

Cops and Robber – When Capturing is not Surrounding

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Submitted: Oct 21, 2024; Accepted: Jul 8, 2025; Published: Aug 22, 2025

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Abstract

We consider “surrounding” versions of the classic Cops and Robber game. The game is played on a connected graph in which two players, one controlling several cops and the other controlling a single robber, take alternating turns. In a turn, each player may move each of their pieces. The robber always moves between adjacent vertices. Regarding the moves of the cops, we distinguish four versions that differ in whether the cops are on the vertices or the edges of the graph and whether the robber may move on/through them. The goal of the cops is to surround the robber, i.e., to occupy all neighbors (vertex version) or incident edges (edge version) of the robber’s current vertex. In contrast, the robber tries to avoid being surrounded indefinitely. Given a graph, the so-called *cop number* denotes the minimum number of cops required to eventually surround the robber.

We relate the different cop numbers of these versions by showing that they are always within a factor of two times the maximum degree of one another. Furthermore, we prove that none of them is bounded by a function of the classical cop number and the maximum degree of the graph, thereby refuting a conjecture by Crytser, Komarov and Mackey [Graphs and Combinatorics, 2020].

Mathematics Subject Classifications: 05C57,91A24,91A46

1 Introduction

Cops and Robber is a well-known combinatorial game played by two players on a graph $G = (V, E)$. The robber player controls a single robber, which we shall denote by r , whereas the cop player controls k cops, denoted c_1, \dots, c_k , for some specified integer $k \geq 1$. The players take alternating turns, and may perform one move with each of their pieces (the single robber or the k cops) in each turn. In the classical game (and also many of its variants), the vertices of G are the possible positions for the pieces, while the edges of G

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model the possible moves. Let us remark that no piece is forced to move, i.e., there is no *zugzwang*. On each vertex there can be any number of pieces.

The game begins with the cop player choosing vertices as the starting positions for the k cops c_1, \dots, c_k . Then, seeing the cops' positions, the robber player places r on a vertex of G as well. The cop player wins if the cops *capture* the robber, which in the classical version means that at least one cop stands on the same vertex as the robber. On the other hand, the robber player wins if the robber can avoid being captured indefinitely.

The *cop number* $c(G)$ of a given connected¹ graph $G = (V, E)$ is the smallest k for which k cops can capture the robber in a finite number of turns. Clearly, every graph satisfies $1 \leq c(G) \leq |V|$.

We consider several versions of the Cops and Robber game. In some of these, the cops are placed on the edges of G and allowed to move to an *adjacent* edge (i.e., an edge sharing an endpoint) during their turn. In all our versions, the robber acts as in the original game but loses the game if he is *surrounded*² by the cops, meaning that they have to occupy all adjacent vertices or incident edges. At all times, let us denote by v_r the vertex currently occupied by the robber. Specifically, we define the following versions of the game, each specifying the possible positions for the cops and the exact surrounding condition:

Vertex Version Cops are positioned on vertices of G (like the robber). They surround the robber if there is a cop on each neighbor of v_r . Let $c_V(G)$ denote the smallest number of cops needed to eventually surround the robber.

Edge Version Cops are positioned on edges of G . A cop on an edge e can move to any edge e' sharing an endpoint with e during its turn. The cops surround the robber if there is a cop on each edge incident to v_r . Let $c_E(G)$ denote the smallest number of cops needed to eventually surround the robber.

In both versions above, the robber sits on the vertices of G and moves along the edges of G . Due to the winning condition for the cops being a full surround, the robber may come very close to, say, a single cop without being threatened. As this can feel counterintuitive, let us additionally consider a restrictive version of each game. Here, we constrain the possible moves for the robber when cops are close by. These *restrictive* versions are given by the following rules:

Restrictive Vertex Version After the robber's turn, there may not be any cop on v_r . In particular, the robber may not move onto a vertex occupied by a cop. Additionally, if a cop moves onto v_r , then the robber must leave that vertex in his next turn.

¹Cops cannot move between different connected components, so the cop number of any graph is the sum over all components. We thus consider connected graphs only.

²To distinguish between the classical and our versions, we use the term *capture* to express that a cop occupies the same vertex as the robber. In contrast, a *surround* always means that all neighbors, respectively incident edges, are occupied.

Restrictive Edge Version. The robber may not move along an edge that is currently occupied by a cop.

We denote the cop numbers of the restrictive versions by putting an additional “r” in the subscript, i.e., the smallest number of cops needed to eventually surround the robber in these versions is $c_{V,r}(G)$ and $c_{E,r}(G)$, respectively.

Clearly, the restrictive versions are favorable for the cops, as they only restrict the robber. Consequently, the corresponding cop numbers are always at most their nonrestrictive counterparts. Thus, for every connected graph G we have

$$c_{V,r}(G) \leq c_V(G) \quad \text{and} \quad c_{E,r}(G) \leq c_E(G). \tag{1}$$

A recent conjecture by Crytser, Komarov and Mackey [7] states that the cop number in the restrictive edge version can be bounded from above by the classical cop number and the maximum degree of the graph:

Conjecture 1 ([7]). For every connected graph G we have $c_{E,r}(G) \leq c(G) \cdot \Delta(G)$.

Prałat [14] verified Conjecture 1 for the random graph $G(n, p)$, i.e., the graph on n vertices where each possible edge is chosen independently with probability p , for some ranges of p . Let us note that Conjecture 1, if true, would strengthen a theorem by Crytser, Komarov and Mackey [7] stating that $c_{E,r}(G) \leq \gamma(G) \cdot \Delta(G)$, where $\gamma(G)$ denotes the size of a smallest dominating set in G .

An extended abstract of this work was presented at the 49th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2023) [10].

1.1 Our Results

Our main contribution is to disprove Conjecture 1. In fact, we prove that there are graphs G for which none of the surrounding cop numbers can be bounded by any function of $c(G)$ and $\Delta(G)$. This proves that the classical game of Cops and Robber is sometimes fundamentally different from all its surrounding versions.

Theorem 2. *There is an infinite family of connected graphs G with classical cop number $c(G) = 2$ and $\Delta(G) = 3$ while neither $c_V(G)$, $c_{V,r}(G)$, $c_E(G)$ nor $c_{E,r}(G)$ can be bounded by any function of $c(G)$ and the maximum degree $\Delta(G)$.*

Additionally, we relate the different surrounding versions to each other. Equation (1) already gives an upper bound for the cop numbers in the restrictive versions in terms of their corresponding nonrestrictive cop numbers. To complete the picture, our second contribution is to prove several lower and upper bounds for different combinations:

Theorem 3. *Each of the following holds (assuming G to be connected):*

1. $\forall G : c_V(G) \leq \Delta(G) \cdot c_{V,r}(G) \quad \text{and} \quad \exists G : c_V(G) \geq \Delta(G) \cdot c_{V,r}(G)$
2. $\forall G : c_E(G) \leq \Delta(G) \cdot c_{E,r}(G) \quad \text{and} \quad \exists G : c_E(G) \geq \Delta(G)/4 \cdot c_{E,r}(G)$

3. $\forall G : c_V(G) \leq 2 \cdot c_E(G)$ *and* $\exists G : c_V(G) \geq 2 \cdot (c_E(G) - 1)$
4. $\forall G : c_{V,r}(G) \leq 2 \cdot c_{E,r}(G)$ *and* $\exists G : c_{V,r}(G) \geq c_{E,r}(G)$
5. $\forall G : c_E(G) \leq \Delta(G) \cdot c_V(G)$ *and* $\exists G : c_E(G) \geq \Delta(G)/12 \cdot c_V(G)$
6. $\forall G : c_{E,r}(G) \leq \Delta(G) \cdot c_{V,r}(G)$ *and* $\exists G : c_{E,r}(G) \geq \Delta(G)/48 \cdot c_{V,r}(G)$

Note that all lower and upper bounds from Theorem 3 are tight up to a small additive or multiplicative constant. We prove the upper bounds in Section 2. The main idea is the same for all six inequalities: Given a winning strategy for a set of cops in one version, we can simulate the strategy in any other version. Afterward, in Section 3, we consider the lower bounds by constructing explicit families of graphs with the desired surrounding cop numbers. While some lower bounds already follow from standard graph families (like complete bipartite graphs), others need significantly more involved constructions. For example, we construct a family of graphs from a set of $k - 1$ mutually orthogonal Latin squares (see Section 3.4 for a definition).

Trivial Bounds. Clearly, if the robber is surrounded at a vertex v_r , then there must be at least $\deg(v_r)$ cops around him in all considered versions. Therefore, the minimum degree $\delta(G)$ of G is an obvious lower bound (stated explicitly for $c_{V,r}(G)$ in [6]). Moreover, if the robber can start at a vertex of *highest* degree and *never move*, i.e., in all but the restrictive vertex version, we get the *maximum degree* $\Delta(G)$ of G as a lower bound (stated explicitly in [7] for $c_{E,r}(G)$):

Observation 4. *For every connected graph $G = (V, E)$, we have*

- $c_{V,r}(G) \geq \delta(G)$ as well as
- $c_V(G) \geq \Delta(G)$, $c_E(G) \geq \Delta(G)$ and $c_{E,r}(G) \geq \Delta(G)$.

1.2 Related Work

The game of Cops and Robber was introduced by Nowakowski and Winkler [13] as well as Quilliot [15] almost forty years ago. Both consider the case where a single cop chases the robber. The version with many cops and therefore also the notion of the cop number $c(G)$ was introduced shortly after by Aigner and Fromme [1], already proving that $c(G) \leq 3$ for all connected planar graphs G . Their version is nowadays considered the standard version of the game, and we refer to it as the *classical version* throughout the paper. The most important open question regarding $c(G)$ is Meyniel's conjecture, stating that a connected n -vertex graph G has $c(G) \in O(\sqrt{n})$ [2, 8]. It is known to be EXPTIME-complete to decide whether $c(G) \leq k$ (for k being part of the input) [12].

