\sqrt{n}/2, \quad m_2 = n - 1 - n/2 - t\sqrt{n}/2.$$

Since $C_{m_1,m_2} \geqslant \max\{m_1, m_2\}$ (see, e.g., [22]), we deduce that for $t > 2x$ it holds

$$\mathbb{P}\big\{C_{n/2+t\sqrt{n}/2,n-1-n/2-t\sqrt{n}/2} \leqslant \frac{n}{2} + x\sqrt{n}\big\} \sim 0.$$

Furthermore, in the range $0 \leqslant t \leqslant 2x$ we obtain from Theorem 7, by setting $\rho = \sqrt{2}t$ and $z = \sqrt{2}(x - t/2)$,

$$\begin{aligned}&\mathbb{P}\big\{C_{n/2+t\sqrt{n}/2,n-1-n/2-t\sqrt{n}/2} \leqslant \frac{n}{2} + x\sqrt{n}\big\}\\ &\qquad \to 1 - \exp\Big(-\sqrt{2}\big(x - \frac{t}{2}\big)\big(\sqrt{2}t + \sqrt{2}(x - \frac{t}{2})\big)\Big) = 1 - \exp\Big(-2x^2 + \frac{t^2}{2}\Big).\end{aligned}$$

This implies that

$$\begin{aligned}F_n(x) &\sim \frac{2}{\sqrt{2\pi}} \cdot \int_0^{2x} e^{-t^2/2}\Big(1 - \exp\Big(-2x^2 + \frac{t^2}{2}\Big)\Big)dt\\ &= \frac{2}{\sqrt{2\pi}} \cdot \int_0^{2x} \Big(e^{-t^2/2} - e^{-2x^2}\Big)dt = \frac{2}{\sqrt{2\pi}} \cdot \Big(\int_0^{2x} e^{-t^2/2}dt - 2xe^{-2x^2}\Big).\end{aligned}$$

Differentiating the last expression with respect to x leads to the desired density function of the limiting r.v. G,

$$f(x) = \frac{2}{\sqrt{2\pi}}\Big(e^{-2x^2} \cdot 2 - 2e^{-2x^2} + 8x^2 \cdot e^{-2x^2}\Big) = \sqrt{\frac{2}{\pi}} \cdot 8x^2 e^{-2x^2}. \qquad \square$$

Next we state the main result of this work, a limit law for the number of correct guesses X_n, as announced in the introduction. The limit law is the same as in Lemma 8, involving the generalized Gamma distribution.

Theorem 10. *The normalized random variable* $Y_n = (X_n - \frac{n}{2})/\sqrt{n}$ *converges in distribution to a generalized gamma distributed random variable* G, $Y_n \xrightarrow{\mathcal{L}} G$, *with density* $f(x) = \sqrt{\frac{2}{\pi}} \cdot 8x^2 e^{-2x^2}$, $x \geqslant 0$.

Remark 11 (A fixed-point equation)*.* Once we know that the limit law exists, one can informally derive the limit law from the distributional equation (4) by omitting asymptotically negligible terms:

$$Y_n \sim I_1 \cdot Y_{n-1}^* + (1 - I_1) \frac{C_{n-1-J_n,J_n} - \frac{n}{2}}{\sqrt{n}},$$

where $I_1 = \mathrm{Be}(0.5)$. Thus, for large n we anticipate a fixed-point equation for the limit law Y of Y_n:

$$Y \sim I_1 \cdot Y^* + (1 - I_1) \cdot G,$$

where $Y^* \stackrel{\mathcal{L}}{=} Y$ denotes an independent copy of Y and G a generalized Gamma distributed random variable, independent of Y, Y^* and I_1. Similarly, we may anticipate that all integer moments of Y are simply the moments of G, as the indicator variables are mutually exclusive:

$$\mathbb{E}(Y^r) = \frac{1}{2}\mathbb{E}\big((Y^*)^r\big) + \frac{1}{2}\mathbb{E}(G^r), \quad \text{such that} \quad \mathbb{E}(Y^r) = \mathbb{E}(G^r), \quad r \geqslant 0.$$

Proof of Theorem 10. According to Theorem 6 we get

$$\begin{aligned}
&\mathbb{P}\big\{X_n \leqslant \frac{n}{2} + x\sqrt{n}\big\} \\
&\quad = \frac{1}{2}\mathbb{P}\big\{X_{n-1} + 1 \leqslant \frac{n}{2} + x\sqrt{n}\big\} + \Big(\frac{1}{2} - \frac{1}{2^n}\Big)\mathbb{P}\big\{C_{n-1-J_n,J_n} \leqslant \frac{n}{2} + x\sqrt{n}\big\}.
\end{aligned}$$

Moreover, by iterating this recursive representation we observe that, for $n \to \infty$,

$$\mathbb{P}\big\{X_n \leqslant \frac{n}{2} + x\sqrt{n}\big\} \sim \sum_{\ell \geqslant 1} \frac{1}{2^\ell} \cdot \mathbb{P}\big\{C_{n-\ell-J_{n+1-\ell},J_{n+1-\ell}} \leqslant \frac{n}{2} + x\sqrt{n}\big\}.$$

As n tends to infinity, Lemma 8 ensures that all the distribution functions occurring converge to the same limit, from which the stated result follows. □

2.4 Moment convergence

Krityakierne and Thanatipanonda [21] provided extremely precise results for the first few (factorial) moments of X_n, as well as for the centered moments $\mathbb{E}((X_n - \mu)^r)$, for

