Alon's Transmitting Problem and Multicolor Beck-Spencer Lemma

Norihide Tokushige

Submitted: Aug 2, 2024; Accepted: May 5, 2025; Published: Aug 8, 2025

Abstract

The Hamming graph $H(n,q)$ is defined on the vertex set $\{1,2,\ldots,q\}^n$ and two vertices are adjacent if and only if they differ in precisely one coordinate. Alon (1992) proved that for any sequence $v_1,\ldots,v_b$ of $b=\lceil\frac{n}{2}\rceil$ vertices of $H(n,2)$, there is a vertex whose distance from v_i is at least $b-i+1$ for all $1\leqslant i\leqslant b$. In this note, we prove that for any $q\geqslant 3$ and any sequence $v_1,\ldots,v_b$ of $b=\lfloor(1-\frac{1}{q})n\rfloor$ vertices of $H(n,q)$, there is a vertex whose distance from v_i is at least $b-i+1$ for all $1\leqslant i\leqslant b$.

Alon used a lemma due to Beck and Spencer (1983) which, in turn, was based on the floating variable method introduced by Beck and Fiala (1981) who studied combinatorial discrepancies. For our proof, we extend the Beck–Spencer Lemma by using a multicolor version of the floating variable method due to Doerr and Srivastav (2003).

Mathematics Subject Classifications: 94A05, 05C35, 90C10

1 Introduction

Alon posed a transmitting problem in [1] and obtained an optimal solution. This problem can be described in terms of the graph burning number (see, e.g., [4]). Let G be a finite graph with the vertex set V. For vertices $u,v\in V$ let $d(u,v)$ denote the distance between u and v. For a non-negative integer k, let $\Gamma_k(v)$ denote the k-neighbors of v, that is, the set of vertices $u\in V$ such that $d(u,v)\leqslant k$. For $v_1,v_2,\ldots,v_b\in V$ we say that $(v_1,v_2,\ldots,v_b)$ is a burning sequence of length b if $\Gamma_{b-1}(v_1)\cup\Gamma_{b-2}(v_2)\cup\cdots\cup\Gamma_0(v_b)=V$. The burning number of G, denoted by $b(G)$, is defined to be the minimum length of the burning sequences. We can think of the vertices as processors. Suppose that there is a sender outside the graph, and it sends a message to v_i at round i. Each processor that receives a message at round j transmits the message to all its neighbors at round $j+1$. Then $b(G)$ is the minimum number of rounds in which all the processors share the message.

College of Education, University of the Ryukyus (hide@cs.u-ryukyu.ac.jp).

Theorem 1 (Alon [1])**.** *Let G be the n-dimensional cube. Then $b(G) = \lceil \frac{n}{2} \rceil + 1$.*

For the n-dimensional cube, let v_1 be an arbitrary vertex, and let v_2 be the antipodal vertex. Then it is easy to check that $\Gamma_{b-1}(v_1) \cup \Gamma_{b-2}(v_2)$ covers all the vertices where $b = \lceil \frac{n}{2} \rceil + 1$. This means that the burning number is at most b. The more difficult part is to show that no matter how $b - 1$ vertices are chosen, they cannot be a burning sequence, that is, after $b - 1$ rounds there is still some vertex that has not received the message yet.

In this note, we extend the above result to Hamming graphs. For positive integers n, q, let $[q] := \{1, 2, \dots, q\}$ and let $[q]^n$ denote the set of n-tuple of elements of $[q]$. The Hamming distance between $u, v \in [q]^n$ is defined to be the number of entries (coordinates) that they differ. The Hamming graph $H(n, q)$ has the vertex set $V = [q]^n$, and two vertices are adjacent if they have Hamming distance one. Note that $H(n, 2)$ is (isomorphic to) the n-dimensional cube. Our main result is the following.

Theorem 2. *Let $q \geqslant 3$ be an integer, and let $G = H(n, q)$ be the Hamming graph. Then*

$$\left\lfloor \left(1 - \tfrac{1}{q}\right) n \right\rfloor + 1 \leqslant b(G) \leqslant \left\lfloor \left(1 - \tfrac{1}{q}\right) n + \tfrac{q+1}{2} \right\rfloor .$$

The upper bound in Theorem 2 is easily verified by construction. Indeed, the sequence $v_i = (i, i, \dots, i)$ for $i = 1, 2, \dots, q$ (and any v_i for $i > q$) works as a corresponding burning sequence (see [7] for more details). This construction is valid for $q = 2$ as well, and in this case the upper bound coincides with the correct value of $b(H(n, 2))$ given by Theorem 1.