By now, countless different versions of the game with their own cop numbers have been considered, see for example these books on the topic [3, 4].

The restrictive vertex version was introduced by Burgess et al. [6]. They prove bounds for $c_{V,r}(G)$ in terms of the clique number $\omega(G)$, the independence number $\alpha(G)$ and the

treewidth $\text{tw}(G)$, as well as considering several interesting graph families. They also show that, for every fixed value of k , it can be decided in polynomial time whether $c_{V,r}(G) \leq k$. The complexity is unknown for k being part of the input. Bradshaw and Hosseini [5] extend the study of $c_{V,r}(G)$ to graphs of bounded genus, proving (among other results) that $c_{V,r}(G) \leq 7$ for every connected planar graph G . See the bachelor's thesis of Schneider [16] for several further results on $c_{V,r}(G)$ (including a version with *zugzwang*).

The restrictive edge version was introduced even more recently by Crytser, Komarov and Mackey [7] (under the name *containment variant*). Besides stating Conjecture 1, which is verified for some graphs by Prałat [14], they give several bounds on $c_{E,r}(G)$ for different families of graphs.

To the best of our knowledge, $c_V(G)$ and $c_E(G)$ were not considered before.

In light of the (restrictive) vertex and edge versions, one might also define a *face version* for embedded planar graphs. Here, the cops occupy the faces, and they surround the robber if they occupy all faces incident to v_r . A restrictive face version could be that the robber must not move along an edge with either one or both incident faces being occupied by a cop. This version was introduced recently by Ha, Jungeblut and Ueckerdt [9]. Despite their similar motivation, the face versions seem to behave differently than the vertex or edge versions.

In each version, one might also add *zugzwang*, i.e., the obligation to actually move during one's turn. We are not aware of any results about this in the literature.

1.3 Outline of the Paper

Section 2 proves the upper bounds from Theorem 3. Then, in Section 3, we give constructions implying the corresponding lower bounds. Finally, in Section 4, we prove Theorem 2, thereby disproving Conjecture 1.

2 Relating the Different Versions

In this section, we prove the upper bounds from Theorem 3. The main idea is always that a sufficiently large group of cops in one version can simulate a single cop in another version. We denote by $N_G(v)$ and $N_G[v]$ the open and closed neighborhood of vertex v in G , respectively.

Proof of Theorem 3 (Upper Bounds). Let G be an arbitrary but connected graph.

1. $c_V(G) \leq \Delta(G) \cdot c_{V,r}(G)$: Let $\mathcal{S}_{V,r}(G)$ be a winning strategy for $k \in \mathbb{N}$ restrictive vertex cops c_1, \dots, c_k in G . For $i \in \{1, \dots, k\}$, replace c_i by a group of $\Delta(G)$ nonrestrictive vertex cops $C_i := \{c_i^1, \dots, c_i^{\Delta(G)}\}$. Initially, all cops in C_i start at the same vertex as c_i and whenever c_i moves to an adjacent vertex, all cops in C_i copy its move.

If the restrictive cops c_1, \dots, c_k arrive in a position where they surround the robber, then he is also surrounded by the groups of cops C_1, \dots, C_k . It remains to consider the case that the robber ends their turn on a vertex v , currently occupied by some

group C_i (a move that would be forbidden in the restrictive version). Then the cops in C_i can spread to the up to $\Delta(G)$ neighbors of v in G , thereby surrounding the robber.

2. $c_E(G) \leq \Delta(G) \cdot c_{E,r}(G)$: Let $\mathcal{S}_{E,r}(G)$ be a winning strategy for $k \in \mathbb{N}$ restrictive edge cops c_1, \dots, c_k in G . We can replace each cop c_i by a group of $\Delta(G)$ nonrestrictive edge cops $C_i := \{c_i^1, \dots, c_i^{\Delta(G)}\}$. Just as above, the cops in C_i copy the moves of c_i . If the c_1, \dots, c_k surround the robber, so do the C_1, \dots, C_k . Further, if the robber moves along an edge occupied by C_i towards a vertex v then the cops in C_i can move to the up to $\Delta(G)$ incident edges in G , thereby surrounding the robber.
3. $c_V(G) \leq 2 \cdot c_E(G)$: Let $\mathcal{S}_E(G)$ be a winning strategy for $k \in \mathbb{N}$ edge cops c_1, \dots, c_k in G . Each edge cop c_i occupying an edge uv can be simulated by two vertex cops c_i^1 and c_i^2 occupying the endpoints u and v of e . If c_i moves to an adjacent edge e' , c_i^1 and c_i^2 can move to the endpoints of e' in their next move. If the robber is surrounded by some of c_1, \dots, c_k , then $\bigcup_{i=1}^k \{c_i^1, c_i^2\} \supseteq N_G[v]$. Thus, the robber is surrounded.
4. $c_{V,r}(G) \leq 2 \cdot c_{E,r}(G)$: This works exactly as in the previous case. However, it remains to verify that two restrictive vertex cops c_i^1 and c_i^2 on the two endpoints of an edge e also simulate the restrictiveness of the edge cop c_i . This is the case, as traversing e would lead the robber onto a vertex currently occupied by a restrictive vertex cop, which is forbidden.
5. $c_E(G) \leq \Delta(G) \cdot c_V(G)$: Let $\mathcal{S}_V(G)$ be a winning strategy for $k \in \mathbb{N}$ vertex cops c_1, \dots, c_k . We replace each vertex cop c_i by a group of $\Delta(G)$ edge cops $C_i := \{c_i^1, \dots, c_i^{\Delta(G)}\}$ that initially position themselves all on an arbitrary edge of G incident to the vertex occupied by c_i . Now, if cop c_i moves along an edge e , then the cops in C_i all move to e in their next move.

When the vertex cops c_1, \dots, c_k surround the robber at a vertex v , then for each neighbor u of v , there is an edge e incident to u occupied by a group C_i . However, in general, the edge cops do not yet surround the robber. If the robber does not move during its next turn, for each edge e incident to v , at least one edge cop can move there from an edge e' adjacent to e , thereby surrounding the robber. Otherwise, if the robber moves to a neighbor u of v , then the $\Delta(G)$ edge cops on some edge incident to u can spread to all edges incident to u in their next turn, thereby surrounding the robber.

6. $C_{E,r}(G) \leq \Delta(G) \cdot c_{V,r}(G)$: Let $\mathcal{S}_{V,r}$ be a winning strategy for $k \in \mathbb{N}$ restrictive vertex cops. They can be simulated by k groups of $\Delta(G)$ restrictive edge cops each, just as in the previous case. However, it remains to prove that the robber also loses if he moves onto a vertex v that would be occupied by a simulated restrictive vertex cop (a move that would be forbidden in $\mathcal{S}_{V,r}$). In this case, there is a group of $\Delta(G)$ edge cops on some edge incident to v and they can spread to all edges incident to v in their next turn, thereby surrounding the robber. \square

Corollary 5. *For every graph G , the surrounding cop numbers $c_V(G)$, $c_E(G)$, $c_{V,r}(G)$ and $c_{E,r}(G)$ are always within a factor of $2\Delta(G)$ of each other.*

Proof. In each of the six upper bounds stated in Theorem 3, the number of cops increases by at most a factor of $\Delta(G)$. In all cases, this is obtained by simulating a winning strategy of one surrounding variant by (groups of) cops in another variant. The only cases where two such simulations need to be combined is when changing both, the cop type (vertex cops/edge cops) and the restrictiveness. It is easy to check that in all but one combination the number of cops increases by at most a factor of $2\Delta(G)$. The only exception is when a winning strategy for restrictive vertex cops is simulated by nonrestrictive edge cops, where the number of cops would increase by a factor of $\Delta(G)^2$. However, looking at the proof of Theorem 3, we can see that both simulations replace a single cop by a group of $\Delta(G)$ cops. In this particular case, it suffices to do this replacement just once. \square

We remark that all upper bounds in Theorem 3 result from simulating a winning strategy of another surrounding version. In the next section we show that these bounds are indeed (asymptotically) tight.

3 Explicit Graphs and Constructions

In this section, we shall mention or construct several families of graphs with some extremal behavior for their corresponding classical and surrounding cop numbers. Together, these graphs prove all lower bounds stated in Theorem 3.

3.1 Complete Bipartite Graphs

We start by considering complete bipartite graphs. By choosing the right parameters, they already serve to prove two of the lower bounds from Theorem 3. Furthermore, they appear again as a building block for slightly more complicated graphs in Section 3.2.

Proposition 6. *For every complete bipartite graph G , we have $c(G) = \min\{2, \delta(G)\}$, $c_{V,r}(G) = \delta(G)$ and $c_V(G) = c_{E,r}(G) = c_E(G) = \Delta(G)$.*

Proof. In the classical version, it is enough to place one cop in each bipartition class. Wherever the robber is, the cop from the other bipartition class can capture him in his next turn, proving $c(G) \leq 2$. Further, if $\delta(G) = 1$, then a single cop suffices: By starting on the single vertex in one bipartition class, the robber is forced to position himself in the other. There, he gets captured in the cops' next turn. This proves $c(G) \leq \delta(G)$. On the other hand, a single cop is not enough in case that $\delta(G) \geq 2$: The robber can always stay in the same bipartition class as the cop (but on a different vertex). We conclude that $c(G) \geq 2$ in this case, and therefore $c(G) = \min\{2, \delta(G)\}$.

The lower bounds of $\delta(G)$ and $\Delta(G)$ for all surrounding versions follow from Observation 4. Therefore, it remains to show upper bounds below.

A group of $\delta(G)$ restrictive vertex cops can initially occupy all vertices of the smaller bipartition class. This forces the robber to position himself on a vertex in the larger bipartition class, where he is surrounded immediately.

