The Complexity of the Greedoid Tutte Polynomial

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Abstract

We consider the Tutte polynomial of three classes of greedoids: those arising from rooted graphs, rooted digraphs and binary matrices. We establish the computational complexity of evaluating each of these polynomials at each fixed rational point (x, y). In each case we show that evaluation is #P-hard except for a small number of exceptional cases when there is a polynomial time algorithm. In the binary case, establishing #P-hardness along one line relies on Vertigan's unpublished result on the complexity of counting bases of a binary matroid. For completeness, we include an appendix providing a proof of this result.

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1 Introduction

Tutte's eponymous polynomial is perhaps the most widely studied two-variable graph and matroid polynomial due to its many specializations, their vast breadth and the richness of the underlying theory. Discussion of the Tutte polynomial and closely related polynomials fills an entire handbook [13]. Tutte first introduced the Tutte polynomial of a graph, as the *dichromate* in [37]. It is closely related to Whitney's rank generating function [43] which Tutte extended from graphs to matroids in his PhD thesis [38]. Crapo [10] later extended the definition of the Tutte polynomial to matroids. See Farr [14] for more on the early history of the Tutte polynomial.

The simplest definition of the Tutte polynomial $T(G; x, y)$ of a graph G is probably in terms of the rank function ρ. Given a graph G and a set A of its edges, we have $\rho(A) = |V(G)| - k(G|A)$, where $V(G)$ is the set of vertices of G and $k(G|A)$ is the number of connected components of the graph $(V(G), A)$.

Definition 1. For a graph G with edge set E, we have

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)}.$$

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By making appropriate substitutions for x and y, a huge number of graph invariants with connections to diverse areas of mathematics may be obtained. We summarise just a few of these evaluations that are particularly relevant later in this paper. A *spanning subgraph* of a graph G is a subgraph including all the vertices of G.

- $T(G;1,1)$ is the number of maximal spanning forests of G. (If G is connected, then this is the number of spanning trees.)
- $T(G;2,1)$ is the number of spanning forests of G.
- $T(G;1,2)$ is the number of spanning subgraphs of G having the same number of components as G.
- $T(G;1,0)$ is the number of acyclic orientations of G with one predefined source vertex per component of G [21][1].

Other evaluations (up to a simple pre-factor) include the reliability polynomial, chromatic polynomial and partition function of the q-state Potts model. For a full list of evaluations see [8, 12, 13].

Given a graph polynomial of this type, a natural question is to determine its complexity, that is to classify the points (a,b) according to whether there is a polynomial time algorithm to evaluate the polynomial at (a,b) or whether the evaluation is computationally intractable. Because of the inherent difficulties of measuring the complexity of algorithms involving arbitrary real numbers, we restrict a and b to being rational. This question was completely resolved in a groundbreaking paper by Jaeger, Vertigan and Welsh [23].

For α in $\mathbb{Q} - \{0\}$, let $H_\alpha = \{(x,y) \in \mathbb{Q}^2 : (x-1)(y-1) = \alpha\}$, and let $H_0^x = \{(1,y) : y \in \mathbb{Q}\}$ and $H_0^y = \{(x,1) : x \in \mathbb{Q}\}$. This family of hyperbolae seems to play a special role in the theory of the Tutte polynomial, both in terms of its evaluations and its complexity. For comparison with our results we restrict a and b to being rational in the statement of the main result from [23] given below.

Theorem 2 (Jaeger, Vertigan, Welsh)**.** *Evaluating the Tutte polynomial of a graph at any fixed point (a,b) in the rational plane is #P-hard apart from when (a,b) lies on H_1 or when (a,b) equals $(-1,-1)$, $(-1,0)$, $(0,-1)$ or $(1,1)$, when there exists a polynomial-time algorithm.*

Later, Vertigan and Welsh [41] proved the theorem below, significantly restricting the class of input graphs while only losing the hardness property along one curve.

Theorem 3 (Vertigan, Welsh)**.** *Evaluating the Tutte polynomial of a bipartite planar graph at any fixed point (a,b) in the rational plane is #P-hard apart from when (a,b) lies on H_1 or H_2, or when (a,b) equals $(-1,-1)$ or $(1,1)$, when there exists a polynomial-time algorithm.*

[1] Greene and Zaslavasky showed in [21, Theorem 7.3] that when G is connected, the number of acyclic orientations of G with one predefined source is, up to sign, equal to the coefficient of λ in its chromatic polynomial $\chi(G;\lambda)$. The connection between the T and χ shows that it is also equal to $T(G;1,0)$. See also, for example, [8]

Roughly speaking, the proof of the hardness part of this result (at least without the planar bipartite restriction) proceeds as follows. By exploiting a result of Brylawski [7], one first shows that for most points (a, b), the existence of a polynomial time algorithm to evaluate $T(G; a, b)$ for every graph G would imply the existence of a polynomial time algorithm to evaluate $T(G; x, y)$ at every point (x, y) in H_α, where $\alpha = (a-1)(b-1)$. Given a graph G, let G^k and G_k denote, respectively, the graph obtained by replacing every edge of G by k parallel edges and the graph obtained by replacing every non-loop of G by a path comprising k edges and every loop by a cycle comprising k edges. The former is known as the *k-thickening* of G and the latter as the *k-stretch* of G. Brylawski gave expressions for the Tutte polynomials of G^k and G_k in terms of the Tutte polynomial of G. By varying k, one may obtain expressions for $T(G; a_k, b_k)$ at a sequence $\{(a_k, b_k)\}$ of points on H_α, and then solve for the coefficients of the one-variable polynomial obtained by restricting the domain of T to H_α. There remain several special cases because the sequence $\{(a_k, b_k)\}$ sometimes contains only a small number of distinct points. The second step proceeds by determining a #P-hard point on each curve H_α. Many of these come from evaluations of the chromatic polynomial.

The Tutte polynomial is essentially a generating function for the number of subsets of the edges of a graph according to their rank and size. Following the work of Jaeger, Vertigan and Welsh, many authors have established corresponding results for a variety of graph polynomials defined in a similar way but using different notions of rank. These include the cover polynomial [3], the Bollobás–Riordan polynomial [4], the interlace polynomial [5], the rank generating function of a graphic 2-polymatroid [32] and the Tutte polynomial of a bicircular matroid [16]. In each case, the proof techniques have some similarities: the bulk of the work is done using a graph operation analogous to the thickening, but there are considerable technical difficulties required to deal with the special cases and to complete the proof. These results provide evidence for Makowsky's Difficult Point Conjecture which states that for an n-variable graph polynomial P that may be defined in monadic second order logic, there is a set S of points with the following properties:

1. For every $\mathbf{x} \in S$, there is a polynomial time algorithm to evaluate $P(\mathbf{x})$;
2. For every $\mathbf{x} \notin S$, it is #P-hard to evaluate $P(\mathbf{x})$;
3. The set S is the finite union of algebraic sets in $\mathbb{C}^n$ each having dimension strictly less than n.

For full details see [30].

In this paper we prove results analogous to Theorem 3 for two graph polynomials, the Tutte polynomials of a rooted graph and a rooted digraph, and a polynomial of binary matrices, the Tutte polynomial of a binary greedoid. Each of these polynomials is a special case of the Tutte polynomial of a greedoid introduced by Gordon and McMahon [18] and the proofs have considerable commonality. (All the necessary definitions are provided in the next sections.) The graph polynomials are the analogue of the Tutte polynomial for rooted graphs and rooted digraphs, and our results provide further evidence for Makowsky's Difficult Point Conjecture.

An overview of the paper is as follows. In Section 2 we provide necessary background on rooted graphs, rooted digraphs, greedoids and computational complexity. In the following section we describe the Tutte polynomial of a greedoid and list some of its evaluations for each of the three classes of greedoid that we work with. Within our hardness proofs we require an analogue of the thickening operation and various other constructions which can be defined for arbitrary greedoids, and may be of independent interest. We describe these in Section 4 and provide analogues of Brylawski's results [7] expressing the Tutte polynomial for these constructions in terms of the Tutte polynomials of their constituent greedoids.

In Section 5, we prove the following result completely determining the complexity of evaluating the Tutte polynomial of a rooted graph at a rational point.

Theorem 4. *Evaluating the Tutte polynomial of a connected, rooted, planar, bipartite graph at any fixed point* (a, b) *in the rational* xy*-plane is #P-hard apart from when* (a, b) *equals* $(1, 1)$ *or when* (a, b) *lies on* H_1*.*

There are polynomial time algorithms to evaluate the Tutte polynomial of a rooted graph at $(1, 1)$ *and at any point lying on* H_1*.*

Compared with Theorem 3, we lose most of the easy points: evaluation at any point on H_2 becomes #P-hard, and evaluation at $(-1, -1)$ is also #P-hard. Perhaps surprisingly, in contrast with the situation for graphs, restricting to bipartite, planar graphs does not create any new points where evaluation becomes polynomial time.

In Section 6, we prove the equivalent result for the Tutte polynomial of a rooted digraph.

Theorem 5. *Evaluating the Tutte polynomial of a root-connected, rooted, bipartite digraph at any fixed point* (a, b) *in the rational* xy*-plane is #P-hard apart from when* (a, b) *equals* $(1, 1)$*, when* (a, b) *lies on* H_1*, or when* $b = 0$*.*

There are polynomial time algorithms to evaluate the Tutte polynomial of a rooted digraph at $(1, 1)$*, at any point lying on* H_1 *and at any point* $(a, 0)$*.*

Compared with Theorem 2, evaluation along the line $y = 0$ becomes easy, but evaluation at the special points $(-1, -1)$ and $(0, -1)$ becomes #P-hard. We have not been able to establish #P-hardness for planar digraphs. Much of our proof in this setting relies on #P-hardness of points on the line $x = 1$, which follows from a result of Provan and Ball [35] proving #P-hardness of counting subgraphs with a certain connectivity property. They mentioned that establishing #P-hardness for the restricted case of planar digraphs was open, and as far as we can tell, this is still the case.

We then determine the complexity of evaluating the Tutte polynomial of a binary greedoid.