To show the lower bound in Theorem 1, Alon used the Beck–Spencer lemma.

Lemma 3 (Beck–Spencer [3])**.** *For $1 \leqslant i \leqslant n$, let $\boldsymbol{a}_i \in \{-1, 1\}^n$ be given. Then there exists $\boldsymbol{x} \in \{-1, 1\}^n$ such that the standard inner product satisfies $|\boldsymbol{a}_i \cdot \boldsymbol{x}| < 2i$ for all $1 \leqslant i \leqslant n$.*

We can think of $\{-1, 1\}^n$ as the vertex set of the n-dimensional cube, and it follows that $\boldsymbol{a} \cdot \boldsymbol{x} = n - 2d(\boldsymbol{a}, \boldsymbol{x})$ as we will see in the next section. Then Lemma 3 is restated as follows.

Lemma 4 (Lemma 3 restated)**.** *For $1 \leqslant i \leqslant n$, let v_i be a given vertex of $H(n, 2)$. Then there exists a vertex w such that $|n - 2d(v_i, w)| < 2i$ for all $1 \leqslant i \leqslant n$.*

For our proof of the lower bound in Theorem 2, we need a multicolor version of the Beck–Spencer lemma.

Lemma 5. *Let $q \geqslant 3$. For $1 \leqslant i \leqslant n$, let v_i be a given vertex of $H(n, q)$. Then there exists a vertex w such that $|(1 - \frac{1}{q})n - d(v_i, w)| < i$ for all $1 \leqslant i \leqslant n$.*

To prove Lemma 3, Beck and Spencer used the so-called *floating variable method* based on [2] where Beck and Fiala studied combinatorial discrepancies. Then Doerr and Srivastav in [5] extended the results in [2] (and many other related results concerning discrepancies) to multicolor settings. We utilize their ideas on vector-coloring to prove Lemma 5, c.f. the proof of Theorem 4.2 in [5]. So we explain a multicolor version of the floating variable method in the next section.

It follows from Lemma 4 that $b(H(n,2)) \geqslant \lfloor \frac{n}{2} \rfloor + 1$. However, this is not the sharp bound if n is odd. Alon made one more twist to improve the bound in [1]. The author was not able to find an appropriate extension of this tricky part for $H(n,q)$ ($q \geqslant 3$), and this suggests that the lower bound in Theorem 2 could be improved.

Conjecture 6. For fixed $q \geqslant 3$ and large enough n, we have

$$b(H(n,q)) = \left\lfloor \left(1 - \frac{1}{q}\right) n + \frac{q+1}{2} \right\rfloor .$$

Since $H(n,q)$ has diameter n, we have a trivial bound $b(H(n,q)) \leqslant n+1$. Thus if the conjecture holds, then we need $n \geqslant \lfloor \frac{q(q-1)}{2} \rfloor$. On the other hand, it follows from the lower bound in Theorem 2 that $b(H(n,q)) = n+1$ if $q \geqslant n \geqslant 3$, see the last section.

2 Proofs

We want to restate Lemma 5 in the form of Lemma 3. For this, we need to assign a vector to a vertex in $H(n,q)$ so that the vector behaves nicely with respect to the inner product and the Hamming distance. To this end, we use vector-coloring introduced by Doerr and Srivastav in [5]. For a qn-dimensional vector $\boldsymbol{z} = (z_1, \dots, z_{qn}) \in \mathbb{R}^{qn}$, let $\bar{z}_j := (z_{q(j-1)+1}, z_{q(j-1)+2}, \dots, z_{qj}) \in \mathbb{R}^q$, and we also write $\boldsymbol{z} = (\bar{z}_1, \dots, \bar{z}_n)$. (We write $\bar{z}$ to indicate the vector is q-dimensional.) For each color $i \in [q]$, we assign a vector $\bar{c}_i = -\frac{1}{q}\bar{1} + \bar{e}_i \in \mathbb{R}^q$, that is, $\bar{c}_i$ has $1 - \frac{1}{q}$ at the i-th entry, and $-\frac{1}{q}$ at the other $q-1$ entries. For example, if $q = 3$ then $\bar{c}_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, $\bar{c}_2 = (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$, $\bar{c}_3 = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$. By definition the sum of all entries of $\bar{c}_i$ equals 0. We also have

$$\bar{c}_i \cdot \bar{c}_j = \begin{cases} 1 - \frac{1}{q} & \text{if } i = j, \\ -\frac{1}{q} & \text{if } i \neq j. \end{cases} \tag{1}$$

