A New Strategy for Finding Spanning Trees without Small Degree Stems

Michitaka Furuya[a] Shoichi Tsuchiya[b]

Submitted: Jun 2, 2024; Accepted: Apr 24, 2025; Published: Aug 8, 2025

Abstract

For an integer $k \geqslant 2$, a spanning tree of a graph without vertices of degree from 2 to k is called a $[2,k]$-*ST* of the graph. The concept of $[2,k]$-STs is a natural extension of a homeomorphically irreducible spanning tree (or HIST), which is a well-studied graph structure. In this paper, we give a new strategy for finding $[2,k]$-STs. By using the strategy, we refine or extend a known degree-sum condition for the existence of a HIST. Furthermore, we also investigate a degree-product condition for the existence of a $[2,k]$-ST.

Mathematics Subject Classifications: 05C05, 05C07

1 Introduction

For a graph G, let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For $u \in V(G)$, let $N_G(u)$ and $d_G(u)$ denote the *neighborhood* and the *degree* of u, respectively; thus $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ and $d_G(u) = |N_G(u)|$. For an integer $i \geqslant 0$, let $V_i(G) = \{u \in V(G) : d_G(u) = i\}$ and $V_{\geqslant i}(G) = \{u \in V(G) : d_G(u) \geqslant i\}$. We let $\delta(G)$ denote the *minimum degree* of G. We define

$$\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G),\ u \neq v,\ uv \notin E(G)\}$$

if G is not complete; $\sigma_2(G) = \infty$ if G is complete. Also, we define

$$\pi_2(G) = \min\{d_G(u)d_G(v) : u, v \in V(G),\ u \neq v,\ uv \notin E(G)\}$$

if G is not complete; $\pi_2(G) = \infty$ if G is complete.

For a tree T, each vertex in $V_{\geqslant 2}$ (resp., $V_1(T)$) is called a *stem* (resp., a *leaf*). For a graph G, a spanning tree of G without vertices of degree 2 is called a *homeomorphically irreducible spanning tree* (or a *HIST*) of G; i.e., a spanning tree T of G is a HIST if and

[a]School of Engineering, Kwansei Gakuin University, Hyogo, Japan (`e-mail:mfuruya@kwansei.ac.jp`).
[b]School of Network and Information, Senshu University, Kanagawa, Japan (`s.tsuchiya@isc.senshu-u.ac.jp`).

only if $V_2(T) = \emptyset$. A structure of HISTs is sometimes used as an essential tool to construct graph classes; for example, in an explicit class of edge-minimal 3-connected plane graphs given by Halin [8], HISTs play a key role. Motivated by such uses, the existence of a HIST (or a large subtree having no vertex of degree 2) has been widely studied (for example, see [1–3, 9, 13]). It is well known that a number of sufficient conditions for the existence of a hamiltonian path have been naturally generalized to those for the existence of a *spanning k-tree*, which is a spanning tree in which every stem has degree lying between 2 and k. Similar to this, the concept of HISTs was naturally extended: A spanning tree T of G is called a $[2, k]$-*ST* of G if $\bigcup_{2\leqslant i\leqslant k} V_i(T) = \emptyset$ (for further historical background and related results, we refer the reader to [6]).

Our aim in this series is to refine and to extend some known degree conditions for the existence of HISTs. In this paper,

(i) we give two results which essentially extend a known degree-sum condition assuring us the existence of a HIST, and

(ii) we focus on a degree-product condition, which seems to be more reasonable for the existence of a HIST, and find a $[2, k]$-ST using such a condition.

We start with a degree-sum condition for the existence of HISTs, which was recently given by Ito and Tsuchiya [10].

Theorem 1 (Ito and Tsuchiya [10]). *Let G be a connected graph of order $n \geqslant 8$. If $\sigma_2(G) \geqslant n - 1$, then G has a HIST.*

They also showed that the bound on σ_2 is best possible, i.e., for each integer $n \geqslant 8$, there exists a graph G of order n with $\sigma_2(G) = n - 2$ having no HIST. Our first result is a refinement of Theorem 1 with a characterization of sharp examples. For integers $k \geqslant 2$ and $n \geqslant 2k + 1$, let $\mathcal{G}_{k,n}$ be the family of graphs G of order n satisfying the following conditions (see Figure 1):

(L1) $V(G)$ is the disjoint union of four non-empty sets L_1, L_2, L_3 and L_4,
(L2) $L_1 \cup L_2$ and L_4 are cliques of G,
(L3) for every $u_1 \in L_1$, $N_G(u_1) \cap (L_3 \cup L_4) = \emptyset$,
(L4) for every $u_2 \in L_2$, $N_G(u_2) \cap L_3 \neq \emptyset$, $N_G(u_2) \cap L_4 = \emptyset$ and $d_G(u_2) \leqslant k$, and
(L5) for every $u_3 \in L_3$, $N_G(u_3) \cap L_2 \neq \emptyset$, $L_4 \subseteq N_G(u_3)$ and $d_G(u_3) \geqslant n - |L_1 \cup L_2| - 1$.

Let $k \geqslant 2$ be an integer, and let $c_k = \sqrt{k(k-1)(k + 2\sqrt{2k} + 2)}$. Let $n_0(k)$ be the smallest positive integer such that $n - 4c_k\sqrt{n} - 2k^2 - 4k - 4 \geqslant 0$ for every integer $n \geqslant n_0(k)$. Our first result is the following.

Theorem 2. *Let $k \geqslant 2$ be an integer. Let G be a connected graph of order $n \geqslant n_0(k)$, and suppose that $\sigma_2(G) \geqslant n - 2$. Then G has a $[2, k]$-ST if and only if G is not isomorphic to any graph in $\mathcal{G}_{k,n}$.*

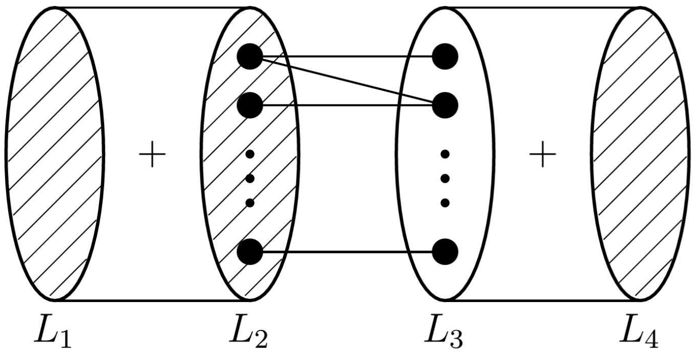

Figure 1: Graphs G belonging to $\mathcal{G}_{k,n}$.

Theorem 2 is a generalization of Theorem 1 for sufficiently large graphs. On the other hand, one can easily calculate that $n_0(2) = 295$, and so it in fact does not cover Theorem 1 when a target graph is small. Actually, in the previous paper [6] of this series, we obtained the same result as Theorem 2 for the case where $k = 2$ and $n \geqslant 10$ in a different way. So the order condition $n \geqslant n_0(k)$ in Theorem 2 is not best possible.

Recently, Shan and Tsuchiya [12] introduced a blocking set, which is a new concept on cutsets closely related to the existence of a HIST. We extend the concept to a $[2, k]$-ST version. Let $k \geqslant 2$ be an integer, and suppose that G is connected. A cutset $U \subseteq V(G)$ of G is *k-blocking set* of G if $U \subseteq \bigcup_{2 \leqslant i \leqslant k} V_i(G)$. If a graph G has a $[2, k]$-ST T, then for a cutset L of G, there exists a vertex $u \in L$ with $d_T(u) \geqslant k + 1$. In particular, if a graph has a $[2, k]$-ST, then the graph has no k-blocking set.

If a graph G satisfies (L1)–(L5), then L_2 is a k-blocking set. Thus, considering Theorem 2, one might expect that the degree-sum condition can be greatly improved if we omit the existence of a k-blocking set. Our second result affirms the expectation. Let $n_1(k)$ be the smallest positive integer such that $\frac{n+2k-2}{4} - 2c_k\sqrt{n} - k^2 - 2k - 1 \geqslant 0$ for every integer $n \geqslant n_1(k)$. Note that $n_1(2) = 1091$.

Theorem 3. *Let $k \geqslant 2$ be an integer. Let G be a connected graph of order $n \geqslant n_1(k)$, and suppose that $\sigma_2(G) \geqslant \frac{n+2k-2}{2}$. Then G has a $[2, k]$-ST if and only if G has no k-blocking set.*

Our third result is to propose a new concept on degree conditions. To explain it in detail, we start with two more natural results on degree conditions for the existence of HISTs (or $[2, k]$-STs). The following theorem is the first result discussing a relationship between a HIST and a degree condition.

Theorem 4 (Albertson, Berman, Hutchinson and Thomassen [1])**.** *Let G be a connected graph of order n, and suppose that $\delta(G) \geqslant 4\sqrt{2n}$. Then G has a HIST.*

Note that $c_2 = 4$ because $c_k = \sqrt{k(k-1)(k + 2\sqrt{2k} + 2)}$. Recently, Theorem 4 was refined and extended in the previous paper of this series as follows.

Theorem 5 (Furuya, Saito and Tsuchiya [6]). *Let $k \geqslant 2$ be an integer. Let G be a connected graph of order n, and suppose that $\delta(G) \geqslant c_k\sqrt{n}$. Then G has a $[2,k]$-ST.*

The coefficient of $\sqrt{n}$ in Theorem 5 might be further improved. On the other hand, for any integers $k \geqslant 2$ and $d \geqslant k-1$ such that $\frac{d}{k-1}$ is an integer, Furuya et al. [6] constructed a connected graph G with $\delta(G) = d = \sqrt{4(k-1)|V(G)| + (2k-1)^2} - 2k + 1$ having no $[2,k]$-ST. Therefore the degree condition in Theorem 5 is asymptotically best possible.

Now we focus on a large gap between Theorems 1 and 4. For example, the following two theorems are well-known, and their degree conditions are best possible:

- Dirac's Theorem [5]: If a graph G of order $n \geqslant 3$ satisfies $\delta(G) \geqslant \frac{n}{2}$, then G has a Hamiltonian cycle.

