On the Number of Generalized Numerical Semigroups

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Abstract

Let r_k be the unique positive root of $x^k-(x+1)^{k-1}=0$. We prove the best known bounds on the number $n_{g,d}$ of d-dimensional generalized numerical semigroups of genus g, in particular that

$$n_{g,d} > C_d^{g^{(d-1)/d}} \mathsf{r}_{2^d}^g$$

for some constant $C_d > 0$, which can be made explicit. To do this, we extend the notion of multiplicity and depth to generalized numerical semigroups and show our lower bound is sharp for semigroups of depth 2. We also show other bounds on special classes of semigroups by introducing *partition labelings*, which extend the notion of Kunz words to the general setting.

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1 Introduction

Let $\mathbb{N}_0$ denote the nonnegative integers. A *numerical semigroup* Λ is a subset of $\mathbb{N}_0$ that has finite complement, contains 0, and is closed under addition. In other words, Λ is a cofinite submonoid of $\mathbb{N}_0$. Given a numerical semigroup Λ, one can define a number of invariants of Λ: its *genus* $g(\Lambda) := \#(\mathbb{N}_0 \setminus \Lambda)$; its *multiplicity* $m(\Lambda) := \min\{\Lambda \setminus \{0\}\}$; its *Frobenius number* $f(\Lambda) := \max\{\mathbb{N}_0 \setminus \Lambda\}$; and its *depth* $q(\Lambda) := \lceil (1 + f(\Lambda))/m(\Lambda) \rceil$, the last of which was recently introduced by Eliahou and Fromentin [10]. We drop the Λ when writing down these invariants if it is clear to which semigroup we are referring.

A great deal of research has been done regarding the enumeration of numerical semigroups after ordering by a specific invariant. Perhaps the most significant result is due to Zhai [23], who in 2011 showed that the number of numerical semigroups with genus g is asymptotic to $C\left(\frac{1+\sqrt{5}}{2}\right)^g$ for some constant C, resolving a conjecture of Bras-Amorós [5]. The asymptotic number of numerical semigroups with fixed Frobenius number [3], multiplicity [13], and recently depth with respect to genus [25] and multiplicity [17] are also

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known. A key ingredient in many of these proofs is that, after ordering, almost all numerical semigroups have small depth. For instance, Zhai showed that almost all numerical semigroups of genus g have depth 2 or 3, verifying a conjecture of Zhao [24, Conj. 2].

Naturally, one can ask similar questions about the analogous objects in $\mathbb{N}_0^d$. A *generalized numerical semigroup* (or *GNS*) Λ^d of dimension d is a cofinite submonoid of $\mathbb{N}_0^d$. In 2016, Failla, Peterson, and Utano [11] initiated the formal study on the number $n_{g,d}$ of generalized numerical semigroups with genus g and dimension d, showing that for fixed d and large g we have

$$n_{g,d} \gtrsim \binom{g+d-1}{d-1} C^d \left(\frac{1+\sqrt{5}}{2}\right)^g$$

by estimating the number of generalized numerical semigroups that contain every point with positive integer coordinates. Their bound implies that $n_{g,d}^{1/g} \gtrsim \frac{1+\sqrt{5}}{2}$ for fixed d. To our knowledge, the above bound is the best known bound on $n_{g,d}$. Currently, the asymptotics of $n_{g,d}$ are wide open; even for $d = 2$, we do not have a Bras-Amorós-like conjecture for the growth rate [14]. However, Cisto, Delgado, and García-Sánchez [7] have developed algorithms to compute $n_{g,d}$ for small values.

In this paper, we constructively show the following main result, which significantly improves the lower bound on $n_{g,d}$.

Theorem 1. *Let r_k be the unique positive root of $x^k - (x+1)^{k-1} = 0$. Then for each d, there is a constant $C_d > 0$ for which*

$$n_{g,d} > C_d^{g^{(d-1)/d}} \mathsf{r}_{2^d}^g$$

for all $g > 0$.

In particular, we have that $n_{g,d}^{1/g} \gtrsim \mathsf{r}_{2^d}$ for fixed d. Hence, Theorem 1 is stronger than the lower bound from Failla, Peterson, and Utano for $d \geqslant 2$. This also implies that $\lim_{g\to\infty}(n_{g,1})^2/n_{g,2} = 0$, since $\left(\frac{1+\sqrt{5}}{2}\right)^2 < \mathsf{r}_4$, negatively answering a question of Cisto, Delgado, and García-Sánchez [7, §8, pg. 16]. We furthermore establish the following superexponential upper bound on $n_{g,d}$.

Theorem 2. *For each d, we have $n_{g,d}^{1/g} < (2e + o(1))(\ln g)^{d-1}$.*

Analyzing $n_{g,d}$ is difficult in part because numerical semigroup invariants are not easy to lift to the general case. For instance, Failla, Peterson, and Utano define the multiplicity and Frobenius gap of a generalized numerical semigroup, but they are not canonical; they depend on a choice of relaxed monomial order on $\mathbb{N}_0^d$ [11, Def. 3.5]. To this end, Cisto, Failla, Peterson, and Utano define the *Frobenius allowable gaps* [8], which Lin and Singhal [21] showed to be the set of maximal elements of $\mathbb{N}_0^d \setminus \Lambda^d$ under the natural partial order and used them to count semigroups with a unique Frobenius allowable gap.

In light of this, a key ingredient of our paper is our extension of multiplicity and depth to the general setting. We define the *multset* to be the set of nonzero minimal elements of Λ^d under the natural partial order. Then, we generalize the notion of depth to generalized numerical semigroups and show that Theorem 1 is sharp for generalized numerical semigroups of depth 2. An advantage of our more flexible definition is that it does not rely on a choice of relaxed monomial order on $\mathbb{N}_0^d$.

An important tool used in the study of numerical semigroups is the Kunz word, which is a word with integer entries satisfying certain linear inequalities that encodes a numerical semigroup. This correspondence allows us to turn the problem of enumerating numerical semigroups into a problem in polyhedral geometry [13, 15] and additive combinatorics [17, 25].

We lift the notion of Kunz words to generalized numerical semigroups by introducing *partition labelings*, which are diagrams labeled with multi-dimensional partitions that satisfy certain additive inequalities that encode the data of a GNS. We use partition labelings to bound the number of semigroups with a fixed multset and the number of semigroups with a multset of minimal size, the latter of which we call *rectangular GNSs*.

1.1 Outline

In Section 2, we establish conventions and review the relevant technical facts about d-dimensional partitions and the constants r_k for our paper. Next, in Section 3, we define the multset, depth, and depth-k regions of a GNS, which generalize numerical semigroup invariants. In Section 4, we prove Theorem 1 and Theorem 2. Then in Section 5, we define partition labelings and use them to bound the size of special classes of semigroups. Finally, we discuss open questions and future lines of work in Section 6.

2 Preliminaries

In this section, we go over some background material on the structure of $\mathbb{N}_0^d$ and d-dimensional partitions, then discuss the constants r_k which appear throughout the paper.

2.1 Points in $\mathbb{N}_0^d$

We denote points in $\mathbb{N}_0^d$ by bolded lowercase letters and their coordinates by unbolded letters with subscripts, e.g., $\mathbf{x} = (x_1, \ldots, x_d)$. The points in $\mathbb{N}_0^d$ have a natural partial order $\leqslant$. Namely, we let $\mathbf{a} \leqslant \mathbf{b}$ if $a_i \leqslant b_i$ for $i = 1, \ldots, d$, with equality if and only if $a_i = b_i$ for all i. Hence, the unique minimal element of $\mathbb{N}_0^d$ is the origin $\mathbf{0} := (0, \ldots, 0)$.

Also, let $\mathbf{e_i^d}$ be the point in $\mathbb{N}_0^d$ with $(e_i^d)_j = 0$ when $i \neq j$ and $(e_i^d)_j = 1$ when $i = j$, so $\mathbf{e_1^d}, \ldots, \mathbf{e_d^d}$ generate $\mathbb{N}_0^d$ as an additive monoid.

2.2 d-dimensional partitions

A *d-dimensional partition* of n is a partition π into nonnegative integer parts $\pi_{\mathbf{x}}$, indexed by $\mathbf{x} \in \mathbb{N}_0^d$, for which

$$n = \sum_{\mathbf{x} \in \mathbb{N}_0^d} \pi_{\mathbf{x}} \qquad \text{and} \qquad \pi_{\mathbf{a}} \geqslant \pi_{\mathbf{b}} \text{ if } \mathbf{a} \leqslant \mathbf{b}.$$

For $d = 0$, these consist of a single number; for $d = 1$, these are the typical partitions of n. The cases $d = 2$ and $d = 3$ are known as *plane partitions* and *solid partitions*, respectively.

One can visually represent a d-dimensional partition in $(d + 1)$-dimensional space by stacking $\pi_{\mathbf{x}}$ hypercubes of dimension $d + 1$ atop the axis-aligned unit d-cell whose least vertex (under the partial order) is at $\mathbf{x}$. This is known as the *Young diagram* of π in the one-dimensional case.

Throughout the paper, we let $p_d(n)$ denote the number of d-dimensional partitions of n and let $P_d(x) := \sum_{n=0}^{\infty} p_d(n)x^n$ be the d-dimensional partition generating function. Recall that

$$P_0(x) := \sum_{n\geqslant 0} x^n = \frac{1}{1-x},$$

$$P_1(x) := \sum_{n\geqslant 0} p_1(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

It is also a classic result of MacMahon [19] that

$$P_2(n) := \sum_{n\geqslant 0} p_2(n)x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k}.$$

From these expressions, one can calculate the following asymptotic expressions for $p_1(n)$ and $p_2(n)$, first calculated in the 1900's by Hardy-Ramanujan [12] and Wright [22], respectively:

$$p_1(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right),$$

$$p_2(n) \sim \frac{\zeta(3)^{7/36}}{\sqrt{12\pi}} \left(\frac{n}{2}\right)^{-25/36} \exp\left(3\zeta(3)^{1/3}\left(\frac{n}{2}\right)^{2/3} + \zeta'(-1)\right),$$

where ζ denotes the Riemann zeta function.

Unfortunately, a closed-form expression for $P_d(n)$ for $d \geqslant 3$ and the precise asymptotics of $p_d(n)$ are unknown. However, of use to us is the following asymptotic result on $p_d(n)$ due to Arora, Bhatia, and Prasad [4].

Theorem 3 (Arora-Bhatia-Prasad, [4, eq. 6])**.** *For each d, there exist $k_d^-, k_d^+ > 0$ for which*

$$k_d^- < \frac{\ln p_d(n)}{n^{d/(d+1)}} < k_d^+.$$

More information on multi-dimensional partitions can be found in [1]. A possible point of confusion is the discrepancy between the dimension of a generalized numerical semigroup and the dimension of a multi-dimensional partition; when working with a d-dimensional GNS, we usually opt to work with $(d-1)$-dimensional partitions. To make this distinction clear, especially with indices, we use the convention $\mathbf{a}, \mathbf{b}, \mathbf{e_i^d}, \mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{N}_0^d$ and $\mathbf{e_i^{d-1}}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{N}_0^{d-1}$.

2.3 Min-sums

In this paper, we introduce the following operation on d-dimensional partitions.

Definition 4. For two d-dimensional partitions π and π', the *min-sum* $\pi \boxplus \pi'$ is the d-dimensional partition τ with entries $\tau_{\mathbf{x}} = \min_{\mathbf{a}+\mathbf{b}=\mathbf{x}} \{\pi_{\mathbf{a}} + \pi'_{\mathbf{b}}\}$.

One can check that $\boxplus$ is commutative and associative. To our knowledge, this operation on multi-dimensional partitions, even in the one-dimensional case, is new.

Example 5. Let $\pi = [4, 2, 1]$ and $\pi' = [3, 2, 2, 1]$ be one-dimensional partitions. Then we have

$$\begin{aligned}
(\pi \boxplus \pi')_0 &= 4 + 3 = 7 \\
(\pi \boxplus \pi')_1 &= \min(4 + 2, 2 + 3) = 5 \\
(\pi \boxplus \pi')_2 &= \min(4 + 2, 2 + 2, 1 + 3) = 4 \\
&\vdots
\end{aligned}$$

Overall, we get $\pi \boxplus \pi' = [7, 5, 4, 3, 2, 2, 1]$.