Let Q be the set of color vectors, that is, $Q = \{\bar{c}_1, \bar{c}_2, \dots, \bar{c}_q\}$. For each vertex $v = (\nu_1, \dots, \nu_n) \in [q]^n$ of $H(n,q)$, we assign a vector $\boldsymbol{z} = (\bar{z}_1, \dots, \bar{z}_n) \in \mathbb{R}^{qn}$ by

$$\bar{z}_i := \bar{c}_{\nu_i} \tag{2}$$

for $1 \leqslant i \leqslant n$. In this case we understand $\boldsymbol{z} \in Q^n$. Define a bijection $\varphi : [q]^n \to Q^n$ by (2) and

$$\varphi(v) := \boldsymbol{z}. \tag{3}$$

Claim 7. *Let $v, w \in [q]^n$, and $\boldsymbol{a} = \varphi(v), \boldsymbol{x} = \varphi(w)$. Then $\boldsymbol{a} \cdot \boldsymbol{x} = (1 - \frac{1}{q})n - d(v,w)$.*

Proof. Let $s := \#\{i \in [n] : \bar{a}_i \neq \bar{x}_i\} = d(v,w)$. It follows from (1) that $\boldsymbol{a} \cdot \boldsymbol{x} = (1 - \frac{1}{q})(n - s) - \frac{1}{q}s = (1 - \frac{1}{q})n - d(v,w)$. □

Now we can restate Lemma 5.

Lemma 8 (Lemma 5 restated)**.** *For $1 \leqslant i \leqslant n$, let $\boldsymbol{a}_i \in Q^n$ be given. Then there exists $\boldsymbol{x} \in Q^n$ such that $|\boldsymbol{a}_i \cdot \boldsymbol{x}| < i$ for all $1 \leqslant i \leqslant n$.*

Proof. Let $\tilde{Q} \subset \mathbb{R}^q$ denote the convex hull of Q, that is,

$$\tilde{Q} := \left\{ \sum_{i=1}^{q} \lambda_i \bar{c}_i : 0 \leqslant \lambda_i \leqslant 1 \, (i \in [q]), \, \sum_{i=1}^{q} \lambda_i \leqslant 1 \right\}.$$

Note that if $\bar{x} \in \tilde{Q}$ then the number of entries of $\bar{x}$ with value $1 - \frac{1}{q}$ is at most one.

Claim 9. *For $\bar{a}, \bar{x} \in Q$ and $\bar{y} \in \tilde{Q}$, we have the following.*

- $\bar{a} \cdot \bar{x} \in \{-\frac{1}{q}, 1 - \frac{1}{q}\}$.
- $-\frac{1}{q} \leqslant \bar{a} \cdot \bar{y} \leqslant 1 - \frac{1}{q}$. *If $\bar{a} \cdot \bar{y} = 1 - \frac{1}{q}$, then $\bar{y} = \bar{a}$.*
- $|\bar{a} \cdot \bar{x} - \bar{a} \cdot \bar{y}| \leqslant 1$. *Moreover, if equality holds, then (i) $\bar{y} \in Q$ or (ii) there exists i such that $\bar{a} = \bar{x} = \bar{c}_i$ and the i-th entry of $\bar{y}$ is $-\frac{1}{q}$.*

Proof. Without loss of generality we may assume that $\bar{a} = \bar{c}_1$. The first item follows from (1).

