# Counting graph orientations with no directed triangles

Pedro Araújo[a] Fábio Botler[b] Guilherme Oliveira Mota[b]

Submitted: Apr 1, 2024; Accepted: Jan 18, 2025; Published: Jul 4, 2025



**Abstract**

Alon and Yuster proved that the number of orientations of any $n$-vertex graph in which every $K_3$ is transitively oriented is at most $2^{\lfloor n^2/4 \rfloor}$ for $n \geqslant 10^4$ and conjectured that the precise lower bound on $n$ should be $n \geqslant 8$. We confirm their conjecture and, additionally, characterize the extremal families by showing that the balanced complete bipartite graph with $n$ vertices is the only $n$-vertex graph for which there are exactly $2^{\lfloor n^2/4 \rfloor}$ such orientations.

**Mathematics Subject Classifications:** 05C35

## 1 Introduction

Given a graph $G$ and an oriented graph $\vec{H}$, we say that $\vec{G}$ is an $\vec{H}$*-free orientation* of $G$ if $\vec{G}$ contains no copy of $\vec{H}$. We denote by $\mathcal{D}(G, \vec{H})$ the family of $\vec{H}$-free orientations of $G$ and we write $D(G, \vec{H}) = |\mathcal{D}(G, \vec{H})|$. In 1974, Erdős [7] posed the problem of determining the maximum number of $\vec{H}$-free orientations of $G$, for every $n$-vertex graph $G$. Formally, we define $D(n, \vec{H}) = \max\{D(G, \vec{H}) : G \text{ is an } n\text{-vertex graph}\}$.

Since every orientation of an $H$-free graph does not contain any orientation $\vec{H}$ of $H$, it is fairly straightforward to see that $D(n, \vec{H}) \geqslant 2^{\mathrm{ex}(n,H)}$, where $\mathrm{ex}(n, H)$ is the maximum number of edges in an $H$-free graph on $n$ vertices. For a tournament $\vec{T}_k$ on $k$ vertices, Alon and Yuster [3] proved that $D(n, \vec{T}_k) = 2^{\mathrm{ex}(n,K_k)}$ for $n \geqslant n_0$ with a very large $n_0$, as they use the Regularity Lemma [9]. For tournaments with three vertices, they avoid using the regularity lemma to prove that $D(n, T_3) = 2^{\lfloor n^2/4 \rfloor}$ for $n \geqslant n_0$, where $n_0$ is slightly less than 10000. Furthermore, for the strongly connected triangle, denoted by $K_3^{\circlearrowright}$, using a computer program they verified that $D(8, K_3^{\circlearrowright}) = 2^{16}$ and $D(n, K_3^{\circlearrowright}) = n!$ for $n \leqslant 7$. In view of this, Alon and Yuster posed the following conjecture.

[a] Institute of Computer Science of the Czech Academy of Sciences, Pod Vodárenskou věží 2, 18207, Prague, Czech Republic (araujo@cs.cas.cz).
[b] Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil (fbotler@ime.usp.br, mota@ime.usp.br).

**Conjecture 1** (Alon and Yuster [3])**.** For $n \geqslant 1$, we have $D(n, K_3^{\circlearrowright}) = \max\{2^{\lfloor n^2/4 \rfloor}, n!\}$.

Using a simple exaustive testing, one can check that $K_{4,4}$ is the only 8-vertex graph that maximizes $D(8, K_3^{\circlearrowright})$. This fact together with the verification made by Alon and Yuster for graphs with at most seven vertices implies the following proposition.

**Proposition 2.** *$D(8, K_3^{\circlearrowright}) = 2^{16}$ and among all graphs with 8 vertices, $D(G, K_3^{\circlearrowright}) = 2^{16}$ if and only if $G \simeq K_{4,4}$. Furthermore, $D(n, K_3^{\circlearrowright}) = n!$ for $1 \leqslant n \leqslant 7$.*

In this paper we prove the following result that confirms Conjecture 1 and states that the balanced complete bipartite graph is the only $n$-vertex graph for which there are exactly $2^{\lfloor n^2/4 \rfloor}$ orientations with no copy of $K_3^{\circlearrowright}$.

**Theorem 3.** *For $n \geqslant 8$, we have $D(n, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$. Furthermore, among all graphs $G$ with $n$ vertices, $D(G, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$ if and only if $G \simeq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

**Overview of the paper.** Our proof is divided into two parts. Proposition 15 deals with graphs with at most 13 vertices, and its proof is given in the appendix (Section 5); and Theorem 3 deals with general graphs (Section 3). The proofs of these results are somehow similar and consist of an analysis of the size of a maximum clique of the given graph. In each step, we partition the vertices of a graph $G$ into a few parts and, using the results presented in Section 2, explore the orientations of the edges between these parts that lead to $K_3^{\circlearrowright}$-free orientations of $G$. Our proof is then reduced to solving a few equations which, in the case of the proof of Proposition 15, can be checked by straightforward computer programs. In Section 4 we present some open problems. The reader is referred to [4, 6] for standard terminology in this paper.

## 2 Extensions of $K_3^{\circlearrowright}$-free orientations

In this section we provide several bounds on the number of ways one can extend a $K_3^{\circlearrowright}$-free orientation of a subgraph of a graph $G$ to a $K_3^{\circlearrowright}$-free orientation of $G$.

Given subgraphs $G_1$ and $G_2$ of $G$, we write $G_1 \cup G_2$ for the subgraph of $G$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Let $\vec{G}_1$ and $\vec{G}_2$ be orientations, respectively, of $G_1$ and $G_2$ with the property that any edge of $E(G_1) \cap E(G_2)$ gets the same orientation in $\vec{G}_1$ and $\vec{G}_2$. We denote by $\vec{G}_1 \cup \vec{G}_2$ the orientation of $G_1 \cup G_2$ following the orientations $\vec{G}_1$ and $\vec{G}_2$.

Let $G$ be a graph and $S \subseteq E(G)$. For simplicity, we say that an orientation of the subgraph $G[S]$ of $G$ induced by the set of edges $S$ is an orientation of $S$. The next definition is a central concept of this paper.

**Definition 4** (Compatible orientations)**.** Given a graph $G$, disjoint sets $S, T \subseteq E(G)$ and orientations $\vec{S}$ of $S$ and $\vec{T}$ of $T$, we say that $\vec{S}$ and $\vec{T}$ are *compatible* if $\vec{S} \cup \vec{T}$ is $K_3^{\circlearrowright}$-free.

Given a graph $G$ and disjoint sets $A, B \subseteq V(G)$, denote by $E_G(A, B)$ the set of edges of $G$ between $A$ and $B$ and by $G[A, B]$ the spanning subgraph of $G$ induced by $E_G(A, B)$.

It is useful to have an upper bound on number of $K_3^{\circlearrowleft}$-free orientations of $E_G(A, B)$ that are compatible with a fixed orientation of $G[A] \cup G[B]$. This quantity is precisely the maximum number of ways one can *extend* a $K_3^{\circlearrowleft}$-free orientation of $G[A] \cup G[B]$ to a $K_3^{\circlearrowleft}$-free orientation of $G[A \cup B]$.

**Definition 5.** Given a graph $G$ and disjoint sets $A$, $B \subseteq V(G)$, let $T = G[A] \cup G[B]$. We define $\mathrm{ext}_G(A, B)$ as follows:

$$\mathrm{ext}_G(A, B) = \max_{\vec{T} \in \mathcal{D}(T, K_3^{\circlearrowleft})} |\{\vec{S} \in \mathcal{D}(G[A, B], K_3^{\circlearrowleft}) \colon \vec{S} \text{ and } \vec{T} \text{ are compatible}\}|.$$

For simplicity, when $A = \{u\}$, we write $\mathrm{ext}_G(u, B)$ instead of $\mathrm{ext}_G(\{u\}, B)$. In the rest of this section we give upper bounds for $\mathrm{ext}_G(A, B)$ for specific graphs $G$ and subgraphs $G[A]$ and $G[B]$. If $A$ induces a complete graph with $k$ vertices, then we remark that any $K_3^{\circlearrowleft}$-free orientation $\vec{S}$ of $G[A]$ is a transitive orientation, which thus induces a unique ordering $(v_1, \ldots, v_k)$ of the vertices of $A$, called *the transitive ordering of* $\vec{S}$, such that every edge $\{v_i, v_j\}$ $(1 \leqslant i < j \leqslant k)$ is oriented from $v_i$ to $v_j$ in $\vec{S}$.