- Ore's Theorem [11]: If a graph G of order $n \geqslant 3$ satisfies $\sigma_2(G) \geqslant n$, then G has a Hamiltonian cycle.

In particular, Ore's Theorem implies Dirac's Theorem. On the other hand, a degree condition in Theorem 1 is much bigger than one in Theorem 4. Considering the fact that the root of the order of a graph appears in Theorem 4, one natural question occurs: Is there a degree-product condition close to the order assuring us the existence of HISTs? We give an affirmative answer for the problem. Recall that $c_k = \sqrt{k(k-1)(k+2\sqrt{2k}+2)}$ for an integer $k \geqslant 2$. For an integer $k \geqslant 2$, let

$$p_k = \frac{5c_k^2 + 3c_k\sqrt{c_k^2 + 4k^2 + 8k + 4}}{2} + k^2 + 2k + 1.$$

Our third result is the following.

Theorem 6. *Let $k \geqslant 2$ be an integer. Let G be a connected graph of order $n \geqslant k+2$, and suppose that $\pi_2(G) \geqslant p_k n$. Then G has a $[2,k]$-ST.*

As with Theorems 4 and 5, we do not know whether the coefficient of n in Theorem 6 is best possible or not. However, the degree-product condition, which is an unprecedented work as we know, seems to be essential for the existence of HISTs (or $[2,k]$-STs).

The proofs of Theorem 2–6 depend on a common strategy. In Section 2, we introduce key lemmas for the strategy. In Section 3, we prove Theorems 2 and 3 at the same time, and discuss the sharpness of the degree-sum condition appearing in Theorem 3. In Section 4, we prove Theorem 6.

1.1 Notations

In this subsection, we prepare the notation required for our proofs. For terms and symbols not defined in this paper, we refer the reader to [4].

Let G be a graph. For $F \subseteq E(G)$, let $V(F) = \{u, v : uv \in F\}$. For a subgraph H of G and a subset F of $E(G)$, let $H + F$ be the subgraph of G with $V(H+F) = V(H) \cup V(F)$ and $E(H+F) = E(H) \cup F$. Let compo(G) be the number of components of G. A vertex

u of G is called a *cut-vertex* of G if $\mathrm{compo}(G-u) > \mathrm{compo}(G)$. Note that cut-vertices are defined in disconnected graphs. Let $\mathrm{cut}(G)$ be the number of cut-vertices of G.

Let $k \geqslant 2$ be an integer. For a tree T and a subset U of $V(T)$, T is (k,U)-*good* if $V(T) \setminus U \subseteq V_1(T) \cup V_{\geqslant k+1}(T)$ and $U \subseteq V_{\geqslant k}(T)$. Note that a spanning $(k,\emptyset)$-good tree of a graph G is a $[2,k]$-ST of G. If a tree is $(k,\{u\})$-good, then the tree is simply said to be (k,u)-*good*.

2 Key lemmas

In this section, we introduce a key lemma for our argument (Lemma 8) and arrange it for the existence of $[2,k]$-STs (Lemmas 9 and 10). Our strategy is that first we take the vertex set S in a graph G consisting of all small degree vertices, where small means half or the root of the degree condition (by the definitions of degree conditions, we can show that S induces a clique). Then we can see that each component of $G-S$ has large minimum degree. In order to take $[2,k]$-STs of G, we guarantee the existence of convenient structures in such components by proving Lemmas 9 and 10.

Lemma 7. *Let G be a graph of order n, and suppose that $\delta(G) \geqslant 2\sqrt{n}$. Then* $\mathrm{cut}(G) + \mathrm{compo}(G) - 1 \leqslant 2\sqrt{n}$.

Proof. We proceed by induction on n. If $n \leqslant 5$, then there is no graph G of order n with $\delta(G) \geqslant 2\sqrt{n}$, and hence the lemma holds. Thus we may assume that $n \geqslant 6$.

Since $\delta(G) \geqslant 2\sqrt{n}$, every component of G contains more than $2\sqrt{n}$ vertices, and hence $\mathrm{compo}(G) < \frac{n}{2\sqrt{n}} = \frac{\sqrt{n}}{2}$. In particular, if G has no cut-vertex, then $\mathrm{cut}(G) + \mathrm{compo}(G) - 1 < 0 + \frac{\sqrt{n}}{2} - 1 < 2\sqrt{n}$, as desired. Thus we may assume that a component G_1 of G has a cut-vertex.

Let L be an *end-block* of G_1, which is a block of G containing exactly one cut-vertex. Let u be the unique cut-vertex of G_1 with $u \in V(L)$. For a vertex $u' \in V(L) \setminus \{u\}$, $2\sqrt{n} \leqslant \delta(G) \leqslant d_G(u') \leqslant |V(L) \setminus \{u'\}| = |V(L)| - 1$. Hence

$$|V(L)| \geqslant 2\sqrt{n} + 1. \tag{1}$$

Since u is a cut-vertex of G_1, $V(G_1) \setminus V(L) \neq \emptyset$. Furthermore, since L is an end-block, all cut-vertices of G_1 other than u are contained in $V(G_1) \setminus V(L)$. Let X be the set of cut-vertices of G_1 other than u such that they are not cut-vertices of $G_1 - V(L)$.

Fix a vertex $v \in X$, and let H be the component of $G_1 - V(L)$ containing v. Then v belongs to exactly two blocks of G_1 and all neighbors of v in one of them have been deleted in $G - V(L)$. This implies that $uv \in E(G)$ and $G_1[\{u,v\}]$ is a block of G_1. In particular, $V(H) \cap X = \{v\}$. Since v is arbitrary, $|X| = |\{H : H \text{ is the component of } G_1 - V(L) \text{ containing a vertex in } X\}| \leqslant \mathrm{compo}(G_1 - V(L))$. This implies that

$$\begin{aligned}
&\mathrm{cut}(G - V(L)) + \mathrm{compo}(G - V(L)) + 1 \\
&\quad = (\mathrm{cut}(G - V(G_1)) + \mathrm{cut}(G_1 - V(L))) + (\mathrm{compo}(G) - |\{G_1\}| + \mathrm{compo}(G_1 - V(L))) + 1 \\
&\quad \geqslant \mathrm{cut}(G - V(G_1)) + (\mathrm{cut}(G_1) - |\{u\} \cup X|) + \mathrm{compo}(G) - 1 + |X| + 1 \\
&\quad = \mathrm{cut}(G) + \mathrm{compo}(G) - 1.
\end{aligned} \tag{2}$$

Since $(2\sqrt{n}-2)^2 - \left(2\sqrt{n-2\sqrt{n}-1}\right)^2 = 8 > 0$, we have

$$2\sqrt{n}-2 > 2\sqrt{n-2\sqrt{n}-1}. \tag{3}$$

By (1) and (3), $\delta(G-V(L)) \geqslant \delta(G)-1 \geqslant 2\sqrt{n}-1 > 2\sqrt{n-2\sqrt{n}-1} \geqslant 2\sqrt{|V(G-V(L))|}$. Hence, by the induction hypothesis on $G-V(L)$, (1) and (3), we have

$$\mathrm{cut}(G-V(L))+\mathrm{compo}(G-V(L)) \leqslant 2\sqrt{|V(G-V(L))|}+1 \leqslant 2\sqrt{n-2\sqrt{n}-1}+1 < 2\sqrt{n}-1,$$

This together with (2) leads to

$$\mathrm{cut}(G)+\mathrm{compo}(G)-1 \leqslant \mathrm{cut}(G-V(L))+\mathrm{compo}(G-V(L))+1 < (2\sqrt{n}-1)+1,$$

as desired. □

Lemma 8. *Let $m \geqslant 0$ be an integer. Let G be a connected graph of order n, and let $u \in V(G)$ and $Y \subseteq V(G) \setminus \{u\}$. If $\delta(G) \geqslant 2\sqrt{n}+m+|Y|$, then there exists a set $X \subseteq N_G(u) \setminus Y$ with $|X| = m$ such that $G-X$ is connected.*

Proof. We proceed by induction on m. If $m = 0$, then the desired conclusion clearly holds. Thus we may assume that $m \geqslant 1$.

Since G is connected, it follows from Lemma 7 that $\mathrm{cut}(G) \leqslant 2\sqrt{n}$. Since $d_G(u) \geqslant 2\sqrt{n}+m+|Y|$, this implies that there exists a vertex $v \in N_G(u) \setminus Y$ which is not a cut-vertex of G. Let $G' = G-v$. Then G' is connected and $\delta(G') \geqslant \delta(G)-1 \geqslant 2\sqrt{n}+(m-1)+|Y| > 2\sqrt{|V(G')|}+(m-1)+|Y|$. Hence by the induction hypothesis on G', there exists a set $X' \subseteq N_{G'}(u) \setminus Y$ $(= N_G(u) \setminus (Y \cup \{v\}))$ with $|X'| = m-1$ such that $G'-X'$ $(= G-(\{v\} \cup X'))$ is connected. Consequently, $X := \{v\} \cup X'$ is a desired subset of $N_G(u) \setminus Y$. □

In the remainder of this section, we implicitly use the fact that $c_k > 2$ for every integer $k \geqslant 2$.

Lemma 9. *Let $k \geqslant 2$ be an integer. Let G be a connected graph of order n, and let $U \subseteq V(G)$ be a set with $U \neq \emptyset$. If $\delta(G) \geqslant c_k\sqrt{n}+(k+1)|U|-1$, then there exists a spanning forest of G consisting of exactly $|U|$ components $F_1, F_2, \dots, F_{|U|}$ such that for every integer i with $1 \leqslant i \leqslant |U|$, $|V(F_i) \cap U| = 1$ and F_i is a $(k, V(F_i) \cap U)$-good tree.*

Proof. Write $U = \{u_1, u_2, \dots, u_t\}$ where $t = |U|$. Since $(k+1)t-(k+t) = k(t-1) \geqslant 0$, we have $\delta(G) \geqslant c_k\sqrt{n}+(k+1)t-1 > 2\sqrt{n}+k+t-1 = 2\sqrt{n}+k+|U \setminus \{u_t\}|$. This together with Lemma 8 with $(m,u,Y) = (k, u_t, U \setminus \{u_t\})$ implies that there exists a set $X \subseteq N_G(u_t) \setminus (U \setminus \{u_t\})$ $(= N_G(u_t) \setminus U)$ with $|X| = k$ such that $G-X$ is connected.