We will use the min-sum operator $\boxplus$ to help us define the multset and depth of a GNS.

Remark 6. There is a connection between $\boxplus$ and tropical geometry. Specifically, let $a \oplus b := \min(a, b)$ and $a \odot b := a + b$ denote tropical addition and multiplication, respectively. Then working in $\mathbb{Z}[x_1, \ldots, x_d]$, we have that

$$\left(\bigoplus_{\mathbf{a}\in\mathbb{N}_0^d} (\pi_{\mathbf{a}} \odot x^{\mathbf{a}})\right) \odot \left(\bigoplus_{\mathbf{a}\in\mathbb{N}_0^d} (\pi'_{\mathbf{a}} \odot x^{\mathbf{a}})\right) = \bigoplus_{\mathbf{a}\in\mathbb{N}_0^d} (\pi \boxplus \pi')_{\mathbf{a}} \odot x^{\mathbf{a}},$$

where $x^{\mathbf{a}} = x_1^{a_1} \cdots x_d^{a_d}$.

We can also interpret the min-sum operator in terms of multiplication of monomial ideals. If we work in the ring $\mathbb{Z}[x_1, \ldots, x_d, y]$ then we have that

$$\left(\sum_{\mathbf{a}\in\mathbb{N}_0^d} (x^{\mathbf{a}} y^{\pi_{\mathbf{a}}})\right) \cdot \left(\sum_{\mathbf{a}\in\mathbb{N}_0^d} (x^{\mathbf{a}} y^{\pi'_{\mathbf{a}}})\right) = \sum_{\mathbf{a}\in\mathbb{N}_0^d} (x^{\mathbf{a}} y^{(\pi\boxplus\pi')_{\mathbf{a}}}),$$

where we are summing principal ideals.

2.4 The constants r_k

It will be helpful to define the following family of constants.

Definition 7. For positive integers k, let the constant r_k be the root of largest magnitude of $x^k - (x+1)^{k-1} = 0$. The values of these constants begin as follows:

$$\begin{aligned}
\mathsf{r}_1 &= 1 \\
\mathsf{r}_2 &= \frac{1+\sqrt{5}}{2} = 1.6180\ldots \\
\mathsf{r}_3 &= 2.1479\ldots \\
\mathsf{r}_4 &= 2.6297\ldots \\
&\vdots
\end{aligned}$$

We verify that r_k is well-defined with the following lemma.

Lemma 8. *The polynomial $x^k - (x+1)^{k-1}$ has a unique root of largest magnitude r_k, and it is real with $\mathsf{r}_k \geqslant 1$.*

Proof. Let $f(x) = x^k - (x+1)^{k-1}$. We prove the following facts, which suffice to prove the claim:

(1) We have that r_k is real and a simple root.

(2) The polynomial f has a unique positive root r_k^+, and $\mathsf{r}_k^+ > 1$.

(3) We must have $\mathsf{r}_k > 0$.

To prove (1), note that any root r of f satisfies $|r|^k = |r+1|^{k-1} \leqslant (|r|+1)^{k-1}$ with equality only if r is a positive real. So if r is non-real, then $|r|^k < (|r|+1)^{k-1}$ and so $|r|$ is less than the largest real root of f. Hence, we must have that the root of largest magnitude r_k is real. Moreover, one can check that $f(x)$ and $f'(x)$ share no common roots, so r_k is a simple root.

To show (2), note that $f(1) \leqslant 0$ and the leading coefficient of f is positive, so f has a positive root $\mathsf{r}_k^+ \geqslant 1$. By Descartes' rule of signs, f has a unique positive root. Thus, if $\mathsf{r}_k > 0$ then $\mathsf{r}_k = \mathsf{r}_k^+$.

Finishing with (3), suppose for the sake of contradiction that $\mathsf{r}_k < 0$. Then we must have that $\mathsf{r}_k \leqslant -|\mathsf{r}_k^+| \leqslant -1$ since r_k is the root of largest magnitude. But note that for $x < -1$, we have $f(x) > 0$ when k is even and $f(x) < 0$ when k is odd. In particular, we have $f(\mathsf{r}_k) \neq 0$, a contradiction. □

While not strictly necessary for the rest of the paper, it is still natural to consider the size of r_k as k goes to infinity. By writing the equation as $x^{k/(k-1)} = x+1$, it is clear that r_k is increasing in k and goes to infinity as k goes to infinity.

Proposition 9. *We have $\mathsf{r}_k \sim k/(\ln k)$.*

Proof. Rewrite the given equation to the form $x = (1+1/x)^{k-1}$, which in turn implies

$$\mathsf{r}_k^{\mathsf{r}_k} = \left(\left(1+\frac{1}{\mathsf{r}_k}\right)^{\mathsf{r}_k}\right)^{k-1} = e^{k(1-o(1))},$$

since as k goes to infinity, so does r_k. The solution to the equation $x^x = e^a$ is $x = e^{W(a)}$, where W is the Lambert W-function. It is known (see, e.g. [9, eq. 4.18]) that $W(x) = \ln x - \ln\ln x + o(1)$, so we have

$$\mathsf{r}_k = e^{\ln k - \ln\ln k + o(1)} = (1 + o(1))\left(\frac{k}{\ln k}\right),$$

as desired. □

3 Invariants of generalized numerical semigroups

In this section, we define a number of invariants that will help us count generalized numerical semigroups.

3.1 Multiplicity and depth

First, we define analogues of multiplicity and depth for generalized numerical semigroups.

Recall that the *multiplicity* $m(\Lambda)$ of a numerical semigroup Λ is the least nonzero integer contained in Λ. In higher dimensions, a GNS may not have a unique least nonzero point. Nevertheless, we generalize the notion of multiplicity to GNSs as follows.

Definition 10. Let Λ^d be a GNS. The *multset* $\mathcal{M}(\Lambda^d)$ is the set of minimal, nonzero points in Λ^d. In other words, $\mathcal{M}(\Lambda^d) := \{\mathbf{m} \in \Lambda^d : \mathbf{m} \not> \mathbf{x} \text{ for all } \mathbf{x} \in \Lambda^d \setminus \{\mathbf{0}\}\}$.

As a reminder, we drop the Λ^d when it is clear to which GNS we are referring. We use the calligraphic $\mathcal{M}$ to remind the reader that the multset is a set, not an integer like in the one-dimensional case. However, note for numerical semigroups Λ, we have $\mathcal{M}(\Lambda) = \{m(\Lambda)\}$. We have the following characterization of possible multsets.

Proposition 11. *A finite set $\mathcal{M} \subset \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ is a possible multset if and only if:*

- *it is an antichain of $\mathbb{N}_0^d$ under $\leqslant$; and*
- *it contains some multiple of $\mathbf{e_i^d}$ for $i = 1, \ldots, d$.*

Proof. We first show the forward direction. If $\mathcal{M}(\Lambda^d)$ contained two comparable elements $\mathbf{x} > \mathbf{y}$, then this contradicts the definition of a multset, since $\mathbf{y} \in \Lambda^d \setminus \{\mathbf{0}\}$. On the other hand, if $\mathcal{M}$ did not contain an element on the i-th coordinate axis, then Λ^d does not contain any point on the axis and is thus not cofinite.

In the reverse direction, suppose $\mathcal{M}$ is an antichain that contains the elements $m_i \cdot \mathbf{e_i^d}$ for indices $i = 1, \ldots, d$. Let

$$\Lambda^d = \{\mathbf{0}\} \sqcup \{\mathbf{x} \in \mathbb{N}_0^d : \mathbf{x} \geqslant \mathbf{m} \text{ for some } \mathbf{m} \in \mathcal{M}\}.$$

Then Λ^d is closed upwards, so it is closed under addition. It is cofinite, since it includes all elements with $x_i \geqslant m_i$ for each i and thus excludes at most $\prod_{i=1}^d m_i$ elements. Finally, it has multset $\mathcal{M}$ by construction. Hence, $\mathcal{M}$ is a valid multset. □

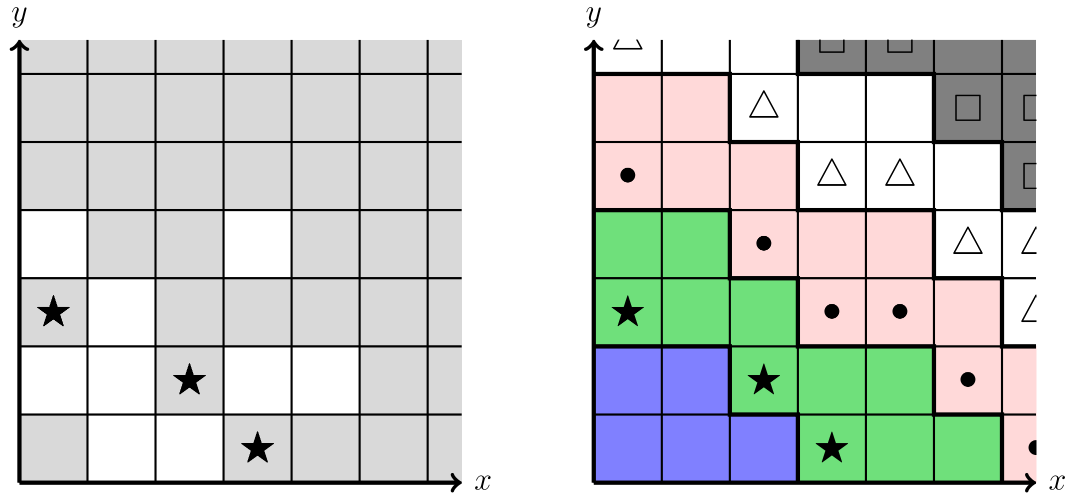

Figure 1: A two-dimensional GNS and multset, along with its depth-k regions.

We also generalize the notion of depth to GNSs. For sets $A, B \subseteq \mathbb{N}_0^d$, define

$$A + B := \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A,\ \mathbf{b} \in B\},$$
$$kA := \underbrace{A + \cdots + A}_{k \text{ times}}.$$

In particular, $kA \neq \{ka : a \in A\}$. One may recognize these as Minkowski sum operators.

Definition 12. The *depth* $q(\Lambda^d)$ of the GNS Λ^d is the least integer q for which

$$\{\mathbf{x} \in \mathbb{N}_0^d : \mathbf{x} \geqslant \mathbf{a} \text{ for some } \mathbf{a} \in q\mathcal{M}\} \subseteq \Lambda^d.$$

Example 13. Let

$$\Lambda^2 = \mathbb{N}_0^2 \setminus \{(0,1), (0,3), (1,0), (1,1), (1,2), (2,0), (3,1), (3,3), (4,1)\}.$$

Then Λ^2 is a GNS with genus $g = 9$, multset $\mathcal{M} = \{(0,2), (2,1), (3,0)\}$, and depth $q = 3$. See the left side of Figure 1, where cells correspond to their bottom left corner (gray if included, white if excluded) and the stars indicate the multset.

3.2 Depth-k regions

Fixing the multset of a GNS is a strong condition that imposes restrictions on what other elements must be included. In the one-dimensional case, we characterize numerical semigroups of multiplicity m by partitioning

$$\mathbb{N}_0 = \{0, 1, \ldots, m-1\} \sqcup \{m, \ldots, 2m-1\} \sqcup \{2m, \ldots 3m-1\} \sqcup \cdots$$

into sets of size m. Then the depth is the least q for which $\{qm, \ldots, (q+1)m-1\} \subseteq \Lambda$, since then every integer above qm is contained in Λ. We somewhat generalize this concept to GNSs as follows.