Given a graph $G$, a vertex $v \in V(G)$ and a clique $W \subseteq V(G) \setminus \{v\}$, we denote by $d_G(v, W)$ the number of neighbors of $v$ in $W$. Consider a $K_3^{\circlearrowleft}$-free orientation $\vec{W}$ of $G[W]$ and note that if we have a transitive ordering $(w_1, \ldots, w_k)$ of $\vec{W}$, then there are exactly $d_G(v, W) + 1$ ways to extend this ordering to a transitive ordering of $v \cup W$, as it depends only on the position in which we place $v$ in $(w_1, \ldots, w_k)$ with respect to its neighbors in $W$ (there are $d_G(v, W) + 1$ such positions). We summarize this discussion in the following proposition.

**Proposition 6.** *Given a graph $G$, $v \in V(G)$ and $W \subseteq V(G) \setminus \{v\}$. If $G[W]$ is a complete graph, then* $\mathrm{ext}_G(v, W) = d_G(v, W) + 1$.

In the next two results, we give an upper bound for $\mathrm{ext}_G(A, B)$ when $A$ induces a complete graph and $B = \{u, v\}$ is an edge. We denote by $d_A(x)$ the neighborhood of $x$ in $A$ and $d_A(x, y)$ denotes the number of common neighbors of $x$ and $y$ in $A$.

**Lemma 7.** *Let $r \geqslant 3$ be an integer and let $G$ be a graph. If $A, B \subseteq V(G)$ induce disjoint cliques with $|A| = r$ and $B = \{u, v\}$ such that $d_A(u, v) \neq 0$, then*

$$\mathrm{ext}_G(A, B) \leqslant (d_A(u) + 1)(d_A(v) + 1) - \binom{d_A(u, v) + 1}{2}.$$

*Proof.* Let $\vec{A}$ and $\vec{B}$ be arbitrary $K_3^{\circlearrowleft}$-free orientations of $G[A]$ and $G[B]$ respectively. Suppose without loss of generality that $\vec{B}$ assigns the orientation of $\{u, v\}$ from $u$ to $v$ and consider the transitive ordering of $\vec{A}$. We estimate in how many ways one can include $u$ and $v$ in the ordering $(v_1, \ldots, v_r)$ while keeping it transitive. Since $\{u\} \cup N_A(u)$ and $\{v\} \cup N_A(v)$ are cliques, by Proposition 6 we have $\mathrm{ext}_G(u, A) \leqslant d_A(u) + 1$ and $\mathrm{ext}_G(v, A) \leqslant d_A(v) + 1$, which gives at most $(d_A(u) + 1)(d_A(v) + 1)$ positions to put the vertices $u$ and $v$ in the transitive ordering of $\vec{A}$. Note that there are $\binom{d_A(u,v)+1}{2}$ ways to place $\{u, v\}$ in the transitive ordering of $\vec{A}$ such that $u$ appears after $v$ and a common neighbor of $u$ and $v$ appears after $v$ and before $u$. But each such ordering induces a $K_3^{\circlearrowright}$. This finishes the proof. $\square$

The following corollary bounds the number of extensions $\text{ext}_G(A, B)$ when $A$ is a maximum clique of $G$ and $B = \{u, v\}$ is an edge.

**Corollary 8.** *Let $r \geqslant 2$ be an integer and let $G$ be a $K_{r+1}$-free graph. If $A, B \subseteq V(G)$ are disjoint cliques with $|A| = r$ and $B = \{x, y\}$, then*

$$\text{ext}_G(A, B) \leqslant r^2 - \binom{r-1}{2}.$$

*Proof.* Let $d_x = d_G(x, A)$ and $d_y = d_G(y, A)$, and put $d = d_x + d_y$. If $d \leqslant r$, then by applying Proposition 6 twice, with $x$ and $y$, we have

$$\text{ext}_G(A, B) \leqslant (d_x + 1)(d_y + 1) \leqslant \frac{d^2}{4} + d + 1 \leqslant \frac{r^2}{4} + r + 1 \leqslant r^2 - \binom{r-1}{2}.$$

Therefore, we assume that $d > r$. Note that since $G$ is $K_{r+1}$-free, we have $d_x, d_y \leqslant r - 1$. Applying Lemma 7 and using the fact that, for $d > r$, we have $d_A(x, y) \geqslant d - r$, we obtain

$$\text{ext}_G(A, B) \leqslant (d_x + 1)(d_y + 1) - \binom{d-r+1}{2} \leqslant \frac{d^2}{4} + d + 1 - \binom{d-r+1}{2}. \tag{1}$$

One can check that the right-hand side of (1) is a polynomial on $d$ of degree 2 with negative leading coefficient and it is a growing function in the interval $(-\infty, 2r + 1)$. Since $d \leqslant 2(r - 1)$, we have

$$\text{ext}_G(A, B) \leqslant (r-1)^2 + 2(r-1) + 1 - \binom{r-1}{2} = r^2 - \binom{r-1}{2}. \qquad \square$$

Given a graph $G$, an edge $e$, and an orientation $\vec{S}$ of $E(G) \setminus \{e\}$, we say that the orientation of $e$ is *forced* if there is only one orientation of $e$ compatible with $\vec{S}$. In the next two lemmas we provide bounds for the number of $K_3^{\circlearrowright}$-free orientations of $K_4$-free graphs. In what follows, denote by $K_4^-$ the graph obtained from $K_4$ by removing one edge.

**Lemma 9.** *Let $G$ be a $K_4$-free graph and let $A$, $B \subseteq V(G)$ be disjoint cliques of size 2. Then $\text{ext}_G(A, B) \leqslant 5$.*

*Proof.* First, note that if $e_G(A, B) \leqslant 2$, then the trivial bound $\text{ext}_G(A, B) \leqslant 2^{e(A,B)}$ implies $\text{ext}_G(A, B) \leqslant 4$. Also, since $G$ is $K_4$-free, we have $e_G(A, B) \leqslant 3$. Thus, we may assume that $e_G(A, B) = 3$, i.e., $G[A \cup B]$ is a $K_4^-$. Let $A = \{u_1, u_2\}$ and $B = \{v_1, v_2\}$ so that $u_2v_2$ is not an edge and consider an arbitrary orientation of $\{u_1, u_2\}$ and $\{v_1, v_2\}$.

If the oriented edges are $u_1u_2$ and $v_1v_2$ (or, by symmetry, $u_2u_1$ and $v_2v_1$), then for the two possible orientations of $\{u_1, v_1\}$, the orientation of one of the two remaining edges in

$E_G(A, B)$ is forced. Thus, since there is only one edge left to orient in $E_G(A, B)$, which can be done in two ways, we have $\text{ext}_G(A, B) \leqslant 4$.

It remains to consider the case where the oriented edges are $u_1u_2$ and $v_2v_1$ (or, by symmetry, $u_2u_1$ and $v_1v_2$). If $\{u_1, v_1\}$ is oriented from $v_1$ to $u_1$, then the orientation of the two remaining edges in $E_G(A, B)$ are forced, which gives us one $K_3^{\circlearrowright}$-free orientation. On the other hand, if $\{u_1, v_1\}$ is oriented from $u_1$ to $v_1$, then one can orient the both remaining edges in $E(A, B)$ in two ways, which in total gives that $\text{ext}_G(A, B) \leqslant 5$. □

**Lemma 10.** *Let $G$ be a $K_4$-free graph and let $u \in V(G)$ and $B \subseteq V(G) \setminus \{u\}$ with $|B| = 4$. If $G[B]$ induces a copy of $K_4^-$, then $\text{ext}_G(u, B) \leqslant 5$.*

*Proof.* Consider an arbitrary orientation of the edges of $G[B]$. We may assume that $d_B(u) \geqslant 3$, as otherwise we have $\text{ext}_G(u, B) \leqslant 4$. Since $G$ is $K_4$-free and $G[B]$ induces a copy of $K_4^-$, the vertex $u$ must have exactly three neighbors in $B$, which span an induced path $v_1v_2v_3$. By symmetry, we assume that $\{v_1, v_2\}$ is oriented from $v_1$ to $v_2$. If we orient $uv_1$ from $u$ to $v_1$, then the orientation of $\{u, v_2\}$ is forced, which leaves two possible orientations for the edge $\{u, v_3\}$. On the other hand, if we orient $uv_1$ from $u$ to $v_1$, we just apply Proposition 6 to conclude that $\text{ext}_G(u, \{v_2, v_3\}) \leqslant 3$. Combining the possible orientations, we obtain $\text{ext}_G(u, B) \leqslant 5$. □

We now provide an upper bound for $\text{ext}_G(A, B)$ (see Lemma 12 below) in a specific configuration of a $K_4$-free graph $G$, and subsets of vertices $A$ and $B$, which is proved using the following proposition.