For the second item, let $\bar{y} = \sum_{i=1}^{q} \lambda_i \bar{c}_i \in \tilde{Q}$. Noting that $\sum_{i=2}^{q} \lambda_i \leqslant 1 - \lambda_1$ and

$$\bar{a} \cdot \bar{y} = \bar{c}_1 \cdot \sum_{i=1}^{q} \lambda_i \bar{c}_i = (1 - \tfrac{1}{q})\lambda_1 - \tfrac{1}{q}(\lambda_2 + \cdots + \lambda_n),$$

we have $-\frac{1}{q} \leqslant \bar{a} \cdot \bar{y} \leqslant 1 - \frac{1}{q}$, because

$$1 - \tfrac{1}{q} \geqslant (1 - \tfrac{1}{q})\lambda_1 \geqslant \bar{a} \cdot \bar{y} \geqslant (1 - \tfrac{1}{q})\lambda_1 - \tfrac{1}{q}(1 - \lambda_1) = \lambda_1 - \tfrac{1}{q} \geqslant -\tfrac{1}{q}. \tag{4}$$

If $\bar{a} \cdot \bar{y} = 1 - \frac{1}{q}$, then $\lambda_1 = 1$, that is, $\bar{y} = \bar{c}_1$.

By the first and second items we have $|\bar{a} \cdot \bar{x} - \bar{a} \cdot \bar{y}| \leqslant 1$. If equality holds, then $(\bar{a} \cdot \bar{x}, \bar{a} \cdot \bar{y}) = (-\frac{1}{q}, 1 - \frac{1}{q})$ or $(1 - \frac{1}{q}, -\frac{1}{q})$. For the former case, we have $\bar{y} = \bar{a} \in Q$ by the second item. For the latter case, we have $\lambda_1 = 0$ and $\sum_{i=2}^{q} \lambda_i = 1$ by (4), and the first entry of $\bar{y}$ is $\lambda_1(1 - \frac{1}{q}) + \sum_{i=2}^{q} \lambda_i(-\frac{1}{q}) = 0 - \frac{1}{q}(1 - \lambda_1) = -\frac{1}{q}$. ☐

Let $\boldsymbol{x} = (x_1, \ldots, x_{qn}) = (\bar{x}_1, \ldots, \bar{x}_n) \in \mathbb{R}^{qn}$ be a variable vector. We describe an algorithm updating $\boldsymbol{x}$ step by step in such a way that at each step $\boldsymbol{x} \in \tilde{Q}^n$, and in the end $\boldsymbol{x} \in Q^n$. We call a variable x_k *floating* if $-\frac{1}{q} < x_k < 1 - \frac{1}{q}$, and *fixed* if $x_k \in \{-\frac{1}{q}, 1 - \frac{1}{q}\}$. Once a variable is fixed, then it stays the same value and it is treated as a constant.

Before going into detail, we explain the algorithm's geometric meaning. Let $\mathcal{P}$ be the polytope obtained by taking the intersection of $\tilde{Q}^n$ and n hyperplanes $\sum_{i=1}^{q} x_{q(j-1)+i} = 0$ $(1 \leqslant j \leqslant n)$, cf. [6]. Starting from the origin, we move along a line until we hit a facet of $\mathcal{P}$, and let $\boldsymbol{x}_1$ be the intersection of the line and the facet. Then, in the facet, starting from $\boldsymbol{x}_1$, we move along another line until we hit a face of $\mathcal{P}$, and let $\boldsymbol{x}_2$ be the intersection. We repeat this procedure so that the dimension of the faces we hit is decreasing. Finally, we stop at a face of dimension 0, and we find one of the vertices $\boldsymbol{x}$ of $\mathcal{P}$, that is, $\boldsymbol{x} \in Q^n$. The choice of lines is crucial for this algorithm to satisfy $|\boldsymbol{a}_i \cdot \boldsymbol{x}| < i$ for all i. To make the right choice, we will solve a system of equations repeatedly as described below.

We consider the following three conditions.

(C1) $\boldsymbol{a}_i \cdot \boldsymbol{x} = 0$,

(C2) $x_{q(j-1)+1} + x_{q(j-1)+2} + \cdots + x_{qj} = 0$,

(C3) $\boldsymbol{x} \in \tilde{Q}^n$.

We will repeatedly solve a system of equations satisfying these conditions according to the following procedure.

Step(0) Consider the following system Eq(0) of equations:

- (C1) for $1 \leqslant i \leqslant n$,
- (C2) for $1 \leqslant j \leqslant n$.