Definition 14. Let $\mathcal{M}$ be a multset. For positive integers k, define the region

$$\mathcal{R}_{\leqslant k}(\mathcal{M}) := \{\mathbf{x} \in \mathbb{N}_0^d : \mathbf{x} \not\geqslant \mathbf{a} \text{ for all } \mathbf{a} \in k\mathcal{M}\}.$$

The *depth-k region* $\mathcal{R}_k(\mathcal{M})$ is the set $\mathcal{R}_{\leqslant k}(\mathcal{M}) \setminus \mathcal{R}_{\leqslant k-1}(\mathcal{M})$. The size of the depth-$k$ region is $s_k(\mathcal{M}) := \#\mathcal{R}_k(\mathcal{M})$. By convention, we let $\mathcal{R}_{\leqslant 0}(\mathcal{M})$ be empty.

Note that depth-k regions are defined in terms of multsets, not GNSs, though we still drop the $\mathcal{M}$ when the argument is clear. These regions serve as a "blueprint" for a possible GNS of a given multset, where we first let Λ^d have multset $\mathcal{M}$, then choose to exclude certain elements from finitely many regions $\mathcal{R}_k(\mathcal{M})$.

Lemma 15. *If* $\mathbf{x} \in \mathcal{R}_k$ *and* $\mathbf{y} \in \mathcal{R}_\ell$*, then* $\mathbf{x} + \mathbf{y} \notin \mathcal{R}_{\leqslant k+\ell-2}$.

Proof. There are elements $\mathbf{a} \in (k-1)\mathcal{M}$ and $\mathbf{b} \in (\ell-1)\mathcal{M}$ for which $\mathbf{x} \geqslant \mathbf{a}$ and $\mathbf{y} \geqslant \mathbf{b}$. Then $\mathbf{x} + \mathbf{y} \geqslant \mathbf{a} + \mathbf{b} \in (k+\ell-2)\mathcal{M}$, so $\mathbf{x} + \mathbf{y}$ cannot be in the region $\mathcal{R}_{\leqslant k+\ell-2}$. □

Corollary 16. *The depth of* Λ^d *is the least integer* q *for which* $\mathcal{R}_{q+1}(\mathcal{M}(\Lambda^d)) \subset \Lambda^d$.

Proof. If q is the depth of Λ^d, then by definition $\mathcal{R}_{q+1} \subset \Lambda^d$, so we show the reverse direction. Assume $\Lambda^d \neq \mathbb{N}_0^d$, so $q \geqslant 1$. It suffices to show that if $\mathcal{R}_k \subset \Lambda^d$, then $\mathcal{R}_{k+1} \subset \Lambda^d$, since then every element bounded below by an element of $q\mathcal{M}$ is in Λ^d.

Suppose that $\mathbf{x} \in \mathcal{R}_{k+1}$, so there exists an $\mathbf{a} \in k\mathcal{M}$ for which $\mathbf{x} \geqslant \mathbf{a}$. Then there exists an element $\mathbf{m} \in \mathcal{M} \subseteq \mathcal{R}_2$ for which $\mathbf{a} - \mathbf{m} \in (k-1)\mathcal{M}$. Now consider the point $\mathbf{x} - \mathbf{m}$. It cannot be in $\mathcal{R}_{\leqslant k-1}$, since $\mathbf{x} - \mathbf{m} \geqslant \mathbf{a} - \mathbf{m} \in (k-1)\mathcal{M}$, and it cannot be in $\mathcal{R}_{k+1}$, since then by Lemma 15 we would have $\mathbf{m} + (\mathbf{x} - \mathbf{m}) \notin \mathcal{R}_{\leqslant k+1}$. Hence, we have that $\mathbf{x} - \mathbf{m} \in \mathcal{R}_k$ is an element of Λ^d, so $\mathbf{x}$ is, too. □

Now, we relate the regions $\mathcal{R}_{\leqslant k}(\mathcal{M})$ with $(d-1)$-dimensional partitions in the following way. For $\mathbf{v} \in \mathbb{N}_0^{d-1}$, let $(\pi^k)_{\mathbf{v}}(\mathcal{M})$ be the least integer t for which $(v_1, \ldots, v_{d-1}, t) \notin \mathcal{R}_{\leqslant k}(\mathcal{M})$.

Lemma 17. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be in* $\mathbb{N}_0^{d-1}$*. If* $\mathbf{u} \leqslant \mathbf{v}$*, then* $(\pi^k)_{\mathbf{u}} \geqslant (\pi^k)_{\mathbf{v}}$.

Proof. Suppose not. Then $(u_1, \ldots, u_{d-1}, (\pi^k)_{\mathbf{u}})$ is at most $(v_1, \ldots, v_{d-1}, (\pi^k)_{\mathbf{v}} - 1)$ in the partial order, but is not included in $\mathcal{R}_{\leqslant k}(\mathcal{M})$. This contradicts the fact that $\mathcal{R}_{\leqslant k}$ is closed downward. □

Hence, for fixed k and $\mathcal{M}$, the integers $(\pi^k)_{\mathbf{v}}(\mathcal{M})$ form a $(d-1)$-dimensional partition $\pi^k(\mathcal{M})$ of $s_1(\mathcal{M}) + \cdots + s_k(\mathcal{M})$.

Example 18. On the right side of Figure 1, we have $\mathcal{M} = \{(0,2), (2,1), (3,0)\}$ and the regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are colored in different shades. We have

$$\begin{aligned} \pi^1(\mathcal{M}) &= [2,2,1], \\ \pi^2(\mathcal{M}) &= [4,4,3,2,2,1], \\ \pi^3(\mathcal{M}) &= [6,6,5,4,4,3,2,2,1], \end{aligned}$$

so $(s_1, s_2, s_3) = (5, 11, 17)$. Also depicted are the sets $\mathcal{M}$ (denoted by ★), $2\mathcal{M}$ (denoted by •), $3\mathcal{M}$ (denoted by △), and $4\mathcal{M}$ (denoted by □).

In fact, we can define the partitions inductively with the min-sum operation (cf. §2).

Proposition 19. *We have* $\pi^k(\mathcal{M}) = \underbrace{\pi^1(\mathcal{M}) \boxplus \cdots \boxplus \pi^1(\mathcal{M})}_{k \text{ times}}$.

Proof. Let $\tau := \pi^1 \boxplus \cdots \boxplus \pi^1$ be π^1 min-summed k times, and define

$$\mathcal{M}' = \{(v_1, \ldots, v_{d-1}, (\pi^1)_{\mathbf{v}}) : \mathbf{v} \in \mathbb{N}_0^{d-1}\}.$$

Note that $\mathcal{M} \subseteq \mathcal{M}'$. Consider the set of points that are not greater than or equal to any element of $k\mathcal{M}$. This is the same as the set of points that are not greater than or equal to any element of $k\mathcal{M}'$. Moreover, the minimal elements of $k\mathcal{M}'$ with respect to $\leqslant$ are precisely the minimal elements of $\{(v_1, \ldots, v_{d-1}, \tau_{\mathbf{v}}) : \mathbf{v} \in \mathbb{N}_0^{d-1}\}$. Hence, $(\pi^k)_{\mathbf{v}} = \tau_{\mathbf{v}}$ and so $\pi^k \simeq \tau$. □

Thus, the partition $\pi^1(\mathcal{M})$ can be used to compute the other partitions $\pi^k(\mathcal{M})$, which characterize $\mathcal{R}_{\leqslant k}(\mathcal{M})$ and the depth-k regions.

Definition 20. The *shape* of Λ^d is the $(d-1)$-dimensional partition $\pi^1(\mathcal{M}(\Lambda^d))$, whose total is the *shape size* $s(\Lambda^d) := s_1(\mathcal{M}(\Lambda^d))$.

Note that for $d = 1$, the shape size $s(\Lambda)$ is exactly the multiplicity $m(\Lambda)$.

4 Improved asymptotic bounds on $n_{g,d}$

Let $n_{g,d}$ denote the number of d-dimensional GNSs with genus g. In this section, we compute some bounds on $n_{g,d}$.

4.1 Lower bound

In this subsection, we show Theorem 1. Specifically, we construct a family of GNSs of depth 2 whose size has the desired growth rate. To this end, let $n_{g,d,q}$ denote the number of generalized numerical semigroups with genus g, dimension d, and depth q. The following result implies Theorem 1. Surprisingly, the growth factor r_{2^d} is sharp for depth 2 GNSs in the following sense.

Theorem 21. *For fixed d, there are constants* $C_d^-, C_d^+ > 0$ *for which*

$$(C_d^-)^{g^{(d-1)/d}} \mathsf{r}_{2^d}^g < n_{g,d,2} < (C_d^+)^{g^{(d-1)/d}} \mathsf{r}_{2^d}^g$$

for all $g > 0$.

First, we give an exact formula for $n_{g,d,2}$ as a sum over the possible multsets $\mathcal{M}$.

Proposition 22. *We have*

$$n_{g,d,2} = \sum_{\mathcal{M}} \binom{s_2(\mathcal{M}) - \#\mathcal{M}}{g + 1 - s_1(\mathcal{M})},$$

where the sum is over all multsets $\mathcal{M}$ *for which* $s_1(\mathcal{M}) \leqslant g$.

Proof. Say Λ^d has depth 2 and multset $\mathcal{M}$, so $\mathcal{R}_k \subset \Lambda^d$ for $k \geqslant 3$. The inclusion of each element of $\mathcal{R}_1$ is predetermined; $\mathbf{0}$ is included and all other elements are excluded. Moreover, every element of $\mathcal{M} \subset \mathcal{R}_2$ is included in Λ^d by definition.

Thus, it remains to choose $g + 1 - s_1$ elements from $\mathcal{R}_2 \setminus \mathcal{M}$. We also must have $s_1 \leqslant g$, otherwise Λ^d has depth 1. We claim that any such subset yields a valid Λ^d. Indeed, by Lemma 15 we have that if $\mathbf{x}, \mathbf{y} \in \mathcal{R}_2$, then $\mathbf{x} + \mathbf{y} \notin \mathcal{R}_{\leqslant 2}$, so in fact $\mathcal{R}_2$ is sum-free. Hence, we have that Λ^d is always closed under addition, and we have the desired claim. $\square$

It turns out we can bound s_2 linearly in terms of s_1, which in turns gives us an exponential upper bound in terms of r_{2^d}. To bound the error term, we use the following formulation of the *Loomis-Whitney inequality*, proved in 1949, which allows us to bound the sum of the entries of a $(d-1)$-dimensional partition along the coordinate hyperplanes.

Theorem 23 (Loomis-Whitney [18, Thm. 2])**.** *For any set of points S in d-space, let S_i be the set of points obtained by projecting S onto the i-th coordinate hyperplane. Then*

$$(\#S)^{d-1} \leqslant \prod_{i=1}^{d} (\#S_i).$$

Lemma 24. *For any multset $\mathcal{M}$ and $d \geqslant 2$, we have $s_2(\mathcal{M}) \leqslant (2^d - 1)s_1(\mathcal{M}) - 2^{d-2}s_1(\mathcal{M})^{(d-1)/d}$.*

Proof. We work with the $(d-1)$-dimensional partitions $\pi^1(\mathcal{M})$ and $\pi^2(\mathcal{M})$ of s_1 and $s_1 + s_2$, respectively. For each point $\mathbf{v} \in \mathbb{N}_0^{d-1}$ and subset $S \subseteq [d-1]$, let $f_S(\mathbf{v}) := (f_{S,1}(v_1), \ldots, f_{S,d-1}(v_{d-1}))$, where

$$f_{S,i}(v) = \begin{cases} \lceil v/2 \rceil & \text{if } i \in S, \\ \lfloor v/2 \rfloor & \text{if } i \notin S. \end{cases}$$