A winning strategy for $\Delta(G)$ nonrestrictive vertex cops is to initially occupy all vertices from the larger bipartition class. If the robber positions himself in the smaller bipartition class, he is surrounded immediately, so he has to position himself on one of the cops. Then in the cops' next turn, any subset of $\delta(G)$ cops can occupy the smaller bipartition class, thereby surrounding the robber.

In the restrictive and nonrestrictive edge versions, we let $\Delta(G)$ cops position themselves such that, in both bipartition classes, each vertex has at least one incident edge occupied by a cop. Then, no matter where the robber positions himself, he gets surrounded by the cops within their next turn. \square

Let us consider two special cases of Proposition 6 for all $\Delta \in \mathbb{N}$: First, the star $K_{\Delta,1}$ has $c_{V,r}(K_{\Delta,1}) = 1$ while $c_V(K_{\Delta,1}) = \Delta$, thus proving the lower bound in Item 1 of Theorem 3. Second, the complete bipartite graph $K_{\Delta,\Delta}$ has $c_{V,r}(K_{\Delta,\Delta}) = c_{E,r}(K_{\Delta,\Delta}) = \Delta$, thus proving the lower bound in Item 4 of Theorem 3.

3.2 Regular Graphs with Leaves

Our first construction takes a connected k -regular graph H and attaches a set of ℓ new degree-1-vertices (*leaves*) to each vertex. Depending on H , k , ℓ and the girth³ of H , we can give several bounds on the surrounding cop numbers of the resulting graph.

Lemma 7. *Let $H = (V_H, E_H)$ be a k -regular connected graph and let $G = (V_G, E_G)$ be the graph obtained from H by attaching to each vertex $v \in V_H$ a set of ℓ new leaves for some $\ell \geq 0$. Then each of the following holds:*

$$1. \ c_V(G) \geq \begin{cases} k(k + \ell - 1) & \text{if } \text{girth}(H) \geq 7 \\ (k + 1)\ell & \text{always} \end{cases}$$

$$2. \ c_{V,r}(G) = \max\{c_{V,r}(H), k + 1\}$$

$$3. \ c_E(G) \geq \begin{cases} k(k + \ell - 1) & \text{if } \text{girth}(H) \geq 6 \\ k\ell & \text{if } \text{girth}(H) \geq 4 \\ \frac{1}{2}(k + 1)\ell & \text{always} \end{cases}$$

$$4. \ c_{E,r}(G) = \max\{c_{E,r}(H), k + \ell\}$$

Proof. Note that most claimed inequalities hold trivially for the case that $\ell = 0$ (many lower bounds become 0, while others follow from $G = H$ in this case). Only the two cases requiring $\text{girth}(H) \geq 6$, respectively $\text{girth}(H) \geq 7$, are not directly clear. However, their proofs below hold for $\ell = 0$ as well. In all other cases, we implicitly assume $\ell \geq 1$ to avoid having to handle additional corner cases.

³The *girth* of H is the length of a shortest cycle in H . If H contains no cycle, we define $\text{girth}(H) = \infty$.

1. To prove the lower bounds on $c_V(G)$, consider any configuration of cops on the vertices of G . For a vertex $v \in V_H$ of G , let A_v be the set consisting of v and all leaves that are attached to it, i.e., $A_v = \{v\} \cup (N_G[v] \setminus V_H)$. We call a vertex $v \in V_H$ *safe* if there are fewer than ℓ cops on A_v in G . Note that if the robber ends his turn on a safe vertex, then the cops cannot surround him in their next turn. Let v_r be the current position of the robber. If the total number of cops is less than $(k+1)\ell$, then at least one of the $k+1$ vertices in the closed neighborhood $N_H[v_r]$ of v_r is safe, as no cop can be in A_v and A_w for $v \neq w$. Thus, the robber always has a safe vertex to move to (or to remain on), giving him a strategy to avoid being surrounded. It follows that $c_V(G) \geq (k+1)\ell$.

Now, if $\text{girth}(H) \geq 7$ and the robber is on $v_r \in V_H$, then we consider for each neighbor v of v_r in V_H additionally the set $B_v = N_G[N_G(v) \setminus \{v_r\}]$, i.e., all vertices w with $\text{dist}(w, v) \leq 2$ except from $N_G[v_r] \setminus \{v\}$. Since $\text{girth}(H) \geq 7$, we have that $B_v \cap B_w = \emptyset$ for distinct $v, w \in N_H(v_r)$. Similar to above, we call $v \in N_H(v_r)$ *safe* if B_v contains fewer than $k + \ell - 1$ cops. Again, if the robber ends his turn on a safe vertex, the cops cannot surround him in their next turn. If the total number of cops is less than $k(k + \ell - 1)$, then at least one of the k neighbors of v_r in H is safe. This would give the robber a strategy to avoid being surrounded. It follows that $c_V(G) \geq k(k + \ell - 1)$ in the case that $\text{girth}(H) \geq 7$.

2. To prove that $c_{V,r}(G) \geq \max\{c_{V,r}(H), k + 1\}$, first note that $c_{V,r}(G) \geq c_{V,r}(H)$ because the leaves do not help the cops. Further, $c_{V,r}(G) \geq k + 1$ holds because the robber versus k cops can easily always stay on the subgraph H of G where it is impossible to surround him (in G) with k cops only (note that G has at least $k + 1$ vertices).

For the upper bound $c_{V,r}(G) \leq \max\{c_{V,r}(H), k + 1\}$, we describe a two-phase winning strategy for the cops. In the first phase, let $c_{V,r}(H)$ cops execute an optimal strategy for surrounding the robber on the subgraph H of G . If during that phase the robber leaves H , i.e., steps on a leaf attached to some $v \in V_H$, act as if the robber stays on v . Hence, after finitely many rounds, the robber is on a vertex in the set $N_v = \{v\} \cup (N_G(v) - V_H)$ for some $v \in V_H$, and k of the $c_{V,r}(H)$ cops occupy all k neighbors of v in V_H . As we are in the restrictive variant, the robber may not leave N_v . Now, in the second phase, let one of the remaining $\max\{c_{V,r}(H), k + 1\} - k \geq 1$ cops go to vertex v . This forces the robber to go to an attached leaf and be surrounded there.

3. To prove the three lower bounds on $c_E(G)$, consider any configuration of cops on the edges of G . For a vertex $v \in V_H$ of G , let E_v be the set of all edges in G incident to v . We call vertex $v \in V_H$ *safe* if there are fewer than ℓ cops in total on E_v in G . Note that if the robber ends his turn on a safe vertex, then the cops can not surround him in their next turn. Let the robber be at vertex $v_r \in V_H$. If all vertices $v \in N_H[v_r]$ were unsafe, then each such v has at least ℓ cops in E_v . Further, each cop is on an edge incident to at most two vertices in $N_H[v_r]$. Thus, at least $\frac{1}{2}(k+1)\ell$ cops are required to make all vertices in $N_H[v_r]$ unsafe, i.e., for fewer than

$\frac{1}{2}(k+1)\ell$ cops there is always a safe vertex for the robber to move onto.

If we further assume $\text{girth}(H) \geq 4$, then no two vertices in $N_H(v_r)$ are connected by an edge, so the k vertices in $N_H(v_r)$ each require their own ℓ cops on incident edges to be unsafe. In other words, fewer than $k\ell$ cops always guarantee the robber a safe vertex to move onto.

Now we even assume that $\text{girth}(G) \geq 6$. Similar to the vertex version, we define for each $v \in N_H(v_r)$ a set B_v containing all edges incident to a vertex in $N_G[v]$ that do not have v_r as an endpoint. We say that v is *safe* if there are at most $k + \ell - 1$ cops in B_v . From $\text{girth}(H) \geq 6$ it follows that $B_v \cap B_w = \emptyset$ for distinct neighbors v, w of v_r . Thus, for all $v \in N_H(v_r)$ to be unsafe, we would need at least $k(k + \ell - 1)$ cops.

4. To prove that $c_{E,r}(G) \geq \max\{c_{E,r}(H), k + \ell\}$, first note that $c_{E,r}(G) \geq c_{E,r}(H)$ because the leaves do not help the cops. Further, $c_{E,r}(G) \geq k + \ell = \Delta(G)$ holds by Observation 4.

Finally, the upper bound $c_{E,r}(G) \leq \max\{c_{E,r}(H), k + \ell\}$ for the restrictive edge version follows mostly along the lines of the restrictive vertex version above. In a first phase, we use $c_{E,r}(H)$ cops to force the robber onto a vertex v or a leaf attached at v , while each of the k incident edges at v in E_H is occupied by a cop. In the second phase, let ℓ of the remaining $\max\{c_{E,r}(H), k + \ell\} - k \geq \ell$ cops go to occupy the remaining ℓ incident edges at v (the ones that are not in E_H). This surrounds the robber, either on v or an attached leaf. \square

Applied to different host graphs H , Lemma 7 yields several interesting bounds, summarized in the following corollaries: Corollary 8 proves the lower bound in Item 2 for even Δ , and Corollary 9 proves the lower bound in Item 3.

Corollary 8. *For every $\Delta \geq 2$ there is a connected graph G with $\Delta(G) = \Delta$ such that $c_{V,r}(G) = \lfloor \frac{\Delta}{2} \rfloor + 1$, $c_V(G) = (\lfloor \frac{\Delta}{2} \rfloor + 1) \lceil \frac{\Delta}{2} \rceil$, $c_{E,r}(G) = \Delta(G)$ and $c_E(G) = \lfloor \frac{\Delta}{2} \rfloor \lceil \frac{\Delta}{2} \rceil$.*

Proof. Let $k = \lfloor \frac{\Delta}{2} \rfloor$, let $H = K_{k,k}$ be the k -regular complete bipartite graph and choose $\ell = \lceil \frac{\Delta}{2} \rceil$. We consider the graph G obtained from H by attaching ℓ new leaves to each vertex. Note that G has maximum degree $\Delta(G) = k + \ell = \Delta$.