Theorem 6. *Evaluating the Tutte polynomial of a binary greedoid at any fixed point* (a, b) *in the rational* xy*-plane is #P-hard apart from when* (a, b) *lies on* H_1*.*

There is a polynomial time algorithm to evaluate the Tutte polynomial of a binary greedoid at any point lying on H_1*.*

One special case of this theorem depends on the binary case of an unpublished result of Vertigan, who proved that the problem of counting the bases of a matroid represented over a fixed field $\mathbb{F}$ is #P-complete. For completeness, in Appendix A, we provide a proof of this result for all fields.

2 Preliminaries

2.1 Rooted graphs and digraphs

All our graphs are allowed to have loops and multiple edges. A *rooted graph* is a graph with a distinguished vertex called the *root*. Most of the graphs we work with will be rooted but occasionally we will work with a graph without a root. For complete clarity, we will sometimes refer to such graphs as *unrooted graphs*. We denote a rooted graph G with vertex set $V(G)$, edge set $E(G)$ and root $r(G)$ by a triple $(V(G), E(G), r(G))$. We omit the arguments when there is no fear of ambiguity. Many of the standard definitions for graphs can be applied to rooted graphs in the natural way. Two rooted graphs (V, E, r) and (V', E', r') are *isomorphic* if the unrooted graphs (V, E) and (V', E') are isomorphic via an isomorphism mapping r to r'. For a subset A of E, the *rooted spanning subgraph* $G|A$ is formed from G by deleting all the edges in $E - A$ (and keeping all the vertices). The *root component* of G is the connected component of G containing the root. A set A of edges of G is *feasible* if the root component of $G|A$ is a tree and contains every edge of A. We define the *rank* $\rho_G(A)$ of A to be

$$\rho_G(A) = \max\{|A'| : A' \subseteq A, A \text{ is feasible}\}.$$

We omit the subscript from ρ_G and similar parameters when the context is clear. We let $\rho(G) = \rho(E)$. Observe that a set A of edges is feasible if and only if $\rho(A) = |A|$. A feasible set is a *basis* if $\rho(A) = \rho(G)$. So A is a basis of G if and only if it is the edge set of a spanning tree of the root component of G.

A *rooted digraph* is a digraph with a distinguished vertex called the *root*. We denote a rooted digraph D with vertex set $V(D)$, edge set $E(D)$ and root $r(D)$ by a triple $(V(D), E(D), r(D))$. Once again we omit the arguments when there is no chance of ambiguity. Two rooted digraphs (V, E, r) and (V', E', r') are *isomorphic* if the unrooted digraphs (V, E) and (V', E') are isomorphic via an isomorphism mapping r to r'. We say that the *underlying rooted graph* of a rooted digraph is the rooted graph we get when we remove all the directions on the edges. A rooted digraph is *bipartite* if its underlying rooted graph is bipartite. For a subset A of E, the *rooted spanning subdigraph* $D|A$ is formed from D by deleting all the edges in $E - A$. The *root component* of D is formed by deleting every vertex v to which there is no directed path from r in D, together with its incident edges. The rooted digraph is *root-connected* if there is a directed path from the root to every other vertex. The rooted digraph D is an *arborescence rooted at* r if D is root-connected and its underlying rooted graph is a tree. Observe that a set A of edges of D is *feasible* if and only if the root component of $D|A$ is an arborescence rooted at r

and contains every edge of A. The *rank* $\rho_D(A)$ of A is defined by

$$\rho_D(A) = \max\{|A'| : A' \subseteq A, D|A' \text{ is feasible}\}.$$

We let $\rho(D) = \rho(E)$. A set A of edges is feasible if and only if $\rho(A) = |A|$. A feasible set is a *basis* if $\rho(A) = \rho(D)$. So A is a basis of D if and only if it is the edge set of an arborescence rooted at r that includes every vertex of the root component of D.

2.2 Greedoids

Greedoids are generalizations of matroids, first introduced by Korte and Lovász in 1981 in [26]. One view of matroids is that they characterize precisely those optimization problems on set systems for which the class of partial solutions is hereditary, that is, closed under taking subsets, and for which the greedy algorithm is guaranteed to find an optimal solution for any weighting of the elements of the set system. For example, the edge sets of forests of a graph form a matroid and the greedy algorithm to find a maximum weight forest (spanning tree if the graph is connected) is Kruskal's algorithm. But a greedy algorithm may still work even on a set system that is not hereditary. For example Prim's algorithm finds a minimum spanning tree in a graph by growing a tree from a specified root vertex, so the class of partial solutions found by the algorithm is not generally hereditary. Greedoids were introduced to generalize the characterization of matroids as hereditary set systems on which the greedy algorithm is guaranteed to determine the optimal member of the set system, but omitting the hereditary condition. Most of the information about greedoids which we summarise below can be found in [2] or [29].

Definition 7 (Greedoid)**.** A *greedoid* Γ is an ordered pair $(E, \mathcal{F})$ consisting of a finite set E and a non-empty collection $\mathcal{F}$ of subsets of E satisfying the following axioms:

(G1) $\emptyset \in \mathcal{F}$.

(G2) For all F and F' in $\mathcal{F}$ with $|F'| < |F|$ there exists some $x \in F - F'$ such that $F' \cup x \in \mathcal{F}$.

The set E is the *ground set* of Γ and the members of $\mathcal{F}$ are the *feasible sets* of Γ. The axioms are the first and third of the usual axioms specifying a matroid in terms of its independent sets, so clearly every matroid is a greedoid, but a greedoid does not necessarily satisfy the hereditary property satisfied by the independent sets of a matroid requiring that the collection of independent sets is closed under taking subsets. The *rank* $\rho_\Gamma(A)$ of a subset A of E is given by

$$\rho_\Gamma(A) = \max\{|A'| : A' \subseteq A, A' \in \mathcal{F}\}$$

and we let $\rho(\Gamma) = \rho_\Gamma(E)$. We omit the subscript when the context is clear. Notice that a set A is feasible if and only if $\rho(A) = |A|$. A feasible set is a *basis* if $\rho(A) = \rho(\Gamma)$. We denote the collection of bases of Γ by $\mathcal{B}(\Gamma)$. Axiom (G2) implies that every basis has the same cardinality. Note that the rank function determines Γ but the collection of bases does not. For example, suppose that a greedoid has ground set $\{1,2\}$ and unique basis $\{1,2\}$. Then its collection of feasible sets could either be $\{\emptyset,\{1\},\{1,2\}\}$, $\{\emptyset,\{2\},\{1,2\}\}$ or $\{\emptyset,\{1\},\{2\},\{1,2\}\}$.

The rank function of a greedoid can be characterized in a similar way to the rank function of a matroid [27].

Proposition 8. *The rank function ρ of a greedoid with ground set E takes integer values and satisfies each of the following.*

(GR1) For every subset A of E, $0 \leqslant \rho(A) \leqslant |A|$;

(GR2) For all subsets A and B of E with $A \subseteq B$, $\rho(A) \leqslant \rho(B)$;

(GR3) For every subset A of E, and elements e and f, if $\rho(A) = \rho(A \cup e) = \rho(A \cup f)$, then $\rho(A) = \rho(A \cup e \cup f)$.

Moreover if E is a finite set and ρ is a function from the subsets of E to the integers, then ρ is the rank function of a greedoid with ground set E if and only if ρ satisfies conditions (GR1)–(GR3) above.

The following lemma is easily proved using induction on $|B|$ and will be useful later.

Lemma 9. *Let (E,ρ) be a greedoid specified by its rank function and let A and B be subsets of E such that for all $b \in B$, $\rho(A \cup b) = \rho(A)$. Then $\rho(A \cup B) = \rho(A)$.*

Two greedoids $\Gamma_1 = (E_1, \mathcal{F}_1)$ and $\Gamma_2 = (E_2, \mathcal{F}_2)$ are *isomorphic*, denoted by $\Gamma_1 \cong \Gamma_2$, if there exists a bijection $f : E_1 \to E_2$ that preserves the feasible sets.

The following two examples of greedoids were introduced in [28]. Let G be a rooted graph. Take $\Gamma = (E, \mathcal{F})$ so that $E = E(G)$ and a subset A of E is in $\mathcal{F}$ if and only if the root component of $G|A$ is a tree containing every edge of A. Then Γ is a greedoid. Any greedoid which is isomorphic to a greedoid arising from a rooted graph in this way is called a *branching greedoid*. The branching greedoid of a rooted graph G is denoted by $\Gamma(G)$.

Similarly suppose we have a rooted digraph D and take $\Gamma = (E, \mathcal{F})$ so that $E = E(D)$ and a subset A of E is in $\mathcal{F}$ if and only if the root component of $D|A$ is an arborescence rooted at r and contains every edge of A. Then Γ is a greedoid. Any greedoid which is isomorphic to a greedoid arising from a rooted digraph in this way is called a *directed branching greedoid*. The directed branching greedoid of a rooted digraph D is denoted by $\Gamma(D)$. (There should be no ambiguity with the overload of notation for a branching greedoid and a directed branching greedoid.)

Notice that for both rooted graphs and digraphs, the concepts of feasible set, basis and rank are compatible with their definitions for the associated branching greedoid or directed branching greedoid in the sense that a set A of edges is feasible in a rooted graph G if and only if it is feasible in $\Gamma(G)$, and similarly for the other concepts.

We now define the class of *binary greedoids*. These are a special case of a much broader class, the *Gaussian elimination greedoids*, introduced by Goecke in [17], motivated by the

Gaussian elimination algorithm. Let M be an $m \times n$ binary matrix. It is useful to think of the rows and columns of M as being labelled by the elements of $[m]$ and $[n]$ respectively, where $[n] = \{1, \ldots, n\}$. If X is a subset of $[m]$ and Y is a subset of $[n]$, then $M_{X,Y}$ denotes the matrix obtained from M by deleting all the rows except those with labels in X and all the columns except those with labels in Y. Take $\Gamma = ([n], \mathcal{F})$, so that

$$\mathcal{F} = \{A \subseteq [n] : \text{ the submatrix } M_{[|A|],A} \text{ is non-singular}\}.$$

By convention, the empty matrix is considered to be non-singular. Then Γ is a greedoid. Any greedoid which is isomorphic to a greedoid arising from a binary matrix in this way is called a *binary greedoid*. The binary greedoid of a binary matrix M is denoted by $\Gamma(M)$.

Example 10. Let

$$M = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

The binary greedoid $\Gamma(M)$ has ground set $\{1, 2, 3, 4\}$ and feasible sets

$$\{\emptyset, \{1\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\}.$$

The following lemma is clear.