We proceed by induction on t. Suppose that $t = 1$, i.e., $U = \{u_1\}$. Then $\delta(G-X) \geqslant \delta(G)-k \geqslant c_k\sqrt{n} > c_k\sqrt{|V(G-X)|}$. Hence by Theorem 5, $G-X$ has a $[2,k]$-ST T_0. Since $X \subseteq N_G(u_1)$, $F_1 := T_0 + \{u_1v : v \in X\}$ is a $[2,k]$-ST of G, and in particular, F_1 is a spanning (k, u_1)-good tree of G, and hence it is a desired forest. Thus we may assume that $t \geqslant 2$.

Let $\mathcal{C}$ be the family of components of $G-(\{u_t\}\cup X)$, and let $\mathcal{C}_1 = \{C \in \mathcal{C} : V(C)\cap U \neq \emptyset\}$ and $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. Since $t \geqslant 2$, we have $\mathcal{C}_1 \neq \emptyset$.

Fix $C \in \mathcal{C}_1$. Let $I_C = \{i : 1 \leqslant i \leqslant t-1,\ u_i \in V(C)\}$, and let $t_C = |I_C|$. Then

$$\begin{aligned}\delta(C) &\geqslant \delta(G) - |\{u_t\} \cup X| \\ &\geqslant c_k\sqrt{n} + (k+1)t - 1 - (k+1) \\ &= c_k\sqrt{n} + (k+1)(t-1) - 1 \\ &> c_k\sqrt{|V(C)|} + (k+1)t_C - 1.\end{aligned}$$

By the induction hypothesis on C, C has a spanning forest consisting of exactly t_C components F_i ($i \in I_C$) such that for every integer $i \in I_C$, $|V(F_i) \cap U| = 1$ and F_i is a $(k, V(F_i) \cap U)$-good tree. Since C is arbitrary, $\sum_{C\in\mathcal{C}_1} t_C$ $(= t-1)$ vertex-disjoint subtrees $F_1, F_2, \dots, F_{t-1}$ of G have been defined. Note that $\bigcup_{1\leqslant i\leqslant t-1} V(F_i) = \bigcup_{C\in\mathcal{C}_1} V(C) = V(G) \setminus (\{u_t\} \cup X \cup (\bigcup_{C'\in\mathcal{C}_2} V(C')))$.

Remark that $\mathcal{C}_2$ might be empty. Assume that $\mathcal{C}_2 \neq \emptyset$ and fix $C' \in \mathcal{C}_2$. Since $G-X$ is connected, there exists a vertex $v_{C'} \in N_G(u_t) \cap V(C')$. Since $(k+1)t-1-(k+1)-k = (k+1)(t-2) \geqslant 0$, $(k+1)t-1-(k+1) \geqslant k$, and hence

$$\delta(C') \geqslant \delta(G) - |\{u_t\} \cup X| \geqslant c_k\sqrt{n} + (k+1)t - 1 - (k+1) > c_k\sqrt{|V(C')|} + k.$$

By the induction hypothesis on C', C' has a spanning $(k, v_{C'})$-good tree $T_{C'}$. Let F' be the subgraph of G with $V(F') = \{u_t\} \cup X$ and $E(F') = \{u_t v : v \in X\}$. Then F' is a (k, u_t)-good subtree of G. Let $F_t = (F' \cup (\bigcup_{C'\in\mathcal{C}_2} T_{C'})) + \{u_t v_{C'} : C' \in \mathcal{C}_2\}$, where $F_t = F'$ if $\mathcal{C}_2 = \emptyset$. Then F_t is a (k, u_t)-good tree, $V(F_t) = \{u_t\} \cup X \cup (\bigcup_{C'\in\mathcal{C}_2} V(C'))$ and $V(F_t) \cap U = \{u_t\}$.

Consequently, the graph $\bigcup_{1\leqslant i\leqslant t} F_i$ is a desired spanning forest of G. □

Lemma 10. *Let $k \geqslant 2$ be an integer. Let G be a connected graph of order n, and let $U \subseteq V(G)$ be a set with $U \neq \emptyset$. If $\delta(G) \geqslant c_k\sqrt{n} + k|U| - 1$, then there exists a spanning (k, U)-good tree of G.*

Proof. Write $U = \{u_1, u_2, \dots, u_t\}$ where $t = |U|$. We recursively define t sets $X_1, X_2, \dots, X_t$ such that for each integer i with $1 \leqslant i \leqslant t$,

- **(A1)** $X_i \subseteq N_G(u_i) \setminus (U \cup (\bigcup_{1\leqslant j\leqslant i-1} X_j))$,
- **(A2)** $|X_i| = k-1$, and
- **(A3)** $G - (\bigcup_{1\leqslant j\leqslant i} X_j)$ is connected

as follows: We let i_0 be an integer with $1 \leqslant i_0 \leqslant t$, and assume that we have defined $i_0 - 1$ sets $X_1, X_2, \dots, X_{i_0-1}$ satisfying (A1)–(A3) for every integer i with $1 \leqslant i \leqslant i_0 - 1$. Let

$G_{i_0} = G - (\bigcup_{1 \leqslant i \leqslant i_0 - 1} X_i)$. Then G_{i_0} is connected and

$$\begin{aligned}\delta(G_{i_0}) &\geqslant \delta(G) - \left| \sum_{1 \leqslant j \leqslant i_0 - 1} X_j \right| \\ &\geqslant c_k\sqrt{n} + kt - 1 - (i_0 - 1)(k-1) \\ &> 2\sqrt{n} + kt - 1 - (t-1)(k-1) \\ &= 2\sqrt{n} + k - 1 + |U \setminus \{u_{i_0}\}|.\end{aligned}$$

This together with Lemma 8 with $(G, m, u, Y) = (G_{i_0}, k-1, u_{i_0}, U \setminus \{u_{i_0}\})$ implies that there exists a set $X_{i_0} \subseteq N_{G_{i_0}}(u_{i_0}) \setminus (U \setminus \{u_{i_0}\})$ $(= N_{G_{i_0}}(u_{i_0}) \setminus U)$ with $|X_{i_0}| = k-1$ such that $G_{i_0} - X_{i_0}$ $(= G - (\bigcup_{1 \leqslant i \leqslant i_0} X_i))$ is connected. Thus $X_1, X_2, \ldots, X_{i_0}$ satisfy (A1)–(A3) for for every integer i with $1 \leqslant i \leqslant i_0$. Consequently, we obtain desired sets.

Let $G' = G - (\bigcup_{1 \leqslant i \leqslant t} X_i)$. By (A3), G' is connected. Since $X_1, X_2, \ldots, X_t$ are pairwise disjoint by (A1), it follows from (A2) that $\delta(G') \geqslant \delta(G) - |\sum_{1 \leqslant i \leqslant t} X_i| \geqslant c_k\sqrt{n} + kt - 1 - t(k-1) > c_k\sqrt{|V(G')|}$. Hence by Theorem 5, G' has a $[2,k]$-ST T. Then $T + \{u_i v : 1 \leqslant i \leqslant t,\ v \in X_i\}$ is a spanning (k, U)-good tree of G. □

3 Proof of Theorems 2 and 3

Proposition 11. *Let $k \geqslant 2$ and $n \geqslant 2k+1$ be integers. Then for every $G \in \mathcal{G}_{k,n}$, G is a connected graph of order n and satisfies $\sigma_2(G) = n-2$.*

Proof. Let $G \in \mathcal{G}_{k,n}$. By the definition of $\mathcal{G}_{k,n}$, it is clear that G is a connected graph of order n. Let L_1, L_2, L_3 and L_4 be subsets of $V(G)$ satisfying (L1)–(L5). Then the following hold.

- For every $u_1 \in L_1$, it follows from (L2) and (L3) that $d_G(u_1) = |L_1 \cup L_2| - 1$.
- For every $u_2 \in L_2$, it follows from (L2) and (L4) that $d_G(u_2) = |L_1 \cup L_2| - 1 + |N_G(u_2) \cap L_3| \geqslant |L_1 \cup L_2|$. Since $d_G(u_2) \leqslant k$ for $u_2 \in L_2$, this implies that $|L_1 \cup L_2| \leqslant k$.
- For every $u_3 \in L_3$, it follows from (L5) that $d_G(u_3) \geqslant n - |L_1 \cup L_2| - 1$.
- For every $u_4 \in L_4$, it follows from (L1)–(L5) that $d_G(u_4) = |L_3 \cup L_4| - 1 = n - |L_1 \cup L_2| - 1$.

Since $n - |L_1 \cup L_2| - 1 \geqslant (2k+1) - k - 1 = k \geqslant |L_1 \cup L_2| - 1$, it follows that

- for $u \in L_1 \cup L_2$ and $u' \in L_3 \cup L_4$ with $uu' \notin E(G)$, $d_G(u) + d_G(u') \geqslant (|L_1 \cup L_2| - 1) + (n - |L_1 \cup L_2| - 1) = n - 2$,
- for $u_1 \in L_1$ and $u_4 \in L_4$, $d_G(u_1) + d_G(u_4) = (|L_1 \cup L_2| - 1) + (n - |L_1 \cup L_2| - 1) = n-2$, and
- for $u_3, u_3' \in L_3$ with $u_3 \neq u_3'$ and $u_3u_3' \notin E(G)$, $d_G(u_3) + d_G(u_3') \geqslant 2(n - |L_1 \cup L_2| - 1) \geqslant (n - |L_1 \cup L_2| - 1) + (|L_1 \cup L_2| - 1) = n-2$.

Consequently, we obtain $\sigma_2(G) = n - 2$. □

Now we prove Theorems 2 and 3.