- Lemma 7 yields that $c_{V,r}(G) = \max\{c_{V,r}(H), k + 1\}$. From Proposition 6, we further get that $c_{V,r}(H) = k$, so actually $c_{V,r}(G) = k + 1 = \lfloor \frac{\Delta}{2} \rfloor + 1$.
- Lemma 7 gives $c_V(G) \geq (k + 1)\ell$. It remains to show that $(k + 1)\ell$ nonrestrictive vertex cops can always surround the robber, i.e., that $c_V(G) \leq (k + 1)\ell$. We give a winning strategy for $(k + 1)\ell$ cops. Let A and B be the two bipartition classes of H . Initially, we place ℓ cops on each of the k vertices in A and the remaining ℓ cops such that each $b \in B$ is occupied by at least one cop (this is always possible as $\ell \geq k$). Now, if the robber starts at a leaf, he is surrounded immediately. If he starts at $a \in A$, then the ℓ cops on a spread across all adjacent leaves, thereby

surrounding the robber. Thus, the robber has to start at $b \in B$. If he stays on b or an adjacent leaf, he can be surrounded by the ℓ cops currently on B in a finite number of turns. Thus, he has to move to $a \in A$ at some point. Then ℓ of the cops on a move to the adjacent leaves and all other cops currently on some vertices in A move to B in a way such that all vertices of B are occupied. (If $k \geq 2$, the ℓ cops on some $a' \in A \setminus \{a\}$ can already occupy all $b \in B$. Otherwise, if $k = 1$, then the ℓ cops not on leaves adjacent to a can always reach the unique $b \in B$.) Then the robber is surrounded.

- By Lemma 7, we get that $c_{E,r}(G) = \max\{c_{E,r}(H), k + \ell\}$, and it follows from Proposition 6 that $c_{E,r}(H) = k + \ell$. Thus, $c_{E,r}(G) = k + \ell = \Delta$ holds.
- Considering $c_E(G)$, we apply Lemma 7 and get $c_E(G) \geq k\ell = \lfloor \frac{\Delta}{2} \rfloor \lceil \frac{\Delta}{2} \rceil$ (using that $\text{girth}(H) \geq 4$, as H is a bipartite graph). It remains to show that $k\ell$ nonrestrictive edge cops suffice. To this end, let k^2 cops initially occupy all edges of H , one cop per edge. For odd Δ , let M be a perfect matching in H and let each of the k remaining cops occupy a single edge from M . Now if the robber starts on a leaf w adjacent to a vertex v of H , he gets surrounded by one cop moving from an edge of H incident to v onto the edge incident to w . Otherwise, if the robber starts on some vertex v of H , the k edges of H incident to v contain ℓ cops, which can spread across the ℓ leaves incident to v . If $k = 1$, there is one remaining cop which can move to the only edge of H , thereby surrounding the robber. In the remaining cases that $k \geq 2$, each of the k neighbors of v in H is incident to at least one other edge of H with a cop on. This cop can move onto the edge incident to v , thereby surrounding the robber. \square

Corollary 9. *For every $\Delta \geq 2$, there is a connected graph G with $\Delta(G) = \Delta$ such that $c_V(G) = 2(\Delta - 1)$ and $c_E(G) = \Delta$.*

Proof. Let $H = K_2$ be a single edge uv , so $k = 1$, and choose $\ell = \Delta - 1$. Then it holds that $\Delta(G) = \Delta$.

- It follows from Lemma 7 that $c_V(G) \geq (k + 1)\ell = 2(\Delta - 1)$. On the other hand, it is easy to see that $2(\Delta - 1)$ vertex cops suffice: Initially, $\Delta - 1$ cops position themselves at u and v each. If the robber positions himself on a leaf, he is surrounded immediately. If he starts on v (the case for u is symmetric), then the cops at v move to the $\Delta - 1$ leaves, thereby surrounding the robber (together with the cops at u).
- To prove that $c_E(G) = \Delta$, first observe that $c_E(G) \geq \Delta$ by Observation 4 (Lemma 7 gives a worse lower bound in this case). A winning strategy for Δ edge cops goes as follows: All cops start on edge uv . If the robber starts on v (or symmetrically u), then one of the cops stays on uv , while the other $\Delta - 1$ occupy the edges incident to the leaves adjacent to v . If the robber starts on a leaf w , a single cop can move onto the edge incident to w . \square

3.3 Line Graphs of Complete Graphs

The *line graph* $L(G)$ of a given graph $G = (V, E)$ is the graph whose vertex set consists of the edges E of G , and two vertices of $L(G)$ are adjacent if their corresponding edges in G share an endpoint. For $n \geq 3$, let K_n denote the complete graph on the set $[n] = \{1, \dots, n\}$. For distinct $x, y \in [n]$, we denote by $\{x, y\}$ the vertex of $L(K_n)$ corresponding to the edge between x and y in K_n . Burgess et al. [6] showed that $c_{V,r}(L(K_n)) = 2(n-2) = \delta(L(K_n))$. This is obtained by placing the cops on all vertices $\{1, x\}$ for $x \in \{2, \dots, n\}$ and $\{2, y\}$ for $y \in \{3, \dots, n-1\}$. The cops can surround the robber in their first move.

Lemma 10. *For every $n \geq 3$ we have $c_V(L(K_n)) = 2(n-2)$, $c_E(L(K_n)) \geq n(n-2)/3$, and $c_{E,r}(L(K_n)) \geq (n^2 - 4)/12$.*

Proof. As $2(n-2) = \Delta(L(K_n)) \leq c_V(L(K_n))$ (Observation 4), it is enough to observe that the strategy for the vertex cops provided by Burgess et al. [6] also works for vertex cops in the nonrestrictive version. In their strategy, the cops start at the vertices $\{1, x\}$ for $x \in \{2, \dots, n\}$ and $\{2, y\}$ for $y \in \{3, \dots, n-1\}$ of $L(K_n)$. They prove that these cops can surround the robber immediately after he chose his start vertex, in particular, before the robber first moves along an edge. Therefore, the only remaining cases to check for the nonrestrictive version are what happens if the robber starts on a cop (which would be forbidden in the restrictive setting):

- If the robber starts on vertex $\{1, 2\}$, then he is surrounded as soon as the cop from $\{1, 2\}$ moves to $\{2, n\}$.
- If the robber starts on a vertex $\{1, x\}$, we first consider the case that $x \in \{3, \dots, n-1\}$. The robber gets surrounded by the cop at $\{1, x\}$ moving to $\{x, n\}$, and each cop from $\{2, y\}$ for $y \in \{3, n-1\} \setminus \{x\}$ moving to $\{x, y\}$. In the case that $x = n$, then the cop on $\{1, 2\}$ moves to $\{n, 2\}$, and each cop on $\{2, y\}$ for $y \in \{3, \dots, n-1\}$ moves to $\{n, y\}$.
- Lastly, if the robber starts on a vertex $\{2, y\}$ for $y \in \{3, \dots, n-1\}$, then the cop on $\{2, y\}$ moves to $\{1, y\}$, and each cop on $\{1, x\}$ for $x \in \{3, \dots, n\}$ moves to $\{y, x\}$.

We now consider the lower bounds on $c_E(L(K_n))$ and $c_{E,r}(L(K_n))$, and describe an evasion strategy for the robber, provided that the number k of edge cops is sufficiently small (an exact bound will be determined later for both versions). To this end, recall that each edge cop occupies an edge of $L(K_n)$ that corresponds to a copy of P_3 (the path on two edges) in K_n . Each such P_3 has a *midpoint* and two *leaves*, each being a vertex of K_n . An edge cop moving one step from e_1 to e_2 in $L(K_n)$ corresponds to P_3^1 and P_3^2 in K_n . Here P_3^1 shares an edge with P_3^2 , and thus, the midpoint of P_3^2 is contained (as a midpoint or leaf) in P_3^1 . At the moment that the robber gets surrounded on a vertex $\{u, v\}$ of $L(K_n)$, the edge cops occupy all $n-2$ edges of $L(K_n)$ corresponding to P_3 's in K_n with midpoint u and leaf v , as well as all $n-2$ edges of $L(K_n)$ corresponding to P_3 's in K_n with midpoint v and leaf u . Thus, in the step immediately preceding the surround, each of u and v was contained in the P_3 of at least $n-2$ edge cops.

With this in mind, for any vertex v of K_n and point in time $t \in \mathbb{N}$, let $p^t(v)$ be the number of edge cops whose corresponding P_3 contains v before the t -th turn of the robber. Similarly, let $q^t(v)$ be the number of edge cops whose corresponding P_3 contains v as a midpoint. For all $v \in [n]$ and $t \geq 2$, it holds that

$$q^t(v) \leq p^{t-1}(v). \quad (2)$$

Further, recall that each cop has exactly one corresponding P_3 , so we have

$$\sum_{v \in [n]} p^t(v) = 3k \quad (3)$$

for all $t \in \mathbb{N}$. The robber cannot be surrounded on a vertex $\{u, v\}$ of $L(K_n)$ (with $u, v \in [n]$, $u \neq v$) if $\min\{p^t(u), p^t(v)\} < n - 2$. Therefore, we call a vertex $w \in [n]$ of K_n *safe (at time t)* if $p^t(w) < n - 2$.