$r = 1, 2, 3$. We state a simplified version of their result:

$$\mu = \mathbb{E}(X_n) = \frac{n}{2} + \sqrt{\frac{2n}{\pi}} - \frac{1}{2} + \mathcal{O}(n^{-1/2}), \quad \mathbb{E}\big((X_n - \mu)^2\big) = \Big(\frac{3}{4} - \frac{2}{\pi}\Big)n + \mathcal{O}(1),$$
$$\mathbb{E}\big((X_n - \mu)^3\big) = \sqrt{\frac{2}{\pi}}\Big(\frac{4}{\pi} - \frac{5}{4}\Big)n^{3/2} + \mathcal{O}(n^{1/2}). \tag{6}$$

First we use above expansions of $\mathbb{E}((X_n - \mu)^r)$ to determine the asymptotics of the first moments of $Y_n = (X_n - \frac{n}{2})/\sqrt{n}$ in a straightforward way. One observes that the limits of $\mathbb{E}\big(Y_n^r\big)$, $r = 1, 2, 3$, are in agreement with the limit law G stated in Theorem 10.

Proposition 12. *Let $Y_n = (X_n - \frac{n}{2})/\sqrt{n}$. The moments $\mathbb{E}(Y_n^r)$ converge for $r = 1, 2, 3$ to the moments of the limit law G:*

$$\mathbb{E}\big(Y_n\big) \to \sqrt{\frac{2}{\pi}} = \mathbb{E}(G), \quad \mathbb{E}\big(Y_n^2\big) \to \frac{3}{4} = \mathbb{E}(G^2), \quad \mathbb{E}\big(Y_n^3\big) \to \sqrt{\frac{2}{\pi}} = \mathbb{E}(G^3).$$

Proof. The result for the expected value $\mathbb{E}(Y_n)$ follows directly from (6). In the following let $\mu = \mathbb{E}(X_n) = \frac{n}{2} + \delta_n$. Due to (6) it holds

$$\delta_n = \sqrt{\frac{2n}{\pi}} - \frac{1}{2} + \mathcal{O}(n^{-1/2}). \tag{7}$$

Consequently, the second centered moment can be rewritten as follows:

$$\mathbb{E}\big((X_n - \mu)^2\big) = \mathbb{E}\big((X_n - \frac{n}{2} - \delta_n)^2\big) = \mathbb{E}\big((X_n - \frac{n}{2})^2\big) - 2\delta_n\mathbb{E}\big(X_n - \frac{n}{2}\big) + \delta_n^2,$$

which gives, by using expansions (6) and (7),

$$\mathbb{E}\big(Y_n^2\big) = \frac{1}{n}\mathbb{E}\big((X_n - \frac{n}{2})^2\big) = \frac{1}{n}\Big[\mathbb{E}\big((X_n - \mu)^2\big) + 2\delta_n\mathbb{E}\big(X_n - \frac{n}{2}\big) - \delta_n^2\Big]$$
$$= \frac{1}{n}\Big[\mathbb{E}\big((X_n - \mu)^2\big) + \delta_n^2\Big] \sim \frac{3}{4}.$$

In a similar way, by rewriting the third centered moment and using (6) and (7), one obtains the stated result for $\mathbb{E}(Y_n^3)$. □

Actually, in the following we are going to show that indeed all integer moments of Y_n converge to the corresponding moments of the limit law G. We want to point out that in general convergence in distribution does not imply moment convergence. To the best of the authors knowledge, to guarantee moment convergence additional uniform integrability conditions on the sequence $(Y_n)_{n\in\mathbb{N}} = \big((X_n - \frac{n}{2})/\sqrt{n}\big)_{n\in\mathbb{N}}$ would be required (for the first moment see, e.g., [4, p. 30ff]), which seems to be much more technical and out of reach. Here, we will deal directly with the generating functions description to obtain the desired moment convergence. Let us first state the moments of the limit law.

Proposition 13. *The integer moments of the generalized gamma distributed random variable G with probability density function as defined in Lemma 8 are given as follows:*

$$\mathbb{E}\big(G^r\big) = \frac{\Gamma\big(\frac{r+3}{2}\big)}{2^{\frac{r}{2}-1}\sqrt{\pi}}, \quad r \geqslant 0.$$

Proof. A straightforward evaluation of the defining integral of the r-th moment of G by means of the Γ-function after substituting $t = 2x^2$ yields the stated result:

$$\mathbb{E}\big(G^r\big) = 8\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty x^{r+2} e^{-2x^2} dx = \frac{1}{2^{\frac{r}{2}-1}\sqrt{\pi}} \cdot \int_0^\infty t^{\frac{r+1}{2}} e^{-t} dt = \frac{\Gamma\big(\frac{r+3}{2}\big)}{2^{\frac{r}{2}-1}\sqrt{\pi}}. \qquad \square$$

Theorem 14. *Let $Y_n = (X_n - \frac{n}{2})/\sqrt{n}$. The r-th integer moments $\mathbb{E}(Y_n^r)$ converge, for arbitrary but fixed r and $n \to \infty$, to the moments of the limit law G:*

$$\mathbb{E}\big(Y_n^r\big) \to \mathbb{E}\big(G^r\big) = \frac{\Gamma\big(\frac{r+3}{2}\big)}{2^{\frac{r}{2}-1}\sqrt{\pi}}, \quad r \geqslant 0.$$

Remark 15. Since the generalized gamma distributed r.v. G is uniquely characterized by its moments (which easily follows, e.g., from simple growth bounds), we note that an application of the moment's convergence theorem of Fréchet and Shohat (see the original work [13], or the book [26, 11.4.C, page 187]) immediately shows convergence in distribution of Y_n to G, thus gives an alternative proof of Theorem 10.