Proof of Theorems 2 and 3. Let $k \geqslant 2$ be an integer. For an integer $n \geqslant n_0(k)$, since $0 \leqslant n - 4c_k\sqrt{n} - 2k^2 - 4k - 4 < n - 2k$, i.e., $n \geqslant 2k + 1$, it follows from Proposition 11 that each element of $\mathcal{G}_{k,n}$ satisfies the assumption of Theorem 2. Furthermore, if a graph G satisfies (L1)–(L5), then L_2 is a cutset of G, and so L_2 is a k-blocking set of G by (L4). As we mentioned in Section 1, if a graph has a k-blocking set, then the graph has no $[2, k]$-ST. Therefore, to complete the proof of Theorems 2 and 3, it suffices to show that a connected graph G of order n satisfying one of the following has a $[2, k]$-ST:

(G1) $n \geqslant n_0(k)$, $\sigma_2(G) \geqslant n - 2$ and G is not isomorphic to any graph in $\mathcal{G}_{k,n}$, or

(G2) (G1) does not hold, $n \geqslant n_1(k)$, $\sigma_2(G) \geqslant \frac{n+2k-2}{2}$ and G has no k-blocking set.

Since $k \geqslant 2$,

$$c_k = \sqrt{k(k-1)(k + 2\sqrt{2k} + 2)} > \sqrt{k(k-1)(k+2)} \geqslant \sqrt{k^3}. \tag{4}$$

By the definition of $n_0(k)$ and $n_1(k)$, we have

(N1) if $n \geqslant n_0(k)$, then $n \geqslant 4c_k\sqrt{n} + 2k^2 + 4k + 4$, and

(N'1) if $n \geqslant n_1(k)$, then $\frac{n+2k-2}{4} \geqslant 2c_k\sqrt{n} + k^2 + 2k + 1$.

By (4), if $k^3 \geqslant n_1(k)$, then

$$\begin{aligned} 0 &\leqslant \frac{k^3 + 2k - 2}{4} - 2c_k\sqrt{k^3} - k^2 - 2k - 1 \\ &< \frac{k^3 + 2k - 2}{4} - 2\sqrt{k^3} \cdot \sqrt{k^3} - k^2 - 2k - 1 \\ &= -\frac{7k^3 + 4k^2 + 6k + 6}{4} \\ &< 0, \end{aligned}$$

which is a contradiction. Thus $n_1(k) > k^3$. In particular, if $n \geqslant n_1(k)$, then

$$n - \frac{n + 2k - 2}{4} = \frac{3n - 2k + 2}{4} > \frac{3k^3 - 2k + 2}{4} > 0,$$

and by (4),

$$c_k\sqrt{n} > \sqrt{k^3} \cdot \sqrt{k^3} = k^3 \geqslant 2k^2. \tag{5}$$

Consequently, it follows from (N'1) that

(N'2) if $n \geqslant n_1(k)$, then $n > \frac{n+2k-2}{4} > c_k\sqrt{n} + 3k^2 + 2k + 1$.

(N'3) if $n \geqslant n_1(k)$, then $\frac{n+2k-2}{4} > 5k^2 + 2k + 1$.

Let $s_0 = n - 2$ if (G1) holds; and let $s_0 = \frac{n+2k-2}{2}$ if (G2) holds. Let $S = \{u \in V(G) : d_G(u) < \frac{s_0}{2}\}$. Then for vertices $u, u' \in S$ with $u \neq u'$, we have $d_G(u) + d_G(u') < s_0 \leqslant \sigma_2(G)$, and hence $uu' \in E(G)$. This implies that S is a clique of G.

If $\delta(G) \geqslant c_k\sqrt{n}$, then by Theorem 5, G has a $[2,k]$-ST. Thus we may assume that $\delta(G) < c_k\sqrt{n}$. If (G1) holds, then by (N1), $\frac{s_0}{2} = \frac{n-2}{2} \geqslant \frac{(4c_k\sqrt{n}+2k^2+4k+4)-2}{2} > c_k\sqrt{n}$; if (G2) holds, by (N'1), $\frac{s_0}{2} = \frac{n+2k-2}{4} \geqslant 2c_k\sqrt{n} + k^2 + 2k + 1 > c_k\sqrt{n}$. In either case, we have $\frac{s_0}{2} > c_k\sqrt{n}$, and hence there exists a vertex $u_0 \in S$ such that $d_G(u_0) = \delta(G)$ $(< c_k\sqrt{n})$. In particular, $S \neq \emptyset$. Since S is a clique of G,

$$|S| = |(N_G(u_0) \cap S) \cup \{u_0\}| \leqslant d_G(u_0) + 1 < c_k\sqrt{n} + 1. \tag{6}$$

Let $\mathcal{Q}$ be the family of components of $G - S$. If $\mathcal{Q} = \emptyset$, then by (6), $n = |S| < c_k\sqrt{n} + 1$, which contradicts (N1) or (N'2). Thus $\mathcal{Q} \neq \emptyset$. Furthermore, for each $Q \in \mathcal{Q}$,

$$\delta(Q) \geqslant \min\{d_G(v) : v \in V(Q)\} - |S| \geqslant \frac{s_0}{2} - |S| = \begin{cases} \frac{n-2}{2} - |S| & \text{(if (G1) holds)} \\ \frac{n+2k-2}{4} - |S| & \text{(if (G2) holds).} \end{cases} \tag{7}$$

Claim 12. *For $u \in S$ and $Q \in \mathcal{Q}$, $V(Q) \setminus N_G(u) \neq \emptyset$.*

Proof. Suppose that $V(Q) \subseteq N_G(u)$. Then $|V(Q)| + |S| - 1 = |V(Q) \cup (S \setminus \{u\})| \leqslant d_G(u) < \frac{s_0}{2}$. On the other hand, for a vertex $v \in V(Q)$, we have $|V(Q)| + |S| - 1 = |(V(Q) \setminus \{v\}) \cup S| \geqslant d_G(v) \geqslant \frac{s_0}{2}$, which is a contradiction. ☐

Take $Q_1 \in \mathcal{Q}$ so that $|V(Q_1)|$ is as small as possible. Then

$$|V(Q_1)| \leqslant \frac{n - |S|}{|\mathcal{Q}|}. \tag{8}$$

Claim 13. (i) *If (G1) holds, then $|\mathcal{Q}| = 1$.*

(ii) *If (G2) holds, then $|\mathcal{Q}| \leqslant 3$.*

(iii) *If (G2) holds and $|S| \leqslant 3k - 2$, then $|\mathcal{Q}| \leqslant 2$.*

Proof. Recall that u_0 is a vertex in S with $d_G(u_0) < c_k\sqrt{n}$. By Claim 12 with $(u, Q) = (u_0, Q_1)$, there exists a vertex $v_1 \in V(Q_1) \setminus N_G(u_0)$. In order to prove statements of the claim, we show that $d(u_0) + d(u_1)$ does not satisfy the degree-sum condition. By (8),

$$\begin{aligned} d_G(v_1) &\leqslant |V(Q_1) \setminus \{v_1\}| + |S \setminus \{u_0\}| \\ &\leqslant \left(\frac{n - |S|}{|\mathcal{Q}|} - 1\right) + (|S| - 1) \\ &= \frac{n + (|\mathcal{Q}| - 1)|S|}{|\mathcal{Q}|} - 2. \end{aligned} \tag{9}$$

To prove (i), (ii) and (iii), we prepare three equations as follows. If $n \geqslant n_0(k)$, then by (N1),

$$\begin{aligned} n-2-\left(c_k\sqrt{n}+\frac{n+c_k\sqrt{n}+1}{2}-2\right) &= \frac{n-3c_k\sqrt{n}-1}{2} \\ &\geqslant \frac{(4c_k\sqrt{n}+2k^2+4k+4)-3c_k\sqrt{n}-1}{2} \\ &\geqslant \frac{c_k\sqrt{n}+2k^2+4k+3}{2} \\ &> 0. \end{aligned} \tag{10}$$

If $n \geqslant n_1(k)$, then by (N'1),

$$\begin{aligned} &\frac{n+2k-2}{2}-\left(c_k\sqrt{n}+\frac{n+3c_k\sqrt{n}+3}{4}-2\right) \\ &= \frac{n+2k-2}{4}-\frac{7c_k\sqrt{n}-2k-3}{4} \\ &\geqslant (2c_k\sqrt{n}+k^2+2k+1)-\frac{7c_k\sqrt{n}-2k-3}{4} \\ &= \frac{c_k\sqrt{n}+4k^2+10k+7}{4} \\ &> 0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\frac{n+2k-2}{2}-\left(c_k\sqrt{n}+\frac{n+6k-4}{3}-2\right) \\ &= \frac{2}{3}\cdot\frac{n+2k-2}{4}-\frac{3c_k\sqrt{n}+4k-8}{3} \\ &\geqslant \frac{2(2c_k\sqrt{n}+k^2+2k+1)}{3}-\frac{3c_k\sqrt{n}+4k-8}{3} \\ &= \frac{c_k\sqrt{n}+2k^2+10}{3} \\ &> 0. \end{aligned} \tag{12}$$

Suppose that (G1) holds and $|\mathcal{Q}| \geqslant 2$. By (6), $\frac{n+(|\mathcal{Q}|-1)|S|}{|\mathcal{Q}|} \leqslant \frac{n+|S|}{2} < \frac{n+c_k\sqrt{n}+1}{2}$. This together with (9) and (10) implies that

$$\sigma_2(G) \leqslant d_G(u_0)+d_G(v_1) < c_k\sqrt{n}+\left(\frac{n+c_k\sqrt{n}+1}{2}-2\right) < n-2,$$

which contradicts the assumption on $\sigma_2(G)$ in (G1).

Suppose that (G2) holds. If $|\mathcal{Q}| \geqslant 4$, then it follows from (6) that $\frac{n+(|\mathcal{Q}|-1)|S|}{|\mathcal{Q}|} \leqslant \frac{n+3|S|}{4} < \frac{n+3(c_k\sqrt{n}+1)}{4}$, and hence by (9) and (11),

$$\sigma_2(G) \leqslant d_G(u_0)+d_G(v_1) < c_k\sqrt{n}+\left(\frac{n+3c_k\sqrt{n}+3}{4}-2\right) < \frac{n+2k-2}{2};$$

if $|S| \leqslant 3k-2$ and $|\mathcal{Q}| = 3$, then $\frac{n+(|\mathcal{Q}|-1)|S|}{|\mathcal{Q}|} \leqslant \frac{n+2(3k-2)}{3}$, and hence by (9) and (12),

$$\sigma_2(G) \leqslant d_G(u_0) + d_G(v_1) < c_k\sqrt{n} + \left(\frac{n+6k-4}{3} - 2\right) < \frac{n+2k-2}{2}.$$

In either case, we obtain a contradiction to the assumption on $\sigma_2(G)$ in (G2). □

A set $S' \subseteq S$ *dominates* $\mathcal{Q}$ if for each $Q \in \mathcal{Q}$, there exists a vertex $u \in S$ such that $N_G(u) \cap V(Q) \neq \emptyset$.