Lemma 11. *Let $E = [n]$, let M be an $m \times n$ binary matrix with columns labelled by E and let M' be obtained from M by adding row i to row j, where $i < j$. Then $\Gamma(M') \cong \Gamma(M)$.*

A consequence of this lemma is that if Γ is a binary greedoid, then there is a binary matrix M with linearly independent rows so that $\Gamma = \Gamma(M)$. With this in mind we easily obtain the following result which will be useful later.

Lemma 12. *Let Γ be a binary greedoid. Then there is a binary matroid M so that $\mathcal{B}(M) = \mathcal{B}(\Gamma)$.*

In contrast with the situation in matroids, where every graphic matroid is binary, it is not the case that every branching greedoid is binary. For example, take G to be the star with four vertices in which the central vertex is the root. Then $\Gamma(G)$ is not binary. The same example but with the edges directed away from the root demonstrates that not every directed branching greedoid is binary.

An element of a greedoid is a *loop* if it does not belong to any feasible set. So if G is a rooted graph then an edge e is a loop of $\Gamma(G)$ if it does not lie on any path from the root and if G is connected then it is just a loop in the normal graph-theoretic sense. Similarly if D is a directed rooted graph then an edge e is a loop of $\Gamma(D)$ if it does not lie on any directed path from the root. As the concepts of loops in greedoids and in rooted graphs and digraphs do not completely coincide, we use the term *greedoid loop* whenever there is potential for confusion.

Let Γ be a greedoid with ground set E and rank function ρ. Elements e and f of E are said to be *parallel* in Γ if for all subsets A of E,

$$\rho(A \cup e) = \rho(A \cup f) = \rho(A \cup e \cup f).$$

As far as we are aware, the following elementary lemma does not seem to have been stated before.

Lemma 13. *Let Γ be a greedoid. Define a relation $\bowtie$ on the ground set of Γ by $e \bowtie f$ if e and f are parallel in Γ. Then $\bowtie$ is an equivalence relation and if Γ has at least one loop, then one of the equivalence classes of $\bowtie$ comprises the set of loops.*

Proof. The only part of the lemma that is not immediately obvious is that $\bowtie$ is transitive. Let ρ be the rank function of Γ and e, f and g be elements of Γ, so that $e \bowtie f$ and $f \bowtie g$. Then for any subset A of elements of Γ, we have $\rho(A \cup e) = \rho(A \cup f) = \rho(A \cup e \cup f)$ and $\rho(A \cup f) = \rho(A \cup g) = \rho(A \cup f \cup g)$. Thus $\rho(A \cup e) = \rho(A \cup g)$. By applying Lemma 9 to $A \cup f$ and elements e and g, we see that $\rho(A \cup e \cup f \cup g) = \rho(A \cup f)$. Thus, by (GR2), $\rho(A \cup f) = \rho(A \cup e \cup f \cup g) \geqslant \rho(A \cup e \cup g) \geqslant \rho(A \cup e)$. But as $\rho(A \cup e) = \rho(A \cup f)$, equality must hold throughout, so $\rho(A \cup e \cup g) = \rho(A \cup e) = \rho(A \cup g)$, as required. $\square$

2.3 Complexity

We assume some familiarity with computational complexity and refer the reader to one of the standard texts such as [15] or [34] for more background. Given two computational problems π_1 and π_2, we say that π_2 is *Turing reducible* to π_1 if there exists a deterministic Turing machine solving π_2 in polynomial time using an oracle for π_1, that is a subroutine returning an answer to an instance of π_1 in constant-time. When π_2 is Turing reducible to π_1 we write $\pi_2 \propto_T \pi_1$ and we say that solving problem π_1 is at least as hard as solving problem π_2. The relation $\propto_T$ is transitive.

Informally, the class #P is the counting analogue of NP, that is, the class of all counting problems corresponding to decision problems in NP. Slightly more precisely, a problem is in #P if it counts the number of accepting computations or "witnesses" of a problem in NP. Consider the decision problem of determining whether a graph has a proper vertex 3-colouring. The obvious non-deterministic algorithm for this problem interprets a "witness" as a colouring of the vertices with 3 colours and verifies that it is a proper colouring. So the corresponding problem in #P would be to determine the number of proper vertex 3-colourings. A computational problem π is said to be *#P-hard* if $\pi' \propto_T \pi$ for all $\pi' \in$ #P, and *#P-complete* if, in addition, $\pi \in$ #P. Counting the number of vertex 3-colourings of a graph is an example of an #P-complete problem. As we consider evaluations of polynomials which are not necessarily positive integers, most of our results prove #P-hardness rather than #P-completeness.

The following lemma is crucial in many of our proofs.

Lemma 14. *There is an algorithm which when given a non-singular integer $n \times n$ matrix A and an integer n-vector b such that the absolute value of every entry of A and b is at most 2^l, outputs the vector x so that $Ax = b$, running in time bounded by a polynomial in n and l.*

One algorithm to do this is a variant of Gaussian elimination known as the Bareiss algorithm [1]. Similar ideas were presented by Edmonds [11]. See also [22].

3 The Tutte Polynomial of a Greedoid

Extending the definition of the Tutte polynomial of a matroid, McMahon and Gordon defined the Tutte polynomial of a greedoid in [18]. The *Tutte polynomial* of a greedoid Γ with ground set E and rank function ρ is given by

$$T(\Gamma; x, y) = \sum_{A \subseteq E} (x-1)^{\rho(\Gamma)-\rho(A)}(y-1)^{|A|-\rho(A)}.$$

When Γ is a matroid, this reduces to the usual definition of the Tutte polynomial of a matroid. For a rooted graph G we let $T(G; x, y) = T(\Gamma(G); x, y)$, for a rooted digraph D we let $T(D; x, y) = T(\Gamma(D); x, y)$ and for a binary matrix M we let $T(M; x, y) = T(\Gamma(M); x, y)$.

Example 15.

1. Let P_k be the rooted (undirected) path with k edges in which the root is one of the leaves. Then
$$T(P_k; x, y) = 1 + \sum_{i=1}^{k} (x-1)^i y^{i-1}.$$
2. Let S_k be the rooted (undirected) star with k edges in which the root is the central vertex. Then
$$T(S_k; x, y) = x^k.$$

The Tutte polynomial of a greedoid retains many of the properties of the Tutte polynomial of a matroid, for example, it has a delete–contract recurrence, although its form is not as simple as that of the Tutte polynomial of a matroid [18]. Moreover, for a greedoid Γ:

- $T(\Gamma; 1, 1)$ is the number of bases of Γ;
- $T(\Gamma; 2, 1)$ is the number of feasible sets of Γ;
- $T(\Gamma; 1, 2)$ is the number of subsets A of elements of Γ so that $\rho(A) = \rho(\Gamma)$.
- $T(\Gamma; 2, 2) = 2^{|E(\Gamma)|}$.

But the Tutte polynomial of a greedoid also differs fundamentally from the Tutte polynomial of a matroid, for instance, unlike the Tutte polynomial of a matroid, the Tutte polynomial of a greedoid can have negative coefficents. For example, $T(\Gamma(P_2); x, y) = x^2y - 2xy + x + y$.

The Tutte polynomial of a rooted graph has some of the same evaluations as the Tutte polynomial of a unrooted graph. Let G be a rooted graph with edge set E.

- $T(G;1,1)$ is the number of spanning trees of the root component of G. (When G is connected, this is just the number of spanning trees of G.)
- $T(G;2,1)$ is the number of subsets A of E, so that the root component of $G|A$ is a tree containing all the edges of A.
- $T(G;1,2)$ is the number of subsets A of E so that the root component of $G|A$ includes every vertex of the root component of G. (When G is connected, this is just the number of subsets A so that $G|A$ is connected.)
- If no component of G other than the root component has edges, then $T(G;1,0)$ is the number of acyclic orientations of G with a unique source. Otherwise $T(G;1,0)=0$.

We record the following proposition stating that the Tutte polynomial of a connected rooted graph G coincides with the Tutte polynomial of the corresponding unrooted graph G' along the line $x=1$. This is easy to prove by noting that $\rho(G)=\rho(G')$ and a subset A of the edges of G satisfies $\rho_G(A)=\rho(G)$ if and only if $\rho_{G'}(A)=\rho(G')$.

Proposition 16. *Let $G=(V,E,r)$ be a connected rooted graph and let $G'=(V,E)$ be the corresponding unrooted graph. Then*

$$T(G;1,y)=T(G';1,y).$$

We list some evaluations of the Tutte polynomial of a digraph. Let D be a rooted digraph with edge set E and root r.

- $T(D;1,1)$ is the number of spanning arborescences of the root component of D rooted at r. (When D is root-connected, this is just its number of spanning arborescences rooted at r.)
- $T(D;2,1)$ is the number of subsets A of E, so that the root component of $D|A$ is an arborescence rooted at r containing every edge of A.
- $T(D;1,2)$ is the number of subsets A of E, so that the root component of $D|A$ includes every vertex of the root component of D. (When D is root-connected, this is just the number of subsets A so that $D|A$ is root-connected.)
- $T(D;1,0)=1$ if D is acyclic and every edge lies in its root component, and 0 otherwise.

The last evaluation will be discussed in more detail in Section 6.

Gordon and McMahon [18] proved that if T_1 and T_2 are rooted arborescences, then $T(T_1;x,y)=T(T_2;x,y)$ if and only if $T_1\cong T_2$.

We list some evaluations of the Tutte polynomial of a binary greedoid. Let M be an $m\times n$ binary matrix with linearly independent rows.

- $T(M;1,1)$ is the number of subsets of the columns of M whose deletion leaves a non-singular matrix.

- $T(M;2,1)$ is the number of subsets A of the columns of M so that the submatrix $M_{[|A|],A}$ is non-singular.
- $T(M;1,2)$ is the number of subsets of the columns of M whose deletion leaves a matrix with rank $r(M)$.