**Proposition 11.** *Let $P$ be a path $abcde$, and let $T = \{ac, bd, ce\}$. Given an orientation $\vec{T}$ of $T$, there are at most eight orientations of $E(P)$ compatible with $\vec{T}$. Moreover, if the edges $\{a, c\}$ and $\{b, d\}$ are oriented, respectively, towards $a$ and $d$, then there are at most 7 such orientations.*

*Proof.* By Proposition 6, there are three orientations of $T_1 = G[\{a, b, c\}]$ (resp. $T_2 = G[\{c, d, e\}]$) compatible with $\vec{T}$, and hence there are at most nine orientations of $E(P)$ compatible with $\vec{T}$. In these orientations, each direction of $\{b, c\}$ and $\{c, d\}$ appears at least once. If $\{b, d\}$ is oriented towards $d$ (resp. towards $b$), then the orientations in which $\{b, c\}$ and $\{c, d\}$ are oriented, respectively, towards $b$ and $c$ (resp. $c$ and $d$) are not compatible with $\vec{T}$. Therefore, there are at most eight orientations of $E(P)$ compatible with $\vec{T}$. Now, suppose that $\{a, c\}$ and $\{b, d\}$ are oriented towards $a$ and $d$. If we orient $\{b, c\}$ towards $c$ (resp. $b$), then $\{a, b\}$ must be oriented towards $a$ (resp. $\{c, d\}$ must be oriented towards $d$), and there are three orientations of $E(T_2)$ (resp. four orientations of $\{\{a, b\}, \{d, e\}\}$) from which we can complete a compatible orientation of $E(P)$. Therefore, there are at most seven orientations of $E(P)$. □

**Lemma 12.** *Let $G$ be a $K_4$-free graph and let $A$, $B \subseteq V(G)$ be disjoint cliques of size 3. Then $\text{ext}_G(A, B) \leqslant 15$.*

*Proof.* Let $A = \{x_1, x_2, x_3\}$ and $B = \{y_1, y_2, y_3\}$. Since $G$ is $K_4$-free, $y_i$ cannot be adjacent to every vertex of $A$, for $i = 1, 2, 3$. This implies that $d_A(y_i) \leqslant 2$, for $i = 1, 2, 3$.

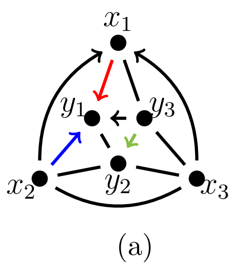


Figure 1: Compatible orientations between two cliques of size 3 in a $K_4$-free graph.

Analogously, $d_B(x_i) \leqslant 2$, for $i = 1, 2, 3$. Thus the set $E$ of edges in $G$ joining $A$ and $B$ induces a set of paths and cycles. Since $G$ is $K_4$-free, $E$ does contain a cycle of length 4. If $|E| \leqslant 3$, then $\mathrm{ext}_G(A, B) \leqslant 2^{|E|} \leqslant 8$, as desired. If $|E| = 4$, then some vertex, say $x_1 \in A$, is incident to two edges of $E$, say $\{x_1, y_1\}$ and $\{x_1, y_2\}$, which implies that $\mathrm{ext}_G(x_1, B) \leqslant 3$, and hence $\mathrm{ext}_G(A, B) \leqslant \mathrm{ext}_G(x_1, B) \cdot 2^{|E \setminus \{\{x_1,y_1\},\{x_1,y_2\}\}|} \leqslant 12$, as desired. If $|E| = 5$, then $E$ induces a path of length 5, say $x_1y_1x_2y_2x_3y_3$. In this case, $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are disjoint cliques of size 2, and hence, by Lemma 9, $\mathrm{ext}_G(\{x_1, x_2\}, \{y_1, y_2\}) \leqslant 5$. Since each edge in $E$ either joins $\{x_1, x_2\}$ to $\{y_1, y_2\}$, or is adjacent to $x_3$, we have $\mathrm{ext}_G(A, B) \leqslant \mathrm{ext}_G(\{x_1, x_2\}, \{y_1, y_2\}) \cdot \mathrm{ext}_G(x_3, B) \leqslant 15$, as desired.

Thus, we may assume $|E| = 6$, and hence $E$ induces the cycle $x_1y_1x_2y_2x_3y_3x_1$. By symmetry, we may assume that $\{x_1, x_2\}$, $\{x_1, x_3\}$ are both oriented towards $x_1$, and $\{y_1, y_3\}$ is oriented towards $y_1$. Suppose $\{y_2, y_3\}$ is oriented towards $y_2$. If we orient $\{x_1, y_1\}$ towards $y_1$, then $x_2y_1$ must be oriented towards $y_1$, and, since $\{x_1, x_3\}$ and $\{y_2, y_3\}$ are oriented towards $x_1$ and $y_2$, by Proposition 11, there are 7 compatible orientations of the edges in the path $x_2y_2x_3y_3x_1$ (see Figure 1a). If we orient $\{x_1, y_1\}$ towards $x_1$, then $\{y_3, x_1\}$ must be oriented towards $x_1$, and by Proposition 11, there are 8 compatible orientations of the edges in the path $y_1x_2y_2x_3y_3$ (see Figure 1b). Thus, there are 15 compatible orientations of $E$, as desired. Thus, we may assume that $\{y_2, y_3\}$ is oriented towards $y_3$, and hence $\{y_1, y_2\}$ must be oriented towards $y_1$. If we orient $\{x_1, y_1\}$ towards $y_1$, then $\{x_2, y_1\}$ must be oriented towards $y_1$, and by Proposition 11, there are 8 compatible orientations of the edges in the path $x_2y_2x_3y_3x_1$ (see Figure 1c). If we orient $\{x_1, y_1\}$ towards $x_1$, then $\{y_3, x_1\}$ must be oriented towards $x_1$, and, since $\{x_1, x_3\}$ and $\{x_1, x_2\}$ are oriented towards $x_1$, regardless of the orientation of $\{x_2, x_3\}$, by Proposition 11, there are 7 compatible orientations of the edges in the path $y_1x_2y_2x_3y_3$ (see Figure 1d) Thus, there are 15 compatible orientations of $E$, as desired. □

## 3 Proof of the main theorem

In this section we prove our main result, Theorem 3. In order to bound the number of $K_3^{\circlearrowright}$-free orientations of a graph $G$, we decompose it into disjoint cliques of different sizes and we use the results of Section 2 to bound the number of extensions of $K_3^{\circlearrowright}$-free orientations between those cliques. Before moving to the proof of the main theorem though, we need bounds on the number of $K_3^{\circlearrowright}$-free orientations of some small graphs. The first one concerns the complete tripartite graph $K_{1,\ell,\ell}$.