Claim 14. *If there exists a vertex $u \in S$ such that $\{u\}$ dominates $\mathcal{Q}$ and $d_G(u) \geqslant k+1$, then G has a $[2,k]$-ST.*

Proof. Since $|N_G(u) \setminus (\bigcup_{Q\in\mathcal{Q}} V(Q))| = |N_G(u) \cap S| = |S| - 1$, there exists a set $X \subseteq N_G(u) \cap (\bigcup_{Q\in\mathcal{Q}} V(Q))$ such that $|X| = \max\{|\mathcal{Q}|, k+1-(|S|-1)\}$ and $X \cap V(Q) \neq \emptyset$ for all $Q \in \mathcal{Q}$. Note that $1 \leqslant |X \cap V(Q)| \leqslant k+1$ for every $Q \in \mathcal{Q}$. Fix $Q \in \mathcal{Q}$. If (G1) holds, then by (N1), (6) and (7),

$$\begin{aligned}
\delta(Q) &\geqslant \frac{n-2}{2} - |S| \\
&> \frac{n-2}{2} - c_k\sqrt{n} - 1 \\
&\geqslant \frac{(4c_k\sqrt{n} + 2k^2 + 4k + 4) - 2}{2} - c_k\sqrt{n} - 1 \\
&= c_k\sqrt{n} + (k+1)^2 - 1 \\
&> c_k\sqrt{|V(Q)|} + (k+1)|X \cap V(Q)| - 1;
\end{aligned}$$

if (G2) holds, then by (N'1), (6) and (7),

$$\begin{aligned}
\delta(Q) &\geqslant \frac{n+2k-2}{4} - |S| \\
&> \frac{n+2k-2}{4} - c_k\sqrt{n} - 1 \\
&\geqslant (2c_k\sqrt{n} + k^2 + 2k + 1) - c_k\sqrt{n} - 1 \\
&= c_k\sqrt{n} + (k+1)^2 - 1 \\
&> c_k\sqrt{|V(Q)|} + (k+1)|X \cap V(Q)| - 1.
\end{aligned}$$

In either case, $\delta(Q) > c_k\sqrt{|V(Q)|} + (k+1)|X \cap V(Q)| - 1$. This together with Lemma 9 with $(G,U) = (Q, X \cap V(Q))$ implies that there exists a spanning forest of Q consisting of exactly $|X \cap V(Q)|$ components $F_{Q,1}, F_{Q,2}, \ldots, F_{Q,|X\cap V(Q)|}$ such that for every integer i with $1 \leqslant i \leqslant |X \cap V(Q)|$, $|V(F_{Q,i}) \cap X| = 1$ and $F_{Q,i}$ is a $(k, V(F_{Q,i}) \cap X)$-good tree. Let

$$T_1 := \left(\bigcup_{Q\in\mathcal{Q}} \left(\bigcup_{1\leqslant i\leqslant |X\cap V(Q)|} F_{Q,i}\right)\right) + \{uv : v \in (S \setminus \{u\}) \cup X\}.$$

Then $d_{T_1}(u) = (|S|-1) + |X| \geqslant (|S|-1) + k + 1 - (|S|-1) = k+1$, and hence T_1 is a $[2,k]$-ST of G. □

By Claim 14, we may assume that

$$\text{if } \{u\} \text{ dominates } \mathcal{Q}, \text{ then } d_G(u) \leqslant k. \tag{13}$$

Note that S dominates $\mathcal{Q}$ because G is connected. Choose a set $\tilde{S} \subseteq S$ dominating $\mathcal{Q}$ so that

(S1) $\tilde{S}$ is minimal, i.e., $\tilde{S} \setminus \{\tilde{u}\}$ does not dominate $\mathcal{Q}$ for every $\tilde{u} \in \tilde{S}$,

(S2) subject to (S1), $|\tilde{S}|$ is as large as possible, and

(S3) subject to (S2), $\sum_{\tilde{u} \in \tilde{S}} d_G(\tilde{u})$ is as large as possible.

If (G1) holds, then by Claim 13(i), $|\tilde{S}| = |\mathcal{Q}| = 1$; if (G2) holds, then by Claim 13(ii), $|\tilde{S}| \leqslant |\mathcal{Q}| \leqslant 3$. Write $\tilde{S} = \{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_s\}$ where $s = |\tilde{S}|$.

For $Q \in \mathcal{Q}$, let $A_Q = \{u \in S : N_G(u) \cap V(Q) \neq \emptyset\}$.

Claim 15. *Let $Q \in \mathcal{Q}$. If (G2) holds and $\max\{d_G(u) : u \in A_Q\} \leqslant k$, then $|\mathcal{Q}| = 1$ and $A_Q = S$.*

Proof. If $|\mathcal{Q}| \geqslant 2$ or $A_Q \neq S$, then A_Q is a cutset of G. Since G has no k-blocking set, this leads to the desired conclusion. □

Now we divide the proof into two cases.

Case 1: $|\tilde{S}| = 1$.

Note that $\tilde{S} = \{\tilde{u}_1\}$. By (13), $d_G(\tilde{u}_1) \leqslant k$.

Claim 16. *We have $|\mathcal{Q}| = 1$, i.e., $\mathcal{Q} = \{Q_1\}$.*

Proof. Suppose that $|\mathcal{Q}| \geqslant 2$. By Claim 13(i), we can assume (G2) holds. By Claim 15, for each $Q \in \mathcal{Q}$, there exists a vertex $u_Q \in A_Q$ with $d_G(u_Q) \geqslant k+1$. Note that $\{u_Q : Q \in \mathcal{Q}\}$ dominates $\mathcal{Q}$. Take a minimal set $\tilde{S}_0 \subseteq \{u_Q : Q \in \mathcal{Q}\}$ dominating $\mathcal{Q}$. Since $|\tilde{S}| = 1$, it follows from (S1) and (S2) that $|\tilde{S}_0| = 1$. However, $d_G(\tilde{u}_1) \leqslant k$ and $d_G(\tilde{u}) \geqslant k+1$ where $\tilde{u}$ is the unique element of $\tilde{S}_0$, which contradicts (S3). Thus $|\mathcal{Q}| = 1$, and so $\mathcal{Q} = \{Q_1\}$. □

Claim 17. *We have $A_{Q_1} = S$.*

Proof. It follows from (13) that

$$\max\{d_G(u) : u \in A_{Q_1}\} \leqslant k. \tag{14}$$

If (G2) holds, then by Claim 15 and (14), $A_{Q_1} = S$. Thus we may assume that (G1) holds.

Suppose that $A_{Q_1} \neq S$. Let $L_1 = S \setminus A_{Q_1}$, $L_2 = A_{Q_1}$, $L_3 = \bigcup_{u \in A_{Q_1}} (N_G(u) \cap V(Q_1))$ and $L_4 = V(Q_1) \setminus L_3$. Note that L_1, L_2 and L_3 are non-empty sets. By Claim 16, $V(G)$ is the disjoint union of L_1, L_2, L_3 and L_4.

For $u_2 \in L_2$ $(= A_{Q_1})$, it follows from (14) that

$$|N_G(u_2) \cap V(Q_1)| \leqslant d_G(u_2) - |L_1| \leqslant k - 1$$

and

$$|L_2| \leqslant |L_1 \cup L_2| - 1 \leqslant (|N_G(u_2) \cup \{u_2\}| - 1) - 1 \leqslant ((k+1) - 1) - 1. \tag{15}$$

In particular, $|L_3| \leqslant \sum_{u_2 \in L_2} |N_G(u_2) \cap V(Q_1)| \leqslant (k-1)^2$. Hence by (N1) and (15), $|L_4| = |V(Q_1) \setminus L_3| = n - |L_1 \cup L_2| - |L_3| \geqslant n - k - (k-1)^2 > 0$. Hence $L_4 \neq \emptyset$.

Let $u \in L_1$. For $v_3 \in L_3$,

$$n - 2 \leqslant \sigma_2(G) \leqslant d_G(u) + d_G(v_3) = |(L_1 \cup L_2) \setminus \{u\}| + d_G(v_3),$$

and hence $d_G(v_3) \geqslant n - |L_1 \cup L_2| - 1$. For $v_4 \in L_4$,

$$\begin{aligned} n - 2 \leqslant \sigma_2(G) &\leqslant d_G(u) + d_G(v_4) \\ &\leqslant |(L_1 \cup L_2) \setminus \{u\}| + |V(Q_1) \setminus \{v_4\}| \\ &= (|L_1 \cup L_2| - 1) + (n - |L_1 \cup L_2| - 1) \\ &= n - 2, \end{aligned}$$

which forces $N_G(v_4) = V(Q_1) \setminus \{v_4\}$. Since v_3 and v_4 are arbitrary, it follows from (14) that (G, L_1, L_2, L_3, L_4) satisfies (L1)–(L5). Consequently, G is isomorphic to a graph in $\mathcal{G}_{k,n}$, which contradicts (G1). □

By Claims 16 and 17, for each $u \in S$, there exists a vertex $v_u \in N_G(u) \cap V(Q_1)$. Let $W = \{v_u : u \in S\}$. Then

$$|W| \leqslant |S| = |S \setminus \{\tilde{u}_1\}| + 1 = (d_G(\tilde{u}_1) - |N_G(\tilde{u}_1) \cap V(Q_1)|) + 1 \leqslant k - 1 + 1 = k. \tag{16}$$

If (G1) holds, then by (N1), (7) and (16),

$$\begin{aligned} \delta(Q_1) &\geqslant \frac{n-2}{2} - |S| \\ &\geqslant \frac{n-2}{2} - k \\ &\geqslant \frac{(4c_k\sqrt{n} + 2k^2 + 4k + 4) - 2}{2} - k \\ &> c_k\sqrt{n} + k^2 - 1 \\ &> c_k\sqrt{|V(Q_1)|} + k|W| - 1; \end{aligned}$$

if (G2) holds, then by (N'2), (7) and (16),

$$\begin{aligned} \delta(Q_1) &\geqslant \frac{n+2k-2}{4} - |S| \\ &\geqslant \frac{n+2k-2}{4} - k \\ &> (c_k\sqrt{n} + 3k^2 + 2k + 1) - k \\ &> c_k\sqrt{n} + k^2 - 1 \\ &> c_k\sqrt{|V(Q_1)|} + k|W| - 1. \end{aligned}$$

In either case, we have $\delta(Q_1) > c_k\sqrt{|V(Q_1)|} + k|W| - 1$. Hence by Lemma 10 with $(G, U) = (Q_1, W)$, there exists a spanning (k, W)-good tree T of Q_1. Then $T + \{uv_u : u \in S\}$ is a $[2, k]$-ST of G.