If a point (a,b) lies on the hyperbola H_1 then we have $(a-1)(b-1)=1$ by definition. Thus the Tutte polynomial of a greedoid Γ evaluated at such a point is given by

$$\begin{aligned} T(\Gamma;a,b) &= \sum_{A\subseteq E(\Gamma)} (a-1)^{\rho(\Gamma)-\rho(A)}(b-1)^{|A|-\rho(A)} \\ &= (a-1)^{\rho(\Gamma)} \sum_{A\subseteq E(\Gamma)} \left(\frac{1}{a-1}\right)^{|A|} = (a-1)^{\rho(\Gamma)-|E(\Gamma)|}a^{|E(\Gamma)|}. \end{aligned}$$

Therefore, given $|E(\Gamma)|$ and $\rho(\Gamma)$, it is easy to compute $T(\Gamma;a,b)$ in polynomial time. For all of the greedoids that we consider, both $|E(\Gamma)|$ and $\rho(\Gamma)$ will be either known or easily computed.

The *characteristic polynomial* of a greedoid was first introduced by Gordon and McMahon in [19] and is a generalization of the characteristic or chromatic polynomial of a matroid. For a greedoid Γ, the *characteristic polynomial* $p(\Gamma;\lambda)$ is defined by

$$p(\Gamma;\lambda) = (-1)^{\rho(\Gamma)}T(\Gamma;1-\lambda,0). \tag{1}$$

4 Greedoid Constructions

In this section we introduce three greedoid constructions and give expressions for the Tutte polynomial of greedoids resulting from these constructions.

The first construction is just the generalization of the k-thickening operation introduced by Brylawski [7] from matroids to greedoids. Given a greedoid $\Gamma=(E,\mathcal{F})$, its k-thickening is the greedoid Γ^k that, informally speaking, is formed from Γ by replacing each element by k parallel elements. More precisely, Γ^k has ground set $E' = E\times[k]$ and collection $\mathcal{F}'$ of feasible sets as follows. Define μ to be the projection operator $\mu: 2^{E\times[k]}\to 2^E$ so that element $e\in\mu(A)$ if and only if $(e,i)\in A$ for some i. Now a subset A is feasible in Γ^k if and only if $\mu(A)$ is feasible in Γ and $|\mu(A)|=|A|$. The latter condition ensures that A does not contain more than one element replacing a particular element of Γ.

It is clear that Γ^k is a greedoid and moreover $\rho_{\Gamma^k}(A)=\rho_\Gamma(\mu(A))$. In particular $\rho(\Gamma^k)=\rho(\Gamma)$. For any element e of Γ the elements (e,i) and (e,j) are parallel. The effect of the k-thickening operation on the Tutte polynomial of a greedoid is given in the following theorem, generalizing the expression for the k-thickening of the Tutte polynomial due to Brylawski [7].

Theorem 17. *Let Γ be a greedoid. The Tutte polynomial of the k-thickening Γ^k of Γ when $y \neq -1$ is given by*

$$T(\Gamma^k; x, y) = (1 + y + \cdots + y^{k-1})^{\rho_G(\Gamma)} T\left(\Gamma; \frac{x + y + \cdots + y^{k-1}}{1 + y + \cdots + y^{k-1}}, y^k\right). \tag{2}$$

When $y = -1$ we have

$$T(\Gamma^k; x, -1) = \begin{cases} (x-1)^{\rho_G(\Gamma)} & \text{if } k \text{ is even;} \\ T(\Gamma; x, -1) & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let Γ^k be the k-thickened greedoid, let E' denote its ground set and let E be the ground set of Γ. Then $E' = E \times [k]$. Let μ be the mapping defined in the discussion at the beginning of this section. To ensure that we do not divide by zero in our calculations, we prove the case when $y = 1$ separately.

For each $A' \subseteq E'$ we have $\rho_{\Gamma^k}(A') = \rho_\Gamma(\mu(A'))$ and furthermore $\rho(\Gamma^k) = \rho(\Gamma)$. The Tutte polynomial of Γ^k when $y \notin \{-1, 1\}$ is thus given by

$$\begin{aligned} T(\Gamma^k; x, y) &= \sum_{A' \subseteq E'} (x-1)^{\rho(\Gamma^k) - \rho_{\Gamma^k}(A')} (y-1)^{|A'| - \rho_{\Gamma^k}(A')} \\ &= \sum_{A \subseteq E} \sum_{\substack{A' \subseteq E': \\ \mu(A') = A}} (x-1)^{\rho(\Gamma) - \rho_\Gamma(\mu(A'))} (y-1)^{|A'| - \rho_\Gamma(\mu(A'))} \qquad (3) \\ &= \sum_{A \subseteq E} (x-1)^{\rho(\Gamma) - \rho_\Gamma(A)} (y-1)^{-\rho_\Gamma(A)} \sum_{\substack{A' \subseteq E': \\ \mu(A') = A}} (y-1)^{|A'|} \\ &= \sum_{A \subseteq E} (x-1)^{\rho(\Gamma) - \rho_\Gamma(A)} (y-1)^{-\rho_\Gamma(A)} (y^k - 1)^{|A|} \\ &= (1 + y + \cdots + y^{k-1})^{\rho(\Gamma)} \sum_{A \subseteq E} \left(\frac{(x-1)(y-1)}{y^k - 1}\right)^{\rho(\Gamma) - \rho_\Gamma(A)} (y^k - 1)^{|A| - \rho_\Gamma(A)} \\ &= (1 + y + \cdots + y^{k-1})^{\rho(\Gamma)} T\left(\Gamma; \frac{x + y + \cdots + y^{k-1}}{1 + y + \cdots + y^{k-1}}, y^k\right). \end{aligned}$$

When $y = 1$ we get non-zero terms in Equation 3 if and only if $|A'| = \rho_\Gamma(\mu(A'))$, which implies that $|A'| = |A|$. For each $A \subseteq E$ there are $k^{|A|}$ choices for A' such that $\mu(A') = A$ and $|A'| = |A|$. Therefore we have

$$\begin{aligned} T(\Gamma^k; x, 1) &= \sum_{\substack{A \subseteq E: \\ \rho_\Gamma(A) = |A|}} (x-1)^{\rho(\Gamma) - \rho_\Gamma(A)} \sum_{\substack{A' \subseteq E': \\ \mu(A') = A, \\ |A'| = |A|}} 1 = \sum_{\substack{A \subseteq E: \\ \rho_\Gamma(A) = |A|}} (x-1)^{\rho(\Gamma) - \rho_\Gamma(A)} k^{\rho_\Gamma(A)} \\ &= \sum_{\substack{A \subseteq E: \\ \rho_\Gamma(A) = |A|}} \left(\frac{x-1}{k}\right)^{\rho(\Gamma) - \rho_\Gamma(A)} k^{\rho(\Gamma)} = k^{\rho(\Gamma)} T\left(\Gamma; \frac{x + k - 1}{k}, 1\right) \end{aligned}$$

which agrees with Equation 2 when $y = 1$.

When $y = -1$ we have

$$\begin{aligned}
T(\Gamma^k; x, -1) &= \sum_{A \subseteq E} \sum_{\substack{A' \subseteq E':\\ \mu(A')=A}} (x-1)^{\rho(\Gamma)-\rho_\Gamma(\mu(A'))}(-2)^{|A'|-\rho_\Gamma(\mu(A'))}\\
&= \sum_{A \subseteq E} (x-1)^{\rho(\Gamma)-\rho_\Gamma(A)}(-2)^{-\rho_\Gamma(A)} \sum_{\substack{A' \subseteq E':\\ \mu(A')=A}} (-2)^{|A'|}\\
&= \sum_{A \subseteq E} (x-1)^{\rho(\Gamma)-\rho_\Gamma(A)}(-2)^{-\rho_\Gamma(A)}((-1)^k-1)^{|A|}\\
&= \begin{cases} (x-1)^{\rho(\Gamma)} & \text{if } k \text{ is even;}\\ T(\Gamma; x, -1) & \text{if } k \text{ is odd.} \end{cases}
\end{aligned}$$

Note that the only contribution to $T(\Gamma^k; x, -1)$ when k is even is from the empty set. □

The second construction is a little more involved. To motivate it we first describe a natural construction operation on rooted graphs. Let G and H be disjoint rooted graphs with G being connected. Then the H-*attachment* of G, denoted by $G \sim H$, is formed by taking G and $\rho(G)$ disjoint copies of H, and identifying each vertex of G other than the root with the root vertex of one of the copies of H. The root of $G \sim H$ is the root of G. See Figure 1 for an illustration of the attachment operation.

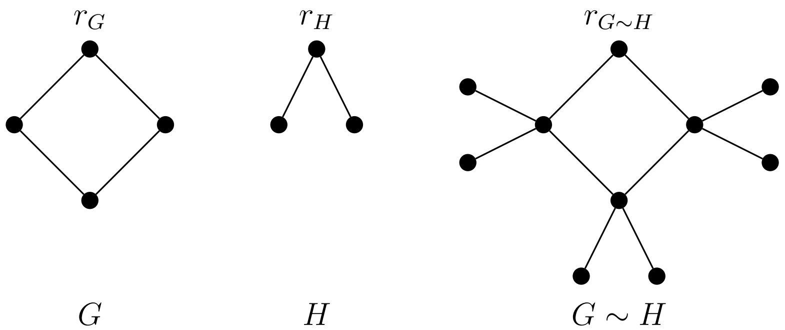

Figure 1: An example of the attachment operation.

Suppose that $V(G) = \{r, v_1, \dots, v_{\rho(G)}\}$, where r is the root of G, let E_0 be the edge set of G and let E_i be the edge set of the copy of H attached at v_i. A set F is feasible in $\Gamma(G \sim H)$ if and only if each of the following conditions holds.

1. $F \cap E_0$ is feasible in $\Gamma(G)$.
2. For all i with $1 \leqslant i \leqslant \rho(G)$, $F \cap E_i$ is feasible in $\Gamma(H)$.
3. For all i with $1 \leqslant i \leqslant \rho(G)$, if v_i is not in the root component of $G|(F \cap E_0)$, then $F \cap E_i = \emptyset$.

In order to extend these ideas to general greedoids, we begin by describing the notion of a closed set, which was first defined for greedoids by Korte and Lovasz [26]. Let Γ be a greedoid with ground set E and rank function ρ. Given a subset A of E, its *closure* $\sigma_\Gamma(A)$ is defined by $\sigma_\Gamma(A) = \{e : \rho(A \cup e) = \rho(A)\}$. We will drop the dependence on Γ whenever the context is clear. Note that it follows from the definition that $A \subseteq \sigma(A)$. Moreover Lemma 9 implies that $\rho(\sigma(A)) = \rho(A)$. Furthermore if $e \notin \sigma(A)$, then $\rho(A \cup e) > \rho(A)$, so axiom (GR2) implies that $\rho(\sigma(A) \cup e) > \rho(\sigma(A))$ and hence $\sigma(\sigma(A)) = \sigma(A)$. A subset A of E satisfying $A = \sigma(A)$ is said to be *closed*. Every subset of E of the form $\sigma(X)$ for some X is closed.