To show Theorem 14 we will again start with the recursive description of $D_n(q)$ given in Lemma 3, but in order to deal with this recurrence we use an alternative approach based on generating functions and basic techniques from analytic combinatorics [12]. Furthermore, we use explicit formulæ for a suitable bivariate generating function of $\mathbb{E}\big(q^{C_{m_1,m_2}}\big)$ and the so-called diagonal as have been derived in [18, 22]. They can be stated in the following form.

Proposition 16 ([18, 22])**.** *The g.f. $\tilde{F}(x, y, q) = \sum_{m_1 \geqslant m_2 \geqslant 0} \binom{m_1+m_2}{m_1} \mathbb{E}\big(q^{C_{m_1,m_2}}\big) x^{m_1} y^{m_2}$ and $\tilde{F}_0(x, y, q) = \sum_{m \geqslant 0} \binom{2m}{m} \mathbb{E}\big(q^{C_{m,m}}\big) x^m y^m$ are given as follows:*

$$\tilde{F}(x, y, q) = \frac{1-y}{1-qx-y} + \frac{qxy(q-(1+q)y)}{(1-qx-y)(1-B(qxy))(1-(1+q)B(qxy))},$$
$$\tilde{F}_0(x, y, q) = \frac{1}{1-(1+q)B(qxy)},$$

where $B(t) = \frac{1-\sqrt{1-4t}}{2} = \sum_{n \geqslant 1} \frac{1}{n}\binom{2n-2}{n-1} t^n$ denotes the g.f. of the shifted Catalan-numbers.

With these results we obtain a generating functions solution of recurrence (2) for $D_n(q)$.

Lemma 17. *The bivariate generating function*

$$D(z,q)=\sum_{n\geqslant 0}D_n(q)z^n=\sum_{n\geqslant 0}2^n\mathbb{E}\big(q^{X_n}\big)z^n$$

is given by the following explicit formula, with $B(t)=\frac{1-\sqrt{1-4t}}{2}$*:*

$$D(z,q)=\frac{1-z}{(1-qz)^2}+\frac{z}{1-qz}\left[\frac{2(1-z)}{1-(1+q)z}\right.$$
$$\left.+\frac{2qz^2(q-(1+q)z)}{(1-(1+q)z)(1-(1+q)B(qz^2))(1-B(qz^2))}-\frac{1}{1-(1+q)B(qz^2)}\right].$$

Proof. Introducing the auxiliary g.f. $F(x,y,q)=\sum_{m_1\geqslant 0}\sum_{m_2\geqslant 0}F_{m_1,m_2}(q)x^{m_1}y^{m_2}$, we obtain from recurrence (2) after multiplying with z^n and summing over integers $n\geqslant 0$ the relation

$$D(z,q)=qzD(z,q)+\frac{1-z}{1-qz}+zF(z,z,q),$$

and further

$$D(z,q)=\frac{1-z}{(1-qz)^2}+\frac{zF(z,z,q)}{1-qz}. \tag{8}$$

Using the relation

$$F(x,y,q)=\tilde{F}(x,y,q)+\tilde{F}(y,x,q)-\tilde{F}_0(x,y,q),$$

which is immediate from the definitions given in (3) and Proposition 16, and the explicit formulæ given in Proposition 16, the stated result follows from (8). □

We are interested in the asymptotic behaviour of the moments of the shifted r.v. $\hat{X}_n:=X_n-n/2=\sqrt{n}\,Y_n$. The corresponding g.f. $\hat{D}(z,q)$ is closely related to $D(z,q)$ as defined in Lemma 17, since we get

$$\hat{D}(z,q)=\sum_{n\geqslant 0}2^n\mathbb{E}\big(q^{\hat{X}_n}\big)z^n=\sum_{n\geqslant 0}2^n\mathbb{E}\big(q^{X_n}\big)q^{-\frac{n}{2}}z^n=D\Big(\frac{z}{\sqrt{q}},q\Big). \tag{9}$$

Actually, we will set $q=1+u$ and use that the coefficients of the probability generating function in a series expansion around $u=0$ yield the factorial moments of $\hat{X}_n$:

$$\mathbb{E}\big((1+u)^{\hat{X}_n}\big)=\sum_{r\geqslant 0}u^r\,\mathbb{E}\Big(\tbinom{\hat{X}_n}{r}\Big)=\sum_{r\geqslant 0}\mathbb{E}\big(\hat{X}_n^{\underline{r}}\big)\frac{u^r}{r!}.$$

Thus one gets

$$\hat{D}(z,1+u)=\sum_{r\geqslant 0}g_r(z)u^r=\sum_{r\geqslant 0}u^r\cdot\frac{1}{r!}\sum_{n\geqslant 0}2^n\mathbb{E}\big(\hat{X}_n^{\underline{r}}\big)z^n, \tag{10}$$

and in order to determine the asymptotic behaviour of the factorial (and raw) moments of $\hat{X}_n$ we carry out a local expansion of the functions $g_r(z)=[u^r]\hat{D}(z,1+u)$ around the dominant singularities followed by basic applications of so-called transfer lemmata.

The next lemma states the relevant properties of the coefficients of $\hat{D}(z,1+u)$.