**Proposition 13.** *For any positive integer $\ell$, we have*

$$D(K_{1,\ell,\ell}, K_3^{\circlearrowright}) = \sum_{i=0}^{\ell}\sum_{j=0}^{\ell}\binom{\ell}{i}\binom{\ell}{j}2^{(\ell-i)j+(\ell-j)i}.$$

*Proof.* Let $K_{1,\ell,\ell}$ be a complete tripartite graph with vertex partition $\{v\} \cup A \cup B$. For $1 \leqslant i, j \leqslant \ell$ there are $\binom{\ell}{i}\binom{\ell}{j}$ orientations of the edges incident to $v$ with exactly $i$ out-neighbors of $v$ in $A$, and $j$ in-neighbors of $v$ in $B$, sets which we denote by $A^+$ and $B^-$ respectively. For each of those orientations, the edges between $A^+$ and $B^-$ and between $A \setminus A^+$ and $B \setminus B^-$ are forced in any $K_3^{\circlearrowright}$-free orientation. Since any of the other $(\ell - i)j + (\ell - j)i$ edges can be oriented in two ways, we sum over $i$ and $j$ to get

$$D(K_{1,\ell,\ell}, K_3^{\circlearrowright}) = \sum_{i=0}^{\ell}\sum_{j=0}^{\ell}\binom{\ell}{i}\binom{\ell}{j}2^{(\ell-i)j+(\ell-j)i}.$$

□

In the rest of the paper, we count the number of $K_3^{\circlearrowright}$-free orientations of a graph by decomposing its vertex set and we often use the following inequality without explicit reference. For a partition of the vertices of a graph $G$ into sets $A$ and $B$ we have, from the definition of $\mathrm{ext}_G(A, B)$, that

$$D(G, K_3^{\circlearrowright}) \leqslant D(G[A], K_3^{\circlearrowright}) \cdot \mathrm{ext}_G(A, B) \cdot D(G[B], K_3^{\circlearrowright}),$$

When $A$ is a clique, we define $m_{A,B} = \max\{|N(v, B)| + 1 : v \in A\}$ and use the bound

$$\mathrm{ext}_G(A, B) \leqslant (m_{A,B})^{|A|}.$$

In the proof of Theorem 3, we first show that $D(G, K_3^{\circlearrowright}) < 2^{\lfloor n^2/4 \rfloor}$ for every graph containing a $K_4$. For $K_4$-free graphs we may use Lemma 12 to bound the number of extensions between two triangles. But when considering graphs with no two disjoint triangles, we need the following result.

**Lemma 14.** *Let $H$ be a $K_4$-free graph with 7 vertices that contains a triangle $T$, a matching $\{e_1, e_2\}$ such that $e_1$ and $e_2$ are not incident to the vertices of $T$, and that does not contains two vertex-disjoint triangles. Then, $D(H, K_3^{\circlearrowright}) < 2^{12}$.*

*Proof.* Let $H$ be as in the statement. Recall that $D(T, K_3^{\circlearrowright}) = 6$ and $D(e_1, K_3^{\circlearrowright}) = D(e_2, K_3^{\circlearrowright}) = 2$. Moreover, $\mathrm{ext}_H(T, e_i) \leqslant 8$ for $i = 1, 2$, by Corollary 8. Also, since $H[e_1 \cup e_2]$ is triangle-free, $E_H(e_1, e_2) \leqslant 2$ and hence $\mathrm{ext}_H(e_1, e_2) \leqslant 4$. Throughout the proof, we use these bounds unless the structure of $H$ allows us to obtain a better bound.

First note that if there is at most one edge between $e_1$ and $e_2$, then $\mathrm{ext}_H(e_1, e_2) \leqslant 2$. In this case we use the bound

$$D(H, K_3^{\circlearrowright}) \leqslant D(T, K_3^{\circlearrowright}) \cdot D(e_1, K_3^{\circlearrowright}) \cdot D(e_2, K_3^{\circlearrowright}) \cdot \mathrm{ext}_H(T, e_1) \cdot \mathrm{ext}_H(T, e_2) \cdot \mathrm{ext}_H(e_1, e_2),$$

to obtain $D(H, K_3^{\circlearrowright}) \leqslant 6 \cdot 2 \cdot 2 \cdot 8 \cdot 8 \cdot 2 < 2^{12}$, which allows us to restrict to graphs $H$ such that $H[e_1 \cup e_2] \simeq K_{2,2}$.

We count the number of orientations by considering different values of $E_H(e_1 \cup e_2, V(T))$. In particular, since $H$ is $K_4$-free, we have that $E_H(e_i, V(T)) \leqslant 4$ for $i = 1, 2$. First note that if $E(e_i, V(T)) = 3$ for $i = 1, 2$, then $\mathrm{ext}_H(e_i, T) \leqslant 6$. Therefore, if there are at most six edges between $H[e_1 \cup e_2]$ and $T$, then either there are at most three edges between each $e_i$ and $T$, which implies $\mathrm{ext}_G(T, e_1), \mathrm{ext}_G(T, e_2) \leqslant 6$; or, without loss of generality, there are at most two edges between $e_1$ and $T$, which implies $\mathrm{ext}_G(T, e_1) \leqslant 4$. In both cases we have that $\mathrm{ext}_G(T, e_1) \cdot \mathrm{ext}_G(T, e_2) \leqslant 36$ and consequently that $D(H, K_3^{\circlearrowright}) \leqslant 6 \cdot 2 \cdot 2 \cdot 36 \cdot 4 < 2^{12}$.

Thus, we assume that $7 \leqslant E_H(e_1 \cup e_2, V(T)) \leqslant 8$. Then, without loss of generality, we have $E_H(e_1, V(T)) = 4$ and, by Turán's Theorem, $H_1 = H[e_1 \cup V(T)]$ is isomorphic to $K_{1,2,2}$. If $E_H(e_2, V(T)) = 3$, the aforementioned bounds and Lemma 13 yields

$$D(H, K_3^{\circlearrowright}) \leqslant D(H_1, K_3^{\circlearrowright}) \cdot \mathrm{ext}_H(e_2, T) \cdot \mathrm{ext}(e_1, e_2) \leqslant 82 \cdot 8 \cdot 4 < 2^{12}.$$

Finally, if $E_H(e_i, V(T)) = 4$ for both $i = 1, 2$, then the graphs $H_i = H[e_i \cup V(T)]$ are isomorphic to $K_{1,2,2}$ with $v_i \in V(T)$ being the vertex of degree 4 in $H_i$. Since $H$ does not contain two disjoint triangles, then $v_1 = v_2$ and since $H[e_1 \cup e_2] \simeq K_{2,2}$, we have in fact $H \simeq K_{1,3,3}$. Finally, Lemma 13 yields $D(H, K_3^{\circlearrowright}) = 2754 < 2^{12}$. □

In the remainder of this section we prove Theorem 3, which follows by induction on the number of vertices. Unfortunately, we need a slightly stronger base of induction than the one given by Proposition 2, which is the content of the next proposition. We present its proof in the Appendix (Section 5). Recall that the *clique number* of a graph $G$, denoted by $\omega(G)$, is the size of a clique in $G$ with a maximum number of vertices.

**Proposition 15.** *Let $G$ be an $n$-vertex graph. If $9 \leqslant n \leqslant 7 + \min\{\omega(G), 8\}$, then $D(G, K_3^{\circlearrowright}) \leqslant 2^{\lfloor n^2/4 \rfloor}$. Furthermore, $D(G, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$ if and only if $G \simeq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

We are now ready to prove Theorem 3, which is rewritten as follows:

*Theorem* (Theorem 3). Let $G$ be an $n$-vertex graph. If $n \geqslant 8$, then $D(G, K_3^{\circlearrowright}) \leqslant 2^{\lfloor n^2/4 \rfloor}$. Furthermore, $D(G, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$ if and only if $G \simeq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

*Proof.* Let $r = \omega(G)$. The proof follows by induction on $n$. By Proposition 2, the statement holds for $n = 8$. If $9 \leqslant n \leqslant 10$, then the result follows from Mantel's Theorem (see [8]) for $r = 2$, and from Proposition 15 for $r \geqslant 3$, as $n \leqslant 7 + \min\{r, 8\}$. Thus, assume $n \geqslant 11$ and suppose that the statement holds for any graph with less than $n$ vertices (but at least 8 vertices).

Let $K$ be a clique of $G$ of size $s = \min\{r, 8\}$. If $n \leqslant 7 + s$, then the result follows from Proposition 15, so we may assume that $n - s \geqslant 8$. Thus, we can apply the induction hypothesis for any subgraph of $G$ with at least $n - s$ vertices.

If $r \geqslant 8$, then we have $s = 8$. By Proposition 2, we have $D(K, K_3^{\circlearrowright}) \leqslant 2^{16}$ and, by Proposition 6, for each vertex $v \in V(G-K)$ we have $\mathrm{ext}_G(v, K) \leqslant 9$. Therefore, applying the induction hypothesis to $G - K$ we have

$$\begin{aligned} D(G, K_3^{\circlearrowright}) &\leqslant D(K, K_3^{\circlearrowright}) \cdot \mathrm{ext}_G(G-K, K) \cdot D(G-K, K_3^{\circlearrowright}) \\ &\leqslant 2^{16} \cdot 9^{n-8} \cdot 2^{(n-8)^2/4} < 2^{\lfloor n^2/4 \rfloor}, \end{aligned} \tag{2}$$

where we used that $n - 8 \geqslant 1$. From now on we assume that $r \leqslant 7$ and consequently that $s = r$. Due to the different structure of the graphs with small clique numbers, we divide the rest of the proof according to the value of $r$.