We now introduce what we call an attachment function. Let Γ be a greedoid with rank function ρ. A function $f : \mathcal{F} \to 2^{[\rho(\Gamma)]}$ is called a Γ-*attachment function* if it satisfies both of the following.

1. For each feasible set F, we have $|f(F)| = \rho(F)$.
2. If F_1 and F_2 are feasible sets and $F_1 \subseteq \sigma(F_2)$ then $f(F_1) \subseteq f(F_2)$.

The following property of attachment functions is needed later.

Lemma 18. *Let Γ be a greedoid and f be a Γ-attachment function. Let A be a subset of the elements of Γ and let F_1 and F_2 be maximal feasible subsets of A. Then $f(F_1) = f(F_2)$.*

Proof. It follows from the axioms for the feasible sets of a greedoid that all maximal feasible subsets of A have the same size. Thus $\rho(F_1) = \rho(F_2) = \rho(A)$. For every element e of A, $\rho(F_1) \leqslant \rho(F_1 \cup e) \leqslant \rho(A)$. As $\rho(F_1) = \rho(A)$, equality must hold throughout. Thus $e \in \sigma(F_1)$. Hence $A \subseteq \sigma(F_1)$, so $F_2 \subseteq \sigma(F_1)$. By symmetry, $F_1 \subseteq \sigma(F_2)$. The result then follows from the second condition satisfied by a Γ-attachment function. □

Given greedoids Γ_1 and Γ_2 with disjoint ground sets, and Γ_1-attachment function f, we define the Γ_2-*attachment of* Γ_1, denoted by $\Gamma_1 \sim_f \Gamma_2$ as follows. The ground set E is the union of the ground set E_0 of Γ_1 together with $\rho = \rho(\Gamma_1)$ disjoint copies $E_1, \dots, E_\rho$ of the ground set of Γ_2. In the following we abuse notation slightly by saying that for $i > 0$, a subset of E_i is feasible in Γ_2 if the corresponding subset of the elements of Γ_2 is feasible. A subset F of E is feasible if and only if each of the following conditions holds.

1. $F \cap E_0$ is feasible in Γ_1.
2. For all i with $1 \leqslant i \leqslant \rho$, $F \cap E_i$ is feasible in Γ_2.
3. For all i with $1 \leqslant i \leqslant \rho$, if $i \notin f(F \cap E_0)$ then $F \cap E_i = \emptyset$.

Proposition 19. *For any greedoids Γ_1 and Γ_2, and Γ_1-attachment function f, the Γ_2-*attachment of Γ_1 *is a greedoid.*

Proof. We use the notation defined above to describe the ground set of $\Gamma_1 \sim_f \Gamma_2$. Clearly the empty set is feasible in $\Gamma_1 \sim_f \Gamma_2$. Suppose that F_1 and F_2 are feasible sets in $\Gamma_1 \sim_f \Gamma_2$ with $|F_2| > |F_1|$. If there is an element e of $F_2 \cap E_0$ which is not in $\sigma_{\Gamma_1}(F_1 \cap E_0)$ then

$(F_1 \cap E_0) \cup e$ is feasible in Γ_1. Moreover $F_1 \cap E_0 \subseteq \sigma_{\Gamma_1}((F_1 \cap E_0) \cup e)$, so $f(F_1 \cap E_0) \subseteq f((F_1 \cap E_0) \cup e)$. Consequently $F_1 \cup e$ is feasible in $\Gamma_1 \sim_f \Gamma_2$.

On the other hand, suppose that $F_2 \cap E_0 \subseteq \sigma_{\Gamma_1}(F_1 \cap E_0)$. Then $f(F_2 \cap E_0) \subseteq f(F_1 \cap E_0)$. Moreover, as there is no element e of $(F_2 \cap E_0) - (F_1 \cap E_0)$ such that $(F_1 \cap E_0) \cup e$ is feasible, we have $|F_2 \cap E_0| \leqslant |F_1 \cap E_0|$. So for some i in $f(F_2 \cap E_0)$, we have $|F_2 \cap E_i| > |F_1 \cap E_i|$. Thus there exists $e \in (F_2 - F_1) \cap E_i$ such that $(F_1 \cap E_i) \cup e$ is feasible in Γ_2. As $i \in f(F_2 \cap E_0)$, we have $i \in f(F_1 \cap E_0)$. Hence $F_1 \cup e$ is feasible in $\Gamma_1 \sim_f \Gamma_2$. $\square$

Every greedoid Γ has an attachment function formed by setting $f(F) = [|F|]$ for each feasible set F. However there are other examples of attachment functions. Let G be a connected rooted graph in which the vertices other than the root are labelled $v_1, \dots, v_\rho$. There is an attachment function f defined on $\Gamma(G)$ as follows. For every feasible set F, define $f(F)$ so that $i \in f(F)$ if and only if v_i is in the root component of $G|F$. It is straightforward to verify that f is indeed an attachment function. Furthermore if H is another rooted graph then $\Gamma(G \sim H) = \Gamma(G) \sim_f \Gamma(H)$.

We now consider the rank function of $\Gamma = \Gamma_1 \sim_f \Gamma_2$. We keep the same notation as above for the elements of Γ. Let A be a subset of $E(\Gamma)$ and let F be a maximal feasible subset of $A \cap E_0$. Then

$$\rho_\Gamma(A) = \rho_{\Gamma_1}(A \cap E_0) + \sum_{i \in f(F)} \rho_{\Gamma_2}(A \cap E_i). \tag{4}$$

Observe that the number of subsets of $E(\Gamma)$ with specified rank, size and intersection with E_0 does not depend on the choice of f. Consequently the Tutte polynomial of $\Gamma_1 \sim_f \Gamma_2$ does not depend on f. We now make this idea more precise by establishing an expression for the Tutte polynomial of an attachment.

Theorem 20. *Let Γ_1 and Γ_2 be greedoids, and let f be an attachment function for Γ_1. Then the Tutte polynomial of $\Gamma_1 \sim_f \Gamma_2$ is given by*

$$T(\Gamma_1 \sim_f \Gamma_2; x, y) = T(\Gamma_2; x, y)^{\rho(\Gamma_1)} T\Big(\Gamma_1; \frac{(x-1)^{\rho(\Gamma_2)+1} y^{|E(\Gamma_2)|}}{T(\Gamma_2; x, y)} + 1, y\Big),$$

providing $T(\Gamma_2; x, y) \neq 0$.

Proof. Let $\Gamma = \Gamma_1 \sim_f \Gamma_2$. We use the notation defined above to describe the ground set of Γ. It is useful to extend the definition of the attachment function f to all subsets of E_0 by setting $f(A)$ to be equal to $f(F)$ where F is a maximal feasible set of A. Lemma 18 ensures that extending f in this way is well-defined. It follows from Equation 4 that $\rho(\Gamma) = \rho(\Gamma_1)(\rho(\Gamma_2) + 1)$. We have

$$\begin{aligned} T(\Gamma; x, y) &= \sum_{A \subseteq E(\Gamma)} (x-1)^{\rho(\Gamma) - \rho_\Gamma(A)} (y-1)^{|A| - \rho_\Gamma(A)} \\ &= \sum_{A_0 \subseteq E_0} (x-1)^{\rho(\Gamma_1) - \rho_{\Gamma_1}(A_0)} (y-1)^{|A_0| - \rho_{\Gamma_1}(A_0)} \cdot \prod_{i \notin f(A_0)} \sum_{A_i \subseteq E_i} (x-1)^{\rho(\Gamma_2)} (y-1)^{|A_i|} \end{aligned}$$

$$\cdot \prod_{i\in f(A_0)} \sum_{A_i\subseteq E_i} (x-1)^{\rho(\Gamma_2)-\rho_{\Gamma_2}(A_i)}(y-1)^{|A_i|-\rho_{\Gamma_2}(A_i)}$$

$$= \sum_{A_0\subseteq E_0} (x-1)^{\rho(\Gamma_1)-\rho_{\Gamma_1}(A_0)}(T(\Gamma_2;x,y))^{\rho_{\Gamma_1}(A_0)}$$

$$\cdot \big((x-1)^{\rho(\Gamma_2)}y^{|E(\Gamma_2)|}\big)^{\rho(\Gamma_1)-\rho_{\Gamma_1}(A_0)}(y-1)^{|A_0|-\rho_{\Gamma_1}(A_0)}$$

$$= (T(\Gamma_2;x,y))^{\rho(\Gamma_1)} \sum_{A_0\subseteq E_0} (y-1)^{|A_0|-\rho_{\Gamma_1}(A_0)}\Big(\frac{(x-1)^{\rho(\Gamma_2)+1}y^{|E(\Gamma_2)|}}{T(\Gamma_2;x,y)}\Big)^{\rho(\Gamma_1)-\rho_{\Gamma_1}(A_0)}$$

$$= T(\Gamma_2;x,y)^{\rho(\Gamma_1)}T\Big(\Gamma_1;\frac{(x-1)^{\rho(\Gamma_2)+1}y^{|E(\Gamma_2)|}}{T(\Gamma_2;x,y)}+1,y\Big). \qquad \square$$

The third construction is called the full rank attachment. Given greedoids $\Gamma_1 = (E_1,\mathcal{F}_1)$ and $\Gamma_2 = (E_2,\mathcal{F}_2)$ with disjoint ground sets, the *full rank attachment of* Γ_2 *to* Γ_1 denoted by $\Gamma_1 \approx \Gamma_2$ has ground set $E_1 \cup E_2$ and a set F of elements is feasible if either of the two following conditions holds.

1. $F \in \mathcal{F}_1$;
2. $F \cap E_1 \in \mathcal{F}_1$, $F \cap E_2 \in \mathcal{F}_2$ and $\rho_{\Gamma_1}(F\cap E_1) = \rho(\Gamma_1)$.

It is straightforward to prove that $\Gamma_1 \approx \Gamma_2$ is a greedoid.