Lemma 18. *Let $\hat{D}(z,q)$ be the g.f. of the shifted r.v. $\hat{X}_n = X_n - \frac{n}{2}$ as defined in* (9)*. Then the functions $g_r(z) = [u^r]\hat{D}(z, 1+u)$ obtained as coefficients in a series expansion of $\hat{D}(z, 1+u)$ around $u = 0$ have radius of convergence $\frac{1}{2}$ and, for $r \geqslant 1$, have the two dominant singularities $\rho_{1,2} = \pm\frac{1}{2}$. Moreover, the local behaviour of $g_r(z)$ around $\rho := \rho_1 = \frac{1}{2}$ is given as follows, with $\mathcal{Z} := \frac{1}{1-2z}$:*

$$g_r(z) = (r+1)\Big(\frac{1}{8}\Big)^{\frac{r}{2}} \mathcal{Z}^{\frac{r}{2}+1} \cdot \big(1 + \mathcal{O}\big(\mathcal{Z}^{-\frac{1}{2}}\big)\big), \quad r \geqslant 0.$$

Remark 19. We remark that a closer inspection shows that the second dominant singularity $\rho_2 = -\frac{1}{2}$ occurring in the functions $g_r(z)$ defined by Lemma 18 yield contributions that do not affect the main terms stemming from the contributions of the singularity $\rho = \rho_1 = \frac{1}{2}$. Since we are here only interested in the main term contribution, we will restrict ourselves to elaborate the expansion around ρ. However, the presence of two dominant singularities is reflected by the fact, that lower order terms of the asymptotic expansions of the r-th moments of X_n are different for n even and n odd, resp., as has been observed in [21].

Proof. Using (9) and the explicit formula of $D(z,q)$ given in Lemma 17, one gets after simple manipulations

$$\hat{D}(z,q) = \frac{\sqrt{q}(\sqrt{q}-z)}{(\sqrt{q}-qz)^2} + \frac{2z(\sqrt{q}-z)}{(\sqrt{q}-(1+q)z)(\sqrt{q}-qz)} + \frac{z\big(\sqrt{q}(q-1) + \big((1+q)z - q^{\frac{3}{2}}\big)\big(1-2B(z^2)\big)\big)}{(1-(1+q)B(z^2))(\sqrt{q}-(1+q)z)(\sqrt{q}-qz)}. \quad (11)$$

We set $q = 1+u$ and carry out a series expansion of the summands of (11) around $u = 0$. Since this is a rather straightforward task using essentially the binomial series, but leads to rather lengthy computations when one intends to be exhaustive in every step, we will here only give a sketch of such computations and are omitting some of the details.

When treating the first summand in (11) and inspecting the coefficients in the series expansion around $u = q - 1 = 0$,

$$\hat{D}^{[1]}(z,q) := \frac{\sqrt{q}\,(\sqrt{q}-z)}{(\sqrt{q}-qz)^2} = \sum_{r\geqslant 0} g_r^{[1]}(z)u^r,$$

one easily observes that the functions $g_r^{[1]}(z)$ are analytic for $|z| < 1$ (to be more precise, the unique dominant singularity is at $z = 1$), which causes exponentially small contributions for the coefficients $[z^r]g_r^{[1]}(z)$ compared to the remaining summands. Thus, these contributions are negligible and do not have to be considered further.

When expanding the second summand of (11) around $u = q - 1 = 0$,

$$\hat{D}^{[2]}(z,q) := \frac{2z(\sqrt{q}-z)}{(\sqrt{q}-(1+q)z)(\sqrt{q}-qz)} = \sum_{r\geqslant 0} g_r^{[2]}(z)u^r, \quad (12)$$

we have to treat with more care the factor $(\sqrt{q}-(1+q)z)^{-1}$. First, by using the binomial series we get

$$\sqrt{q}-(1+q)z=\sqrt{1+u}-(2+u)z=(1-2z)\Big(1+\frac{u}{2}+\sum_{k\geqslant 2}\frac{c_k}{1-2z}u^k\Big),$$

with $c_k=\binom{\frac{1}{2}}{k}$, and further, by using the geometric series,

$$\begin{aligned}\frac{1}{\sqrt{q}-(1+q)z}&=\frac{1}{(1-2z)\big(1+\frac{u}{2}\big(1+\sum_{k\geqslant 1}\frac{2c_{k+1}}{1-2z}u^k\big)\big)}\\&=\mathcal{Z}\Big(1+\sum_{\ell\geqslant 1}(-\frac{1}{2})^\ell u^\ell\big(1+\sum_{k\geqslant 1}2c_{k+1}\mathcal{Z}u^k\big)^\ell\Big).\end{aligned}\tag{13}$$

From this expansion it is apparent that all the coefficients of u^r in the series expansion, considered as functions in z, have a unique dominant singularity at $z=\rho=\frac{1}{2}$. Furthermore, for $\ell\geqslant 1$ we obtain the following expansion in powers of u and locally around $z=\rho$, i.e., $\mathcal{Z}=\infty$:

$$\begin{aligned}\big(1+\sum_{k\geqslant 1}2c_{k+1}\mathcal{Z}u^k\big)^\ell&=1+\ell(2c_2)\mathcal{Z}u+\sum_{j=2}^{\ell}\binom{\ell}{j}(2c_2)^j\mathcal{Z}^j\big(1+\mathcal{O}(\mathcal{Z}^{-1})\big)u^j\\&\quad+\ell(2c_2)^{\ell-1}(2c_3)\mathcal{Z}^\ell\big(1+\mathcal{O}(\mathcal{Z}^{-1})\big)u^{\ell+1}+\sum_{k\geqslant \ell+2}\mathcal{O}(\mathcal{Z}^\ell)u^k,\end{aligned}$$

which, after plugging into (13) and using $c_2=-\frac{1}{8}$, $c_3=\frac{1}{16}$ leads to the required expansion:

$$\begin{aligned}\frac{1}{\sqrt{q}-(1+q)z}=\mathcal{Z}-\frac{\mathcal{Z}}{2}u+\sum_{\ell\geqslant 1}\Bigg[&(\frac{1}{8})^\ell\mathcal{Z}^{\ell+1}(1+\mathcal{O}(\mathcal{Z}^{-1}))u^{2\ell}\\&-\frac{1}{2}(\frac{1}{8})^\ell(2\ell+1)\mathcal{Z}^{\ell+1}(1+\mathcal{O}(\mathcal{Z}^{-1}))u^{2\ell+1}\Bigg].\end{aligned}\tag{14}$$