By definition, we have $f_S(\mathbf{v}) + f_{[d-1]\setminus S}(\mathbf{v}) = \mathbf{v}$. Moreover, Proposition 19 tells us $\pi^2 = \pi^1 \boxplus \pi^1$, and so

$$\begin{aligned} (\pi^2)_{\mathbf{v}} &= \min_{\mathbf{u}+\mathbf{u}'=\mathbf{v}} \left((\pi^1)_{\mathbf{u}} + (\pi^1)_{\mathbf{u}'} \right) \\ &\leqslant \min_{S \subseteq [d-1]} \left((\pi^1)_{f_S(\mathbf{v})} + (\pi^1)_{f_{[d-1]\setminus S}(\mathbf{v})} \right) \\ &\leqslant \frac{1}{2^{d-1}} \sum_{S \subseteq [d-1]} \left((\pi^1)_{f_S(\mathbf{v})} + (\pi^1)_{f_{[d-1]\setminus S}(\mathbf{v})} \right) \\ &= \frac{1}{2^{d-2}} \sum_{S \subseteq [d-1]} (\pi^1)_{f_S(\mathbf{v})}; \end{aligned}$$

$$s_1 + s_2 = \sum_{\mathbf{v} \in \mathbb{N}_0^{d-1}} (\pi^2)_{\mathbf{v}}$$

$$\leqslant \sum_{\mathbf{v} \in \mathbb{N}_0^{d-1}} \frac{1}{2^{d-2}} \sum_{S \subseteq [d-1]} (\pi^1)_{f_S(\mathbf{v})}.$$

For each i, the i-th entry of $f_S(\mathbf{v})$ is determined by the choice of v_i and whether or not i is in S. If we fix the i-th entry to be w_i, there are exactly 3 such choices that make $w_i = 0$ and 4 choices if $w_i \geqslant 1$. Hence, for each $\mathbf{w}$ there are 4^{d-1} choices of $(\mathbf{v}, S)$ for which $f_{S,i}(\mathbf{v}) = \mathbf{w}$ if $\mathbf{w} \geqslant (1, 1, \ldots, 1)$ (i.e., if all entries are nonzero) and at most $3 \cdot 4^{d-2}$ otherwise. In particular,

$$\begin{aligned} s_1 + s_2 &\leqslant \frac{1}{2^{d-2}} \left(4^{d-1} \sum_{\mathbf{w} \in \mathbb{N}_0^{d-1}} (\pi^1)_{\mathbf{w}} - 4^{d-2} \sum_{\substack{\mathbf{w} \in \mathbb{N}_0^{d-1} \\ \mathbf{w} \not\geqslant (1,1,\ldots,1)}} (\pi^1)_{\mathbf{w}} \right) \\ &= 2^d s_1 - 2^{d-2} \cdot \sum_{\substack{\mathbf{w} \in \mathbb{N}_0^{d-1} \\ \mathbf{w} \not\geqslant (1,1,\ldots,1)}} (\pi^1)_{\mathbf{w}}. \end{aligned}$$

The summation is equal to the number of points in $\mathcal{R}_1$ that are on the i-th coordinate hyperplane for some $i < d$. Pick the i which maximizes the summation. We then have that

$$\sum_{\substack{\mathbf{w} \in \mathbb{N}_0^{d-1} \\ \mathbf{w} \not\geqslant (1,1,\ldots,1)}} (\pi^1)_{\mathbf{w}} \geqslant \left(\prod_{i=1}^{d} (\#(\mathcal{R}_1)_i) \right)^{1/d} \geqslant s_1^{(d-1)/d},$$

so $s_1 + s_2$ is at most $2^d s_1 - 2^{d-2} s_1^{(d-1)/d}$, as desired. □

Remark 25. The choice of $\mathcal{M}$ in the proof of Lemma 33 shows that $(2^d - 1)s_1 - O\left(s_1^{(d-1)/d}\right)$ is the best possible bound. This multset is provably best for $d = 2$; this is the third problem of the Team Selection Test for the 2023 United States International Math Olympiad team, posed by the author.

Corollary 26. *For each d, there is a constant $C_d^+ > 0$ for which $n_{g,d,2} < (C_d^+)^{g^{(d-1)/d}} \mathsf{r}_{2^d}^g$ for all $g > 0$.*

Proof. Since π^1 is a $(d-1)$-dimensional partition, there are exactly $p_{d-1}(s)$ multsets $\mathcal{M}$ with $s_1(\mathcal{M}) = s$. Using Lemma 24, we have

$$\begin{aligned} n_{g,d,2} &= \sum_{s=1}^{g} \sum_{\substack{\mathcal{M} \text{ multset} \\ s_1(\mathcal{M}) = s}} \binom{s_2(\mathcal{M}) - \#\mathcal{M}}{g + 1 - s} \\ &\leqslant \sum_{s=1}^{g} \sum_{\substack{\mathcal{M} \text{ multset} \\ s_1(\mathcal{M}) = s}} \binom{(2^d - 1)s}{g + 1 - s} \\ &\leqslant p_{d-1}(g+1) \cdot \sum_{s=1}^{g} \binom{(2^d - 1)s}{g + 1 - s}. \end{aligned}$$

By Theorem 3, there is a constant K for which $p_{d-1}(g+1) < K^{g^{(d-1)/d}}$. On the other hand, we can verify by expansion that the summation is the coefficient of x^{g+1} in the generating function

$$\sum_{s=0}^{\infty} x^s (x+1)^{(2^d-1)s} = \frac{1}{1-x(x+1)^{2^d-1}}.$$

The coefficients of the generating function follow a linear recurrence with characteristic polynomial $x^{2^d} - (x+1)^{2^d-1}$, and thus have growth $O(\mathsf{r}_{2^d}^g)$. The result follows. ☐

It turns out that the constant $2^d - 1$ in Lemma 24 is sharp, which suggests that we can choose a specific $\mathcal{M}$ to give a sufficient lower bound on $n_{g,d,2}$. The proof gives us intuition for the near-equality cases: we should have $\pi^1_{\mathbf{u}} + \pi^1_{\mathbf{v}}$ be roughly constant for fixed $\mathbf{u} + \mathbf{v}$. We use a specific near-equality case to show the lower bound.

First, we need the following analytic lemmas.

Definition 27. For a positive integer k, define the rational function

$$F_k(x) := \frac{k^k (1-x) x^k}{((k+1)x-1)^{k+1}}$$

and let c_k be the largest positive root of $F_k(x) - 1 = 0$.

Lemma 28. *The root c_k is the unique real root of $F_k(x) - 1$ larger than $1/(k+1)$, and $\mathsf{c}_k < 1$.*

Proof. The claim is easily seen to be true for $k = 1$ since the only positive root is $\mathsf{c}_1 = 1$, so assume $k \geqslant 2$. Let $f(x) = k^k(1-x)x^k$ and $g(x) = ((k+1)x-1)^{k+1}$. The graph of f has critical points at $x = 0, k/(k+1)$, and is strictly increasing on the interval $(0, k/(k+1))$ and decreasing on the interval $(k/(k+1), \infty)$. It is concave down for $x > (k-1)/(k+1)$.

Meanwhile, the graph of g has a single critical point at $x = 1/(k+1)$, and strictly increases thereafter; moreover, it is concave up.

First, check that $f(1/(k+1)) > 0 = g(1/(k+1))$ and

$$g\left(\frac{k}{k+1}\right) = (k-1)^{k+1} > \frac{k^{2k}}{(k+1)^{k+1}} = f\left(\frac{k}{k+1}\right).$$

Since f and g are both increasing on the interval $(1/(k+1), k/(k+1))$, one concave up and the other concave down, the unique value of x for which $f(x) = g(x)$ on this interval is c_k. Moreover, $g > f$ on $[k/(k+1), \infty)$, since f is decreasing on that interval. We thus have the desired claim. ☐

Definition 29. For positive integers g, k, and $x \leqslant g$, define $G_k(x, g) := \binom{kx}{g+1-x}$. For fixed g, let $x = r_k(g)$ be the value of x which maximizes $G_k(x, g)$.

Lemma 30. *We have $G_k(r_k(g), g) \geqslant K/g \cdot \mathsf{r}_{k+1}^g$ for some constant $K > 0$.*

Proof. By a standard characteristic polynomial argument as in Corollary 26, we have that

$$\sum_{s=1}^{g} G_k(s,g) = \Omega(\mathsf{r}_{k+1}^g).$$

But the left-hand side is at most $g \cdot G_k(r_k(g), g)$, which yields the desired bound. $\square$

Lemma 31. *Let $s(1), s(2), \ldots$ be a sequence of integers with $s(g) \leqslant g$. As g goes to infinity,*

$$\frac{G_k(s(g)+1,g)}{G_k(s(g),g)} \sim F_k(s(g)/g).$$

Proof. Abbreviate $s := s(g)$. Rewrite

$$\frac{G_k(s+1,g)}{G_k(s,g)} = \frac{(g+1-s)\prod_{i=1}^{k}(ks+i)}{\prod_{i=0}^{k}((k+1)s-g+i)} = \frac{(1-s/g+1/g)\prod_{i=1}^{k}(ks/g+i/g)}{\prod_{i=0}^{k}((k+1)s/g-1+i/g)}.$$

Then the right-hand side tends to $F_k(s/g)$, as desired. $\square$

Corollary 32. *As g approaches infinity, the ratio $r_k(g)/g$ approaches c_k.*

Proof. Abbreviate $r := r_k(g)$. Our choices of r dictate that

$$\frac{G_k(r+1,g)}{G_k(r,g)} < 1 \quad \text{and} \quad \frac{G_k(r,g)}{G_k(r-1,g)} > 1.$$

For large g, the two quantities both approach $F_k(r/g)$. Hence, by the squeeze theorem, we have that $F_k(r/g)$ approaches 1, so r/g approaches some root of $F_k(x) - 1$. We must have $kr > g - r$ or $r/g > 1/(k+1)$, so the only possibility is for r/g to approach c_k, as desired. $\square$

Lemma 33. *For each positive integer d, there exists a constant $C_d^- > 0$ for which $n_{g,d,2} > (C_d^-)^{g^{(d-1)/d}} \mathsf{r}_{2^d}^g$.*

Proof. Let $\mathcal{M}_k$ denote the lattice points on the plane $x_1 + \cdots + x_d = k$. Then $\mathcal{M}_k$ is an antichain with points on the axes and is thus a valid multset, with

$$\mathcal{R}_1(\mathcal{M}_k) = \{(x_1, \ldots, x_d) : x_1 + \cdots + x_d < k\},$$
$$\mathcal{R}_2(\mathcal{M}_k) \setminus \mathcal{M}_k = \{(x_1, \ldots, x_d) : k < x_1 + \cdots + x_d < 2k\},$$

whose cardinalities are

$$s_{1,k,d} := \binom{k+d-1}{d} = \frac{1}{d!}k^d + O(k^{d-1}),$$
$$s_{2,k,d} := \binom{2k+d-1}{d} - \binom{k+d}{d} = \frac{1}{d!}(2^d-1)k^d + O(k^{d-1}).$$

Thus, there are $t^k_{g,d} := \binom{s_{2,k,d}}{g+1-s_{1,k,d}}$ GNSs with depth 2, genus g, and multset $\mathcal{M}_k$. It suffices to show we can pick a $k = k(g)$ for which $t^k_{g,d}$ exceeds the desired bound.

Let $r := r_{2^d-1}(g)$. Since $s_{1,k,d}$ is a polynomial of degree d in k, there is a constant $A = A(g) > 0$ for which we can always select k such that $0 < s_{1,k,d} - r < A \cdot k^{d-1}$. The idea is that we can approximate $t^k_{g,d}$ closely by $G_{2^d-1}(s_{1,k,d}, g)$, which in turn is close to $G_{2^d-1}(r, g)$, the last of which is large.

Let $\varepsilon > 0$ be a threshold which we later send to 0. We first compare the quantities $t^k_{g,d}$ and $G_{2^d-1}(s_{1,k,d}, g)$. Let $D = (2^d-1)s_{1,k,d} - s_{2,k,d}$. Since D is a polynomial of degree $d-1$ in k with positive leading coefficient, we know $0 < D < B \cdot k^{d-1}$ for some $B > 0$. Note that

$$\frac{t^k_{g,d}}{G_{2^d-1}(s_{1,k,d}, g)} = \prod_{j=1}^{D} \frac{2^d s_{1,k,d} - g - j}{(2^d-1)s_{1,k,d} + 1 - j} = \prod_{j=1}^{D} \frac{2^d s_{1,k,d}/g - 1 - j/g}{(2^d-1)s_{1,k,d}/g + 1/g - j/g}.$$

Since $s_{1,k,d} - r < A \cdot k^{d-1} \ll g$, we have that $s_{1,k,d}/g \sim r/g$ which tends to c_{2^d-1} by Corollary 32. Hence, the right-hand side is bounded below by

$$\left(\frac{2^d \mathsf{c}_{2^d-1} - 1}{(2^d-1)\mathsf{c}_{2^d-1}} - \varepsilon\right)^D > \left(\frac{2^d \mathsf{c}_{2^d-1} - 1}{(2^d-1)\mathsf{c}_{2^d-1}} - \varepsilon\right)^{B \cdot k^{d-1}}$$

for sufficiently large g.