- We start with the nonrestrictive edge version, and choose $k < n(n - 2)/3$. Then, Equation (3) implies that

$$\#\{w \in [n] \mid p^t(w) \geq n - 2\} \leq \frac{3k}{n - 2} < \frac{3n(n - 2)}{3(n - 2)} = n$$

for every $t \in \mathbb{N}$. In particular, there is always a safe vertex $w \in [n]$. For $t = 1$, i.e., in his first turn, the robber can go to $\{v, w\}$ with an arbitrary $v \in [n] \setminus \{w\}$. For $t > 1$, let $\{u, v\}$ be the robber's current position. Then, as K_n is a complete graph, the robber can move to $\{v, w\}$ in a single step. In both cases, the robber avoids being surrounded because $p^t(v, w) \leq p^t(w) < n - 2$.

- We now consider the restrictive edge version. We prove that the robber can avoid being surrounded by ending his t -th turn (for $t \in \mathbb{N}$) on a vertex $\{u, v\}$ of $L(K_n)$ with $p^t(u, v) < (n - 2)/2$. It follows from $n \geq 3$ that $(n - 2)/2 < n - 2$, so at least one of u or v is safe at time t .

Let $k < (n^2 - 4)/12$. For $t = 1$, i.e., the robber's first move, it suffices to identify a single $u \in [n]$ with $p^1(u) < (n - 2)/2$. Then, the robber can start at $\{u, v\}$ with an arbitrary $v \in [n] \setminus \{u\}$. Such a u always exists because

$$\#\left\{w \in [n] \mid p^1(w) \geq \frac{n - 2}{2}\right\} \leq \frac{3k}{(n - 2)/2} = \frac{6k}{n - 2} < \frac{n + 2}{2} < n. \quad (4)$$

Here, the first inequality follows from Equation (3), the second from the choice of k , and the third from $n \geq 3$.

For $t > 1$, let $\{u, v\}$ be the current robber position, chosen in turn $t - 1$ with $p^{t-1}(u, v) < (n - 2)/2$. We assume without loss of generality that $p^{t-1}(u) < (n - 2)/2$, and therefore $q^t(u) < (n - 2)/2$ (see Equation (2)). Then, less than $(n - 2)/2$ cops correspond to P_3 's in K_n with midpoint u . This means that less than $(n - 2)/2$ of

the $n - 2$ edges of the form $\{\{u, v\}, \{u, w\}\}$ (with $w \in [n] \setminus \{u, v\}$) are blocked by a cop. In particular, more than $(n - 2)/2$ of these edges can be taken by the robber to move to a vertex $\{u, w\}$ of $L(K_n)$.

It remains to prove that at least one of these $w \in [n]$ is safe, i.e., has $p^t(w) < (n - 2)/2$. This is the case because, by Equation (4), there are more than $n - (n + 2)/2 = (n - 2)/2$ vertices $w \in [n]$ with $p^t(w) < (n - 2)/2$. If u or v are among these, i.e., $\min\{p^t(u), p^t(v)\} < (n - 2)/2$, then the robber stays at his current position. Otherwise, more than half of the $w \in [n] \setminus \{u, v\}$ are safe. At the same time, more than half of the edges $\{\{u, v\}, \{u, w\}\}$ are free for the robber. We conclude, that there is at least one w^* with $p^t(w^*) < (n - 2)/2$ such that the robber can move to $\{u, w^*\}$. \square

The bounds from Lemma 10 are stated in terms of the number of vertices n . Stating them in terms of their maximum degree $\Delta := \Delta(L(K_n)) = 2(n - 2)$, we obtain the claimed lower bounds in Items 5 and 6 of Theorem 3.

$$\begin{aligned} c_V(L(K_n)) &= \Delta & c_E(L(K_n)) &\geq \frac{\Delta^2 + 4\Delta}{12} \\ c_{V,r}(L(K_n)) &= \Delta & c_{E,r}(L(K_n)) &\geq \frac{\Delta^2 + 8\Delta}{48} \end{aligned}$$

With these, all lower bounds from Theorem 3 are proven.

3.4 Graphs from Mutually Orthogonal Latin Squares

Burgess et al. [6] notice that, for graphs G of many families with a “large” value of $c_{V,r}(G)$, the classical cop number $c(G)$ was “low” (often even constant). In fact, they only provide a single family of graphs (constructed from finite projective planes) where $c(G) \approx c_{V,r}(G)$. They ask (Question 7 in [6]) whether graphs with large $c_{V,r}(G)$ inherently possess some property that implies that $c(G)$ is low. In this section we construct graphs G_k with $c(G_k), c_{V,r}(G_k) \in \{k, k + 1\}$. We interpret this as evidence that there is no such property.

A *Latin square* of order $k \geq 1$ is a $(k \times k)$ -array filled with numbers from $[k] = \{1, \dots, k\}$ such that each row and each column contains each number from $[k]$ exactly once. Formally, a Latin square L is a partition $L(1) \cup \dots \cup L(k)$ of $A = [k] \times [k]$ such that row i (with $i \in [k]$) is $A[i, \cdot] = \{(i, j) \in A \mid j \in [k]\}$, and for every number $n \in [k]$ we have $|A[i, \cdot] \cap L(n)| = 1$, and symmetrically for the columns. See the left of Figure 1 for two different Latin squares.

Let L_1 and L_2 be two Latin squares of order k . Their *juxtaposition* $L_1 \otimes L_2$ is the Latin square of order k that contains in each cell the ordered pair of the entries of L_1 and L_2 in that cell. We say that L_1 and L_2 are *orthogonal* if each ordered pair appears exactly once in $L_1 \otimes L_2$, i.e., if for every two distinct $n_1, n_2 \in [k]$ we have $|L_1(n_1) \cap L_2(n_2)| = 1$. It is well known that $k - 1$ *mutually orthogonal Latin squares (MOLS)* L_1, \dots, L_{k-1} (meaning that L_s and L_t are orthogonal whenever $s \neq t$) exist if and only if k is a prime power [11]. The two Latin squares in Figure 1 (left) are indeed orthogonal, as can be seen by their juxtaposition below.

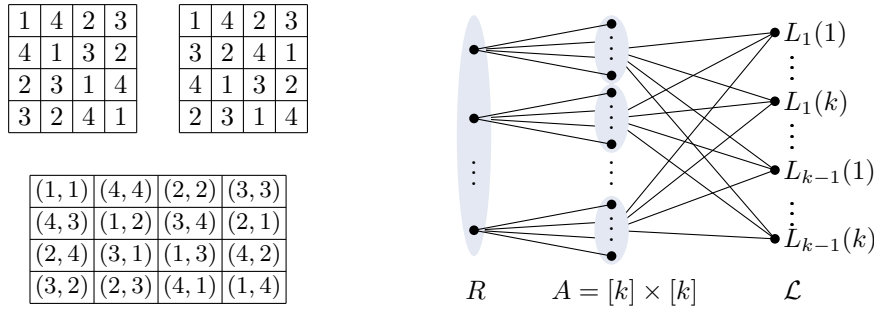


Figure 1: Left: Two Latin squares and their juxtaposition, proving that they are orthogonal. Right: The graph G_k created from $k - 1$ MOLS of order k . The vertices in R correspond to the rows of A , the middle vertices correspond to the cells of A (ordered row by row in the drawing) and the vertices in \mathcal{L} correspond to the parts of the MOLS.

Let k be a prime power and L_1, \dots, L_{k-1} a set of $k - 1$ mutually orthogonal Latin squares of order k . We construct a graph G_k . Let $A = [k] \times [k]$ denote the set of all positions, $R = \{A[i, \cdot] \mid i \in [k]\}$ denote the set of all rows in A , and $\mathcal{L} = \{L_s(n) \mid s \in [k - 1] \wedge n \in [k]\}$ denote the set of all parts of the Latin squares L_1, \dots, L_{k-1} . Then, $G_k = (V, E)$ is the graph with

$$V = A \cup R \cup \mathcal{L} \quad \text{and} \quad E = \{pS \mid p \in A, S \in R \cup \mathcal{L}, p \in S\}.$$

We observe that G_k is a k -regular bipartite graph with $|A| + |R \cup \mathcal{L}| = k^2 + (k + k(k - 1)) = 2k^2$ vertices. It has an edge between position $p \in A$ and a set $S \in R \cup \mathcal{L}$ if and only if p is in set S . See also the right of Figure 1 for a schematic drawing.

For the next lemma, recall that $\text{girth}(G)$ is the length of a shortest cycle in G .

Lemma 11. *For a prime power k , graph G_k has $\text{girth}(G_k) \geq 6$.*

Proof. It suffices to show that G_k has no 4-cycle, since G_k is bipartite (and therefore has no odd cycles). First note that every 4-cycle must be of the form (p_1, S_1, p_2, S_2) , in particular, it must contain two vertices p_1 and p_2 from A . However, there is neither a 4-cycle

- with $S_1, S_2 \in R$, since every position is in at most one row,
- nor with $S_1 \in R$ and $S_2 \in \mathcal{L}$, since every part $L_s(n)$ of a Latin square contains only one position of each row,
- nor with $S_1, S_2 \in \mathcal{L}$ being parts of the same Latin square L_s , since every position is in only one part of L_s ,
- nor with $S_1, S_2 \in \mathcal{L}$ being parts of different Latin squares L_s and L_t , since L_s and L_t are orthogonal (and hence no two distinct positions are in the same parts in L_s and in L_t). \square

Lemma 12. For a prime power k , graph G_k has $c(G_k) = k$ and $c_{V,r}(G_k) \leq k + 1$.

Proof. • To prove $c(G_k) \geq k$, we use a theorem from Aigner and Fromme [1] stating that any graph with girth at least 5 has $c(G) \geq \delta(G)$. As G_k is k -regular and has $\text{girth}(G_k) \geq 6$ (Lemma 11), we get $c(G_k) \geq k$.