Then $\boldsymbol{x}_0 := \boldsymbol{0}$ is a solution to Eq(0), and it satisfies (C3) as well because $\boldsymbol{0} \in \mathbb{R}^q$ is the center of $\tilde{Q}$, and $\boldsymbol{0} \in \mathbb{R}^{rq}$ is inside $\tilde{Q}^n$.

Step(1) Consider the following system Eq(1) of equations:

- (C1) for $1 \leqslant i \leqslant n-1$,
- (C2) for $1 \leqslant j \leqslant n$.

Eq(1) has qn variables and $2n-1$ equations. Thus we have a non-trivial solution $\boldsymbol{y} \neq \boldsymbol{0}$. For every $\lambda \geqslant 0$, $\lambda \boldsymbol{y}$ also satisfies Eq(1). Moreover, if $\lambda = 0$, then it satisfies (C3) as well.

Increase λ continuously starting from 0. Then at some point at least one of the entries of $\lambda \boldsymbol{y}$ becomes $-\frac{1}{q}$ or $1 - \frac{1}{q}$ for the first time. Using this $\lambda > 0$, let $\boldsymbol{x}_1 := \boldsymbol{x}_0 + \lambda \boldsymbol{y}$. Then $\boldsymbol{x}_1$ satisfies Eq(1) and (C3). This $\boldsymbol{x}_1$ is not yet determined and may be updated. Let $s = 1$ and go to **Step(s)** below.

Step(s) We have a temporary vector $\boldsymbol{x}_s = (\bar{x}_1, \ldots, \bar{x}_n)$ which satisfies (C1) for $1 \leqslant i \leqslant n-s$, (C2) for $1 \leqslant j \leqslant n$, and (C3). For $r \in \{0, 1, \ldots, q\}$ let

$$F_r(\boldsymbol{x}_s) := \{j \in [n] : \text{the number of fixed entries in } \bar{x}_j \text{ is } r\}.$$

We claim that $F_{q-1}(\boldsymbol{x}_s) = \emptyset$. Suppose, to the contrary, that $j \in F_{q-1}(\boldsymbol{x}_s)$. Recall that the entries of $\bar{x}_j$ can contain $1 - \frac{1}{q}$ at most once. If $\bar{x}_j$ contains $1 - \frac{1}{q}$, then the other $q-2$ fixed entries are $-\frac{1}{q}$. Then, by (C2), the remaining entry is $-\frac{1}{q}$, a contradiction. If $\bar{x}_j$ does not contain $1 - \frac{1}{q}$, then all the $q-1$ fixed entries are $-\frac{1}{q}$, and the remaining entry is $1 - \frac{1}{q}$, a contradiction.

If $|F_q(\boldsymbol{x}_s)| \geqslant s$, then we determine $\boldsymbol{x}_s$ and complete Step(s). Moreover, if $s = n$, then exit the procedure, otherwise let $\boldsymbol{x}_{s+1} := \boldsymbol{x}_s$ and proceed to Step($s+1$). Note that $\boldsymbol{x}_{s+1}$ satisfies (C1) for $1 \leqslant i \leqslant n-(s+1)$, (C2) for $1 \leqslant j \leqslant n$, and (C3).

From now on, we deal with the case $|F_q(\boldsymbol{x}_s)| \leqslant s-1$. Since $|F_q(\boldsymbol{x}_s)|$ is non-decreasing in s, and $|F_q(\boldsymbol{x}_{s-1})| \geqslant s-1$ holds when $\boldsymbol{x}_{s-1}$ is determined, we may assume that

$|F_q(\boldsymbol{x}_s)| = s-1$. Let $f_r = |F_r(\boldsymbol{x}_s)|$. Since $f_{q-1} = 0$ and $\sum_{r=0}^{q-2} f_r = n - f_q$, it follows that $\sum_{r=1}^{q-2} r f_r \leqslant (q-2)\sum_{r=1}^{q-2} f_r = (q-2)(n-f_q)$.

Recall that we treat a fixed variable as a constant. Let $[qn] = I_c \cup I_v$ be a partition, where I_c is the set of indices for constant entries of $\boldsymbol{x}_s = (x_1, x_2, \ldots, x_{qn})$, and I_v is the set of indices for floating variables of $\boldsymbol{x}_s$. Then $|I_v| = qn - \sum_{r=1}^{q} r f_r$.