Suppose that $\Gamma = \Gamma_1 \approx \Gamma_2$ and that A is a subset of $E(\Gamma)$. Then

$$\rho(A) = \begin{cases} \rho(A\cap E_1) & \text{if } \rho(A\cap E_1) < \rho(\Gamma_1),\\ \rho(A\cap E_1)+\rho(A\cap E_2) & \text{if } \rho(A\cap E_1) = \rho(\Gamma_1).\end{cases}$$

This observation enables us to prove the following identity for the Tutte polynomial.

Theorem 21. *Let* Γ_1 *and* Γ_2 *be greedoids, and let* $\Gamma = \Gamma_1 \approx \Gamma_2$*. Let* E*,* E_1 *and* E_2 *denote the ground sets of* Γ*,* Γ_1 *and* Γ_2 *respectively. Then*

$$T(\Gamma_1 \approx \Gamma_2;x,y) = T(\Gamma_1;x,y)(x-1)^{\rho(\Gamma_2)}y^{|E_2|} + T(\Gamma_1;1,y)(T(\Gamma_2;x,y)-(x-1)^{\rho(\Gamma_2)}y^{|E_2|}).$$

Proof. We have

$$T(\Gamma_1 \approx \Gamma_2;x,y)$$
$$= \sum_{A\subseteq E}(x-1)^{\rho(\Gamma)-\rho_\Gamma(A)}(y-1)^{|A|-\rho_\Gamma(A)}$$
$$= \sum_{\substack{A_1\subseteq E_1:\\ \rho_{\Gamma_1}(A_1)<\rho(\Gamma_1)}} (x-1)^{\rho(\Gamma_1)-\rho_{\Gamma_1}(A_1)}(y-1)^{|A_1|-\rho_{\Gamma_1}(A_1)} \sum_{A_2\subseteq E_2}(x-1)^{\rho(\Gamma_2)}(y-1)^{|A_2|}$$
$$+ \sum_{\substack{A_1\subseteq E_1:\\ \rho_{\Gamma_1}(A_1)=\rho(\Gamma_1)}} (y-1)^{|A_1|-\rho_{\Gamma_1}(A_1)} \sum_{A_2\subseteq E_2}(x-1)^{\rho(\Gamma_2)-\rho_{\Gamma_2}(A_2)}(y-1)^{|A_2|-\rho_{\Gamma_2}(A_2)}$$

$$
\begin{aligned}
&= \sum_{A_1 \subseteq E_1} (x-1)^{\rho(\Gamma_1)-\rho_{\Gamma_1}(A_1)}(y-1)^{|A_1|-\rho_{\Gamma_1}(A_1)}(x-1)^{\rho(\Gamma_2)}y^{|E_2|} \\
&\quad + \sum_{\substack{A_1 \subseteq E_1: \\ \rho_{\Gamma_1}(A_1)=\rho(\Gamma_1)}} (y-1)^{|A_1|-\rho_{\Gamma_1}(A_1)} \\
&\quad \cdot \Big(\sum_{A_2 \subseteq E_2} (x-1)^{\rho(\Gamma_2)-\rho_{\Gamma_2}(A_2)}(y-1)^{|A_2|-\rho_{\Gamma_2}(A_2)} - (x-1)^{\rho(\Gamma_2)}y^{|E_2|} \Big) \\
&= T(\Gamma_1; x, y)(x-1)^{\rho(\Gamma_2)}y^{|E_2|} + T(\Gamma_1; 1, y)\big(T(\Gamma_2; x, y) - (x-1)^{\rho(\Gamma_2)}y^{|E_2|}\big). \qquad \square
\end{aligned}
$$

This construction will be useful later in Section 7 when Γ_1 and Γ_2 are binary greedoids with $\Gamma_1 = \Gamma(M_1)$ and $\Gamma_2 = \Gamma(M_2)$, where M_1 has full row rank. Then $\Gamma_1 \approx \Gamma_2 = \Gamma(M)$ where M has the form

$$
M = \left(\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_2 \end{array} \right).
$$

5 Rooted Graphs

Throughout the remainder of the paper we focus on three computational problems. Let $\mathbb{G}$ denote either the class of branching greedoids, directed branching greedoids or binary greedoids. Our first problem is computing all the coefficients of the Tutte polynomial for a greedoid in the class $\mathbb{G}$.

$\pi_1[\mathbb{G}]$: #Rooted Tutte Polynomial
Input: $\Gamma \in \mathbb{G}$.
Output: The coefficients of $T(\Gamma; x, y)$.

The second problem involves computing the Tutte polynomial along a plane algebraic curve L. We restrict our attention to the case where L is a rational curve given by the parametric equations

$$
x(t) = \frac{p(t)}{q(t)} \quad \text{and} \quad y(t) = \frac{r(t)}{s(t)},
$$

where p, q, r and s are polynomials over $\mathbb{Q}$. More precisely, we compute the coefficients of the one-variable polynomial obtained by restricting T to the curve L.

$\pi_2[\mathbb{G}, L]$: #Rooted Tutte Polynomial Along L
Input: $\Gamma \in \mathbb{G}$.
Output: The coefficients of the rational function of t given by evaluating $T(\Gamma; x(t), y(t))$ along L.

Most of the time, L will be one of the hyperbolae H_α. We will frequently make a slight abuse of notation by writing $L = H_\alpha$.

The final problem is the evaluation of the Tutte polynomial at a fixed rational point (a, b).

$\pi_3[\mathbb{G}, a, b]$: #Rooted Tutte Polynomial At (a, b)
Input: $\Gamma \in \mathbb{G}$.
Output: $T(\Gamma; a, b)$.

It is straightforward to see that for each possibility for $\mathbb{G}$, we have

$$\pi_3[\mathbb{G}, a, b] \propto_T \pi_2[\mathbb{G}, H_{(a-1)(b-1)}] \propto_T \pi_1[\mathbb{G}].$$

Our results in the remainder of the paper will determine when the opposite reductions hold.

In this section we prove Theorem 4. We let $\mathcal{G}$ be the class of branching greedoids of connected, rooted, planar, bipartite graphs and take $\mathbb{G} = \mathcal{G}$. It is, however, more convenient to take the input to each problem to be a connected, rooted, planar, bipartite graph rather than its branching greedoid.

We begin by reviewing the exceptional points of Theorem 4. If a point (a, b) lies on the hyperbola H_1 then, following the remarks at the end of Section 3, $T(G; a, b)$ is easily computed. We noted in Section 3 that for a connected rooted graph G, $T(G; 1, 1)$ is equal to the number of spanning trees of G. That this can be evaluated in polynomial time follows from Kirchhoff's Matrix–Tree theorem [25]. Hence there are polynomial time algorithms to evaluate the Tutte polynomial of a connected rooted graph at $(1, 1)$ and at any point lying on H_1. It is easy to extend this to all rooted graphs because every edge belonging to a component that does not include the root is a loop in the corresponding branching greedoid.

We will now review the hard points of Theorem 4. A key step in establishing the hardness part of Theorem 4 for points lying on the line $y = 1$ is to strengthen a result of Jerrum [24]. Given an unrooted graph $G = (V, E)$, a *subtree* of G is a subgraph of G which is a tree. (We emphasize that the subgraph does not have to be an induced subgraph.) Jerrum [24] showed that the following problem is #P-complete.

#Subtrees
Input: Planar unrooted graph G.
Output: The number of subtrees of G.

Consider the restriction of this problem to bipartite planar graphs.

#Bisubtrees
Input: Bipartite, planar, unrooted graph G.
Output: The number of subtrees of G.

We shall show that #Bisubtrees is #P-complete. We say that an edge of a graph G is *external* in a subtree T of G if it is not contained in $E(T)$. Let $t_{i,j}(G)$ be the number of subtrees of G with i external edges having precisely one endvertex in T and j external edges having both endvertices in T.

Recall that the k-stretch of an unrooted graph G is obtained by replacing each loop by a cycle with k edges and every other edge by a path of length k. Let $t(G)$ denote the number of subtrees of G.

Proposition 22. *For every unrooted graph G, the number of subtrees of the k-stretch G_k of G is given by*

$$t(G_k) = \left(\sum_{i,j \geqslant 0} t_{i,j}(G) k^i \binom{k+1}{2}^j \right) + \frac{k(k-1)|E|}{2}.$$

Proof. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$ and let E_t be the set of edges replacing e_t in G_k for $1 \leqslant t \leqslant m$. Thus $E(G_k) = \bigcup_{t=1}^m E_t$. We can think of the vertices of G_k as being of two types: those corresponding to the vertices of G and the extra ones added when G_k is formed. We construct a function f that maps every subtree T of G_k to a graph T' which is either a subtree of G or an empty graph with no vertices or edges. We let $V(T')$ comprise all the vertices of $V(T)$ corresponding to vertices in G. The edge set $E(T')$ is defined so that $e_t \in E(T')$ if and only if $E_t \subseteq E(T)$.

Let T' be a subtree of G with at least one vertex, i external edges having precisely one endvertex in T' and j external edges having both endvertices in T'.

If $T \in f^{-1}(T')$ then it must contain all of the edges in G_k that replace the edges in $E(T')$. Suppose there is an edge $e_t = v_1v_2$ in G that is external in T' with $v_1 \in V(T')$ and $v_2 \notin V(T')$. Then there are k possibilities for the subset of E_t appearing in T. Now suppose there exists an edge $e_t = v_1v_2$ in G that is external in T' with $v_1, v_2 \in V'$. Then there are $\binom{k+1}{2}$ choices for the subset of E_t appearing in T. Therefore,

$$|f^{-1}(T'_{i,j})| = k^i \binom{k+1}{2}^j.$$

It remains to count the subtrees of G_k mapped by f to a graph with no vertices. Such a subtree does not contain any vertices corresponding to vertices in G. There are $(k-1)|E(G)|$ subtrees of G_k comprising a single vertex not in $V(G)$ and no edges, and $\binom{k-1}{2}|E(G)|$ subtrees of G_k with at least one edge but not containing any vertex in $V(G)$. Hence

$$t(G_k) = \left(\sum_{i,j \geqslant 0} t_{i,j}(G) k^i \binom{k+1}{2}^j \right) + \frac{k(k-1)}{2}|E(G)|. \qquad \square$$

We can now show that #Bisubtrees is #P-complete.