Case 2: $|\tilde{S}| \in \{2, 3\}$.

By Claim 13(i), we can assume (G2) holds. For the moment, we suppose that $|S| \geqslant (k-1)|\tilde{S}| + 2$. Recall that $s = |\tilde{S}|$. Since $|S \setminus \tilde{S}| \geqslant (k-1)|\tilde{S}| - (|\tilde{S}| - 2)$, there exists a partition $\{S_1, S_2, \ldots, S_s\}$ of $S \setminus \tilde{S}$ such that

- if $|\tilde{S}| = 2$, then $|S_i| \geqslant k - 1$ for $i \in \{1, 2\}$, and
- if $|\tilde{S}| = 3$, then $|S_i| \geqslant k - 1$ for $i \in \{1, 3\}$ and $|S_2| \geqslant k - 2$.

Fix $Q \in \mathcal{Q}$. Since $\tilde{S}$ dominates $\mathcal{Q}$, we can take an edge $u_Q v_Q \in E(G)$ with $u_Q \in \tilde{S}$ and $v_Q \in V(Q)$. By (N'1), (6) and (7),

$$\begin{aligned}\delta(Q) &\geqslant \frac{n + 2k - 2}{4} - |S| \\ &> \frac{n + 2k - 2}{4} - c_k\sqrt{n} - 1 \\ &\geqslant (2c_k\sqrt{n} + k^2 + 2k + 1) - c_k\sqrt{n} - 1 \\ &> c_k\sqrt{n} + k - 1 \\ &> c_k\sqrt{|V(Q)|} + k|\{v_Q\}| - 1.\end{aligned}$$

Hence by Lemma 10 with $(G, U) = (Q, \{v_Q\})$, there exists a spanning (k, v_Q)-good tree T_Q of Q. Let $P = \tilde{u}_1\tilde{u}_2 \cdots \tilde{u}_s$ be a path on $\tilde{S}$, and let

$$T_1^* = \left(\left(\bigcup_{Q \in \mathcal{Q}} T_Q\right) \cup P\right) + (\{u_Q v_Q : Q \in \mathcal{Q}\} \cup \{\tilde{u}_i u : 1 \leqslant i \leqslant s,\ u \in S_i\}).$$

For $\tilde{u}_i \in \tilde{S}$, $|S_i| + d_P(\tilde{u}_i) \geqslant k$ and, by (S1), there exists $Q \in \mathcal{Q}$ such that $\tilde{u}_i = u_Q$, and hence $d_{T_1^*}(\tilde{u}_i) = d_P(\tilde{u}_i) + |\{Q \in \mathcal{Q} : \tilde{u}_i = u_Q\}| + |S_i| \geqslant k + 1$. This implies that T_1^* is a $[2, k]$-ST of G. Thus we may assume that $|S| \leqslant (k-1)|\tilde{S}| + 1$ $(\leqslant (k-1)|\mathcal{Q}| + 1)$.

If $|\mathcal{Q}| = 3$, then $|S| \leqslant 3k - 2$, which contradicts Claim 13(iii). Thus $|\mathcal{Q}| = |\tilde{S}| = 2$. In particular,

$$2 = |\tilde{S}| \leqslant |S| \leqslant 2(k-1) + 1 = 2k - 1. \tag{17}$$

Write $\mathcal{Q} \setminus \{Q_1\} = \{Q_2\}$. We may assume that $N_G(\tilde{u}_i) \cap V(Q_i) \neq \emptyset$ for each $i \in \{1, 2\}$. Then by (S1),

$$\text{for } i \in \{1, 2\},\ N_G(\tilde{u}_i) \cap V(Q_{3-i}) = \emptyset, \text{ i.e., } N_G(\tilde{u}_i) \subseteq (S \setminus \{\tilde{u}_i\}) \cup V(Q_i). \tag{18}$$

Claim 18. *If $d_G(u) \leqslant 2k - 1$ for every $u \in S$, then G has a $[2, k]$-ST.*

Proof. It follows from (18) that $N_G(\tilde{u}_2) \cap V(Q_1) = \emptyset$. Hence by (17) and the assumption of the claim, we have

$$\begin{aligned}\left|\bigcup_{u\in S}(N_G(u)\cap V(Q_1))\right| &\leqslant \sum_{u\in S\setminus\{\tilde{u}_2\}} |N_G(u)\cap V(Q_1)| \\ &\leqslant \sum_{u\in S\setminus\{\tilde{u}_2\}} (d_G(u) - |S\setminus\{u\}|) \\ &\leqslant (|S|-1)(2k-1-(|S|-1)) \\ &= (|S|-1)(2k-|S|) \\ &\leqslant (2k-2)^2.\end{aligned}$$

On the other hand, it follows from (N'3), (7) and (17) that

$$\begin{aligned}|V(Q_1)| &\geqslant \delta(Q_1)+1 \\ &\geqslant \frac{n+2k-2}{4} - |S| + 1 \\ &\geqslant \frac{n+2k-2}{4} - (2k-1) + 1 \\ &> (5k^2+2k+1) - (2k-1) + 1 \\ &> (2k-2)^2.\end{aligned}$$

Thus $V(Q_1) \setminus (\bigcup_{u\in S} N_G(u)) \neq \emptyset$. Let $v^* \in V(Q_1) \setminus (\bigcup_{u\in S} N_G(u))$. Recall that we choose Q_1 so that $|V(Q_1)|$ is as small as possible. Then by (8), $d_G(v^*) = |N_G(v^*) \cap V(Q_1)| \leqslant |V(Q_1) \setminus \{v^*\}| \leqslant \frac{n-|S|}{2} - 1$.

Fix $i \in \{1, 2\}$. Let $p_i = |N_G(\tilde{u}_i) \cap V(Q_i)|$. Since $\tilde{u}_i v^* \notin E(G)$,

$$\frac{n+2k-2}{2} \leqslant \sigma_2(G) \leqslant d_G(\tilde{u}_i) + d_G(v^*) \leqslant d_G(\tilde{u}_i) + \frac{n-|S|}{2} - 1,$$

and hence $d_G(\tilde{u}_i) \geqslant k + \frac{|S|}{2}$. This together with (18) implies that

$$p_i = d_G(\tilde{u}_i) - |S \setminus \{\tilde{u}_i\}| \geqslant k + \frac{|S|}{2} - (|S|-1) = k - \frac{|S|}{2} + 1. \tag{19}$$

Let $S'_1 \subseteq S \setminus \tilde{S}$ be a set with $|S'_1| = \lceil \frac{|S\setminus\tilde{S}|}{2} \rceil$, and let $S'_2 = S \setminus (\tilde{S} \cup S'_1)$. Note that $|S'_1| \geqslant |S'_2| = \lfloor \frac{|S\setminus\tilde{S}|}{2} \rfloor = \lfloor \frac{|S|-2}{2} \rfloor \geqslant \frac{|S|-3}{2}$. Let $W_i = (N_G(\tilde{u}_i) \cap V(Q_i)) \cup S'_i$. Then by (19),

$$|W_i| = p_i + |S'_i| \geqslant \left(k - \frac{|S|}{2} + 1\right) + \frac{|S|-3}{2} = k - \frac{1}{2}.$$

Since $|W_i|$ is an integer, this implies that $|W_i| \geqslant k$. By (18) and the assumption of the claim, we have

$$|N_G(\tilde{u}_i) \cap V(Q_i)| = p_i = d_G(\tilde{u}_i) - |N_G(\tilde{u}_i) \cap S| \leqslant (2k-1) - |\{\tilde{u}_{3-i}\}| = 2k-2.$$

Hence by (N'2), (7) and (17),

$$\begin{aligned}\delta(Q_i) &\geqslant \frac{n+2k-2}{4} - |S| \\ &\geqslant \frac{n+2k-2}{4} - (2k-1) \\ &> (c_k\sqrt{n} + 3k^2 + 2k + 1) - 2k + 1 \\ &> c_k\sqrt{n} + (k+1)(2k-2) - 1 \\ &> c_k\sqrt{|V(Q_i)|} + (k+1)|N_G(\tilde{u}_i) \cap V(Q_i)| - 1.\end{aligned}$$