**Case $\mathbf{r \in \{5, 6, 7\}}$.** Let $G' = G - K$. Since $G$ is $K_{r+1}$-free, every vertex $v$ of $V(G')$ is adjacent to at most $r - 1$ vertices of $K$. Then, by Proposition 6, we have $\mathrm{ext}_G(v, K) \leqslant r$ for every $v \in V(G')$. Therefore, the following holds for $r \in \{5, 6, 7\}$ and $n \geqslant 9$.

$$\begin{aligned} D(G, K_3^{\circlearrowright}) &\leqslant D(K, K_3^{\circlearrowright}) \cdot \mathrm{ext}_G(G', K) \cdot D(G', K_3^{\circlearrowright}) \\ &\leqslant r! \cdot r^{n-r} \cdot D(G', K_3^{\circlearrowright}) \\ &\leqslant r! \cdot 2^{(n-r)\log_2 r} \cdot 2^{(n-r)^2/4} \\ &< 2^{\frac{r^2+2r(n-r)-1}{4} + \frac{(n-r)^2}{4}} \leqslant 2^{\lfloor n^2/4 \rfloor}. \end{aligned} \tag{3}$$

**Case $\mathbf{r = 4}$.** Let $G' = G - K$. By the induction hypothesis, for any $u \in V(G)$, we have $D(G-u, K_3^{\circlearrowright}) \leqslant 2^{\lfloor (n-1)^2/4 \rfloor}$. If $G$ contains a vertex $u$ for which $d(u) < (n-1)/2$, then

$$D(G, K_3^{\circlearrowright}) < D(G-u, K_3^{\circlearrowright}) \cdot 2^{d(u)} \leqslant 2^{\lfloor n^2/4 \rfloor}. \tag{4}$$

Thus, we may assume that $\delta(G) \geqslant (n-1)/2$. Since $n \geqslant 11$, we have $\delta(G) \geqslant 5$ and since $G$ is $K_5$-free, each vertex in $V(G')$ contains at most 3 neighbors in $K$. Hence, we have $\delta(G') \geqslant 2$. Therefore, since $|V(G')| \geqslant 7$, there is a matching with at least two edges in $G'$.

Let $y \geqslant 2$ be the size of a maximum matching $M$. By Lemma 7, we have $\mathrm{ext}_G(e, K) \leqslant 13$, for every $e \in E(G')$. Moreover, since every vertex in $V(G')$ has at most 3 neighbors in $K$, by Proposition 6, we have $\mathrm{ext}_G(v, K) \leqslant 4$ for every $v \in V(G') \setminus V(M)$. Therefore, as $3 \cdot (13/16)^2 \leqslant 2^{3/4}$, we have

$$\begin{aligned} D(G, K_3^{\circlearrowright}) &\leqslant D(K, K_3^{\circlearrowright}) \cdot \mathrm{ext}_G(V(M), K) \cdot \mathrm{ext}_G(V(G') \setminus V(M), K) \cdot D(G', K_3^{\circlearrowright}) \\ &\leqslant 4! \cdot 13^y \cdot 4^{n-4-2y} \cdot 2^{\lfloor (n-4)^2/4 \rfloor} \\ &\leqslant 3 \cdot \left(\frac{13}{16}\right)^y \cdot 2^3 \cdot 2^{2(n-4)} \cdot 2^{(n-4)^2/4} < 2^{\lfloor n^2/4 \rfloor}. \end{aligned} \tag{5}$$

**Case $\mathbf{r = 3}$.** Let $\mathcal{T}$ be a maximum collection of vertex-disjoint triangles of $G$. Set $G' = G - \cup_{T \in \mathcal{T}} V(T)$, let $M$ be a maximum matching in $G'$, and let $Z = V(G') \setminus V(M)$.

Clearly, $G'$ is a $K_3$-free graph and $Z$ is an independent set. Set $x = |\mathcal{T}|$, $y = |M|$ and $z = |Z|$ and note that $n = 3x + 2y + z$.

By Lemma 12, we have $\text{ext}_G(T_1, T_2) \leqslant 15$ for every $T_1, T_2 \in \mathcal{T}$ and by Lemma 7 we have $\text{ext}_G(\{u, v\}, T) \leqslant 8$ for every $\{u, v\} \in M$ and every $T \in \mathcal{T}$. Moreover, since $G$ is $K_4$-free, by Proposition 6, we have $\text{ext}_G(v, T) \leqslant 3$ for every $v \in Z$ and every $T \in \mathcal{T}$. Since $G'$ is $K_3$-free, no vertex in $Z$ is adjacent to two vertices of the same edge in $M$, and hence $\text{ext}_G(u, \{v, w\}) \leqslant 2$ for every $u \in Z$ and $\{u, v\} \in M$. Finally, note that $D(T, K_3^{\circlearrowright}) \leqslant 6$ for every $T \in \mathcal{T}$, and since $G'$ is $K_3$-free, we have $D(G[M], K_3^{\circlearrowright}) \leqslant 2^{(2y)^2/4} = 2^{y^2}$. Therefore, we have $D(G, K_3^{\circlearrowright}) \leqslant 6^x \cdot 15^{\binom{x}{2}} \cdot 8^{xy} \cdot 2^{y^2} \cdot 3^{xz} \cdot 2^{yz} = f(x, y, z)$.

**Claim 16.** *$f(x, y, z) < 2^{\lfloor n^2/4 \rfloor}$ when (i) $x \geqslant 3$ or (ii) $z \geqslant 2$.*

*Proof.* Since $n = 3x + 2y + z$, we have that

$$\frac{n^2 - 1}{4} = \frac{9x^2}{4} + 3xy + y^2 + \frac{3}{2}xz + yz + -\frac{z^2 - 1}{4}.$$

We are left to prove that $x \log_2 6 + \binom{x}{2} \log_2 15 + xz \log_2 3 \leqslant 9x^2/4 + 3xz/2 - (z^2 - 1)/4$. By using the bounds $\log_2 15 \leqslant 3.95$, $\log_2 6 \leqslant 2.6$ and $\log_2 3 \leqslant 1.6$ and multiplying the previous equation by 4, we are left with the following inequality:

$$1.1x^2 - 2.5x - 0.4xz + z^2 - 1 > 0. \tag{6}$$

Note that $z^2 - 0.4xz \geqslant -0.04x^2$ and, moreover, that $x^2 - 2.5x - 1 > 0$ for every $x \geqslant 3$. Finally, we are left with the case $x \in \{1, 2\}$ and $z \geqslant 2$, which can be done by replacing each value of $x$ in (6) and using that $z \geqslant 2$. □

Therefore, we may assume $x \leqslant 2$ and $z \leqslant 1$. In this case, we need to explore the structure of the graph $G$ carefully. Recall that $y = |M|$ and $z = |Z|$, where $M$ is a maximum matching of $G'$ and $Z = V(G') \setminus V(M)$. Since $n \geqslant 11$, we have $M \neq \emptyset$.

Suppose first that $x = 2$ and let $T_1$ and $T_2$ be the triangles in $\mathcal{T}$. Let $e$ be an edge of $M$ and $H = G[V(T_1) \cup V(T_2) \cup e \cup Z]$. Since $|V(H)| \in \{8, 9\}$ and $H$ is not a balanced complete bipartite graph, by Proposition 15, we have $D(H, K_3^{\circlearrowright}) < 2^{16+4z}$. By Lemmas 6 and 7 and the fact that $G'$ is $K_3$-free, we have for every $e' \in M \setminus \{e\}$, that $\text{ext}_G(e', Z) \leqslant 2^z$, $\text{ext}_G(e', e) \leqslant 4$ and $\text{ext}_G(e', T_i) \leqslant 8$ for $i = 1, 2$. We conclude that $\text{ext}_G(e', H) \leqslant 2^z \cdot 4 \cdot 8 \cdot 8 = 2^{8+z}$ for every $e' \in M \setminus \{e\}$. Finally, we have $\text{ext}_G(e', e'') \leqslant 4$ for every two edges $e'$ and $e''$ of $M$, and there are 2 ways to orient each one of the $y - 1$ edges of $M \setminus \{e\}$. Therefore,

$$D(G, K_3^{\circlearrowright}) < 2^{16+4z} \cdot 2^{(8+z)(y-1)} \cdot 4^{\binom{y-1}{2}} \cdot 2^{y-1} = 2^{((6+2y+z)^2 - z)/4} = 2^{\lfloor n^2/4 \rfloor}, \tag{7}$$

where we used that $z^2 = z$.