Consider the following system Eq(s) of equations:

- (C1) for $1 \leqslant i \leqslant n-s = n-f_q-1$,
- (C2) for $j \in F_0(\boldsymbol{x}_s) \cup F_1(\boldsymbol{x}_s) \cup \cdots \cup F_{q-2}(\boldsymbol{x}_s)$.

We have $|I_v|$ variables, and $(n-f_q-1)+(n-f_q) = 2n-2f_q-1$ equations. Thus the dimension of the solution space is

$$|I_v| - (2n-2f_q-1) = (q-2)(n-f_q) - \sum_{r=1}^{q-2} r f_r + 1 \geqslant 1,$$

and we obtain a line of solution $\boldsymbol{x}_s + \lambda\boldsymbol{y}$, where $\lambda \in \mathbb{R}$ and $y_i = 0$ for $i \in I_c$. By continuously increasing λ from 0, we get $\boldsymbol{x}_s + \lambda\boldsymbol{y}$ such that one of the entries in I_v becomes $-\frac{1}{q}$ or $1-\frac{1}{q}$ for the first time. Using this $\lambda > 0$, let $\boldsymbol{x} := \boldsymbol{x}_s + \lambda\boldsymbol{y}$. Then $\boldsymbol{x}$ satisfies (C1) for $1 \leqslant i \leqslant n-s$, (C2) for $1 \leqslant j \leqslant n$, and (C3). Update $\boldsymbol{x}_s := \boldsymbol{x}$, and return to the beginning of **Step(s)**.

Completion of procedure When **Step(n)** is completed, we obtain $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. By construction, we have the following conditions.

- $|F_q(\boldsymbol{x}_s)| \geqslant s$ and $\boldsymbol{x}_s \in \tilde{Q}^n$ for $1 \leqslant s < n$.
- $|F_q(\boldsymbol{x}_n)| = n$, that is, $\boldsymbol{x}_n \in Q^n$.
- $\boldsymbol{a}_i \cdot \boldsymbol{x}_{n-i} = 0$ for $1 \leqslant i \leqslant n$. (Indeed $\boldsymbol{a}_i \cdot \boldsymbol{x}_s = 0$ for $1 \leqslant s < n$, $1 \leqslant i \leqslant n-s$.)

By running this algorithm, determine $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, and let $\boldsymbol{x} := \boldsymbol{x}_n = (\bar{x}_1, \ldots, \bar{x}_n) \in Q^n$.

We have $\boldsymbol{a}_1 \cdot \boldsymbol{x}_{n-1} = 0$. If $\boldsymbol{x} = \boldsymbol{x}_{n-1}$, then $|\boldsymbol{a}_1 \cdot \boldsymbol{x}| = 0$. If $\boldsymbol{x} \neq \boldsymbol{x}_{n-1}$, then $|F_q(\boldsymbol{x}_{n-1})| = n-1$, and there exists precisely one j such that $j \notin F_q(\boldsymbol{x}_{n-1})$. Then, writing $\boldsymbol{a}_1 = (\bar{a}_1, \ldots, \bar{a}_n)$ and $\boldsymbol{x}_{n-1} = (\bar{y}_1, \ldots, \bar{y}_n)$, we have $\bar{a}_j, \bar{x}_j \in Q$, $\bar{y}_j \in \tilde{Q} \setminus Q$, and by Claim 9,

$$|\boldsymbol{a}_1 \cdot \boldsymbol{x}| = |\boldsymbol{a}_1 \cdot \boldsymbol{x} - \boldsymbol{a}_1 \cdot \boldsymbol{x}_{n-1}| = |\bar{a}_j \cdot \bar{x}_j - \bar{a}_j \cdot \bar{y}_j| \leqslant 1.$$

Moreover, if $|\bar{a}_j \cdot \bar{x}_j - \bar{a}_j \cdot \bar{y}_j| = 1$, then there exists i such that $\bar{a}_j = \bar{x}_j = \bar{c}_i$, and the i-th entry of $\bar{y}_j$ is $-\frac{1}{q}$. But then the i-th entry of $\bar{y}_j$ is fixed and remains unchanged thereafter. This contradicts the fact that the i-th entry of $\bar{x}_j = \bar{c}_i$ is $1-\frac{1}{q}$. Consequently we have $|\boldsymbol{a}_1 \cdot \boldsymbol{x}| = |\bar{a}_j \cdot \bar{x}_j - \bar{a}_j \cdot \bar{y}_j| < 1$.