Now, we compare $G_{2^d-1}(s_{1,k,d}, g)$ and $G_{2^d-1}(r, g)$. Note that

$$\frac{G_{2^d-1}(s_{1,k,d}, g)}{G_{2^d-1}(r, g)} = \prod_{x=r}^{s_{1,k,d}-1} \frac{G_{2^d-1}(x+1, g)}{G_{2^d-1}(x, g)}.$$

Every factor in the product converges uniformly to 1 by Lemma 31, so in particular the right-hand side is at least $(1-\varepsilon)^{s_{1,k,d}-r} > (1-\varepsilon)^{A \cdot k^{d-1}}$ for sufficiently large g.

In summary, we have

$$t^k_{g,d} > \left(\frac{2^d \mathsf{c}_{2^d-1} - 1}{(2^d-1)\mathsf{c}_{2^d-1}} - \varepsilon\right)^{B \cdot k^{d-1}} (1-\varepsilon)^{A \cdot k^{d-1}} G_{2^d-1}(r, g).$$

But we have $G_{2^d-1}(r, g) > K/g \cdot \mathsf{r}^g_{2^d-1}$ by Lemma 30, and moreover $k = O(g^{1/d})$ since $r = O(g)$ by Corollary 32. The result follows. □

Note that the proof can be modified slightly to produce an explicit value of C_d, but we do not do that here. Corollary 26 and Lemma 33 collectively imply Theorem 21, which in turn implies Theorem 1.

4.2 Upper bound

In this section, we provide the first upper bound on $n_{g,d}$ by proving Theorem 2. In 2007, Bras-Amorós and de Mier [6] gave the first upper bound on the number n_g of numerical semigroups Λ with genus g. They noted that if $n \notin \Lambda$, then at least one of $(k, n-k)$ must also be excluded from Λ for each k, and thus $g \geqslant (n+1)/2$. In particular, $f \notin \Lambda$, so $f \leqslant 2g - 1$.

We use similar logic to develop an upper bound.

Definition 34. For each g, let $\mathcal{A}_g := \{(x_1, \ldots, x_d) \in \mathbb{N}_0^d : \prod_{i=1}^d (x_i + 1) \leqslant 2g\}$.

Proposition 35. *If Λ^d is a GNS with dimension d and genus g, then $\mathbb{N}_0^d \setminus \Lambda^d \subseteq \mathcal{A}_g$.*

Proof. Suppose $\mathbf{x} \notin \Lambda^d$. Then for each $\mathbf{a} \leqslant \mathbf{x}$, one of $(\mathbf{a}, \mathbf{x} - \mathbf{a})$ must be also excluded from Λ^d. In particular, we must exclude at least half of the elements in the prism of points bounded by $\mathbf{x}$, which contains $\prod_{i=1}^d (x_i + 1)$ elements. Hence, $g \geqslant \frac{1}{2} \prod_{i=1}^d (x_i + 1)$, as desired. □

We now establish our upper bound. Recall the harmonic sum $H_n := \sum_{k=1}^n 1/k$.

Proof of Theorem 2. Note that we have

$$\#\mathcal{A}_g \leqslant \sum_{0 \leqslant x_1, \ldots, x_{d-1} < 2g} \left\lfloor \frac{2g}{\prod_{i=1}^{d-1} (x_i + 1)} \right\rfloor \leqslant \sum_{0 \leqslant x_1, \ldots, x_{d-1} < 2g} \frac{2g}{\prod_{i=1}^{d-1} (x_i + 1)} = 2g H_{2g}^{d-1}.$$

Hence, if Λ^d has genus g, we choose to exclude g elements from $\mathcal{A}_g$, and thus

$$n_{g,d} \leqslant \binom{2g H_{2g}^{d-1}}{g} \leqslant \frac{1}{g!} (2g H_{2g}^{d-1})^g.$$

Stirling's approximation tells us $g! > ((1/e - o(1))g)^g$ for sufficiently large g, while we have the bound $H_{2g} \leqslant \ln(2g) + 1$ on the harmonic sum. The result follows. □

5 Partition labelings

In this section, we define *partition labelings* that somewhat generalize the notion of Kunz words to generalized numerical semigroups.

Given a numerical semigroup Λ with multiplicity m, recall the *Kunz word* $\mathcal{K}(\Lambda)$ is the word $w_1 \cdots w_{m-1}$, where $m \cdot w_i + i$ is the least element in Λ congruent to $i \pmod{m}$. One can easily recover Λ from $\mathcal{K}(\Lambda)$, and we can read off many invariants of Λ, such as its genus and depth, from its Kunz word.

We say a word $w_1 \cdots w_{m-1}$ is a *valid Kunz word* if it is the Kunz word of some numerical semigroup. In 1987, Kunz showed the valid Kunz words are exactly those which follow certain additive inequalities known as the *Kunz conditions*:

Proposition 36 (Kunz [16, 20, §2])**.** *A word $w_1 \cdots w_{m-1}$ is a valid Kunz word if and only if:*

- *$w_i + w_j \geqslant w_{i+j}$ for all i, j with $i + j < m$;*

- $w_i + w_j + 1 \geqslant w_{i+j-m}$ *for all* i, j *with* $i + j \geqslant m + 1$.

These properties make Kunz words a powerful tool to study numerical semigroups. For more on Kunz words, see [17, 25].

In the setting of generalized numerical semigroups, we generalize Kunz words to an object that we call a *partition labeling*, formed by labeling elements within a prism with certain $(d-1)$-dimensional partitions. To begin, we make the following definition.

Definition 37. For a multset $\mathcal{M}$ and index $i = 1, \dots, d$, let $m_i(\mathcal{M})$ be the integer m_i for which $m_i \cdot \mathbf{e_i^d} \in \mathcal{M}$. The *volume* $V(\mathcal{M})$ of a multset is the quantity $\prod_{i=1}^{d} m_i$.

Recall from Proposition 11 that such an m_i exists and is unique, so our definition is sound.

Definition 38. Given a GNS Λ^d, the *partition labeling* $\mathscr{L}(\Lambda^d)$ of Λ^d is composed of the points $\mathbf{x} \leqslant (m_1 - 1, \dots, m_d - 1)$, each labeled with a partition $L^{\mathbf{x}}$ as follows:

- if $\mathbf{x} \in \Lambda^d$, then $\mathbf{x}$ is labeled with the empty (zero) partition;
- if $\mathbf{x} \notin \Lambda^d$, then $\mathbf{x}$ is labeled with the partition $L^{\mathbf{x}}$, where

$$(L^{\mathbf{x}})_{\mathbf{v}} = \min\{\ell : \mathbf{x} + (m_1 v_1, \dots, m_{d-1} v_{d-1}, m_d \ell) \in \Lambda^d\}.$$

We elucidate the above definition with an example.

Example 39. As in Example 13, let

$$\Lambda^2 = \mathbb{N}_0^2 \setminus \{(0,1), (0,3), (1,0), (1,1), (1,2), (2,0), (3,1), (3,3), (4,1)\}.$$

Then we have $m_1 = 3$ and $m_2 = 2$.

We first compute the entries of $L^{(0,1)}$. To compute $\left(L^{(0,1)}\right)_0$, we wish to find the least ℓ for which

$$(0,1) + (3 \cdot 0, 2 \cdot \ell) \in \Lambda^2.$$

This is false for $\ell = 0, 1$, but true for $\ell = 2$. Hence, $\left(L^{(0,1)}\right)_0 = 2$.

Similarly, to compute $\left(L^{(0,1)}\right)_1$, we wish to find the least ℓ for which

$$(0,1) + (3 \cdot 1, 2 \cdot \ell) \in \Lambda^2.$$

Once again, this is false for $\ell = 0, 1$ but true for $\ell = 2$. Thus, $\left(L^{(0,1)}\right)_1 = 2$ as well.

Continuing in this fashion, we get that $\mathscr{L}(\Lambda^d)$ is as follows:

$$\begin{aligned} L^{(0,1)} &= [2,2], & L^{(1,1)} &= [1,1], & L^{(2,1)} &= [\quad] \\ L^{(0,0)} &= [\quad], & L^{(1,0)} &= [\ 2\ ], & L^{(2,0)} &= [\ 1\ ]. \end{aligned}$$

This is depicted (perhaps more intuitively) with Young diagrams in Figure 2. For instance, to recover $L^{(0,1)}$, we look at the top left corner of each of the bolded 3×2 boxes on the left-hand side of Figure 2 and see which elements are omitted from Λ^2.

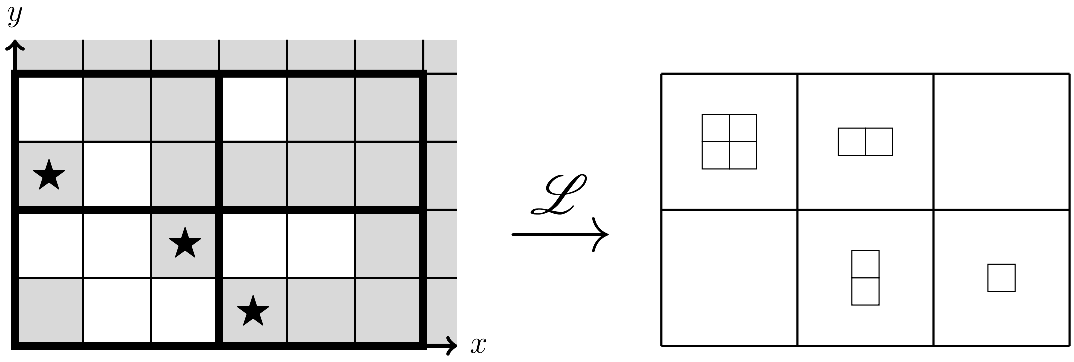

Figure 2: Partition labeling of Λ^2.

We first check that the $L^{\mathbf{x}}$ are indeed partitions.

Proposition 40. *If* $\mathbf{u} \leqslant \mathbf{v}$*, then* $(L^{\mathbf{x}})_{\mathbf{u}} \geqslant (L^{\mathbf{x}})_{\mathbf{v}}$*.*

Proof. Suppose $(L^{\mathbf{x}})_{\mathbf{u}} < (L^{\mathbf{x}})_{\mathbf{v}}$. Then $\mathbf{a} := \mathbf{x} + (m_1 u_1, \ldots, m_{d-1} u_{d-1}, m_d((L^{\mathbf{x}})_{\mathbf{v}} - 1))$ is in Λ^d, but $\mathbf{b} := \mathbf{x} + (m_1 v_1, \ldots, m_{d-1} v_{d-1}, m_d((L^{\mathbf{x}})_{\mathbf{v}} - 1))$ is not. However, since $m_i \mathbf{e}_{\mathbf{i}}^{\mathbf{d}}$ is in Λ^d, we have that $\mathbf{a} + (m_1(v_1 - u_1), \ldots, m_{d-1}(v_{d-1} - u_{d-1}), 0) = \mathbf{b}$ is in Λ^d, which is a contradiction. □

Note we can easily recover Λ^d from $\mathscr{L}(\Lambda^d)$, so $\mathscr{L}$ is an injective map from GNSs to valid partition labelings. Thus, we naturally investigate which partition labelings give rise to valid GNSs. We first offer the following operation on partitions, which will help us extend the one-dimensional Kunz condition $w_x + w_y + 1 \geqslant w_{x+y-m}$ to the general case.

Definition 41. Given a $(d-1)$-dimensional partition π and a subset $X \subseteq [d]$, let $\mathrm{sh}_X(\pi)$ be the partition sh_X given by

$$(\mathrm{sh}_X)_{\mathbf{v}} = \begin{cases} \pi_{\mathbf{v}(X)} & \text{if } d \notin X, \\ \max\left(\pi_{\mathbf{v}(X)} - 1, 0\right) & \text{if } d \in X, \end{cases}$$

where $\mathbf{v}(X) := \mathbf{v} + \sum_{i \in X \cap [d-1]} \mathbf{e}_{\mathbf{i}}^{\mathbf{d-1}}$.