Thus, we may assume that $x = 1$. Let $T$ be the triangle in $\mathcal{T}$. Since $n \geqslant 11$, we have $y \geqslant 2$. Let $e_1$ and $e_2$ be edges of $M$ and put $H = G[V(T) \cup e_1 \cup e_2]$. By Lemma 14, we have $D(H, K_3^{\circlearrowright}) < 2^{12}$. For every $e \in M \setminus \{e_1, e_2\}$ we have $\text{ext}_G(e, e_1) \leqslant 4$ and $\text{ext}_G(e, e_2) \leqslant 4$, and, by Lemma 7, we have $\text{ext}_G(e, T) \leqslant 8$, and hence $\text{ext}_G(e, H) \leqslant \text{ext}_G(e, K) \cdot \text{ext}_G(e, e_1) \cdot$

$\text{ext}_G(e, e_2) = 128$. Also, by Proposition 6, for every vertex $u \notin V(T) \cup V(M)$ we have $\text{ext}_G(u, T) \leqslant 3$, and since $G'$ is $K_3$-free, $\text{ext}_G(u, e) \leqslant 2$ for every $e \in M$. Therefore,

$$D(G, K_3^{\circlearrowright}) < 2^{12} \cdot 128^{y-2} \cdot 2^{(2y-4)^2/4} \cdot (3 \cdot 2^y)^z \leqslant 2^{((3+2y+z)^2-1)/4} < 2^{\lfloor n^2/4 \rfloor}. \tag{8}$$

**Case r = 2.** Since $G$ is triangle-free, we have $D(G, K_3^{\circlearrowright}) = 2^{|E(G)|}$. Thus, by Mantel's Theorem, if $G$ is not isomorphic to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, we have

$$D(G, K_3^{\circlearrowright}) < 2^{\lfloor n^2/4 \rfloor}. \tag{9}$$

Furthermore, $D(G, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$ if and only if $G \simeq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. This completes the proof that for any $n$-vertex graph $G$ with $n \geqslant 8$, we have $D(G, K_3^{\circlearrowright}) \leqslant 2^{\lfloor n^2/4 \rfloor}$. Since inequalities (2)–(9) are strict, we get that $D(G, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$ if and only if $G \simeq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, which concludes the proof of the theorem. □

# 4 Open problems

In this section we discuss some open problems and directions for future research. Given an oriented graph $\vec{H}$, recall that $D(n, \vec{H})$ denotes the maximum number of $\vec{H}$-free orientations of $G$, for all $n$-vertex graphs $G$.

## 4.1 Avoiding an oriented graph

In this paper we determine $D(n, K_3^{\circlearrowright})$ for every possible $n$. A natural problem is to extend our result to estimate the number of orientations of graphs avoiding strongly connected cycles $C_k^{\circlearrowright}$ for $k \geqslant 4$. As far as we know, the following problem is open even for large $n$.

**Problem 17.** Let $k \geqslant 4$. Determine $D(n, C_k^{\circlearrowright})$ for every $n \geqslant 1$.

An interesting problem is to determine $D(n, \vec{H})$ for any oriented graph $\vec{H}$. For a tournament $\vec{T}_k$ on $k$ vertices, $D(n, \vec{T}_k)$ was determined for sufficiently large $n$ by Alon and Yuster [3]. For a moment, we consider edge colorings of graphs. Denote by $F(n, k)$ the maximum number of 2-edge colorings of a graph $G$ with no monochromatic $K_k$, among all graphs $G$ on $n$ vertices. The following result was proved by Yuster [10] (for $k = 3$) and Alon, Balogh, Keevash and Sudakov [2] (for $k \geqslant 4$).

**Lemma 18.** *For any $k \geqslant 3$, there is $n_0$ such that for all $n \geqslant n_0$ we have $F(n, k) = 2^{\lfloor n^2/4 \rfloor}$.*

Consider now the transitively oriented tournament $K_k^{\rightarrow}$ with $k$ vertices. Using a simple argument, Alon and Yuster [3] used Lemma 18 to prove that $D(n, K_3^{\rightarrow}) = 2^{\lfloor n^2/4 \rfloor}$ for $n \geqslant 1$. For $k \geqslant 4$, they proved that $D(n, K_k^{\rightarrow}) = 2^{\lfloor n^2/4 \rfloor}$ for a (very) large $n$. Thus, the following problem remains open.

**Problem 19.** Let $k \geqslant 4$. Determine $D(n, K_k^{\rightarrow})$ for every $n \geqslant 1$.

### 4.2 Avoiding families of oriented graphs

Another direction of research arises when, instead of forbidding a fixed oriented graph, we forbid families of oriented graphs. For example, one may consider orientations of graphs that avoid non-transitive tournaments. Denote by $T_k(n)$ the maximum number of orientations of a graph $G$ in which *every* copy of $K_k$ is transitively oriented, for every $n$-vertex graph $G$. The following problem generalizes Theorem 3.

**Problem 20.** Let $k \geqslant 4$. Determine $T_k(n)$ for every $n \geqslant 1$.

Consider the number of orientations of graphs that avoids strongly connected tournaments. We denote by $S_k(n)$ the maximum number of orientations of a graph $G$ in which *no copy* of $K_k$ is strongly connected, for every $n$-vertex graph $G$.

**Problem 21.** Let $k \geqslant 4$. Determine $S_k(n)$ for every $n \geqslant 1$.

Note that Problem 21 also generalizes Theorem 3. For related problems in the context of random graphs, the reader is referred to [1, 5].

## 5 Appendix

Here we prove Proposition 15, which states that for an $n$-vertex graph $G$ with $9 \leqslant n \leqslant 7 + \min\{\omega(G), 8\}$ we have $D(G, K_3^{\circlearrowright}) \leqslant 2^{\lfloor n^2/4 \rfloor}$ and, furthermore, $D(G, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$ if and only if $G \simeq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Similarly to the proof of Theorem 3, we explore the structure of the graph $G$ depending on the size of its maximum clique. By Mantel's Theorem, we have $D(G, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$ when $G$ is the balanced complete bipartite graph. We show that if this is not the case, then $D(G, K_3^{\circlearrowright}) < 2^{\lfloor n^2/4 \rfloor}$. To show that this holds we use straightforward computer methods to check some inequalities, namely, inequalities (10)–(19).

*Proof of Proposition 15.* Let $G$ be an $n$-vertex graph and for simplicity put $r = \omega(G)$. Suppose $9 \leqslant n \leqslant 7 + \min\{r, 8\}$ and let $W$ be a clique of size $|W| = \min\{r, 8\}$ in $G$. Put $G' = G \setminus W$. Note that if $|W| = 8$, then Proposition 2 implies $D(G', K_3^{\circlearrowright}) \leqslant (n-8)!$ and $D(G[W], K_3^{\circlearrowright}) \leqslant 2^{16}$, and Proposition 6 implies $\text{ext}_G(v, W) \leqslant 9$ for every $v \in V(G')$. Therefore, for every $9 \leqslant n \leqslant 15 = 7 + \min\{r, 8\}$ we have

$$D(G, K_3^{\circlearrowright}) \leqslant (n-8)! \cdot 9^{n-8} \cdot 2^{16} < 2^{\lfloor n^2/4 \rfloor}. \tag{10}$$

From now on we assume that $|W| \leqslant 7$, which implies $|W| = r$ and, from Proposition (2), we have $D(G', K_3^{\circlearrowright}) \leqslant (n-r)!$ and $D(G[W], K_3^{\circlearrowright}) \leqslant r!$. Note that since $G$ has no clique of size $r+1$, for each $v \in V(G')$ we have $d_G(v, W) \leqslant r - 1$, which implies from Proposition 6 that $\text{ext}_G(v, W) \leqslant r$ for every $v \in V(G')$. Combining these facts, for $r \in \{6, 7\}$ and $9 \leqslant n \leqslant 7 + r$ we have

$$D(G, K_3^{\circlearrowright}) \leqslant (n-r)! \cdot r^{n-r} \cdot r! < 2^{\lfloor n^2/4 \rfloor}. \tag{11}$$

Therefore, we may assume that $r \leqslant 5$. Due to the different structure of the graphs with small clique numbers, we divide the rest of the proof according to the value of $r$.