This together with Lemma 9 with $(G, U) = (Q_i, N_G(\tilde{u}_i) \cap V(Q_i))$ implies that there exists a spanning forest of Q_i consisting of exactly p_i components $F'_{i,1}, F'_{i,2}, \ldots, F'_{i,p_i}$ such that for every integer j with $1 \leqslant j \leqslant p_i$, $|V(F'_{i,j}) \cap N_G(\tilde{u}_i)| = 1$ and $F'_{i,j}$ is a $(k, V(F'_{i,j}) \cap N_G(\tilde{u}_i))$-good tree. Then

$$\left(\bigcup_{i\in\{1,2\}} \left(\bigcup_{1\leqslant j\leqslant p_i} F'_{i,j}\right)\right) + (\{\tilde{u}_i v : i \in \{1,2\},\ v \in W_i\} \cup \{\tilde{u}_1\tilde{u}_2\})$$

is a $[2, k]$-ST of G. □

By Claim 18, we may assume that $\max\{d_G(u) : u \in S\} \geqslant 2k$. Since $|S| \leqslant 2k-1$, a vertex $u' \in S$ with $d_G(u') = \max\{d_G(u) : u \in S\}$ satisfies $N_G(u') \setminus S \neq \emptyset$. This together with (13) and (S3) forces $d_G(\tilde{u}_{i_0}) \geqslant 2k$ for some $i_0 \in \{1,2\}$. Furthermore, it follows from Claim 15 that $\max\{d_G(u) : u \in A_{Q_{3-i_0}}\} \geqslant k+1$. Hence by (13) and (S2), we have $d_G(\tilde{u}_{3-i_0}) \geqslant k+1$. Take a set $Z_{3-i_0} \subseteq N_G(\tilde{u}_{3-i_0}) \setminus \{\tilde{u}_{i_0}\}$ such that $|Z_{3-i_0}| = k$ and $Z_{3-i_0} \cap V(Q_{3-i_0}) \neq \emptyset$. Then by (18), $|N_G(\tilde{u}_{i_0}) \cap Z_{3-i_0}| = |Z_{3-i_0}| - |Z_{3-i_0} \cap V(Q_{3-i_0})| \leqslant k-1$, and hence $|N_G(\tilde{u}_{i_0}) \setminus (Z_{3-i_0} \cup \{\tilde{u}_{3-i_0}\})| \geqslant 2k - ((k-1)+1) = k$. In particular, we can take a set $Z_{i_0} \subseteq N_G(\tilde{u}_{i_0}) \setminus (Z_{3-i_0} \cup \{\tilde{u}_{3-i_0}\})$ such that $1 \leqslant |Z_{i_0} \cap V(Q_{i_0})| \leqslant k \leqslant |Z_{i_0}|$ and $S \setminus (Z_{3-i_0} \cup \tilde{S}) \subseteq Z_{i_0}$.

Fix $i \in \{1,2\}$. Then by the definition of Z_i, we have $q_i := |Z_i \cap V(Q_i)| \leqslant k$. By (N'2), (7) and (17),

$$\begin{aligned}\delta(Q_i) &\geqslant \frac{n+2k-2}{4} - |S| \\ &\geqslant \frac{n+2k-2}{4} - (2k-1) \\ &> (c_k\sqrt{n} + 3k^2 + 2k + 1) - 2k + 1 \\ &> c_k\sqrt{n} + (k+1)k - 1 \\ &> c_k\sqrt{|V(Q_i)|} + (k+1)|Z_i \cap V(Q_i)| - 1.\end{aligned}$$

Hence by Lemma 9 with $(G, U) = (Q_i, Z_i \cap V(Q_i))$, there exists a spanning forest of Q_i consisting of exactly q_i components $F_{i,1}, F_{i,2}, \ldots, F_{i,q_i}$ such that for every integer i with

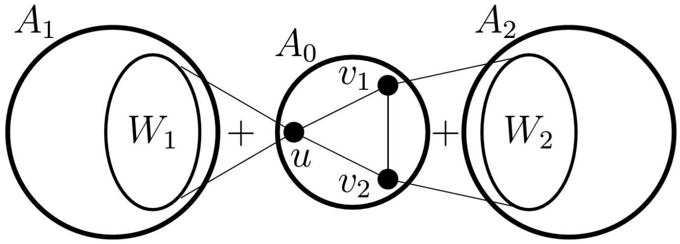

Figure 2: Graph $G'_{k,n}$.

$1 \leqslant j \leqslant q_i$, $|V(F_{i,j}) \cap Z_i| = 1$ and $F_{i,j}$ is a $(k, V(F_i) \cap Z_i)$-good tree. Then

$$\left(\bigcup_{i\in\{1,2\}} \left(\bigcup_{1\leqslant j\leqslant q_i} F_{i,j} \right) \right) + (\{\tilde{u}_i v : i \in \{1,2\},\ v \in Z_i\} \cup \{\tilde{u}_1\tilde{u}_2\})$$

is a $[2,k]$-ST of G.

This completes the proof of Theorem 3. □

The degree-sum condition in Theorem 3 is best possible in a sense: Let $k \geqslant 2$ be an integer, and let n be an odd integer with $n \geqslant n_1(k)$. Since $n \geqslant 2k^2+1$ by (N'2), we have $\frac{n-3}{2}-(k+1) = \frac{n-2k-5}{2} \geqslant \frac{(2k^2+1)-2k-5}{2} = (k+1)(k-2) \geqslant 0$, i.e., $\frac{n-3}{2} \geqslant k+1$. Let A_0, A_1 and A_2 be vertex-disjoint complete graphs with $|V(A_0)| = 3$ and $|V(A_1)| = |V(A_2)| = \frac{n-3}{2}$. Write $V(A_0) = \{u, v_1, v_2\}$, and for each $i \in \{1,2\}$, take a set $W_i \subseteq V(A_i)$ with $|W_i| = k-1$. Let $G'_{k,n} = (\bigcup_{0\leqslant i\leqslant 2} A_i) + (\{uw : w \in W_1\} \cup \{v_1w, v_2w : w \in W_2\})$ (see Figure 2). Then $G'_{k,n}$ is a connected graph of order n and $\sigma_2(G'_{k,n}) = \frac{n+2k-3}{2}$. Furthermore, $G'_{k,n}$ has no k-blocking set. Thus the following proposition gives a sharpness of Theorem 3.

Proposition 19. *There exists no $[2,k]$-ST of $G'_{k,n}$.*

Proof. Suppose that $G'_{k,n}$ has a $[2,k]$-ST T. Since u is a cut-vertex of $G'_{k,n}$, we have $d_T(u) = k+1$; since $\{v_1, v_2\}$ is a cutset of $G'_{k,n}$, we have $d_T(v_i) = k+1$ for some $i \in \{1,2\}$. This implies that uv_1v_2u is a cycle of T, which contradicts the fact that T is a tree. □

4 Proof of Theorem 6

Since $p_k = \left(\frac{3c_k+\sqrt{c_k^2+4k^2+8k+4}}{2}\right)^2$, we have

$$\sqrt{p_k} = \frac{3c_k + \sqrt{c_k^2 + 4k^2 + 8k + 4}}{2} > 2c_k. \tag{20}$$

By (20), we obtain the following:

(M1) Since $c_k \geqslant 4$, we have $\sqrt{p_k} > c_k + \frac{1}{c_k}$, and hence

$$c_k(\sqrt{p_k} - c_k) - 1 > c_k\left(\left(c_k + \frac{1}{c_k}\right) - c_k\right) - 1 = 0.$$

(M2) Since $\sqrt{p_k} > \frac{c_k+\sqrt{c_k^2+4k}}{2}$, we have

$$\begin{aligned} p_k - \frac{k\sqrt{p_k}}{\sqrt{p_k} - c_k} &= \frac{\sqrt{p_k}(\sqrt{p_k}(\sqrt{p_k} - c_k) - k)}{\sqrt{p_k} - c_k} \\ &> \frac{\sqrt{p_k}\left(\frac{c_k+\sqrt{c_k^2+4k}}{2}\left(\frac{c_k+\sqrt{c_k^2+4k}}{2} - c_k\right) - k\right)}{\sqrt{p_k} - c_k} \\ &= 0. \end{aligned}$$

(M3) We have

$$\begin{aligned} &(\sqrt{p_k} - 2c_k)(\sqrt{p_k} - c_k) \\ &\quad = \left(\frac{3c_k + \sqrt{c_k^2 + 4k^2 + 8k + 4}}{2} - 2c_k\right)\left(\frac{3c_k + \sqrt{c_k^2 + 4k^2 + 8k + 4}}{2} - c_k\right) \\ &\quad = k^2 + 2k + 1. \end{aligned}$$

Let $S = \{u \in V(G) : d_G(u) < \sqrt{p_k n}\}$. Then for vertices $u, u' \in S$ with $u \neq u'$, we have $d_G(u)d_G(u') < p_k n \leqslant \pi_2(G)$, and hence $uu' \in E(G)$. This implies that S is a clique of G.

If $\delta(G) \geqslant c_k\sqrt{n}$, then it follows from Theorem 5 that G has a $[2,k]$-ST. Thus we may assume that $\delta(G) < c_k\sqrt{n}$. This together with (20) implies that $\delta(G) < \sqrt{p_k n}$. In particular, $S \neq \emptyset$. Since S is a clique of G, for a vertex $u_0 \in S$ with $d_G(u_0) = \delta(G)$, we have

$$|S| = |(N_G(u_0) \cap S) \cup \{u\}| \leqslant d_G(u_0) + 1 < c_k\sqrt{n} + 1. \tag{21}$$

Let $\mathcal{Q}$ be the family of components of $G - S$. If $\mathcal{Q} = \emptyset$, i.e., $G = G[S]$, then G is a complete graph of order at least $k+2$, and hence G has a $[2,k]$-ST. Thus we may assume that $\mathcal{Q} \neq \emptyset$. By (20) and (21), for each $Q \in \mathcal{Q}$,

$$\delta(Q) \geqslant \min\{d_G(v) : v \in V(Q)\} - |S| > \sqrt{p_k n} - (c_k\sqrt{n} + 1). \tag{22}$$

Claim 20. *For $u \in S$ and $Q \in \mathcal{Q}$, $V(Q) \setminus N_G(u) \neq \emptyset$.*

Proof. Suppose that $V(Q) \subseteq N_G(u)$. Then $|V(Q)| + |S| - 1 = |V(Q) \cup (S \setminus \{u\})| \leqslant d_G(u) < \sqrt{p_k n}$. On the other hand, for a vertex $v \in V(Q)$, we have $|V(Q)| + |S| - 1 = |(V(Q) \setminus \{v\}) \cup S| \geqslant d_G(v) \geqslant \sqrt{p_k n}$, which is a contradiction. □

Take $Q_1 \in \mathcal{Q}$ so that $|V(Q_1)|$ is as small as possible. Then

$$|V(Q_1)| \leqslant \frac{n-|S|}{|\mathcal{Q}|} < \frac{n}{|\mathcal{Q}|}. \tag{23}$$