On the other hand, k cops suffice: Initially, the cops occupy the k vertices in R , so the robber has to position himself in A or \mathcal{L} . If he starts at a vertex $v \in A$, then this vertex must be in some row and the cop on the corresponding vertex in R can capture him in his next move. Otherwise, if the robber starts on some $L_i(n) \in \mathcal{L}$ (for $i \in [k - 1]$ and $n \in [k]$), then $L_i(n)$ has exactly k neighbors in A , one per row. The cops can move to $N_{G_k}(L_i(n))$, thereby surrounding the robber (and thereby capturing him in their next move).

- To show that $k + 1$ restrictive vertex cops suffice to surround the robber, we start by placing one cop on each of the k vertices in R . Thus, the robber may not enter R . Now, if the robber ends his turn on a vertex in \mathcal{L} corresponding to the part $L_s(n)$ of the Latin square L_s for some $n \in [k]$, then the k cops on R can move to the vertices in A corresponding to the positions in $L_s(n)$, and thus surround the robber. This works because the elements in $L_s(n)$ are from all k different rows (because L_s is a Latin square) and there is one cop responsible for every row. Thus, we may assume that the robber starts on a vertex in A and never moves. But then, it is enough to use the $(k + 1)$ -th cop to go to the robber vertex and force him to move (recall that we are in the restrictive version). The only possible move is to a vertex from \mathcal{L} where he gets surrounded in the next move of the cops as seen above. \square

Remark. We can easily see that $c_E(G_k) \geq k(k - 1)$ for the graph G_k in Lemma 12. In fact, G_k is k -regular and has $\text{girth}(G) \geq 6$ by Lemma 11. Attaching $\ell = 0$ leaves to each vertex, Lemma 7 yields $c_E(G) \geq k(k + \ell - 1) = k(k - 1)$.

4 When Capturing is not Surrounding

This section is devoted to the proof of Theorem 2, i.e., that none of the four surrounding cop numbers can be bounded by any function of the classical cop number and the maximum degree of the graph. In particular, we shall construct, for infinitely many integers $k \geq 1$, a graph G_k with $c(G_k) = 2$ and $\Delta(G_k) = 3$, but $c_{V,r}(G_k) \geq k$. Theorem 3 then implies that $c_V(G_k)$, $c_E(G_k)$ and $c_{E,r}(G_k)$ are unbounded as well for growing k .

The construction of G_k is quite intricate. We divide it into several steps.

The Graph $H[s]$. Let $s \geq 1$ be an integer and let $k = 2^{s-1}$. We start with a graph $H[s]$, which we call the *base graph*, that has the following properties:

- $H[s]$ is $2k$ -regular, i.e., every vertex of $H[s]$ has degree $2k$.
- $H[s]$ has girth at least 5.

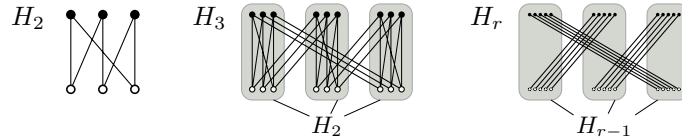


Figure 2: Iterating $C_6 = H_2$ to obtain r -regular (bipartite) graphs H_r with $\text{girth}(H_r) \geq 5$.

There are many ways to construct such graphs $H[s]$, one being our graphs in Section 3.4 constructed from $2^s - 1$ mutually orthogonal Latin squares. An alternative construction (not relying on non-trivial tools) is an iteration of the 6-cycle, as illustrated in Figure 2. We additionally endow $H[s]$ with an orientation such that each vertex has exactly $k = 2^{s-1}$ outgoing and exactly $k = 2^{s-1}$ incoming edges. (For example, orient the edges according to an Eulerian tour in $H[s]$.)

The Graph $H[s, \ell]$. Let $\ell \geq 1$ be another integer⁴. We define a graph $H[s, \ell]$ based on $H[s]$ and its orientation. See Figure 3 for an illustration with $s = 3$ and $\ell = 4$. For each vertex a in $H[s]$, take a complete balanced binary tree $T(a)$ of height $s = \log_2(k) + 1$ with root $r(a)$ and $2^s = 2k$ leaves. Let $r^{\text{in}}(a)$ and $r^{\text{out}}(a)$ denote the two children of $r(a)$ in $T(a)$, and let $T^{\text{in}}(a)$ and $T^{\text{out}}(a)$ denote the subtrees rooted at $r^{\text{in}}(a)$ and $r^{\text{out}}(a)$, respectively. We associate each of the $k = 2^{s-1}$ leaves in $T^{\text{in}}(a)$ with one of the k incoming edges at a in $H[s]$, and each of the k leaves in $T^{\text{out}}(a)$ with one of the k outgoing edges at a in $H[s]$. Finally, for each edge ab in $H[s]$ oriented from a to b , connect the associated leaf in $T^{\text{out}}(a)$ with the associated leaf in $T^{\text{in}}(b)$ by a path $P(ab)$ of length $2\ell + 1$, i.e., on 2ℓ new inner vertices. This completes the construction of $H[s, \ell]$.

Establishing some notation and properties of $H[s, \ell]$, observe that the following holds for any edge ab of the base graph $H[s]$:

$$\text{dist}_{H[s, \ell]}(r(a), r(b)) = 2s + 2\ell + 1 \quad (5)$$

Let $R = \{r(a) \mid a \in V(H[s])\}$, and call the elements of R the *roots* in $H[s, \ell]$. For any root $r \in R$ let us define the *ball around r* by

$$B(r) = \{v \in V(H[s, \ell]) \mid \text{dist}_{H[s, \ell]}(v, r) \leq s + \ell\},$$

i.e., $B(r)$ consists of all vertices in $H[s, \ell]$ that are closer to r than to any other root in R . Note that every vertex of $H[s, \ell]$ lies in the ball around exactly one root. Further, for distinct $a, b \in V(H[s])$ with $ab \notin E(H[s])$, let $v \in B(r(a))$ and $w \in B(r(b))$. Then, we have

$$\text{dist}_{H[s, \ell]}(v, w) > 2\ell. \quad (6)$$

For each edge ab in $H[s]$, let $e(ab)$ denote the unique middle edge of $P(ab)$ in $H[s, \ell]$. Then, $e(ab)$ connects a vertex in $B(r(a))$, which we denote by $v(a, b)$, with a vertex in $B(r(b))$, which we denote by $v(b, a)$. See also Figure 3.

⁴We shall choose $\ell \gg s$ later. So you may think of s as “short” and of ℓ as “long”.

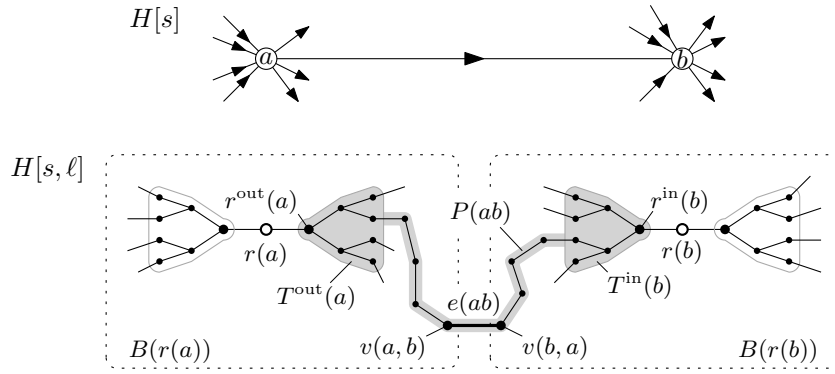


Figure 3: Construction of $H[s, \ell]$ based on $H[s]$. A directed edge ab in $H[s]$ and the corresponding trees $T^{\text{out}}(a)$, $T^{\text{in}}(b)$, and path $P(ab)$ with middle edge $e(ab)$ in $H[s, \ell]$.

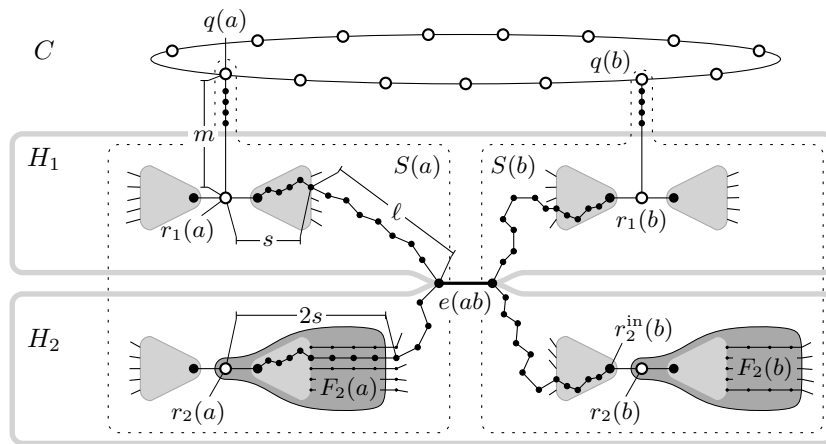


Figure 4: Construction of $H[s, \ell, m]$ based on two copies of $H[s, \ell]$. A directed edge ab in $H[s]$ and the corresponding sets $S(a)$, $S(b)$, $F_2(a)$, etc. and edge $e(ab)$ in $H[s, \ell, m]$.

The Graph $H[s, \ell, m]$. Let $m \geq 1$ be yet another integer⁵. We start with two vertex-disjoint copies H_1 and H_2 of $H[s, \ell]$, and transfer our notation (such as R , $r(a)$, $v(a, b)$, etc.) for $H[s, \ell]$ to H_i for $i \in \{1, 2\}$ by adding the subscript i , e.g., R_1 , $r_2(a)$, or $v_1(a, b)$. We connect H_1 and H_2 as follows: For each edge ab in $H[s]$, we identify the edge $e_1(ab)$ in H_1 with the edge $e_2(ab)$ in H_2 such that $v_1(a, b) = v_2(a, b)$ and $v_1(b, a) = v_2(b, a)$. Next, for each vertex a of $H[s]$, use the vertex $r_1(a)$ in H_1 as an endpoint for a new path $Q(a)$ of length m , and denote the other endpoint of $Q(a)$ by $q(a)$. Note that we do this only for the roots in H_1 .