Proposition 23. *The problem* #Bisubtrees *is #P-complete.*

Proof. It is clear that #Bisubtrees belongs to #P. To establish hardness, first note that $G_2, \ldots, G_{4|E(G)|+2}$ are all bipartite and may be constructed from G in polynomial time. We have $\max_{i,j \geqslant 0}\{i + 2j : t_{i,j}(G) > 0\} \leqslant \max_{i,j \geqslant 0}\{i + 2j : i + j \leqslant |E(G)|\} = 2|E(G)|$. Therefore, by Proposition 22, $t(G_k)$ is a polynomial in k of degree at most $2|E(G)|$. So we can write

$$t(G_k) = \sum_{p=0}^{2|E(G)|} a_p k^p.$$

Thus, if we compute $t(G_k)$ for $k = 2, \ldots, 4|E(G)| + 2$, then we can apply Lemma 14 to recover a_i for all i and then determine $t(G) = t(G_1)$ in polynomial time. Therefore we have shown that SUBTREES $\propto_T$ #BISUBTREES. □

We now present three propositions which together show that at most fixed rational points (a, b), evaluating the Tutte polynomial of a connected, bipartite, planar, rooted graph at (a, b) is just as hard as evaluating it along the curve $H_{(a-1)(b-1)}$. The k-thickening operation is crucial. Notice that $\Gamma(G^k) \cong (\Gamma(G))^k$, so we may apply Theorem 17 to obtain an expression for $T(G^k)$. The first proposition deals with the case when $a \neq 1$ and $b \notin \{-1, 0, 1\}$.

Proposition 24. *Let $L = H_\alpha$ for some $\alpha \in \mathbb{Q} - \{0\}$. Let (a, b) be a point on L such that $b \notin \{-1, 0\}$. Then*

$$\pi_2[\mathcal{G}, L] \propto_T \pi_3[\mathcal{G}, a, b].$$

Proof. For a point (x, y) on L we have $y \neq 1$. Therefore $z = y - 1 \neq 0$ and so $\alpha/z = x - 1$. Let G be in $\mathcal{G}$. Along L the Tutte polynomial of G has the form

$$T(G; x, y) = T(G; 1 + \alpha/z, 1 + z) = \sum_{A \subseteq E(G)} \left(\frac{\alpha}{z}\right)^{\rho(G)-\rho(A)} z^{|A|-\rho(A)} = \sum_{i=-\rho(G)}^{|E(G)|} t_i z^i,$$

for some $t_{-\rho(G)}, \ldots, t_{|E(G)|}$.

We now show that we can determine all of the coefficients t_i from the evaluations $T(G^k; a, b)$ for $k = 1, \ldots, |E(G)| + \rho(G) + 1$ in time polynomial in $|E(G)|$. For each such k, G^k may be constructed from G in time polynomial in $|E(G)|$ and is bipartite, planar and connected. By Theorem 17, we have

$$T(G^k; a, b) = (1 + b + \ldots + b^{k-1})^{\rho(G)} T\left(G; \frac{a + b + \ldots + b^{k-1}}{1 + b + \ldots + b^{k-1}}, b^k\right).$$

Since $b \neq -1$, we have $1 + b + \ldots + b^{k-1} \neq 0$. Therefore we may compute

$$T\left(G; \frac{a + b + \ldots + b^{k-1}}{1 + b + \ldots + b^{k-1}}, b^k\right)$$

from $T(G^k; a, b)$. The point $\left(\frac{a+b+\ldots+b^{k-1}}{1+b+\ldots+b^{k-1}}, b^k\right)$ will also be on the curve L since

$$\left(\frac{a + b + \ldots + b^{k-1}}{1 + b + \ldots + b^{k-1}} - 1\right)(b^k - 1) = (a - 1)(b - 1).$$

As $b \notin \{-1, 0, 1\}$, for $k = 1, 2, \ldots, |E(G)| + \rho(G) + 1$, the points $\left(\frac{a+b+\ldots+b^{k-1}}{1+b+\ldots+b^{k-1}}, b^k\right)$ are pairwise distinct. Therefore by evaluating $T(G^k; a, b)$ for $k = 1, \ldots, |E(G)| + \rho(G) + 1$, we obtain $\sum_{i=-\rho(G)}^{|E(G)|} t_i z^i$ for $|E(G)| + \rho(G) + 1$ distinct values of z. This gives us $|E(G)| + \rho(G) + 1$ linear equations for the coefficients t_i. The matrix of the equations is a Vandermonde matrix and clearly non-singular. So, we may apply Lemma 14 to compute t_i for all i in time polynomial in $|E(G)|$. □

The next proposition deals with the case when $a = 1$. Recall $H_0^x = \{(1, y) : y \in \mathbb{Q}\}$ and $H_0^y = \{(x, 1) : x \in \mathbb{Q}\}$.

Proposition 25. *Let $L = H_0^x$ and let $b \in \mathbb{Q} - \{-1, 0, 1\}$. Then*

$$\pi_2[\mathcal{G}, L] \propto_T \pi_3[\mathcal{G}, 1, b].$$

Proof. Let G be in $\mathcal{G}$. Along L the Tutte polynomial of G has the form

$$T(G; 1, y) = \sum_{\substack{A \subseteq E(G):\\ \rho(A)=\rho(G)}} (y-1)^{|A|-\rho(G)} = \sum_{i=0}^{|E(G)|} t_i y^i,$$

for some $t_0, \ldots, t_{|E(G)|}$. (Note that the restriction that $\rho(A) = \rho(G)$ ensures that $|A| - \rho(G) \geqslant 0$.)

The proof now follows in a similar way to that of Proposition 24 by computing $T(G^k; 1, b)$ for $k = 1, \ldots, |E(G)| + 1$ and then determining each coefficient t_i in time polynomial in $|E(G)|$. □

The following proposition deals with the case when $b = 1$.

Proposition 26. *Let $L = H_0^y$ and $a \in \mathbb{Q} - \{1\}$. Then*

$$\pi_2[\mathcal{G}, L] \propto_T \pi_3[\mathcal{G}, a, 1].$$

Proof. Let G be in $\mathcal{G}$. Along L the Tutte polynomial of G has the form

$$T(G; x, 1) = \sum_{\substack{A \subseteq E(G):\\ \rho(A)=|A|}} (x-1)^{\rho(G)-\rho(A)} = \sum_{i=0}^{\rho(G)} t_i x^i,$$

for some $t_0, \ldots, t_{\rho(G)}$.

We now show that we can determine all of the coefficients t_i from the evaluations $T(G^k; a, 1)$ for $k = 1, \ldots, \rho(G) + 1$ in time polynomial in $|E(G)|$. For each such k, G^k may be constructed from G in time polynomial in $|E(G)|$ and is bipartite, planar and connected. By Theorem 17, we have

$$T(G^k; a, 1) = k^{\rho(G)} T\left(G; \frac{a+k-1}{k}, 1\right).$$

Therefore we may compute $T\left(G; \frac{a+k-1}{k}, 1\right)$ from $T(G^k; a, 1)$. Clearly $\left(\frac{a+k-1}{k}, 1\right)$ lies on H_0^y. Since $a \neq 1$, the points $\left(\frac{a+k-1}{k}, 1\right)$ are pairwise distinct for $k = 1, 2, \ldots, \rho(G) + 1$. Therefore by evaluating $T(G^k; a, 1)$ for $k = 1, \ldots, \rho(G) + 1$, we obtain $\sum_{i=0}^{|\rho(G)|} t_i z^i$ for $\rho(G) + 1$ distinct values of z. This gives us $\rho(G) + 1$ linear equations for the coefficients t_i. Again the matrix of the equations is a Vandermonde matrix and clearly non-singular. So, we may apply Lemma 14 to compute t_i for all i in time polynomial in $|E(G)|$. □

We now summarize the three preceding propositions.

Proposition 27. *Let L be either H_0^x, H_0^y, or H_α for $\alpha \in \mathbb{Q} - \{0\}$. Let (a,b) be a point on L such that $(a,b) \neq (1,1)$ and $b \notin \{-1,0\}$. Then*

$$\pi_2[\mathcal{G}, L] \propto_T \pi_3[\mathcal{G}, a, b].$$

We now consider the exceptional case when $b = -1$. For reasons that will soon become apparent, we recall from Example 15 that $T(P_2; x, y) = x^2y - 2xy + x + y$ and $T(S_k; x, y) = x^k$.

Proposition 28. *Let L be the line $y = -1$. For $a \notin \{\frac{1}{2}, 1\}$ we have*

$$\pi_2[\mathcal{G}, L] \propto_T \pi_3[\mathcal{G}, a, -1].$$

Proof. Let G be in $\mathcal{G}$ and let $z = x - 1$. Along L the Tutte polynomial of G has the form

$$T(G; x, -1) = \sum_{A \subseteq E(G)} z^{\rho(G)-\rho(A)}(-2)^{|A|-\rho(A)} = \sum_{i=0}^{\rho(G)} t_i z^i$$

for some $t_0, \ldots, t_{\rho(G)}$.

We now show that, apart from a few exceptional values of a, we can determine all of the coefficients t_i in polynomial time from $T(G \sim S_k; a, -1)$, for $k = 0, 1, \ldots, \rho(G)$, in time polynomial in $|E(G)|$. For each such k, $G \sim S_k$ may be constructed from G in time polynomial in $|E(G)|$ and is bipartite, planar and connected.

By Theorem 20 we have, for $a \neq 0$,

$$T(G \sim S_k; a, -1) = a^{k\rho(G)} T\left(G; \frac{(a-1)^{k+1}(-1)^k}{a^k} + 1, -1\right).$$

Providing $a \neq 0$ we may compute $T\left(G; \frac{(a-1)^{k+1}(-1)^k}{a^k} + 1, -1\right)$ from $T(G \sim S_k; a, -1)$. For $a \notin \{0, \frac{1}{2}, 1\}$ the points $\left(\frac{(a-1)^{k+1}(-1)^k}{a^k} + 1, -1\right)$ are pairwise distinct for $k = 0, 1, \ldots, \rho(G)$. Therefore by evaluating $T(G \sim S_k; a, -1)$ for $k = 0, 1, 2, \ldots, \rho(G)$ where $a \notin \{0, \frac{1}{2}, 1\}$, we obtain $\sum_{i=0}^{\rho(G)} t_i z^i$ for $\rho(G) + 1$ distinct values of z. This gives us $\rho(G) + 1$ linear equations for the coefficients t_i. Again the matrix corresponding to these equations is a Vandermonde matrix and clearly non-singular. So, we may apply Lemma 14 to compute t_i for all i in time polynomial in $|E(G)|$. Hence for $a \notin \{0, \frac{1}{2}, 1\}$, $\pi_2[\mathcal{G}, L] \propto \pi_3[\mathcal{G}, a, -1]$.

