Restricted van der Waerden Theorem for Nilprogressions

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Submitted: Oct 22, 2024; Accepted: May 10, 2025; Published: Aug 8, 2025

Abstract

In [*Adv. Math.*, 321: 269–286, 2017], using the theory of ultrafilters, J. H. Johnson Jr., and F. K. Richter proved the nilpotent polynomial Hales-Jewett theorem. Using this result they proved the restricted version of the van der Waerden theorem for nilprogressions of rank 2 and conjectured that this result must hold for arbitrary rank. In this article, we give an affirmative answer to their conjecture.

Mathematics Subject Classifications: 05D10, 05C55

1 Introduction

Arithmetic Ramsey theory deals with the monochromatic patterns found in any given finite coloring of the integers or of the positive integers $\mathbb{N}$. Here, "coloring" means disjoint partition, and a set is called "monochromatic" if it is included in one piece of the partition. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. If for every finite coloring of $\mathbb{N}$, there exists a monochromatic member of $\mathcal{F}$, then such a family $\mathcal{F}$ is called a partition regular family. So basically, Ramsey theory is the study of the classifications of partition regular families. Arguably, one of the first substantial developments in this area of research was due to Van der Waerden in 1927 when he proved the following theorem.

Theorem 1 (**Van der Waerden theorem**, [8])**.** *For any finite coloring of the natural numbers, one always finds arbitrarily long monochromatic arithmetic progressions. In other words, the set of all arithmetic progressions of finite length is partition regular.*

In [6], Spencer (independently in [5], by Nešetřil, and V. Rödl) proved a restricted version of the Van der Waerden theorem: there exists a set $V \subset \mathbb{N}$ containing no $k+1$ term arithmetic progressions and such that for any partition of V into finitely many classes, some class must contain a k-term arithmetic progression. To address the conjecture for nilprogressions we need to recall some notions. For $k \geqslant 2$, define $\sum_{<k}$ to be the

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collection of all words $w(*_1, *_2, \ldots, *_d)$, (where $d \in \mathbb{N}$) in the letters $*_1, *_2, \ldots, *_d$ such that every letter $*_i$ appears at most $(k-1)$ times. For any given word $w(*_1, *_2, \ldots, *_d)$ and elements $x_1, x_2, \ldots, x_d$ in a group $(G, \cdot)$, let $w(x_1, x_2, \ldots, x_d)$ be the group element of G obtained by replacing all occurrences of the variable $*_i$ in the word $w(*_1, *_2, \ldots, *_d)$ with x_i. A **nilprogression of step s, length k and rank d** is a set of the form $A = \left\{w(x_1, x_2, \ldots, x_d)a : w \in \sum_{<k+1}\right\}$, where $a, x_1, x_2, \ldots, x_d$ are elements in an s-step nilpotent group G. If $|A| = |\sum_{<k+1}|$, then A is called **non-degenerated nilprogression.** In [4] Johnson Jr. and Richter posed the following conjecture.

Conjecture 2 (**Restricted van der Waerden theorem for nilprogressions**). For every $k \geqslant 1$, and $d \geqslant 2$ there exists a k-step nilpotent group $(G, \cdot)$ with d generators and a set $V \subset G$ with the property that V does not contain any non-degenerated nilprogressions of step k, length $k+1$ and rank d but for any partition of V into finitely many classes, some class contains a non-degenerated nilprogressions of step k, length k and rank d.

In [4, Theorem B], they proved this conjecture for $d = 2$ using the Nilpotent polynomial Hales-Jewett theorem ([4, Theorem A]), which was a conjecture of Bergelson and Leibman [2]. In this article, we construct a k-step nilpotent group with d generators, where we can explicitly calculate the evaluations of all words and then, proceeding similarly to their proofs, one can obtain the desired result. To avoid unnecessary technical difficulties, we omit the entire literature on the Polynomial Hales-Jewett theorem (see [1, 4]) and the algebra of Stone-Čech compactification (see [3]). That's why we also omit the full proof, but rather we limit ourselves to construct our required groups, and then one may adapt the technique of the rest of the proof as in the proof of [4, Theorem B].

2 Our proof

For a long time, the cartesian (or tensor) product trick has been proven to be a very useful technique to estimate bounds in additive combinatorics (see [7]). Our proof is inspired by this trick.

Proof of Conjecture 2: We divide the proof into two cases where d is even or odd.