Finally, we connect the vertices $\{q(a) \mid a \in V(H[s])\}$ by a cycle C of length $|V(H[s])|$. This completes the construction of $H[s, \ell, m]$. See Figure 4 for an illustration. Note that $H[s, \ell, m]$ has maximum degree 3 and degeneracy 2.

⁵We shall choose $\ell \gg m \gg s$ later. So you may think of m as “medium”.

Establishing more notation for $H[s, \ell, m]$, for each vertex a in $H[s]$, let us define

$$S(a) = B(r_1(a)) \cup B(r_2(a)) \cup Q(a),$$

see again Figure 4 for an illustration. Let H_- denote the graph $H[s, \ell, m] - E(C)$ obtained from $H[s, \ell, m]$ by removing all edges of the cycle C . This completes the construction.

Lemma 13. *For every $s, m \geq 1$ and $\ell > |V(H[s])| + m + s$, it holds that $c(H[s, \ell, m]) \leq 2$.*

Before starting with the formal proof, let us already give a vague idea of the winning strategy for two cops. Two cops could easily capture the robber on the cycle C (including the attached paths $Q(a)$ for $a \in V(H[s])$). Thus, they force him to “flee” to H_1 at some point. In a second phase, they can even force him to flee to H_2 . Loosely speaking, cop c_1 stays on C to prevent the robber from getting back on C , while cop c_2 always moves towards the robber in H_1 . Whenever the robber traverses one of the long paths $P_1(ab)$ for some $ab \in E(H[s])$, say from $T_1(a)$ towards $T_1(b)$, then c_1 can go in $|V(C)| + m + s < \ell$ steps along C to $q(b)$ and along $Q(b)$ and through $T_1(b)$ to arrive at the other end of $P_1(ab)$ before the robber. However, to escape c_2 , the robber must traverse a path $P_1(ab)$ eventually, with his only way to escape being to turn into H_2 at the middle edge $e(ab)$. At some point, we enforce the situation that the robber occupies some vertex v of H_2 and one of the cops, say c_1 , occupies the corresponding copy of v in H_1 . Now, in a third phase, the robber moves in H_2 while c_1 always copies his moves in H_1 . This prohibits the robber to ever walk along a middle edge $e(ab)$ for some $ab \in E(H[s])$. But without these edges, H_2 is a forest, and thus cop c_2 can capture the robber in the tree component in which the robber is located.

Proof of Lemma 13. We describe a strategy for two cops, the first cop c_1 and the second cop c_2 , to capture the robber in the classical game played on $H[s, \ell, m]$. They both start (anywhere) on the cycle C . There are three phases, each eventually ensuring a particular configuration of the game after the cops’ turn.

We start with a convenient definition and observation: A cop *guards the robber on C* if, for some vertex a in $H[s]$, the robber occupies some vertex $w \in Q(a) \cup T_1(a) \cup T_2(a)$ and the cop occupies the corresponding vertex $q(a)$ on C . Observe that if a cop already guards the robber on C , then the cop can maintain guarding the robber on C by always moving along C towards the vertex $q(b)$ for which the robber occupies some vertex in $S(b)$. In fact, by Equation (6), we have $\text{dist}_{H_-}(v, w) > 2\ell$ if $v \in S(a)$ and $w \in S(b)$ with $a \neq b$ and $ab \notin E(H[s])$. Also, if ab is an edge in $H[s]$, then the distance in H_- between $T_1(a) \cup T_2(a)$ and $T_1(b) \cup T_2(b)$ is equal to the length of $P_i(ab)$, which is $2\ell + 1 > 2\ell$. On the other hand, $\text{dist}_C(q(a), q(b)) \leq |V(C)| = |V(H[s])| \leq \ell$, and hence the cop is fast enough on C in both cases.

First Phase. The target configuration of the first phase is that both cops guard the robber on C . The cops can achieve this as follows: After both starting at the same vertex q on C , cop c_1 moves along C to decrease its distance to the robber (while staying on C). The second cop c_2 stays at q , which forces the robber to eventually leave the cycle C .

After finitely many steps, c_1 guards the robber on C and continues to maintain this property. This prevents the robber from ever coming back to C without being captured. Then, cop c_2 can move along C towards the first cop to eventually reach the target configuration.

Second Phase. The target configuration of the second phase is that the robber occupies some vertex w_2 in H_2 , and one cop occupies the corresponding copy w_1 of w_2 in H_1 . To this end, c_1 starts by maintaining the guarding property, whereas c_2 leaves C and always goes towards the robber along a shortest path in H_- . Whenever the robber does not move or goes back and forth along an edge (i.e., is at the same vertex as two moves ago), then c_2 reduces its distance to the robber. As this may happen only a bounded number of times without the robber being captured by c_2 , we may assume that the robber always “moves forward”, i.e., never moves along the same edge in two consecutive turns, and thus eventually occupies one of the two vertices of the middle edge $e_1(ab) = e_2(ab)$ corresponding to some edge ab in $H[s]$. Moreover, the robber continues along $P_1(ab)$ or $P_2(ab)$ towards one of $T_1(a)$, $T_2(a)$, $T_1(b)$ or $T_2(b)$; say towards $T_i(b)$ for some $i \in \{1, 2\}$.

At this point, cop c_1 goes along C to $q(b)$, then along $Q(b)$ to $r_1(b)$, and along $T_1(b)$ to the leaf w_1 of $T_1(b)$ that is the endpoint of $P_1(ab)$. This takes c_1 at most $|V(C)| + m + s < \ell$ steps. On the other hand, the robber needs at least ℓ steps to reach $T_1(b) \cup T_2(b)$. This way, the robber (always moving forward) is either captured by c_1 on w_1 , reaches the copy w_2 of w_1 in H_2 , or the copy in H_2 of the vertex occupied by c_2 in H_1 , which is the desired target configuration.

Third Phase. Finally, we shall capture the robber. Without loss of generality, let c_1 be the cop in H_1 that copies the robber’s moves in H_2 . Cop c_2 continues to go towards the robber along a shortest path in H_- , forcing him to keep moving forward. As $H_2 - \{e_2(ab) \mid ab \in E(H[s])\}$ is a forest, the robber must eventually occupy an endpoint of $e_2(ab)$ for some $ab \in E(H[s])$, where he is captured by c_1 . \square

In contrast to the bounded classical cup number, the surrounding cop numbers are unbounded:

Lemma 14. *For every $s \geq 1$, $m > 2s + 1$, and $\ell > 3s + 1$, it holds that $c_{V,r}(H[s, \ell, m]) \geq k = 2^{s-1}$.*

Again, we start by giving a vague idea how the robber can avoid being surrounded by $k - 1$ vertex cops: The robber stays solely in H_2 , and he moves between roots $r_2(x)$ for $x \in V(H[s])$, always from a root $r_2(a)$ to the next root $r_2(b)$ for which the edge ab in $H[s]$ is directed from a to b . In $H[s]$, vertex a has k outgoing neighbors, and we show that at least one such neighbor b is always “safe” for the robber to escape to. However, it is quite tricky to identify this safe neighbor. Indeed, the robber has to start moving in the “right” direction down the tree $T_2(a)$ always observing the cops’ response, before he can be absolutely sure which outgoing neighbor b of a is safe. With suitable choices of s , ℓ and m , the robber is fast enough at $r_2(b)$ to then choose his next destination from there.

The crucial point is that the cops can “join” the robber when he traverses the middle edge $e(ab)$, but they can never be one step ahead of the robber on $P_2(ab)$; and thus never surround him.

Proof of Lemma 14. We provide a strategy for the robber to avoid getting surrounded indefinitely against any strategy for $k - 1$ cops on $H[s, \ell, m]$. Our strategy ensures that the robber always moves onto a vertex that is not occupied by a cop, and thus works for the restrictive vertex version of the game. For this strategy, the robber shall exclusively stay on H_2 in $H[s, \ell, m]$. Recall that for any vertex a of $H[s]$, there is a corresponding binary tree $T_2(a)$ with root $r_2(a)$, and two subtrees $T_2^{\text{in}}(a)$ and $T_2^{\text{out}}(a)$ rooted at $r_2^{\text{in}}(a)$ and $r_2^{\text{out}}(a)$, respectively. The leaves of $T_2^{\text{out}}(a)$ and $T_2^{\text{in}}(a)$ correspond to the outgoing, respectively incoming, edges at a in $H[s]$, and for every directed edge from a to b in $H[s]$ there is a path $P_2(ab)$ of length $2\ell + 1$ from a leaf in $T_2^{\text{out}}(a)$ to a leaf in $T_2^{\text{in}}(b)$.

For our strategy, we call a configuration of the game after the robber’s turn a *good situation* if there is a vertex a in $H[s]$ such that

- the robber occupies $r_2(a)$ in $H[s, \ell, m]$ and
- each cop has distance at least $2s$ to $r_2(a)$ in graph $H[s, \ell, m] - \{\{r_2(a), r_2^{\text{in}}(a)\}\}$, obtained by removing only the edge between $r_2(a)$ and $r_2^{\text{in}}(a)$.

For convenience, let us denote by $F_i(a)$ (with $i \in \{1, 2\}$) the set of all vertices in $H[s, \ell, m]$ at distance less than $2s$ to $r_i(a)$ in $H[s, \ell, m] - \{\{r_i(a), r_i^{\text{in}}(a)\}\}$. Thus, we have a good situation with the robber at $r_2(a)$ if no cop occupies a vertex of $F_2(a)$. Since $\ell > s$, we have $F_1(a), F_2(a) \subset S(a)$ for any $a \in V(H[s])$. Together with $|V(H[s])| > 2k$, this implies that the robber can start the game in a good situation, i.e., on a vertex $r_2(a)$ with no cop in $F_2(a)$ (as there are only $k - 1$ cops).