Similarly, for $1 \leqslant i \leqslant n$, we have $\boldsymbol{a}_i \cdot \boldsymbol{x}_{n-i} = 0$ and $|F_q(\boldsymbol{x}_{n-i})| \geqslant n-i$. Thus, letting $J = [n] \setminus F_q(\boldsymbol{x}_{n-i})$, we have $|J| \leqslant i$ and

$$|\boldsymbol{a}_i \cdot \boldsymbol{x}| = |\boldsymbol{a}_i \cdot \boldsymbol{x} - \boldsymbol{a}_i \cdot \boldsymbol{x}_{n-i}| = \left| \sum_{j \in J} (\bar{a}_j \cdot \bar{x}_j - \bar{a}_j \cdot \bar{y}_j) \right| \leqslant \sum_{j \in J} |\bar{a}_j \cdot \bar{x}_j - \bar{a}_j \cdot \bar{y}_j| < i,$$

where we write $\boldsymbol{a}_i = (\bar{a}_1, \ldots, \bar{a}_n)$ and $\boldsymbol{x}_{n-i} = (\bar{y}_1, \ldots, \bar{y}_n)$. This completes the proof of Lemma 8. □

Proof of Theorem 2. As stated in the discussion following the statement of Theorem 2, the upper bound is proved in [7]. Here we prove the lower bound. Let $m := \lfloor (1 - \frac{1}{q})n \rfloor$. For arbitrary m vertices $v_1, v_2, \ldots, v_m \in [q]^n$, we show that there exists a vertex $w \in [q]^n$ such that $d(v_i, w) \geqslant m + 1 - i$ for all $1 \leqslant i \leqslant m$.

Recall the definition of φ from (3). Let $\boldsymbol{a}_i = \varphi(v_i) \in Q^n$ for $1 \leqslant i \leqslant n$. By Lemma 8 and Claim 7, we get $\boldsymbol{x} \in Q^n$ such that

$$|\boldsymbol{a}_i \cdot \boldsymbol{x}| = \left| (1 - \tfrac{1}{q})n - d(v_i, w) \right| < i$$

for all $1 \leqslant i \leqslant n$, where $w = \varphi^{-1}(\boldsymbol{x})$. Thus we have

$$d(v_i, w) > (1 - \tfrac{1}{q})n - i$$

for all $1 \leqslant i \leqslant n$. Let $n = qk + r$, $0 \leqslant r < q$. Then,

$$(1 - \tfrac{1}{q})n = (q-1)k + r - \tfrac{r}{q}.$$

This together with the fact that the distance is an integer yields

$$\begin{aligned} d(v_i, w) &\geqslant \begin{cases} (q-1)k - i + 1 & \text{if } r = 0, \\ (q-1)k + r - i & \text{if } 1 \leqslant r < q \end{cases} \\ &= m + 1 - i. \end{aligned}$$

Thus $w \notin \Gamma_{m-1}(v_1) \cup \cdots \cup \Gamma_0(v_m)$, which means that $b(H(n,q)) \geqslant m + 1$. □

3 Concluding remarks

In this subsection, we also write $b(n,q)$ to mean $b(H(n,q))$.

3.1 The case $q \geqslant n$

Recall that $H(n,q)$ has diameter n, so $b(n,q) \leqslant n + 1$ for all n, q.

Claim 10. *Let $q \geqslant 3$ and $s \leqslant n$. If $b(n, q-1) \geqslant s$ then $b(n,q) \geqslant s + 1$.*

Proof. Suppose that $b(n, q-1) \geqslant s$ but $b(n,q) \leqslant s$. Then we have a burning sequence for $H(n,q)$ of length s, starting from, without loss of generality, $v_1 = q\mathbf{1}$. Since $[q]^n \setminus \Gamma_{s-1}(v_1) \supset [q-1]^n$, we need to cover $H(n, q-1)$ in less than s rounds, which is impossible because $b(n, q-1) \geqslant s$. □

Proposition 11. *If $q \geqslant n \geqslant 3$ then $b(H(n,q)) = n + 1$.*

Proof. Since $b(n, q+1) \geqslant b(n,q)$, it suffices to show that $b(n,n) = n + 1$. This is true if $n = 3$. Indeed we have $b(3,2) = 3$ by Theorem 1, and so $b(3,3) \geqslant 4$ by Claim 10.