***d* <u>is even:</u>** Let $\mathbb{Z}[x]$ denote the collection of all polynomials with integer coefficients. For each $i \in \{1, \ldots, d/2\}$, define $R_i, S_i : \mathbb{Z}[x]^{d/2} \to \mathbb{Z}[x]^{d/2}$ as

1. $R_i(P_1(x), \ldots, P_{d/2}(x)) = (P_1(x), \ldots, P_{i-1}(x), P_i(x) + x^k, P_{i+1}(x), \ldots, P_{d/2}(x))$
2. $S_i(P_1(x), \ldots, P_{d/2}(x)) = (P_1(x), \ldots, P_{i-1}(x), P_i(x+1), P_{i+1}(x), \ldots, P_{d/2}(x))$

Let G be the group generated by $\{R_i, S_i : 1 \leqslant i \leqslant d/2\}$. For each $i \in \{1, 2, \ldots, d/2\}$, let G_i be the group generated by R_i and S_i. Let $R, S : \mathbb{Z}[x] \to \mathbb{Z}[x]$ be the map defined by $R(p(x)) = p(x) + x^k$ and $S(p(x)) = p(x+1)$. As R_i and S_i are two maps which are nothing but R and S resp. on the i^{th} coordinate and are identity over the rest of the coordinates, we have G_i is isomorphic to the group generated by R and S. But the group generated by R and S is a k-step nilpotent group[1], and so our $G_i's$ are. It is worth noting that if $i \neq j$, then each of R_i, S_i commutes with each of R_j, S_j. Again each G_i is a k-step nilpotent group, and it follows that G is a (internal) direct product of $G_1, G_2, \dots, G_{d/2}$. Hence it follows that G is a k-step nilpotent group.

Let $\sum_{<k+1}$ denote the collection of all words in the letters $*_1, \dots, *_d$, where each $*_i$ appears at most k times. Our claim is that for all $w_1, w_2 \in \sum_{<k+1}$, if $w_1(R_1, S_1, \dots, R_{d/2}, S_{d/2}) = w_2(R_1, S_1, \dots, R_{d/2}, S_{d/2})$ then

$$w_1(*_1, \dots, *_d) = w_2(*_1, \dots, *_d).$$

Note that for any $w \in \sum_{<k+1}$, one can write

$$w(R_1, S_1, \dots, R_{d/2}, S_{d/2}) = w^1(R_1, S_1) \cdots w^{d/2}(R_{d/2}, S_{d/2})$$

where each w^i is a a word in G_i with length k, rank 2.

Now

$$w_1(R_1, S_1, \dots, R_{d/2}, S_{d/2})(x^k, \dots, x^k) = w_2(R_1, S_1, \dots, R_{d/2}, S_{d/2})(x^k, \dots, x^k)$$

implies

$$(w_1^1(R_1, S_1)x^k, \dots, w_1^{d/2}(R_{d/2}, S_{d/2})x^k) = (w_2^1(R_1, S_1)x^k, \dots, w_2^{d/2}(R_{d/2}, S_{d/2})x^k).$$

But then in light of the proof of [4, Theorem B], we have $w_1 = w_2$. This shows that G admits non-degenerated nilprogressions of length k and rank d. As each rank d word in G is a product of rank 2 words of G_i's (at least one of them G_1), and as in [4, Theorem B], it has been proven that they do not admit non-degenerated nilprogressions of length $(k+1)$ and rank 2, our group does not contain nilprogressions of length $(k+1)$ and rank d.

Now the rest of the proof is similar to the proof of [4, Theorem B], so we leave it to the reader.

***d* is odd:** Whenever d is odd, we need a little modification. For each $i \in \{1, \dots, (d-1)/2\}$, define $R_i, S_i, R_d : \mathbb{Z}[x]^{(d+1)/2} \to \mathbb{Z}[x]^{(d+1)/2}$ as

1. $R_i(P_1(x), \dots, P_{(d+1)/2}(x)) = (P_1(x), \dots, P_{i-1}(x), P_i(x)+x^k, P_{i+1}(x), \dots, P_{(d+1)/2}(x))$
2. $S_i(P_1(x), \dots, P_{(d+1)/2}(x)) = (P_1(x), \dots, P_{i-1}(x), P_i(x+1), P_{i+1}(x), \dots, P_{(d+1)/2}(x))$
3. $R_d(P_1(x), \dots, P_{(d+1)/2}(x)) = (P_1(x), \dots, P_{(d+1)/2}(x) + x^k)$.

Now consider the group G generated by $\{R_i, S_i : 1 \leqslant i \leqslant (d-1)/2\} \cup \{R_d\}$. We agree that the group generated by R_d is commutative, but the other R_i, S_i's generates k-step nilpotent groups. Now proceed similarly to the above proof. □

[1]This is exactly the same group that was considered in the proof of [4, Theorem B]

Acknowledgements

The author gratefully acknowledges the support of the NBHM Postdoctoral Fellowship with Ref. No. 0204/27/(27)/2023/R & D-II/11927. I thank F. K. Richter for valuable feedback on an earlier draft of this article. I am also grateful to the anonymous referee for the helpful comments and suggestions on the previous version of the paper.

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