Next, it is easy to see that the coefficients in the expansion around $u=0$ of the remaining factors of $\hat{D}^{[2]}(z,q)$ are functions in z with radius of convergence 1, and one gets

$$\frac{2z(\sqrt{q}-z)}{\sqrt{q}-qz}=1+\mathcal{O}(\mathcal{Z}^{-1})+(1+\mathcal{O}(\mathcal{Z}^{-1}))u+\sum_{r\geqslant 2}\mathcal{O}(\mathcal{Z}^0)u^r.\tag{15}$$

Combining the expansions (14) and (15), we obtain that the functions $g_r^{[2]}(z)$ in expansion (12) have a unique dominant singularity at $z=\rho$ and allow there the local expansions

$$g_r^{[2]}(z)=\begin{cases}(\frac{1}{8})^\ell\mathcal{Z}^{\ell+1}(1+\mathcal{O}(\mathcal{Z}^{-1})), & \text{for } r=2\ell \text{ even},\\ -\frac{1}{2}(\frac{1}{8})^\ell(2\ell-1)\mathcal{Z}^{\ell+1}(1+\mathcal{O}(\mathcal{Z}^{-1})), & \text{for } r=2\ell+1 \text{ odd}.\end{cases}\tag{16}$$

Finally, we consider an expansion in powers of $u = q - 1$ of the third summand of (11),

$$\hat{D}^{[3]}(z,q) := \frac{z\big(\sqrt{q}(q-1) + \big((1+q)z - q^{\frac{3}{2}}\big)\big(1-2B(z^2)\big)\big)}{(1-(1+q)B(z^2))(\sqrt{q}-(1+q)z)(\sqrt{q}-qz)}. \tag{17}$$

Let us define $\tilde{\mathcal{Z}} = \frac{1}{1-4z^2}$. Since $B(z^2) = \frac{1}{2}(1 - \tilde{\mathcal{Z}}^{-\frac{1}{2}})$, we get

$$1 - (1+q)B(z^2) = \tilde{\mathcal{Z}}^{-\frac{1}{2}}\Big(1 - \frac{1}{2}(\tilde{\mathcal{Z}}^{\frac{1}{2}} - 1)\Big)$$

and thus

$$\frac{1}{1-(1+q)B(z^2)} = \frac{\tilde{\mathcal{Z}}^{\frac{1}{2}}}{1 - \frac{1}{2}(\tilde{\mathcal{Z}}^{\frac{1}{2}} - 1)u} = \tilde{\mathcal{Z}}^{\frac{1}{2}}\Big(1 + \sum_{r \geqslant 1}\Big(\frac{1}{2}(\tilde{\mathcal{Z}}^{\frac{1}{2}} - 1)u\Big)^r\Big). \tag{18}$$

Therefore, for this factor of $\hat{D}^{[3]}(z,q)$ we obtain that the coefficients of u^r are functions in z with two dominant singularities $\rho_{1,2} = \pm\frac{1}{2}$. However, as already pointed out in Remark 19, the contributions stemming from the singularity $\rho_2 = -\frac{1}{2}$ do not affect the main term contributions and thus they are not considered any further. Since $\tilde{\mathcal{Z}} = \frac{1}{(1-2z)(1+2z)} = \frac{1}{2}\mathcal{Z}(1 + \mathcal{O}(\mathcal{Z}^{-1}))$, we thus obtain from (18) the local expansion around $z = \rho$:

$$\frac{1}{1-(1+q)B(z^2)} = \sum_{r \geqslant 0}(\frac{1}{2})^{\frac{3r+1}{2}}\mathcal{Z}^{\frac{r+1}{2}}(1 + \mathcal{O}(\mathcal{Z}^{-\frac{1}{2}}))u^r. \tag{19}$$

In a similar fashion one obtains the expansion

$$\begin{aligned}\frac{z\big(\sqrt{q}(q-1) + \big((1+q)z - q^{\frac{3}{2}}\big)\big(1-2B(z^2)\big)\big)}{\sqrt{q}-qz}& \\ = -2^{\frac{1}{2}}\mathcal{Z}^{-\frac{3}{2}}(1 + \mathcal{O}(\mathcal{Z}^{-1})) + (1 + \mathcal{O}(\mathcal{Z}^{-\frac{1}{2}}))u + \sum_{r \geqslant 2}\mathcal{O}(\mathcal{Z}^0)u^r,&\end{aligned} \tag{20}$$

whereas the last factor of $\hat{D}^{[3]}(z,q)$ has been treated already in (14). Combining expansions (19), (20) and (14), we get

$$\begin{aligned}\hat{D}^{[3]}(z,q) = &\Big(\sum_{r \geqslant 0}(\frac{1}{8})^{\frac{r}{2}}\mathcal{Z}^{\frac{r}{2}}(1 + \mathcal{O}(\mathcal{Z}^{-\frac{1}{2}}))u^r\Big) \\ &\cdot\Big(\sum_{\ell \geqslant 0}(\frac{1}{8})^{\ell}\mathcal{Z}^{\ell}(1 + \mathcal{O}(\mathcal{Z}^{-1}))u^{2\ell} + (-\frac{1}{2})(\frac{1}{8})^{\ell}(2\ell+1)\mathcal{Z}^{\ell}(1 + \mathcal{O}(\mathcal{Z}^{-1}))u^{2\ell+1}\Big) \\ &\cdot\Big(-(1 + \mathcal{O}(\mathcal{Z}^{-1})) + (\frac{1}{2})^{\frac{1}{2}}\mathcal{Z}^{\frac{3}{2}}(1 + \mathcal{O}(\mathcal{Z}^{-\frac{1}{2}}))u + \sum_{r \geqslant 2}\mathcal{O}(\mathcal{Z}^{\frac{3}{2}})u^r\Big).\end{aligned} \tag{21}$$