Claim 21. *We have $|\mathcal{Q}| < \frac{\sqrt{n}}{\sqrt{p_k}-c_k}$, and in particular, $\sqrt{n} > \sqrt{p_k} - c_k$.*

Proof. By Claim 20, there exists a vertex $v \in V(Q_1)$ such that $S \not\subseteq N_G(v)$, and hence

$$|V(Q_1) \cup S| \geqslant d_G(v) + 2 \geqslant \sqrt{p_k n} + 2. \tag{24}$$

If $|\mathcal{Q}| \geqslant \frac{\sqrt{n}}{\sqrt{p_k}-c_k}$, then it follows from (21) and (23) that

$$|V(Q_1) \cup S| < \frac{n}{|\mathcal{Q}|} + |S| < \frac{n}{\frac{\sqrt{n}}{\sqrt{p_k}-c_k}} + (c_k\sqrt{n}+1) = \sqrt{p_k n} + 1,$$

which contradicts (24). □

A set $S' \subseteq S$ *dominates* $\mathcal{Q}$ if for each $Q \in \mathcal{Q}$, there exists a vertex $u \in S$ such that $N_G(u) \cap V(Q) \neq \emptyset$. Note that S dominates $\mathcal{Q}$ because G is connected. Take a minimum set $\tilde{S} \subseteq S$ dominating $\mathcal{Q}$, and write $\tilde{S} = \{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_s\}$ where $s = |\tilde{S}|$. By Claim 21 and the minimality of $\tilde{S}$,

$$s \leqslant |\mathcal{Q}| < \frac{\sqrt{n}}{\sqrt{p_k} - c_k}. \tag{25}$$

Claim 22. *For each $\tilde{u} \in \tilde{S}$, $d_G(\tilde{u}) \geqslant sk + 1$.*

Proof. Suppose that $d_G(\tilde{u}) \leqslant sk$. By Claim 20, there exists a vertex $v \in V(Q_1)$ such that $v\tilde{u} \notin E(G)$. By (23) and (25), $d_G(v) < |V(Q_1) \cup S| \leqslant \frac{n-|S|}{|\mathcal{Q}|} + |S| \leqslant \frac{n-|S|}{s} + |S|$. This together with (M1), (M2), (21) and (25) leads to

$$\begin{aligned}
\pi_2(G) &\leqslant d_G(\tilde{u})d_G(v) \\
&< sk\left(\frac{n-|S|}{s} + |S|\right) \\
&= k(n + (s-1)|S|) \\
&< k\left(n + \left(\frac{\sqrt{n}}{\sqrt{p_k}-c_k} - 1\right)(c_k\sqrt{n}+1)\right) \\
&= k\left(\left(1 + \frac{c_k}{\sqrt{p_k}-c_k}\right)n - \frac{(c_k(\sqrt{p_k}-c_k)-1)\sqrt{n}}{\sqrt{p_k}-c_k} - 1\right) \\
&< \frac{k\sqrt{p_k}n}{\sqrt{p_k}-c_k} \\
&< p_k n,
\end{aligned}$$

which is a contradiction. □

By Claim 22, $|N_G(\tilde{u}) \setminus \tilde{S}| \geqslant sk + 1 - (s-1) = sk - s + 2$ for every $\tilde{u} \in \tilde{S}$. Hence there exist s disjoint subsets $W_1, W_2, \ldots, W_s$ of $V(G) \setminus \tilde{S}$ such that for each integer i with $1 \leqslant i \leqslant s$, $W_i \subseteq N_G(\tilde{u}_i)$ and

$$|W_i| = \begin{cases} k+1 & (\text{if } s = 1) \\ k & (\text{if } s \geqslant 2 \text{ and } i \in \{1, s\}) \\ k-1 & (\text{if } s \geqslant 2 \text{ and } 2 \leqslant i \leqslant s-1). \end{cases}$$

Let $W = \bigcup_{1 \leqslant i \leqslant s} W_i$. Note that $|W| = sk - s + 2$.

By the minimality of $\tilde{S}$, for each integer i with $1 \leqslant i \leqslant s$, there exists $D_i \in \mathcal{Q}$ such that $(\bigcup_{v \in V(D_i)} N_G(v)) \cap \tilde{S} = \{\tilde{u}_i\}$. We may assume that $V(D_i) \cap W \neq \emptyset$. For each $Q \in \mathcal{Q}$, if $Q \in \{D_i : 1 \leqslant i \leqslant s\}$, then $1 \leqslant |V(Q) \cap W| \leqslant k + 1$ $(\leqslant k + (s-1)(k-2) + 1)$; otherwise, $|V(Q) \cap W| \leqslant |W| - \sum_{1 \leqslant i \leqslant s} |V(D_i) \cap W| \leqslant sk - 2s + 2$. In either case, we have $|V(Q) \cap W| \leqslant sk - 2s + 3$. Let $\mathcal{Q}_1 = \{Q \in \mathcal{Q} : V(Q) \cap W = \emptyset\}$. For each $Q \in \mathcal{Q}_1$, it follows from the definition of $\tilde{S}$, there exists an edge $\tilde{u}_Q v_Q$ $(\tilde{u}_Q \in \tilde{S},\ v_Q \in V(Q))$ of G. Let $W' = W \cup \{v_Q : Q \in \mathcal{Q}_1\}$.

Fix $Q \in \mathcal{Q}$. Then $1 \leqslant |W' \cap V(Q)| \leqslant sk - 2s + 3$, and hence by Claim 21, (M3), (22) and (25),

$$\begin{aligned}
\delta(Q) &> (\sqrt{p_k} - c_k)\sqrt{n} - 1 \\
&= c_k\sqrt{n} + (\sqrt{p_k} - 2c_k)\sqrt{n} - 1 \\
&> c_k\sqrt{|V(Q)|} + (\sqrt{p_k} - 2c_k)\sqrt{n} - 1 \\
&= c_k\sqrt{|V(Q)|} + \frac{(k+1)^2\sqrt{n}}{\sqrt{p_n} - c_k} - 1 \\
&= c_k\sqrt{|V(Q)|} + \frac{(k+1)((k-2)\sqrt{n} + 3\sqrt{n})}{\sqrt{p_n} - c_k} - 1 \\
&> c_k\sqrt{|V(Q)|} + \frac{(k+1)((k-2)\sqrt{n} + 3(\sqrt{p_n} - c_k))}{\sqrt{p_n} - c_k} - 1 \\
&> c_k\sqrt{|V(Q)|} + (k+1)(s(k-2) + 3) - 1 \\
&\geqslant c_k\sqrt{|V(Q)|} + (k+1)|W' \cap V(Q)| - 1.
\end{aligned}$$

This together with Lemma 9 with $(G, U) = (Q, W' \cap V(Q))$ implies that there exists a spanning forest of Q consisting of exactly $|W' \cap V(Q)|$ components $F_{Q,1}, F_{Q,2}, \ldots, F_{Q,|W' \cap V(Q)|}$ such that for every integer i with $1 \leqslant i \leqslant |W' \cap V(Q)|$, $|V(F_{Q,i}) \cap W'| = 1$ and $F_{Q,i}$ is a $(k, V(F_{Q,i}) \cap W')$-good tree. Let H be the graph obtained from the path $\tilde{u}_1\tilde{u}_2 \cdots \tilde{u}_s$ by joining $\tilde{u}_1$ and all vertices in $S \setminus (\tilde{S} \cup W')$. Then

$$\left(\left(\bigcup_{Q \in \mathcal{Q}} \left(\bigcup_{1 \leqslant i \leqslant |W' \cap V(Q)|} F_{Q,i}\right)\right) \cup H\right) + (\{\tilde{u}_i w : 1 \leqslant i \leqslant s,\ w \in W_i\} \cup \{\tilde{u}_Q v_Q : Q \in \mathcal{Q}_1\})$$

is a $[2, k]$-ST of G.

This completes the proof of Theorem 6.

Acknowledgments

The authors would like to thank Professor Akira Saito for helpful comments. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. This work is also supported by JSPS KAKENHI Grant numbers, 23K03204 (to M.F.), 25K07107 (to S.T.) and research grant of Senshu University 2025 (to S.T.).

References

[1] M.O. Albertson, D.M. Berman, J.P. Hutchinson and C. Thomassen, Graphs with homeomorphically irreducible spanning trees, J. Graph Theory **14** (1990), 247–258.

[2] G. Chen, H. Ren and S. Shan, Homeomorphically irreducible spanning trees in locally connected graphs, Combin. Probab. Comput. **21** (2012), 107–111.

[3] G. Chen and S. Shan, Homeomorphically irreducible spanning trees, J. Combin. Theory Ser. B **103** (2013), 409–414.

[4] R. Diestel, "Graph Theory" (5th edition), Graduate Texts in Mathematics **173**, Springer, Berlin (2017).

[5] G.A. Dirac, Some theorems on abstract graphs, Proc. Lond. Math. Soc. **2** (1952), 69–81.

[6] M. Furuya, A. Saito and S. Tsuchiya, Refinements of degree conditions for the existence of a spanning tree without small degree stems, Discrete Math. **348** (2025), 114307.

[7] M. Furuya and S. Tsuchiya, Forbidden subgraphs and the existence of a spanning tree without small degree stems, Discrete Math. **313** (2013), 2206–2212.

[8] R. Halin, Studies on minimally n-connected graphs, In *Combinatorial Mathematics and its Applications*, Academic Press, London (1969) pp. 129–136.

[9] A. Hoffmann-Ostenhof, K. Noguchi and K. Ozeki, On homeomorphically irreducible spanning trees in cubic graphs, J. Graph Theory **89** (2018), 93–100.

[10] T. Ito and S. Tsuchiya, Degree sum conditions for the existence of homeomorphically irreducible spanning trees, J. Graph Theory **99** (2022), 162–170.

[11] O. Ore, Note on Hamilton circuits, Amer. Math. Monthly **67** (1960), 67.

[12] S. Shan and S. Tsuchiya, Characterization of graphs of diameter 2 containing a homeomorphically irreducible spanning tree, J. Graph Theory **104** (2023), 886–903.

[13] S. Zhai, E. Wei, J. He and D. Ye, Homeomorphically irreducible spanning trees in hexangulations of surfaces, Discrete Math. **342** (2019), 2893–2899.