One can think of sh_X as a geometric operation on multi-dimensional Young diagrams that “shaves off” the blocks along the hyperplanes indexed by X.

Example 42. The one-dimensional partition $\pi = [4, 3, 2, 2, 1, 1]$ has

$$\begin{aligned} \mathrm{sh}_{\{1\}}(\pi) &= [3, 2, 2, 1, 1], \\ \mathrm{sh}_{\{2\}}(\pi) &= [3, 2, 1, 1], \\ \mathrm{sh}_{\{1,2\}}(\pi) &= [2, 1, 1]. \end{aligned}$$

Figure 3 shows the Young diagrams of these partitions in relation to each other.

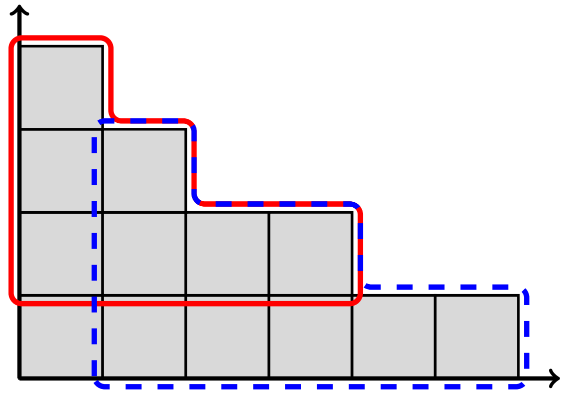

Figure 3: The partitions $\mathrm{sh}_X(\pi)$ for $\pi = [4,3,2,2,1,1]$.

For two $(d-1)$-dimensional partitions π and π', we say $\pi \geqslant \pi'$ if $\pi_{\mathbf{v}} \geqslant \pi'_{\mathbf{v}}$ for all $\mathbf{v} \in \mathbb{N}_0^{d-1}$.

Theorem 43. *A partition labeling* $\mathscr{L} = \{L^{\mathbf{x}} : \mathbf{x} \leqslant (m_1 - 1, \ldots, m_d - 1)\}$ *corresponds to a valid GNS if and only if*

- $L^{\mathbf{0}} = \varnothing$*; and*
- *for all* $\mathbf{x}$, $\mathbf{y} \leqslant (m_1 - 1, \ldots, m_d - 1)$ *we have* $L^{\mathbf{x}} \boxplus L^{\mathbf{y}} \geqslant \mathrm{sh}_X(L^{\mathbf{z}})$*, where* X *consists of the indices* j *for which* $x_j + y_j \geqslant m_j$*, and moreover* $z_i := (x_i + y_i) \mod m_i$*.*

Proof. Let $\mathscr{L}$ be a partition labeling and define $\Lambda^d = \{\mathbf{x} + (m_1 v_1, \ldots, m_{d-1} v_{d-1}, m_d \ell) : \mathbf{x} \leqslant (m_1 - 1, \ldots, m_d - 1),\ \mathbf{v} \in \mathbb{N}_0^{d-1},\ \ell \leqslant (L^{\mathbf{x}})_{\mathbf{v}}\}$. Then we wish to determine when Λ^d is a GNS.

The set of points Λ^d corresponding to $\mathscr{L}$ form a valid GNS if and only if the set is closed under addition. Any nonzero point in Λ^d is expressible as $\mathbf{x} + (m_1 v_1, \ldots, m_{d-1} v_{d-1}, m_d \ell)$ where $\mathbf{x} \leqslant (m_1 - 1, \ldots, m_d - 1)$ and $\ell \geqslant (L^{\mathbf{x}})_{\mathbf{v}}$.

Now, select $\mathbf{x} + (m_1 v_1, \ldots, m_{d-1} v_{d-1}, m_d \ell)$ and $\mathbf{y} + (m_1 w_1, \ldots, m_{d-1} w_{d-1}, m_d k)$ from Λ^d. These must satisfy $\ell \geqslant (L^{\mathbf{x}})_{\mathbf{v}}$ and $k \geqslant (L^{\mathbf{y}})_{\mathbf{w}}$. We require that their sum

$$\begin{aligned} &\mathbf{x} + \mathbf{y} + (m_1(v_1 + w_1), \ldots, m_{d-1}(v_{d-1} + w_{d-1}), m_d(\ell + k)) \\ &\qquad = \mathbf{z} + \left(\sum_{i \in X} m_i \mathbf{e_i^d}\right) + (m_1(v_1 + w_1), \ldots, m_{d-1}(v_{d-1} + w_{d-1}), m_d(\ell + k)) \end{aligned}$$

is in Λ^d, too. Let $\mathbf{a} := (v_1, \ldots, v_{d-1}, \ell) + (w_1, \ldots, w_{d-1}, k) + \sum_{i \in X} \mathbf{e_i^d}$. The above is true if and only if $a_d \geqslant (L^{\mathbf{z}})_{(a_1, \ldots, a_{d-1})}$, or

- $\ell + k \geqslant (L^{\mathbf{z}})_{(a_1, \ldots, a_{d-1})}$ if $d \notin X$, and
- $\ell + k \geqslant (L^{\mathbf{z}})_{(a_1, \ldots, a_{d-1})} - 1$ if $d \in X$.

Note the right-hand side is exactly $(\mathrm{sh}_X(L^{\mathbf{z}}))_{\mathbf{v}+\mathbf{w}}$.

So essentially, the condition that Λ^d is closed under addition is equivalent to the following: if $\ell \geqslant (L^{\mathbf{x}})_{\mathbf{v}}$ and $k \geqslant (L^{\mathbf{y}})_{\mathbf{w}}$, then $\ell + k \geqslant (\mathrm{sh}_X(L^{\mathbf{z}}))_{\mathbf{v}+\mathbf{w}}$. Fixing $\mathbf{u} := \mathbf{v} + \mathbf{w}$, we thus have that

$$\min_{\mathbf{u}=\mathbf{v}+\mathbf{w}} ((L^{\mathbf{x}})_{\mathbf{v}} + (L^{\mathbf{y}})_{\mathbf{w}}) \geqslant (\mathrm{sh}_X(L^{\mathbf{z}}))_{\mathbf{u}}.$$

The left-hand side is precisely $(L^{\mathbf{x}} \boxplus L^{\mathbf{y}})_{\mathbf{u}}$, so the desired claim follows. □

Evidently, the sum of the entries over all partitions $L^{\mathbf{x}} \in \mathscr{L}(\Lambda^d)$ is the genus $g(\Lambda^d)$.

5.1 GNSs with fixed multset

In this section, we study the number of GNSs of fixed multset, and thus fixed shape. The following result is immediate.

Proposition 44. *The number of GNSs with multset* $\{\mathbf{e_1^d}, \ldots, \mathbf{e_{i-1}^d}, 2\cdot\mathbf{e_i^d}, \mathbf{e_{i+1}^d}, \ldots, \mathbf{e_d^d}\}$ *and genus* g *is* $p_{d-1}(g)$.

Proof. Suppose Λ^d has genus g and the given multset. Then the partition labeling $\mathscr{L}(\Lambda^d)$ has $L^{\mathbf{0}} = \varnothing$ and $L^{\mathbf{e_i^d}}$ being some partition of g. There are exactly $p_{d-1}(g)$ choices of $L^{\mathbf{e_i^d}}$, all of which give rise to valid GNSs, so the result follows. □

Corollary 45. *The number of* d*-dimensional GNSs with shape size* 2 *is* $d \cdot p_{d-1}(g)$.

Proof. If $s(\Lambda^d) = 2$, then $\mathcal{M}(\Lambda^d)$ has d elements and consists of $2\mathbf{e_i^d}$ for some i, and $\mathbf{e_j^d}$ for $j \neq i$. Then apply Proposition 44 for each i. □

However, it is significantly more difficult to count the number of GNSs with shape size at least 3. For instance, in the case of $d = 2$, the possible shapes of size 3 are:

- $[3]$, which corresponds to $\mathcal{M} = \{(0,3),(1,0)\}$;
- $[2,1]$, which corresponds to $\mathcal{M} = \{(0,2),(1,1),(2,0)\}$; and
- $[1,1,1]$, which corresponds to $\mathcal{M} = \{(0,1),(3,0)\}$.

One can check that

- the number $n_{g,2}^{[3]}$ of GNSs with shape $[3]$ is equal to the number of pairs of partitions π, π', whose total is g, such that $\pi \boxplus \pi \geqslant \pi'$ and $\pi' \boxplus \pi' \geqslant \mathrm{sh}_{\{2\}}(\pi)$;
- the number $n_{g,2}^{[2,1]}$ of GNSs with shape $[2,1]$ is equal to the number of pairs of partitions π, π', whose total is g, such that $\pi \geqslant \mathrm{sh}_{\{1\}}(\pi')$ and $\pi' \geqslant \mathrm{sh}_{\{2\}}(\pi)$; and
- we have $n_{g,2}^{[1,1,1]} = n_{g,2}^{[3]}$.

We have computed $n_{g,2}^{[3]}$ and $n_{g,2}^{[2,1]}$ for $g \leqslant 50$, shown in Table 1. These sequences do not yet appear on the OEIS and do not appear to have a well-behaved closed form.

Nevertheless, we can write down the following coarse asymptotic bound on the number of GNSs with fixed multset.

g	$n_{g,2}^{[3]}$	$n_{g,2}^{[2,1]}$	g	$n_{g,2}^{[3]}$	$n_{g,2}^{[2,1]}$	g	$n_{g,2}^{[3]}$	$n_{g,2}^{[2,1]}$	g	$n_{g,2}^{[3]}$	$n_{g,2}^{[2,1]}$
1	0	0	14	1028	1675	27	92415	105990	40	3546174	2908311
2	1	1	15	1526	2422	28	125261	139819	41	4587402	3671626
3	4	4	16	2241	3462	29	168974	183648	42	5918389	4623480
4	8	10	17	3251	4900	30	227020	240224	43	7615125	5807744
5	14	22	18	4691	6874	31	303674	312984	44	9773454	7277974
6	27	43	19	6697	9560	32	404646	406255	45	12512191	9099348
7	45	76	20	9503	13198	33	537092	525424	46	15980127	11351083
8	73	129	21	13387	18092	34	710360	677201	47	20361285	14129340
9	118	210	22	18747	24636	35	936150	869940	48	25885096	17550599
10	189	331	23	26074	33344	36	1229632	1113989	49	32834413	21755722
11	293	510	24	36073	44873	37	1609732	1422136	50	41560508	26914894
12	454	771	25	49595	60058	38	2100858	1810194			
13	684	1144	26	67874	79977	39	2733427	2297616			

Table 1: Values of $n_{g,2}^{[3]}$ and $n_{g,2}^{[2,1]}$ for $g \leqslant 50$.

Proposition 46. *For each d, there is a constant $K_d > 0$ for which the number of GNSs with genus g and multset $\mathcal{M}$ is at most*

$$\binom{V(\mathcal{M})+g-1}{g} \cdot K_d^{g^{(d-1)/d} V(\mathcal{M})^{1/d}}$$

for all $g > 0$ and multsets $\mathcal{M}$.