To compute the Cauchy product of the first two factors of (21) we use (with some coefficients $\alpha_r, \beta_r \in \mathbb{R}$):

$$\Big(\sum_{r\geqslant 0} \alpha_r \mathcal{Z}^{\frac{r}{2}}(1+\mathcal{O}(\mathcal{Z}^{-\frac{1}{2}}))u^r\Big)\cdot\Big(\sum_{r\geqslant 0} \beta_r \mathcal{Z}^{\lfloor\frac{r}{2}\rfloor}(1+\mathcal{O}(\mathcal{Z}^{-\frac{1}{2}}))u^r\Big)$$
$$= \sum_{r\geqslant 0}\gamma_r \mathcal{Z}^{\frac{r}{2}}(1+\mathcal{O}(\mathcal{Z}^{-\frac{1}{2}}))u^r, \qquad \text{with} \quad \gamma_r = \sum_{\ell=0}^{\lfloor\frac{r}{2}\rfloor}\beta_{2\ell}\,\alpha_{r-2\ell}.$$

In particular, for $\alpha_r = (1/8)^{\frac{r}{2}}$ and $\beta_{2\ell} = (1/8)^{\ell}$ one gets $\gamma_r = (1/8)^{\frac{r}{2}}(\lfloor r/2\rfloor + 1)$, which eventually shows that the coefficients $g_r^{[3]}(z)$ in the expansion of $\hat{D}^{[3]}(z,q)$ around $u = q-1 = 0$ are given as follows:

$$g_r^{[3]}(z) = \begin{cases} -(1+\mathcal{O}(\mathcal{Z}^{-1})), & \text{for } r=0,\\ 2(\frac{1}{8})^{\frac{r}{2}}(\lfloor\frac{r-1}{2}\rfloor+1)\mathcal{Z}^{\frac{r}{2}+1}(1+\mathcal{O}(\mathcal{Z}^{-\frac{1}{2}})), & \text{for } r\geqslant 1.\end{cases} \tag{22}$$

Thus, combining (16) and (22) one obtains, after simple manipulations, the stated local expansion of the coefficients $g_r(z) = g_r^{[1]}(z) + g_r^{[2]}(z) + g_r^{[3]}(z)$ in the series expansion of $\hat{D}(z,q)$ around $u = q-1 = 0$. □

The expansion of $\hat{D}(z,q)$ stated in Lemma 18 easily yields the asymptotic behaviour of the moments of Y_n.

Proof of Theorem 14. According to the definition of $\hat{X}_n$ and relation (10) we get for the factorial moments:

$$\mathbb{E}\big(\hat{X}_n^{\underline{r}}\big) = \frac{r![z^n u^r]\hat{D}(z,1+u)}{2^n} = \frac{r![z^n]g_r(z)}{2^n},$$

with $g_r(z)$ as defined in Lemma 18. Since the dominant singularity of $g_r(z)$ relevant for the asymptotic behaviour of the main term is at $z=\rho=\frac{1}{2}$ (see Remark 19) with a local expansion stated in above lemma, we can apply basic transfer lemmata [12] to obtain for the coefficients:

$$[z^n]g_r(z) = [z^n](r+1)(\frac{1}{8})^{\frac{r}{2}}\frac{1}{(1-2z)^{\frac{r}{2}+1}}\cdot\big(1+\mathcal{O}(\sqrt{1-2z})\big)$$
$$= (r+1)(\frac{1}{8})^{\frac{r}{2}}\frac{2^n n^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+1)}\cdot\big(1+\mathcal{O}(n^{-\frac{1}{2}})\big).$$

Thus, the asymptotic behaviour of the factorial moments is given by

$$\mathbb{E}\big(\hat{X}_n^{\underline{r}}\big) = \frac{(r+1)!(\frac{1}{8})^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+1)}n^{\frac{r}{2}}\cdot\big(1+\mathcal{O}(n^{-\frac{1}{2}})\big), \quad r\geqslant 0. \tag{23}$$

Since the r-th integer moments can be obtained by a linear combination of the factorial moments of order $\leqslant r$, due to $\mathbb{E}\big(\hat{X}_n^r\big) = \mathbb{E}\big(\hat{X}_n^{\underline{r}}\big) + \mathcal{O}\big(\mathbb{E}\big(\hat{X}_n^{\underline{r-1}}\big)\big) = \mathbb{E}\big(\hat{X}_n^{\underline{r}}\big)\cdot\big(1+\mathcal{O}(n^{-\frac{1}{2}})\big)$

the same asymptotic behaviour (23) also holds for the raw moments. An application of the duplication formula for the Γ-function gives then the alternative representation

$$\mathbb{E}\big(\hat{X}_n^r\big) = \frac{\Gamma\big(\frac{r+3}{2}\big)}{2^{\frac{r}{2}-1}\sqrt{\pi}}\, n^{\frac{r}{2}} \cdot \big(1 + \mathcal{O}(n^{-\frac{1}{2}})\big), \quad r \geqslant 0. \tag{24}$$