Observe that the robber is not surrounded in a good situation because there is no cop on the neighbor $r_2^{\text{out}}(a)$ of $r_2(a)$. Recall that $H[s]$ is endowed with an orientation with outdegree and indegree exactly k at each vertex a . Let $N_{H[s]}^+(a)$ denote the set of all outgoing neighbors of a in $H[s]$. It is then enough to show that, starting in a good situation at vertex $r_2(a)$, the robber can directly move from $r_2(a)$ to $r_2(b)$ for some $b \in N_{H[s]}^+(a)$ in $2s + 2\ell + 1$ steps (along $T_2^{\text{out}}(a)$, then $P_2(ab)$, then $T_2^{\text{in}}(b)$, cf. (5)) to be again in a good situation at vertex $r_2(b)$ and without being surrounded at any time. In fact, it will be enough to ensure that no cop reaches $F_2(b)$ before the robber reaches $r_2(b)$. (Recall that from a good situation, the cops go first.)

Claim 15. *Let a be a vertex in $H[s]$ and $b, b' \in N_{H[s]}^+(a)$ be two distinct outgoing neighbors of a in $H[s]$. Then, there is no vertex w in $H[s, \ell, m] - S(a)$ that has distance at most $2\ell + 2s + 1$ to $F_2(b)$ and distance at most $2\ell + 2s + 1$ to $F_2(b')$.*

Proof of Claim. Let w be any fixed vertex in $H[s, \ell, m] - S(a)$. We have to show that w has distance greater than $2\ell + 2s + 1$ to $F_2(b)$ or $F_2(b')$, or both.

First, observe that a shortest path from $F_2(b)$ to the cycle C uses $\ell - s$ edges of $P_2(bc)$ for some $c \in N_{H[s]}(b)$, ℓ edges of $P_1(bc)$, s edges of $T_1(b)$, and m edges of $Q(b)$, for a total

of $(\ell - s) + \ell + s + m = 2\ell + m > 2\ell + 2s + 1$ edges. Thus, we are done in case $w \in V(C)$. Furthermore, if a shortest path from w to $F_2(b)$ (respectively $F_2(b')$) contains a vertex in C , then w has distance greater than $2\ell + 2s + 1$ to $F_2(b)$ (respectively $F_2(b')$), and we are done for such w as well.

As $H[s]$ has girth at least 5, we know that b and b' are not adjacent in $H[s]$. Furthermore, we have $N_{H[s]}(b) \cap N_{H[s]}(b') = \{a\}$, i.e., a is the only common neighbor of b and b' in $H[s]$. As $w \in S(c)$ for some vertex $c \neq a$ in $H[s]$, we can assume without loss of generality that $c \neq b$ and c is not adjacent to b in $H[s]$. Now consider a shortest path P from w to $F_2(b)$ in $H[s, \ell, m]$. We already know that P contains no vertex in C . As c and b are not adjacent in $H[s]$, the path P starts in $S(c)$, ends in $S(b)$, and traverses $S(d)$ for some vertex $d \in N_{H[s]}(b)$. But then P contains at least 2ℓ vertices that are in $S(d)$ in order to traverse $S(d)$. Together with the at least $\ell - s$ vertices in $S(b)$ (to reach $F_2(b)$) we see that w has distance at least $3\ell - s > 2\ell + 2s + 1$ to $F_2(b)$. This proves the claim. \square

Recall that we are in a good situation with the robber at $r_2(a)$, and thus no cop occupies a vertex in $F_2(a)$. We seek to move the robber directly towards $r_2(b)$ for some $b \in N_{H[s]}^+(a)$. Guided by the above claim, we say that a cop occupying vertex w in $H[s, \ell, m]$ blocks vertex $b \in N_{H[s]}^+(a)$ if w has distance at most $2\ell + 2s + 1$ to $F_2(b)$ and $w \notin F_1(a)$. Note that with this definition, the above claim implies that each cop blocks at most one $b \in N_{H[s]}^+(a)$. Let $X \subset N_{H[s]}^+(a)$ be the (possibly empty) subset of all outgoing neighbors of a that are blocked by some cop.

There are $k - 1$ cops in total, hence $|X| \leq k - 1$, while at the same time $|N_{H[s]}^+(a)| = k$. Thus, we have $N_{H[s]}^+(a) - X \neq \emptyset$. However, we shall make a definite decision for a vertex $b \in N_{H[s]}^+(a) - X$ only after the first s steps. This is due to the cops in $F_1(a)$ with distance at most $2\ell + 2s + 1$ to some vertex in $N_{H[s]}^+(a) - X$. We choose each move for the robber inside $T_2(a)$ according to the locations of the cops in $F_1(a)$ at that moment. As the first step (after the cops have moved), the robber moves to $r_2^{\text{out}}(a)$, the root of $T_2^{\text{out}}(a)$.

Now consider the situation after step j of the robber, $j \in \{1, \dots, s\}$. Let the robber occupy the non-leaf vertex v_2 in $T_2^{\text{out}}(a)$ at distance $j - 1$ from the root $r_2^{\text{out}}(a)$. Consider the copy v_1 of v_2 in $T_1^{\text{out}}(a)$, and the subtree T of $T_1^{\text{out}}(a)$ below v_1 with exactly 2^{s-j} leaves. We call a vertex $b \in N_{H[s]}^+(a)$ *available* if b is not blocked ($b \notin X$) and $P_1(ab)$ starts at a leaf of T , and denote by Y the set of all available vertices. We maintain the invariant that after step j of the robber, strictly less than $|Y|$ cops occupy a vertex in

$$U = \{v_2\} \cup V(T) \cup \bigcup_{b \in Y} P_1(ab),$$

meaning that there are more available vertices than cops below v_1 . Observe that a cop blocking some $b \in X$ starts at distance at least s to T and cannot occupy a vertex in U after j steps. In particular, this implies that the invariant holds for $j = 1$, as there are less than k cops in total, and thus less than $k - |X| = 2^{s-j} - |X| = |Y|$ cops in U .

After the next move of the cops, we shall choose the next step of the robber. Consider the two subtrees T' and T'' below v_1 , let x' and x'' be their respective number of leaves

corresponding to blocked vertices, and let y' and y'' be their respective number of leaves corresponding to available vertices. By our invariant, there are now (after the cops' move) less than $y' + y'' = 2^{s-j} - (x' + x'')$ cops in $U - \{v_2\}$. Thus, there are less than $y' = 2^{s-j-1} - x'$ cops in the direction of T' , or less than $y'' = 2^{s-j-1} - x''$ cops in the direction of T'' (or both). In particular, there is a move for the robber in $T_2^{\text{out}}(a)$ to maintain our invariant.

After s moves, the robber occupies some leaf v_2 of $T_2^{\text{out}}(a)$. By our invariant, v_2 is the first vertex of the path $P_2(ab)$ for some vertex $b \notin X$ and there is no cop ("less than 1") in the set $U = P_1(ab)$. Thus, the robber can move along $P_2(ab)$ in $2\ell + 1 + s$ further steps to reach $r_2(b)$ before any of the cops can. Finally, if the cops could surround the robber on the way, then at least one cop could also arrive at $r_2(b)$ before the robber, which we just excluded. Hence, we can guarantee a good situation at $r_2(b)$, which concludes the proof. \square

Finally, Lemmas 13 and 14 and theorem 3 immediately give the following corollary, which proves Theorem 2.

Corollary 16. *For any $s \geq 1$, $k = 2^{s-1}$, $m \geq 2s + 1$, and $\ell \geq |V(H[s])| + m + s$, the graph $G_k = H[s, \ell, m]$ has $\Delta(G_k) = 3$ and*

$$c(G_k) \leq 2, \quad c_V(G_k) \geq k, \quad c_{V,r}(G_k) \geq k, \quad c_{E,r}(G_k) \geq \frac{k}{2} \quad \text{and} \quad c_E(G_k) \geq \frac{k}{2}.$$

5 Conclusion

We considered the cop numbers of four different surrounding versions of the well-known Cops and Robber game on a graph G , namely $c_V(G)$, $c_{V,r}(G)$, $c_E(G)$ and $c_{E,r}(G)$. Here, index "V" denotes the vertex versions, while index "E" denotes the edge versions, i.e., whether the cops occupy the vertices or the edges of the graph (recall that the robber always occupies a vertex). An additional index "r" stands for the corresponding restrictive version, meaning that the robber must not end his turn on a cop or move along an edge occupied by a cop.

Only the two restrictive cop numbers have recently been considered in the literature, the vertex version $c_{V,r}(G)$ in [5, 6] (denoted by $\sigma(G)$ and $s(G)$) and the edge version $c_{E,r}(G)$ in [7, 14] (denoted by $\xi(G)$).

In this paper, we related the four different versions to each other, showing that all of them lie within a factor of (at most) $2\Delta(G)$ to each other. We proved that this is tight (up to small additive or multiplicative constants) for all combinations by presenting explicit graph families. It is an interesting open question to identify the exact constant factors for the lower and upper bounds in Theorem 3. We conjecture that all six presented upper bounds are tight (up to small *additive* constants). This would mean that optimal strategies for the cops in one surrounding version can indeed be obtained by simulating optimal strategies from different surrounding versions.

As our second main result, we disproved a conjecture by Crytser, Komarov and Mackey [7] by constructing a family of graphs with maximum degree 3 in which the

classical cop number is bounded whereas the cop number in all four surrounding versions is unbounded. It remains open to find other parameters that can be used to bound the surrounding cop numbers from above in combination with the classical cop number and the maximum degree.

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