Proof. By Theorem 3, there is a constant $k^+ > 0$ for which $p_{d-1}(n) < \exp(k^+ n^{(d-1)/d})$ for all n. A partition labeling $\mathscr{L}(\Lambda^d)$ consists of at most $V(\mathcal{M})$ nonzero partition labels whose sum is equal to g. Namely, if we require that $L^{\mathbf{x}}$ is a partition of size $n^{\mathbf{x}}$, then there are at most

$$\prod_{\mathbf{x} \leqslant (m_1-1,\ldots,m_d-1)} p_{d-1}(n^{\mathbf{x}}) \leqslant \exp\left(k^+ \sum_{\mathbf{x} \leqslant (m_1-1,\ldots,m_d-1)} (n^{\mathbf{x}})^{(d-1)/d}\right)$$

choices of $\mathscr{L}(\Lambda^d)$. Since $g = \sum n^{\mathbf{x}}$ and $f(x) = x^{(d-1)/d}$ is concave, by Jensen's inequality on concave functions we have that the right-hand side is at most

$$\exp\left(k^+ (g/V(\mathcal{M}))^{(d-1)/d} V(\mathcal{M})\right).$$

There are at most $\binom{V(\mathcal{M})+g-1}{g}$ choices of $n^{\mathbf{x}}$, so the result follows. □

5.2 Rectangular GNSs

We say that a GNS Λ^d is *rectangular* if $\mathcal{M}(\Lambda^d)$ has size d. In particular, this means that $\mathcal{M}(\Lambda^d) = \{m_1 \mathbf{e_1^d}, \ldots, m_d \mathbf{e_d^d}\}$. We denote this by using the square symbol □ in the exponent. The partition labeling of a rectangular GNS $\Lambda^{d,\square}$ naturally encodes the depth q in the following way.

Proposition 47. *The depth of a rectangular GNS $\Lambda^{d,\square}$ is exactly*

$$\max_{L^{\mathbf{x}}\in\mathscr{L}(\Lambda^{d,\square})} \max_{\substack{\mathbf{v}\in\mathbb{N}_0^d\\ (L^{\mathbf{x}})_{\mathbf{v}}>0}} (v_1+\cdots+v_{d-1}+(L^{\mathbf{x}})_{\mathbf{v}}).$$

Proof. If $\mathcal{M}=\{m_1\mathbf{e_1^d},\dots,m_d\mathbf{e_d^d}\}$, then we have that

$$k\mathcal{M}=\{(a_1m_1,\dots,a_dm_d) : a_1+\cdots+a_d=k\}.$$

Thus, for each $\mathbf{x}\leqslant(m_1-1,\dots,m_d-1)$, the point $\mathbf{x}+(m_1v_1,\dots,m_{d-1}v_{d-1},m_d\ell)\notin\Lambda^d$ is part of $\mathcal{R}_k$ if and only if $v_1+\cdots+v_{d-1}+\ell=k-1$. The maximal value of the left-hand sum over all choices of $(\mathbf{v},\ell)$ is exactly the expression given in the theorem. But by Corollary 16, the depth q is the greatest integer k for which $\mathcal{R}_k\not\subseteq\Lambda^{d,\square}$, from which the result follows after letting $\mathbf{x}$ vary. □

Recall that $P_d(x):=\sum_{n\geqslant 0}p_d(n)x^n$ is the multi-dimensional partition generating function.

Theorem 48. *Let r'_d be the unique positive root of $P_{d-1}(1/x)=2$. Then we have*

$$\mathsf{r}_d\leqslant\limsup_{g\to\infty}\left(n_{g,d}^{\square}\right)^{1/g}\leqslant\mathsf{r}'_d.$$

Proof. We show the lower and upper bounds separately. Let $n_{g,d}^{\mathcal{M}}$ be the number of d-dimensional GNSs with genus g and multset $\mathcal{M}$. In each case, we estimate $n_{g,d}^{\mathcal{M}}$ for fixed $\mathcal{M}$ of size d, then we sum over all $\mathcal{M}$.

Suppose Λ^d has multset $\mathcal{M}=\{m_1\mathbf{e_1^d},\dots,m_d\mathbf{e_d^d}\}$.

Lower bound. We construct a family of valid partition labelings $\mathscr{L}(\Lambda^d)$ that match the lower bound. Let τ denote the $(d-1)$-dimensional partition given by $\tau_{\mathbf{0}}=1$ and $\tau_{\mathbf{v}}=0$ for $\mathbf{v}>\mathbf{0}$. If every partition in $\mathscr{L}$ (other than $L^{\mathbf{0}}$) is nonzero and less than or equal to $\tau\boxplus\tau$, then evidently $\mathscr{L}$ is a valid partition labeling.

There are exactly 2^d nonempty partitions less than or equal to $\tau\boxplus\tau$, of which $\binom{d}{s-1}$ have sum $s\geqslant 1$. Hence, we have that

$$\sum_{g=1}^{\infty} n_{g,d}^{\mathcal{M}}x^g\geqslant\left(x(x+1)^d\right)^{V(\mathcal{M})-1}$$

for any $x>0$. By summing over all $\mathcal{M}$, we can show that

$$\sum_{g=1}^{\infty} n_{g,d}^{\square}x^g=\sum_{\substack{\mathcal{M}\text{ multset}\\|\mathcal{M}|=d}}\sum_{g=1}^{\infty} n_{g,d}^{\mathcal{M}}x^g\geqslant\sum_{\substack{\mathcal{M}\text{ multset}\\|\mathcal{M}|=d}}\left(x(x+1)^d\right)^{V(\mathcal{M})-1}\geqslant\sum_{V=0}^{\infty}\left(x(x+1)^d\right)^{V}.$$

By the root test, the right-hand side converges if $x(x+1)^d\leqslant 1$ or $x<1/\mathsf{r}_d$. But then the left-hand side converges when $x<\left(\limsup_{g\to\infty}\left(n_{g,d}^{\square}\right)^{1/g}\right)^{-1}$, so the radius of convergence for the left-hand side is at least $1/\mathsf{r}_d$.

(In the language of depth-2 regions, cf. §4, we are counting the number of depth-2 rectangular GNSs. However, we choose to work with partition labelings here to mirror the proof of the upper bound below.)

Upper bound. If Λ^d has multset $\mathcal{M} = \{m_1\mathbf{e_1^d}, \ldots, m_d\mathbf{e_d^d}\}$, then $\mathscr{L}(\Lambda^d)$ consists of a $m_1 \times \cdots \times m_d$ prism whose nonzero entries are all labeled with nonzero $(d-1)$-dimensional partitions. In particular, we have that

$$\sum_{g=1}^{\infty} n_{g,d}^{\mathcal{M}} x^g \leqslant \left(\sum_{k=1}^{\infty} p_{d-1}(k)x^k\right)^{V(\mathcal{M})-1} = (P_{d-1}(x)-1)^{V(\mathcal{M})-1}$$

for any $x > 0$. By summing over all $\mathcal{M}$, we get that

$$\begin{aligned}\sum_{g=1}^{\infty} n_{g,d}^{\square} x^g &= \sum_{\substack{\mathcal{M} \text{ multset} \\ |\mathcal{M}|=d}} \sum_{g=1}^{\infty} n_{g,d}^{\mathcal{M}} x^g \\ &\leqslant \sum_{\substack{\mathcal{M} \text{ multset} \\ |\mathcal{M}|=d}} (P_{d-1}(x)-1)^{V(\mathcal{M})-1} \\ &\leqslant \sum_{V=1}^{\infty} \sigma_0(V)^d \left(P_{d-1}(x)-1\right)^{V-1},\end{aligned}$$

since there are at most $\sigma_0(V)^d$ choices of $m_1, m_2, \ldots, m_d$ which multiply to V. (Here, $\sigma_0(V)$ denotes the number of divisors of V.) By the root test, the right-hand side converges only if

$$\limsup_{V\to\infty} \left(\sigma_0(V)^d \left(P_{d-1}(x)-1\right)^V\right)^{1/V} = \limsup_{V\to\infty} (P_{d-1}(x)-1) \leqslant 1,$$

which is not true if $P_{d-1}(x) > 2$. Then we finish in a similar fashion to the lower bound. □

6 Future directions

In this section, we discuss possible lines of future work by sharpening bounds on $n_{g,d}$ and better understanding partition labelings.

6.1 Sharpening asymptotics

In Section 4, we show an exponential lower bound and a superexponential upper bound on $n_{g,d}$ with respect to g. It is natural to ask whether we can reconcile these bounds, since it is somewhat unclear whether $n_{g,d}$ grows exponentially or superexponentially.

Question 49. Is the quantity $n_{g,d}^{1/g}$ bounded?

Part of the difficulty of this question is the disparity between the sizes of $s_1(\mathcal{M})$ and $V(\mathcal{M})$; the excluded elements of a generalized numerical semigroup can be "skinny" along each of the coordinate axes, which yields a small shape size but a large volume for $d \geqslant 2$. For instance, this behavior is exhibited in the set $\mathcal{A}_g$ of possible points excluded from a genus g GNS (cf. §4.2), which has shape size $O(g(\ln g)^{d-1})$ but volume $O(g^d)$. If $n_{g,d}$ does grow exponentially, it is also natural to ask whether our lower bound of r_{2^d} is sharp, and if so, whether the subexponential factor is $C_d^{g^{(d-1)/d}}$.

Question 50. Does the limit $\lim_{g\to\infty} \mathsf{r}_{2^d}^{-g^{1/d}} n_{g,d}^{g^{-(d-1)/d}}$ exist?

A key ingredient of Zhai's proof is the conjecture of Zhao that almost all numerical semigroups have small depth after ordering by genus [23, Conj. 2]. This is no longer true for the general case, at least in terms of exponential growth and for our definition of depth. Take the following example, which also exhibits the aforementioned "axial skinniness."

Example 51. We will construct a large family of GNSs with depth q as follows.

Let $\mathcal{M}_k$, $s_{1,k,d}$, and $s_{2,k,d}$ be defined as in the proof of Lemma 33. Suppose Λ^d has genus g and satisfies the following properties:

- we have $t \cdot \mathbf{e_1^d} \notin \Lambda^2$ for every positive integer $t \leqslant (q-1)k+1$ and $k \nmid t$;
- but it contains every other point in a depth-k region for $k \geqslant 3$.

One can check that Λ^d must be a GNS as follows. Suppose $\mathbf{a}, \mathbf{b} \in \Lambda^d$ are nonzero.

- *Case 1*: Both $\mathbf{a}$ and $\mathbf{b}$ are multiples of $\mathbf{e_1^d}$. Note that $S = \mathbb{N}_0 \setminus \{t \in \mathbb{N}_0 : t \leqslant (q-1)k+1,\ k \nmid t\}$ is a numerical semigroup. Moreover, we have $s \cdot \mathbf{e_1^d} \in \Lambda^d$ if and only if $s \in S$. Hence, since $a_1, b_1 \in S$, we have $a_1 + b_1 \in S$, ergo $\mathbf{a} + \mathbf{b} \in \Lambda^d$.
- *Case 2*: Either $\mathbf{a}$ or $\mathbf{b}$ are not multiples of $\mathbf{e_1^d}$. Then $\mathbf{a} + \mathbf{b}$ is in a region of at least 3 by Lemma 15 and also is not a multiple of $\mathbf{e_1^d}$. This guarantees $\mathbf{a} + \mathbf{b} \in \Lambda^d$ by construction.

Then there are $g - s_{1,k,d} - (q-2)(k-1)$ more points to exclude from $\mathcal{R}_2(\mathcal{M}_k)$, which has $s_{2,k,d} - k + 1$ remaining elements. Hence, there are $\binom{s_{2,k,d}-k+1}{g-s_{1,k,d}-(q-2)(k-1)}$ GNSs that satisfy the given requirements. By holding q constant and letting g, k grow, this quantity has exponential growth factor r_{2^d}. However, all of these GNSs have depth q. For an example of a GNS with $k = 3$ and $q = 5$, see Figure 4.

However, the data from small cases suggests that most GNSs are depth 2 or 3, akin to the one-dimensional case. A natural guess for $d \geqslant 2$ would be that all $n_{g,d,q}$ have exponential growth rate r_{2^d}, but the subexponential growth factor $\mathsf{r}_{2^d}^{-g^{1/d}} n_{g,d,q}^{g^{-(d-1)/d}}$ is largest for $q = 3$. In this paper, we describe a large class of depth 2 GNSs; it would be interesting to examine large classes of depth 3 GNSs, analogous to the resuls of Zhao for $d = 1$ [24].

Question 52. Do d-dimensional GNSs almost all have depth 3 for $d \geqslant 2$?