**Case $\mathbf{r = 5}$.** Let $M$ be a maximum matching of $G'$, say with $x$ edges ($0 \leqslant x \leqslant \lfloor (n-5)/2 \rfloor$), and note that $G'' = G'[V(G') \setminus V(M)]$ is an independent set with $n - 5 - 2x$ vertices. By Corollary 8, we have $\text{ext}_G(e, W) \leqslant 19$ for every $e \in M$. Therefore, for $9 \leqslant n \leqslant 12$ and $2 \leqslant x \leqslant \lfloor (n-5)/2 \rfloor$, we have

$$D(G, K_3^{\circlearrowright}) \leqslant (n-5)! \cdot 19^x \cdot 5^{n-5-2x} \cdot 5! < 2^{\lfloor n^2/4 \rfloor}. \tag{12}$$

Thus, we may assume that $x \leqslant 1$. This implies $G'$ is a star with at most $n-6$ edges or $G'$ is composed of one triangle and $n-8$ isolated vertices. Hence, $D(G', K_3^{\circlearrowright}) \leqslant 2^{n-6}$. Therefore, for $9 \leqslant n \leqslant 12$ and $0 \leqslant x \leqslant 1$, we have

$$D(G, K_3^{\circlearrowright}) \leqslant 5! \cdot 19^x \cdot 5^{n-5-2x} \cdot 2^{n-6} < 2^{\lfloor n^2/4 \rfloor}. \tag{13}$$

**Case $\mathbf{r = 4}$.** First, suppose that $G'$ contains a clique $K$ with 4 vertices. Let $G'' = G'[V(G') \setminus K]$ (note that $G''$ has $n-8$ vertices) and let $x$ be the number of edges in a maximum matching of $G''$ ($0 \leqslant x \leqslant 1$). From Proposition 2 we have $D(G[W \cup K], K_3^{\circlearrowright}) <$

$2^{16}$ and from Proposition 6, since $G$ has no $K_5$, for every $v \in V(G'')$ we have $\text{ext}_G(v, K) \leqslant 4$ and $\text{ext}_G(v, W) \leqslant 4$. Furthermore, by Corollary 8, for any edge $\{u, v\}$ of $G''$, we have $\text{ext}_G(\{u, v\}, K) \leqslant 13$ and $\text{ext}_G(\{u, v\}, W) \leqslant 13$. Therefore, for $9 \leqslant n \leqslant 11$ and $0 \leqslant x \leqslant 1$ we have

$$D(G, K_3^{\circlearrowright}) < (n-8)! \cdot 13^{2x} \cdot 4^{2(n-8-2x)} \cdot 2^{16} < 2^{\lfloor n^2/4 \rfloor}. \tag{14}$$

Thus we may assume that $G'$ contains no copy of $K_4$. This allows us to use Lemma 12. Suppose that $G'$ contains two vertex-disjoint triangles. In this case, we have $n \geqslant 10$. Let $V_1$ and $V_2$ be the vertex sets of these triangles, say $V_2 = \{u, v, w\}$, and note that, since $n \leqslant 11$, there is one vertex that do not belong to $V_1 \cup V_2$ in $G'$ if and only if $n = 11$. If $n = 11$, let $z$ be this vertex. In this case, from Proposition 6, we have $\text{ext}_G(z, V_1 \cup V_2 \cup W) \leqslant 3 \cdot 3 \cdot 4 = 36$. Since $\text{ext}_G(\{u, v\}, W) \leqslant 13$ and $\text{ext}_G(w, W) \leqslant 4$, we obtain that $\text{ext}_G(V_2, W) \leqslant 52$. Note that $D(G[W \cup V_1], K_3^{\circlearrowright}) \leqslant 7!$ and $D(G[V_2], K_3^{\circlearrowright}) \leqslant 6$ and, from Lemma 12 we obtain $\text{ext}_G(V_1, V_2) \leqslant 15$. Combining the above facts, we have

$$D(G, K_3^{\circlearrowright}) \leqslant 6 \cdot 15 \cdot 52 \cdot 7! < 2^{\lfloor n^2/4 \rfloor}, \qquad \text{for } n = 10; \tag{15}$$

$$D(G, K_3^{\circlearrowright}) \leqslant 6 \cdot 36 \cdot 15 \cdot 52 \cdot 7! < 2^{\lfloor n^2/4 \rfloor}, \qquad \text{for } n = 11. \tag{16}$$

Thus, we may assume that $G'$ contains no two vertex-disjoint triangles. If $G'$ contains a triangle $K$, then let $G'' = G'[V(G') \setminus K]$ (note that $G''$ has $n-7$ vertices) and let $x$ be the number of edges in a maximum matching of $G''$ $(0 \leqslant x \leqslant \lfloor (n-7)/2) \rfloor)$. Therefore, for $9 \leqslant n \leqslant 11$ we have

$$D(G, K_3^{\circlearrowright}) \leqslant (2^x \cdot 4^{\binom{x}{2}}) \cdot 13^x \cdot 8^x \cdot 3^{n-7-2x} \cdot 4^{n-7-2x} \cdot 2^{x(n-7-2x)} \cdot 7! < 2^{\lfloor n^2/4 \rfloor}. \tag{17}$$

Finally, assume that $G'$ contains no triangles. Then, similarly as before, letting $x$ be the number of edges in a maximum matching of $G'$ $(0 \leqslant x \leqslant \lfloor (n-4)/2) \rfloor)$, for $9 \leqslant n \leqslant 11$ we have

$$D(G, K_3^{\circlearrowright}) \leqslant (2^x \cdot 4^{\binom{x}{2}}) \cdot 13^x \cdot 4^{n-4-2x} \cdot 2^{x(n-4-2x)} \cdot 4! < 2^{\lfloor n^2/4 \rfloor}. \tag{18}$$

**Case $\mathbf{r = 3}$.** In this case the graph $G$ has $9 \leqslant n \leqslant 10$ vertices. We start by noticing that if $G$ contains three vertex-disjoint triangles, then there are six possible orientations of the edges of each triangle and, by Lemma 12, there are at most fifteen ways to orient the edges between the triangles. Let $y$ be the number of vertices that are not in these triangles. Note that $0 \leqslant y \leqslant 1$ and $y = 1$ if and only if $n = 10$. Since $G$ is $K_4$-free, in case $y = 1$, Proposition 6 implies that there are 3 ways to orient the edges between the vertex outside the triangles and each of the triangles. Therefore, for $9 \leqslant n \leqslant 10$ we have

$$D(G, K_3^{\circlearrowright}) \leqslant 6^3 \cdot 15^3 \cdot 3^{3y} < 2^{\lfloor n^2/4 \rfloor}. \tag{19}$$

From the above discussion, we may assume that $G$ contains at most two vertex-disjoint triangles. For the rest of the proof we have to analyze the structure of $G$ carefully. Thus we consider two possible cases, depending on the number of vertices of $G$.

**Subcase $\mathbf{n = 9}$.** First suppose that $\delta(G) \leqslant 4$. Let $u$ be a vertex of minimum degree and note that if $u$ is contained in a triangle, then $\text{ext}_G(u, G-u) \leqslant 3 \cdot 2^2 < 2^4$ and by

Proposition 2, we have $D(G-u, K_3^{\circlearrowleft}) \leqslant 2^{16}$. In case no triangle contains $u$, Proposition 2 gives $D(G-u, K_3^{\circlearrowleft}) < 2^{16}$ and $\text{ext}_G(u, G-u) \leqslant 2^4$. Therefore, we obtain

$$D(G, K_3^{\circlearrowleft}) \leqslant D(G-u, K_3^{\circlearrowleft}) \cdot \text{ext}_G(u, G-u) < 2^{20} = 2^{\lfloor n^2/4 \rfloor}. \tag{20}$$