Since $\hat{X}_n = X_n - n/2 = \sqrt{n}\, Y_n$, equation (24) implies $\mathbb{E}(Y_n^r) \to \mathbb{E}(G^r)$ as stated. □

3 First pure luck guess

So far, we have been interested in the total number of correct guesses. As the guesser follows the optimal strategy, the chances of a correct guess are always greater or equal 50 percent. Starting with a deck of n cards, we might be interested in the number of cards P_n, divided by two, remaining in the deck when the first "pure luck guess" with only a 50 percent success chance occurs. By Proposition 2 and Theorem 6, this can only happen after the "first phase" of always guessing the smallest number remaining in the deck has failed and thus has been finished and so the "two-color card guessing process" has been started already. Similar to Theorem 6 we obtain for $P := P_n$ the distributional equation

$$P_n \stackrel{\mathcal{L}}{=} I_1 \cdot P_{n-1}^* + (1 - I_1)(1 - I_2) \cdot H_{n-1-J_n, J_n},$$

where $I_1 \stackrel{\mathcal{L}}{=} \mathrm{Be}(0.5)$, $I_2 \stackrel{\mathcal{L}}{=} \mathrm{Be}(0.5^{n-1})$, and H_{m_1,m_2} denotes the number of cards present, divided by two, in a two-color card guessing game when for the first time a pure luck guess occurs. Additionally, P_{n-1}^* is an independent copy of P defined on $n-1$ cards. Moreover, as in Theorem 6, $J_n \stackrel{\mathcal{L}}{=} \mathrm{B}^*(n-1, p)$ denotes a truncated binomial distribution:

$$\mathbb{P}(J_n = j) = \binom{n-1}{j} / (2^{n-1} - 1), \quad 0 \leqslant j \leqslant n-2.$$

All random variables I_1, I_2, J_n, as well as H_{m_1,m_2} are mutually independent.

We use a limit law for H_{m_1,m_2}, for a certain regime of m_1, m_2 when both parameters are tending to infinity, relying on results of [22, 23].

First, we require a new distribution, a functional of a Lévy distributed random variable $L = \text{Lévy}(c)$, $c > 0$, with density

$$f_L(x) = \sqrt{\frac{c}{2\pi}}\,\frac{e^{-c/(2x)}}{x^{3/2}}, \quad x > 0. \tag{25}$$

Definition 20 (Reciprocal of a shifted Lévy distribution)**.** Let $L = \text{Lévy}(c)$, $c > 0$. Then, let $R = R(c)$ denote the reciprocal of the shifted random variable $1 + L$:

$$R = \frac{1}{1+L}, \quad \text{with support } (0, 1).$$

The density of R is given by

$$f_R(x) = \sqrt{\frac{c}{2\pi}} \cdot \frac{1}{(1-x)^{3/2} x^{1/2}} \cdot e^{-\frac{cx}{2(1-x)}}, \quad 0 < x < 1.$$

This random variable in terms of the above density function has been appeared already in several applications. See for example [23] for the limit law of the hitting time in sampling without replacement or [15, 16] for its occurrence in the limit law of an uncover process for random trees. Moreover, this random variable has appeared earlier in context of the standard additive coalescent, where also the relation to the Lévy distribution has been observed by Aldous and Pitman [1, Corollary 5 and Theorem 6]. We further note that the random variable also appears as the limit law of random dynamics on the edges of a uniform Cayley tree, a so-called "fire on tree" model [3]. In contrast to the Lévy distribution, the random variable R has integer moments of all orders. In the special case of $c = 1$ the moments have a particularly interesting structure [3, Lemma 3]:

$$\mathbb{E}(R^k) = \mathbb{E}\big(\exp(-\chi(2k))\big),$$

where $\chi(2k)$ is a chi-distributed random variable with $2k$ degrees of freedom and density

$$\frac{2^{1-k}}{(k-1)!}x^{2k-1}\exp(-x^2/2), \quad x \geqslant 0.$$

Furthermore, the random variable encodes the hitting time of a Brownian bridge starting at level $y = 0$ and ending at a fixed level [27]. Finally, we note that it is easy to see that R has the stated density function:

$$\begin{aligned} F_R(x) &= \mathbb{P}\{R \leqslant x\} = \mathbb{P}\Big\{\frac{1}{1+L} \leqslant x\Big\} = \mathbb{P}\Big\{\frac{1}{x} \leqslant 1 + L\Big\} \\ &= \mathbb{P}\Big\{L \geqslant \frac{1}{x} - 1\Big\} = 1 - \mathbb{P}\Big\{L < \frac{1-x}{x}\Big\}. \end{aligned}$$

Consequently,

$$f_R(x) = -f_L\big((1-x)/x\big) \cdot (-1) \cdot x^{-2} = \sqrt{\frac{c}{2\pi}}\frac{e^{-cx/(2(1-x))}x^{3/2}}{(1-x)^{3/2}} \cdot \frac{1}{x^2},$$

immediately leading to the stated density.

Next, we use the following result.