Now let $n \geqslant 4$. By Theorem 2 we have

$$b(n, n-2) \geqslant \left\lfloor \left(1 - \tfrac{1}{n-2}\right) n \right\rfloor + 1 = \left\lfloor n - 1 - \tfrac{2}{n-2} \right\rfloor + 1 = n - 1. \tag{5}$$

Applying Claim 10 to (5) twice, we get $b(n,n) \geqslant n + 1$. □

3.2 Possible extension of Lemma 5

For the proof of Theorem 1, Alon observed the following. In Lemma 3 we have $|\boldsymbol{a}_i \cdot \boldsymbol{x}| = |\boldsymbol{a}_i \cdot (-\boldsymbol{x})|$, and so the inequality on $d(v_i, w)$ in Lemma 4 also holds if we replace v_i with its antipodal vertex. It would be nice if a similar extension was possible for Lemma 5, which would improve Theorem 2. For simplicity, let $q = 3$ and $n = 3k + 1 \geqslant 4$ be fixed. For $v = (x_1, \ldots, x_n) \in \{0, 1, 2\}^n$, let $v' = (x_1+1, \ldots, x_n+1)$ and $v'' = (x_1+2, \ldots, x_n+2)$, where addition is done in modulo 3. For vertices u and v, let $f(u, v) := |\frac{2}{3}(3k+1)-d(u, v)|$, and $g(u, v) := \max\{f(u, v), f(u, v'), f(u, v'')\}$.

Problem 12. For $1 \leqslant i \leqslant 3k + 1$, let u_i be a given vertex of $H(3k + 1, 3)$ on $\{0, 1, 2\}^n$. Then is it true that there exists a vertex w such that $g(u_i, w) < i$ for all $1 \leqslant i \leqslant 3k + 1$?

If this is true, then it implies that $b(3k + 1, 3) = 2k + 2$. To see this, assume that the answer to the problem is affirmative, and let vertices $v_1, v_2, \ldots, v_{2k+1}$ be given in $H(3k+1, 3)$. As in [1], we apply the existence of w to $u_1 := v_2, u_2 := v_3, \ldots, u_{2k} := v_{2k+1}$. Then we have $g(v_i, w) < i - 1$ for $i = 2, 3, \ldots, 2k + 1$. Thus all of w, w' and w'' have distance at least $2k+2-i$ from v_i for all $2 \leqslant i \leqslant 2k+1$. Finally we use the fact that one of w, w', and w'' has distance at least $2k+1$ from v_1 because $d(v_1, w)+d(v_1, w')+d(v_1, w'') = 2(3k + 1)$. Consequently, there is $w^* \in \{w, w', w''\}$ such that $d(v_i, w^*) \geqslant 2k + 2 - i$ for all $1 \leqslant i \leqslant 2k + 1$, that is, $b(3k + 1, 3) \geqslant 2k + 2$.

Acknowledgments

The author thanks Noga Alon for valuable comments, in particular, for pointing out an error in the conjecture in the earlier version, Hajime Tanaka for providing information on eigenpolytopes and [6], and Naoki Matsumoto for stimulating discussions. He also thanks the referees who read the manuscript very carefully and corrected some errors.

The author was supported by JSPS KAKENHI Grant Number JP23K03201.

References

[1] N. Alon. Transmitting in the n-dimensional cube. *Discrete Appl. Math.* 37/38 (1992) 9–11.

[2] J. Beck, T. Fiala. "Integer-making" theorems. *Discrete Appl. Math.* 3 (1981) 1–8.

[3] J. Beck, J. Spencer. Balancing matrices with line shifts. *Combinatorica* 3 (1983) 299–304.

[4] A. Bonato. A survey of graph burning. *Contrib. Discrete Math.* **16** (2021) 185–197.

[5] B. Doerr, A. Srivastav. Multicolour discrepancies. *Combin. Probab. Comput.* 12 (2003) 365–399.

[6] C. D. Godsil. Eigenpolytopes of distance regular graphs. *Canad. J. Math.* **50** (1998) 739–755.

[7] N. Tokushige. Burning Hamming graphs. *Graphs Combin.* **40** (2024) 111–113.