Thus we may assume $\delta(G) \geqslant 5$. Suppose that $G$ contains two vertex-disjoint triangles with vertex sets $V_1$ and $V_2$ (recall that $G$ contains at most two vertex-disjoint triangles). Let $G'$ be the subgraph of $G$ induced by the vertices that are not in $V_1$ or $V_2$. Thus, since $G$ is $K_4$-free, each vertex of $G'$ has at most two neighbors in each of $V_1$ and $V_2$. Since $\delta(G) \geqslant 5$ and $G'$ is triangle-free, $G'$ is an induced path of length 2, say $uvw$. Moreover, each of the vertices $u$ and $w$ has two neighbors in $V_1$ and also in $V_2$. The vertex $v$ has two neighbors in one of the triangles, say in the set $V_1$. Since $G$ is $K_4$-free, $u$ and $v$ have only one common neighbor in $V_1$, which implies that the subgraph $H$ of $G$ induced by the vertices $V_1 \cup \{u, v\}$ is a $K_{1,2,2}$. Thus, by Proposition 13, we have $D(H, K_3^{\circlearrowleft}) \leqslant 82$. Also, $H' = G[V_2 \cup \{w\}]$ is a copy of $K_4^-$, and hence $D(H', K_3^{\circlearrowleft}) \leqslant 6 \cdot 3 = 18$. Finally, applying Lemmas 6, 10 and 12, we obtain $\text{ext}_G(u, V_2) \leqslant 3$, $\text{ext}_G(w, V_1) \leqslant 3$, $\text{ext}_G(V_1, V_2) \leqslant 15$, $\text{ext}_G(v, V(H')) \leqslant 5$, and hence

$$D(G, K_3^{\circlearrowleft}) \leqslant 82 \cdot 18 \cdot 15 \cdot 3 \cdot 3 \cdot 5 < 2^{20} = 2^{\lfloor n^2/4 \rfloor}. \tag{21}$$

Assume that $G$ contains one triangle with vertex set $V_1 = \{u_1, u_2, u_3\}$, but does not contain two vertex-disjoint triangles. Let $G'$ be the subgraph of $G$ induced by the vertices that are not in $V_1$. Since $\delta(G) \geqslant 5$ and no vertex in $G'$ is adjacent to more than two vertices in $V_1$, we have $\delta(G') \geqslant 3$. Thus, by Mantel's Theorem $G'$ is isomorphic to $K_{3,3}$. It is not hard to show that, since $G$ is $K_4$-free and $\delta(G) \geqslant 5$, the graph $G$ is isomorphic to the graph $K_{1,4,4}$. Therefore, by Proposition 13, we have

$$D(G, K_3^{\circlearrowleft}) = 271614 < 2^{20} = 2^{\lfloor n^2/4 \rfloor}. \tag{22}$$

**Subcase $\mathbf{n = 10}$.** We proceed similarly as in the case above. First suppose that $\delta(G) \leqslant 5$ and let $u$ be a vertex of minimum degree. If $u$ is contained in a triangle, then the previous subcase for graphs with 9 vertices gives $\text{ext}_G(u, G-u) \leqslant 3 \cdot 2^3 < 2^5$ and also by the previous case (or Mantel's Theorem in case $G-u$ is $K_3$-free) we have $D(G-u, K_3^{\circlearrowleft}) \leqslant 2^{20}$. On the other hand, if there is no triangle that contains $u$, then the previous subcase for graphs with 9 vertices gives $D(G-u, K_3^{\circlearrowleft}) < 2^{20}$ because $G-u$ contains a triangle, and hence we have $\text{ext}_G(u, G-u) \leqslant 2^5$. Therefore, we have

$$D(G, K_3^{\circlearrowleft}) \leqslant D(G-u, K_3^{\circlearrowleft}) \cdot \text{ext}_G(u, G-u) < 2^{25} = 2^{\lfloor n^2/4 \rfloor}. \tag{23}$$

Thus, we may assume that $\delta(G) \geqslant 6$. Suppose that $G$ contains two vertex-disjoint triangles with vertex sets $V_1$ and $V_2$ (recall that $G$ contains at most two vertex-disjoint triangles). Let $G'$ be the subgraph of $G$ induced by the vertices that are not in $V_1$ or $V_2$. Note that since $G$ is $K_4$-free and $G'$ is triangle-free, the graph $G'$ is a cycle and each vertex of $G'$ has exactly two neighbors in each of $V_1$ and $V_2$. Let $a_1b_1$ and $a_2b_2$ be two non-incident edges of $G'$ and put $H_i = G[V_i \cup \{a_i, b_i\}]$, for $i \in \{1, 2\}$. Note that $H_1$ and $H_2$

are isomorphic to $K_{1,2,2}$. Then, by Proposition 13, we have $D(H_1, K_3^{\circlearrowright}), D(H_2, K_3^{\circlearrowright}) \leqslant 82$. Analogous to the subcase for graphs with 9 vertices, we have $\mathrm{ext}_G(a_ib_i, V_{3-i}) \leqslant 8$ for $1 \leqslant i \leqslant 2$, $\mathrm{ext}_G(a_1b_1, a_2b_2) \leqslant 4$, and $\mathrm{ext}_G(T_1, T_2) \leqslant 15$. Therefore, we have

$$D(G, K_3^{\circlearrowright}) \leqslant 82^2 \cdot 15 \cdot 8^2 \cdot 4 < 2^{25} = 2^{\lfloor n^2/4 \rfloor}. \tag{24}$$

Assume that $G$ contains one triangle with vertex-set $V_1$, but does not contain two vertex-disjoint triangles. Let $G' = G - V_1$ and note that $G'$ is a triangle-free graph with seven vertices. Since $\delta(G) \geqslant 6$, and every vertex of $G'$ has at most two neighbors in $V_1$, we have $\delta(G') \geqslant 4$, which implies $|E(G')| \geqslant 14$. On the other hand, by Mantel's Theorem, $|E(G')| \leqslant 12$, a contradiction.

**Case $\mathbf{r = 2}$.** Since $G$ is triangle-free, we have $D(G, K_3^{\circlearrowright}) = 2^{|E(G)|}$. Thus, by Mantel's Theorem, if $G$ is not isomorphic to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, we have

$$D(G, K_3^{\circlearrowright}) < 2^{\lfloor n^2/4 \rfloor}. \tag{25}$$

Furthermore, $D(G, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$ if and only if $G \simeq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, which completes the proof that for any $n$-vertex graph $G$ with $9 \leqslant n \leqslant 7 + \min\{\omega(G), 8\}$ we have $D(G, K_3^{\circlearrowright}) \leqslant 2^{\lfloor n^2/4 \rfloor}$. Since inequalities (10)–(25) are strict, we get that $D(G, K_3^{\circlearrowright}) = 2^{\lfloor n^2/4 \rfloor}$ if and only if $G \simeq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, which concludes the proof of the proposition. □

**Acknowledgements**

P. Araújo was partially supported by CNPq; F. Botler was partially supported by CNPq (Proc. 304315/2022-2), FAPERJ (Proc. 201.334/2022); G. O. Mota was partially supported by CNPq (315916/2023-0, 406248/2021-4) and FAPESP (2018/04876-1, 2019/13364-7). The research that led to this paper started in WoPOCA 2019, which was financed by FAPESP (2015/11937-9) and CNPq (425340/2016-3, 423833/2018-9). This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brasil (CAPES), Finance Code 001. FAPERJ is the Rio de Janeiro Research Foundation. FAPESP is the São Paulo Research Foundation. CNPq is the National Council for Scientific and Technological Development of Brazil.

## References

[1] P. Allen, Y. Kohayakawa, G. O. Mota, and R. F. Parente. On the number of orientations of random graphs with no directed cycles of a given length. *Electron. J. Combin.*, 21(1):#P1.52, 13, 2014.

[2] Noga Alon, József Balogh, Peter Keevash, and Benny Sudakov. The number of edge colorings with no monochromatic cliques. *J. London Math. Soc. (2)*, 70(2):273–288, 2004.

[3] Noga Alon and Raphael Yuster. The number of orientations having no fixed tournament. *Combinatorica*, 26(1):1–16, 2006.

[4] Béla Bollobás. *Extremal graph theory*, volume 11 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978.

[5] Maurício Collares, Yoshiharu Kohayakawa, Robert Morris, and Guilherme Oliveira Mota. Counting restricted orientations of random graphs. *Random Structures & Algorithms*, 56:1016–1030, 2020.

[6] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, 4 edition, 2010.

[7] P. Erdős. Some new applications of probability methods to combinatorial analysis and graph theory. In *Proceedings of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1974)*, pages 39–51. Congressus Num., No. X, Winnipeg, Man., 1974. Utilitas Math.

[8] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10:60–61, 1907.

[9] Endre Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.

[10] Raphael Yuster. The number of edge colorings with no monochromatic triangle. *J. Graph Theory*, 21(4):441–452, 1996.