Lemma 21 (Hitting time and first pure luck guess). *Let H_{m_1,m_2} denote the random variable counting the number of remaining cards, divided by two, when for the first time a pure luck guess happens in the two-color card guessing game, starting with m_1 red and m_2 black cards. Assume further that $m_1, m_2 \to \infty$ and $m_2 = m_1 - \rho\sqrt{m_1}$, with $\rho > 0$. Then,*

$$\frac{H_{m_1,m_2}}{m_1} \xrightarrow{\mathcal{L}} R(\rho^2/2).$$

Proof. We combine arguments of [22, 23]: by the results of [22], the weighted sample paths of the two-color card guessing game coincide with the sample paths of the sampling without replacement urn (see also Remark 5). In particular, this holds with respect to the hitting position of the diagonal $x = y$, as a crossing of the diagonal without hitting cannot happen. In [23] such hitting positions have been studied in a general setting for paths starting at (m_1, m_2), with $m_1 \geqslant tm_2 + s$, and absorbing lines $y = x/t - s/t$, for $t \in \mathbb{N}$ and $s \in \mathbb{N}_0$. For our purpose we set $t = 1$ and $s = 0$ in [23, Theorem 2 (4)], which gives for $0 < x < 1$:

$$\mathbb{P}\Big\{\frac{H_{m_1,m_2}}{m_1} \leqslant x\Big\} \sim \int_0^x \frac{\rho}{\sqrt{2}\sqrt{2\pi}} \frac{1}{\sqrt{u}\,(1-u)^{\frac{3}{2}}} \cdot e^{-\frac{\rho^2 u}{4(1-u)}} du = \int_0^x f_R(u) du,$$

with $f_R(x)$ the density of the reciprocal of a shifted Lévy distribution with parameter $c = \rho^2/2$. Thus, this shows the stated limit law. □

In order to obtain the limit law of P_n we require the limit law of H_{n-1-J_n,J_n}, which will be determined next.

Lemma 22. *The random variable H_{n-1-J_n,J_n} has an Arcsine limit law $\beta(\frac{1}{2}, \frac{1}{2})$:*

$$\frac{H_{n-1-J_n,J_n}}{\frac{n}{2}} \xrightarrow{\mathcal{L}} \beta\Big(\frac{1}{2}, \frac{1}{2}\Big),$$

i.e., after suitable scaling, it converges in distribution to a Beta-distributed r.v. with parameters $1/2$ and $1/2$ that has the probability density function

$$f_\beta(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}, \quad 0 < x < 1.$$

In a way analogous to the proof of Theorem 10, this lemma readily leads to the main result of this section.

Theorem 23. *The random variable P_n counting the number of remaining cards, divided by two, when the first pure luck guess with only a 50 percent success chance occurs, starting with n ordered cards and performing a single riffle shuffle, has a $\beta\big(\frac{1}{2}, \frac{1}{2}\big)$ limit law, a so-called Arcsine distribution:*

$$\frac{P_n}{\frac{n}{2}} \to \beta\Big(\frac{1}{2}, \frac{1}{2}\Big).$$

Proof of Lemma 22. We proceed similar to the proof of Lemma 8. We study the distribution function $F(k) = \mathbb{P}\{H_{n-1-J_n,J_n} \leqslant k\}$ and obtain

$$F(k) = \sum_{j=0}^{n-2} \frac{\binom{n-1}{j}}{2^{n-1}-1} \mathbb{P}\{H_{n-1-j,j} \leqslant k\}.$$

We use the symmetry of the binomial distribution around $\lfloor n/2 \rfloor$ as well as $H_{m_1,m_2} = H_{m_2,m_1}$ and approximate the binomial distribution using the de Moivre-Laplace theorem. This leads to

$$F(k) \sim 2 \int_{\lfloor n/2 \rfloor}^{n} e^{-\frac{(j-\mu_n)^2}{2\sigma_n^2}} \cdot \frac{1}{\sigma_n \sqrt{2\pi}} \cdot \mathbb{P}\{H_{j,n-1-j} \leqslant k\} dj,$$

where $\mu_n = n/2$ and $\sigma_n = \sqrt{n}/2$. Changing the range of integration and the choice $k = x \cdot n/2$, with $0 < x < 1$, leads then, together with Lemma 21, to the improper integral

$$F(k) = \mathbb{P}\Big\{\frac{H_{n-1-J_n,J_n}}{n/2} \leqslant x\Big\} \sim 2\int_0^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int_0^x \frac{t}{\sqrt{2\pi u}(1-u)^{3/2}} \cdot e^{-\frac{t^2 u}{2(1-u)}} du\ dt.$$

Derivation with respect to x gives then the desired density function, where the arising improper integral is readily evaluated:

$$\int_0^\infty e^{-t^2/2} \cdot t \cdot e^{-t^2 g/2} dt = \frac{1}{1+g}, \qquad \text{for } g \geqslant 0.$$

Setting $g = x/(1-x)$ immediately yields the Arcsine law density function

$$f_\beta(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}, \quad 0 < x < 1. \qquad \square$$

Finally, we note that the number of guesses with success probability one, i.e., where the guesser knows in advance to be correct, can be treated in a similar way.

4 Summary and outlook

We obtained a weak limit law for the number X_n of correct guesses with full feedback in a once riffle-shuffled deck of n cards. Moreover, we also derived a limit law for the size of the deck P_n when a pure luck guess happens for the first time. Concerning generalizations, it seems difficult to generalize our results to more than one shuffles. A main difficulty is that the descriptions of the optimal strategy become rather involved. Also, another difficulty is that the analysis of the three-color card guessing game (or m-color guessing game) is very involved by itself. However, the authors are currently investigating into a once riffle shuffled deck with a so-called 3-shuffle [2, 6], where we initially split the deck into three packs, instead of only two. Again, it seems involved to describe the optimal strategy, but there is still a direct link to the three-color card guessing game. The authors plan to report on this research direction elsewhere.

Declarations of interest and Acknowledgments

The authors declare that they have no competing financial or personal interests that influenced the work reported in this paper. The authors thank the referee for the feedback on this work.

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