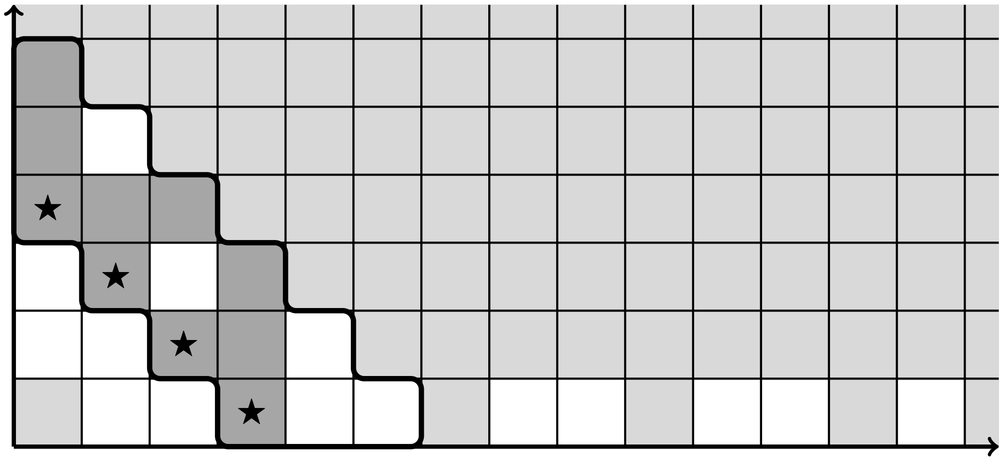

Figure 4: A GNS with multset $\mathcal{M}_3$ and depth 5, with $\mathcal{R}_2(\mathcal{M}_3)$ outlined.

g	$n_{g,2,1}$	$n_{g,2,2}$	$n_{g,2,3}$	$n_{g,2,4}$	$n_{g,2,5}$
1	2	0	0	0	0
2	3	4	0	0	0
3	5	14	4	0	0
4	7	48	12	4	0
5	11	143	44	8	4
6	15	412	163	36	8
7	22	1176	550	106	28
8	30	3332	1751	333	86
9	42	9287	5514	1009	254
10	56	25630	17080	3065	737
11	77	70574	52028	9128	2133
12	101	194290	156358	26985	6053
13	135	534127	465726	78983	16992
14	176	1465245	1377185	228727	47225

Table 2: Values of $n_{g,2,q}$ for $g \leqslant 14$ and $q \leqslant 5$.

g	$n_{g,2,2}$	g	$n_{g,2,2}$	g	$n_{g,2,2}$
0	0	30	12459909670309	60	80985430675574711412980916
1	0	31	33519288444409	61	215509768877495232586787465
2	4	32	90136456224494	62	573383202769145098057975309
3	14	33	242283690207403	63	1525226173996843571825323845
4	48	34	650936600796631	64	4056291288735430727151164447
5	143	35	1747891377256538	65	10785145844024419432004114254
6	412	36	4690642296534889	66	28669745119349640035022238173
7	1176	37	12580211126984860	67	76194552501074658365836459077
8	3332	38	33720107313956188	68	202455550832885616509159776241
9	9287	39	90333780254836434	69	537831233961624997213173542362
10	25630	40	241874514915972126	70	1428495078136679841557819365161
11	70574	41	647335685418582083	71	3793447898643022179662596244366
12	194290	42	1731773886602728051	72	10072060467737818893614010324770
13	534127	43	4631250509157734047	73	26738498106822231994902593485746
14	1465245	44	12381460478034483318	74	70973288195363677225963531535048
15	4011126	45	33092335174560159808	75	188363010271347363103428460974784
16	10961060	46	88424351052896671941	76	499851837500292856875731277058977
17	29903045	47	236212572399447537141	77	1326272602033306568840724593782556
18	81429566	48	630827866930313644489	78	3518599430142665518024919482166660
19	221325445	49	1684152607151129735036	79	9333642921927341197807452053383505
20	600659520	50	4494703368297811355435	80	24755693987767914166837735101399289
21	1628709545	51	11991135688827147388952	81	65650741449233606049989435056291703
22	4414300344	52	31978416951800296071831	82	174077611139574752854463708279935997
23	11958683448	53	85250406896754816152086	83	461512525629540684214148624663888012
24	32372736224	54	227191018857947112334513	84	1223377600065175892800725928261667064
25	87541376014	55	605282191834901220600054	85	3242455319972149681281785135048236895
26	236440731005	56	1612185156193460856587117	86	8592605228187134388298469836076911868
27	637862590414	57	4293176639427000769790008	87	22767484181294798508811998075481662904
28	1719101643609	58	11430408760122793960003154		
29	4629525846179	59	30427812808611490639896278		

Table 3: Values of $n_{g,2,2}$ for $g \leqslant 87$.

We have explicitly calculated $n_{g,2,q}$ for the small cases of $g \leqslant 14$ and $q \leqslant 5$, shown in Table 2, whose numerics support our conjecture. It is not difficult to show that $n_{g,d,1} = p_d(g+1)$ and $n_{g,d,g} = d^2$ for $g \geqslant 2$. However, the columns of the table are not yet on the OEIS, so it would be interesting to see if these sequences have other combinatorial significance.

Theorem 21 implies that the quantity $\mathsf{r}_{2^d}^{-g^{1/d}} n_{g,d,2}^{g^{-(d-1)/d}}$ for depth 2 GNSs is bounded. By implementing Proposition 22, we have calculated the values of $n_{g,2,2}$ for $n \leqslant 87$, shown in Table 3. The values of $\mathsf{r}_4^{-\sqrt{g}} n_{g,2,2}^{1/\sqrt{g}}$ are graphed against $1/g$ in Figure 5, which suggests that this quantity converges to a constant near 1.2.

6.2 Partition labelings

In Section 5, we generalize the notion of Kunz words to partition labelings. In the one-dimensional setting, Kunz words allow us to reinterpret the enumeration of numerical semigroups as a polytopal [15] and additive-combinatorial [2] problem, which allow us to use tools such as Ehrhart theory [13] and graph homomorphisms [17, 25] to count semigroups.

Thus, it is natural to ask whether these methods can be extended to partition labelings.

Question 53. Can we view partition labelings in a polytopal or additive-combinatorial setting?

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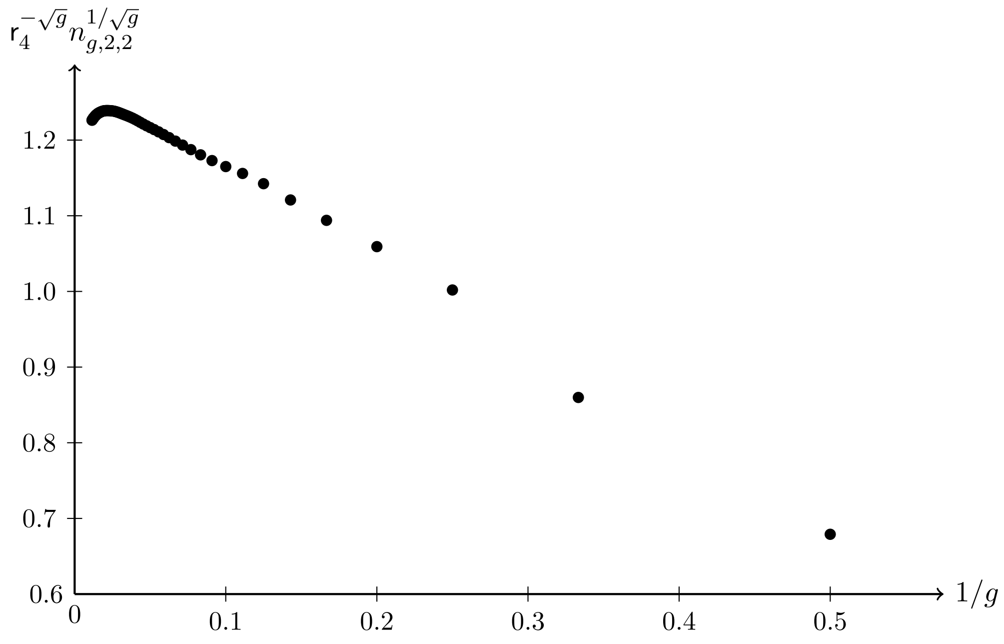

Figure 5: Plot of $\mathsf{r}_4^{-\sqrt{g}} n_{g,2,2}^{1/\sqrt{g}}$ against $1/g$ for $2 \leqslant g \leqslant 87$.

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References

[1] G. E. Andrews. *The theory of partitions.* Cambridge University Press (1998).

[2] R. Bacher. Generic numerical semigroups (2019). arXiv:2105.04200.

[3] J. Backelin. On the number of semigroups of natural numbers. *Mathematica Scandinavica*, **66** (1990), 197–215.

[4] D. P. Bhatia, M. A. Prasad, and D. Arora. Asymptotic results for the number of multidimensional partitions of an integer and directed compact lattice animals. *Journal of Physics A Mathematical General*, **30**(7) (1997), 2281–2285.

[5] M. Bras-Amorós. Fibonacci-like behavior of the number of numerical semigroups of a given genus. *Semigroup Forum*, **76**(2) (2007), 379–384.

[6] M. Bras-Amoros and A. de Mier. Representation of numerical semigroups by Dyck paths. *Semigroup Forum*, **75**(3) (2007), 676–681.

[7] C. Cisto, M. Delgado, and P. A. García-Sánchez. Algorithms for generalized numerical semigroups. *Journal of Algebra and Its Applications*, **20**(5) (2021).

[8] C. Cisto, G. Failla, C. Peterson, and R. Utano. Irreducible generalized numerical semigroups and uniqueness of the Frobenius element. *Semigroup Forum*, **99**(2) (2019), 481–495.

[9] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert W-function. *Advances in Computational Mathematics*, **5**(1) (1996), 329–359.

[10] S. Eliahou and J. Fromentin. Gapsets and numerical semigroups. *Journal of Combinatorial Theory, Series A*, **169** (2020), 105–129.

[11] G. Failla, C. Peterson, and R. Utano. Algorithms and basic asymptotics for generalized numerical semigroups in $\mathbb{N}^d$. *Semigroup Forum*, **92** (2016), 460–473.

[12] G. H. Hardy and S. Ramanujan. Asymptotic formulae in combinatory analysis. *Proceedings of the London Mathematical Society*, **s2-17**(1) (1918), 75–115.

[13] N. Kaplan. Counting numerical semigroups by genus and some cases of a question of Wilf. *Journal of Pure and Applied Algebra*, **216**(5) (2012), 1016–1032.

[14] N. Kaplan. Counting numerical semigroups. *The American Mathematical Monthly*, **124**(9) (2017), 862–875.

[15] N. Kaplan and C. O'Neill. Numerical semigroups, polyhedra, and posets I: the group cone. *Combinatorial Theory*, **1**(19) (2021), 1–23.

[16] E. Kunz. *Über die Klassifikation Numerischer Halbgruppen*. Fakultät fur Mathematik der Universität (1987).

[17] S. Li. Counting numerical semigroups by Frobenius number, multiplicity, and depth. *Combinatorial Theory*, **3**(3) (2023).

[18] L. H. Loomis and H. Whitney. An inequality related to the isoperimetric inequality. *Bulletin of the American Mathematical Society*, **55**(10) (1949), 961–962.

[19] P. A. MacMahon. *Combinatory analysis*. Cambridge Univ. Pr (1915).

[20] J. Rosales and P. García-Sánchez. *Numerical semigroups*. Springer-Verlag (2009).

[21] D. Singhal and Y. Lin. Frobenius allowable gaps of generalized numerical semigroups. *The Electronic Journal of Combinatorics*, **29**(4) #P4.12 (2022), 12.

[22] E. M. Wright. Asymptotic partition formulae: I. plane partitions. *The Quarterly Journal of Mathematics*, **2**(1) (1931), 177–189.

[23] A. Zhai. Fibonacci-like growth of numerical semigroups of a given genus. *Semigroup Forum*, **86**(3) (2012), 634–662.

[24] Y. Zhao. Constructing numerical semigroups of a given genus. *Semigroup Forum*, **80**(2) (2010), 242–254.

[25] D. G. Zhu. Sub-Fibonacci behavior in numerical semigroup enumeration. *Combinatorial Theory*, **3**(2) (2023).