We now look at the case when $a = 0$. Note that $T(P_2; 0, -1) = -1$. Applying Theorem 20 to G and P_2 gives

$$T(G \sim P_2; 0, -1) = (-1)^{\rho(G)} T\left(G; \frac{(-1)^3(-1)^2}{-1} + 1, -1\right) = (-1)^{\rho(G)} T(G; 2, -1).$$

Therefore we have the reductions

$$\pi_2[\mathcal{G}, L] \propto_T \pi_3[\mathcal{G}, 2, -1] \propto_T \pi_3[\mathcal{G}, 0, -1].$$

Since the Turing reduction relation is transitive, this implies that evaluating the Tutte polynomial at the point $(0, -1)$ is at least as hard as evaluating it along the line $y = -1$. This completes the proof. □

We now begin to classify the complexity of π_3. The next results will establish hardness for a few special cases, namely when $b \in \{-1, 0, 1\}$.

Proposition 29. *The problem* $\pi_3[\mathcal{G}, 1, b]$ *is #P-hard apart from when* $b = 1$*, in which case it has a polynomial time algorithm.*

Proof. The hardness part follows directly from Theorem 3 and Proposition 16. We have already noted the existence of a polynomial time algorithm to solve $\pi_3[\mathcal{G}, 1, 1]$. □

Proposition 30. *The problem* $\pi_3[\mathcal{G}, a, -1]$ *is #P-hard apart from when* $a = 1/2$*, in which case it has a polynomial time algorithm.*

Proof. First note that there is a polynomial time algorithm for $\pi_3[\mathcal{G}, a, -1]$ because $(\frac{1}{2}, -1)$ lies on H_1. Now let L be the line $y = -1$. By Proposition 28 we have

$$\pi_2[\mathcal{G}, L] \propto_T \pi_3[\mathcal{G}, a, -1]$$

for $a \notin \{\frac{1}{2}, 1\}$. So

$$\pi_3[\mathcal{G}, 1, -1] \propto_T \pi_3[\mathcal{G}, a, -1]$$

for $a \neq 1/2$. By Proposition 29 we know that $\pi_3[\mathcal{G}, 1, -1]$ is #P-hard. So the result follows. □

Proposition 31. *The problem* $\pi_3[\mathcal{G}, a, 0]$ *is #P-hard apart from when* $a = 0$*, in which case it has a polynomial time algorithm.*

Proof. Let G be in $\mathcal{G}$. First note that evaluating the Tutte polynomial of G at the point $(0, 0)$ is easy since $(0, 0)$ lies on the hyperbola H_1.

The rooted graph $G \sim S_1$ may be constructed from G in time polynomial in $|E(G)|$ and is bipartite, planar and connected. Applying Theorem 20 to G and S_1 gives

$$T(G \sim S_1; a, 0) = a^{\rho(G)} T(G; 1, 0).$$

Since $a \neq 0$ we may compute $T(G; 1, 0)$ from $T(G \sim S_1; a, 0)$. Therefore $\pi_3[\mathcal{G}, 1, 0] \propto \pi_3[\mathcal{G}, a, 0]$. By Proposition 29, $\pi_3[\mathcal{G}, 1, 0]$ is #P-hard, and the result follows. □

Recall from Equation 1 that along $y = 0$ the Tutte polynomial of a rooted graph specializes to the characteristic polynomial. Therefore we have the following corollary.

Corollary 32. *Computing the characteristic polynomial* $p(G; k)$ *of a connected rooted graph* G *is #P-hard for all* $k \in \mathbb{Q} - \{1\}$*. When* $k = 1$*, there is a polynomial time algorithm.*

Proof. Let k be in $\mathbb{Q}$. We have

$$p(G;k) = (-1)^{\rho(G)}T(G;1-k,0).$$

By Proposition 31 evaluating $T(G;1-k,0)$ is #P-hard providing $k \neq 1$. Furthermore when $k = 1$ we have

$$p(G;1) = (-1)^{\rho(G)}T(G;0,0) = \begin{cases} 1 & \text{if } G \text{ is edgeless;} \\ 0 & \text{otherwise,} \end{cases}$$

and so it is easy to compute (as expected since $(0,0)$ lies on H_1). □

We now consider points along the line $y = 1$.

Proposition 33. *The problem* $\pi_3[\mathcal{G}, a, 1]$ *is #P-hard when* $a \neq 1$.

Proof. Let G be a connected, planar, bipartite, unrooted graph with $V(G) = \{v_1, \dots, v_n\}$. Now for $1 \leqslant j \leqslant n$, let G_j be the graph in $\mathcal{G}$ obtained from G by choosing v_j to be the root. Let ρ_j denote the rank function of G_j and $a_i(G_j)$ be the number of subsets A of the edges of G_j having size i so that the root component of $G|A$ is a tree. Then

$$T(G_j;x,1) = \sum_{\substack{A \subseteq E: \\ \rho_j(A)=|A|}} (x-1)^{\rho(G_j)-|A|} = \sum_{i=0}^{\rho(G_j)} a_i(G_j)(x-1)^{\rho(G_j)-i}.$$

Let $a_i(G)$ denote the number of subtrees of G with i edges. Then

$$a_i(G) = \sum_{j=1}^{n} \frac{a_i(G_j)}{i+1}.$$

This is because every subtree T of G with i edges has $i+1$ vertices and its edge set is one of the sets A contributing to $a_i(G_j)$ for the $i+1$ choices of j corresponding to its vertices.

Given an oracle for $\pi_2[\mathcal{G}, H_0^y]$, we can compute $a_i(G_j)$ for $i = 0, \dots, |E(G)|$ and $1 \leqslant j \leqslant n$ in time polynomial in $|E(G)|$. So we can compute $a_i(G)$ and consequently the number of trees of G in time polynomial in $|E(G)|$. Thus

$$\#\text{BISUBTREES} \propto_T \pi_2[\mathcal{G}, H_0^y].$$

By Proposition 27 we have

$$\#\text{BISUBTREES} \propto_T \pi_2[\mathcal{G}, H_0^y] \propto_T \pi_3[\mathcal{G}, a, 1]$$

for $a \neq 1$. The result now follows from Proposition 23. □

We now summarize our results and prove Theorem 4.

Proof of Theorem 4. Let (a,b) be a point on H_α for some α in $\mathbb{Q}-\{0,1\}$. By Proposition 27 we have $\pi_2[\mathcal{G},H_\alpha] \propto_T \pi_3[\mathcal{G},a,b]$ providing $b \notin \{-1,0\}$. The hyperbola H_α crosses the x-axis at the point $(1-\alpha,0)$. By Proposition 31 the problem $\pi_3[\mathcal{G},1-\alpha,0]$ is #P-hard since $\alpha \neq 1$. This gives us a #P-hard point on each of these curves and therefore implies $\pi_2[\mathcal{G},H_\alpha]$ is #P-hard for $\alpha \in \mathbb{Q}-\{0,1\}$. Hence $\pi_3[\mathcal{G},a,b]$ is #P-hard for $(a,b)\in H_\alpha$ with $\alpha \in \mathbb{Q}-\{0,1\}$ and $b \neq -1$. The rest of the proof now follows directly by Propositions 29, 30 and 33, and the discussion concerning the easy points at the beginning of the section. □

6 Rooted Digraphs

In this section we take $\mathbb{G}$ to be the class of directed branching greedoids of root-connected, rooted, bipartite digraphs, a class we denote by $\mathcal{D}$. We consider the same three problems as in the previous section. Again, it is more convenient to think of the input as being a root-connected, rooted, bipartite digraph rather than its directed branching greedoid. We present analogous results to those in the previous section by finding the computational complexity of evaluating the Tutte polynomial of a root-connected, rooted, bipartite digraph at a fixed rational point, eventually proving Theorem 5.

We begin the proof by examining the easy points. Let D be a rooted digraph with edge set E and rank function ρ. If a point (a,b) lies on the hyperbola H_1 then, following the remarks at the end of Section 3, $T(D;a,b)$ is easily computed. We now show that evaluating $T(D;a,0)$ is easy for all $a\in\mathbb{Q}$. A *sink* in a digraph is a non-isolated vertex with no outgoing edges. Suppose that D is a root-connected, rooted digraph with s sinks. Then Gordon and McMahon [20] have shown that its characteristic polynomial p satisfies the following.

$$p(D;\lambda)=\begin{cases}(-1)^{\rho(D)}(1-\lambda)^s & \text{if } D \text{ is acyclic;}\\ 0 & \text{if } D \text{ has a directed cycle.}\end{cases}$$

Using the relation $T(D;1-\lambda,0)=(-1)^{\rho(D)}p(D;\lambda)$ we see that

$$T(D;x,0)=\begin{cases}x^s & \text{if } D \text{ is acyclic;}\\ 0 & \text{if } D \text{ has a directed cycle.}\end{cases}$$

It is easy to count the sinks in a digraph. Moreover, every edge of a component of a rooted digraph other than the root component is a greedoid loop, so if D has such an edge then $T(D;1-\lambda,0)=0$. Furthermore, the addition or removal of isolated vertices makes no difference to $T(D)$. So $T(D;a,0)$ can be computed in polynomial time for the class of all rooted digraphs.

We noted in Section 3 that $T(D;1,1)$ is the number of spanning arborescences of the root component of D rooted at r. This can be computed in polynomial time using the Matrix–Tree theorem for directed graphs [6, 36].

We now move on to consider the hard points. The k-thickening operation will again be crucial: the *k-thickening* D^k of a root-connected digraph D is obtained by replacing every edge e in D by k parallel edges that have the same direction as e. We have $\Gamma(D^k) \cong (\Gamma(D))^k$, so Theorem 17 can be applied to give an expression for $T(D^k)$.

The proof of the following proposition is omitted as it is analogous to that of Proposition 27.

Proposition 34. *Let L be either H_0^x, H_0^y, or H_α for $\alpha \in \mathbb{Q} - \{0\}$. Let (a, b) be a point on L such that $(a, b) \neq (1, 1)$ and $b \notin \{-1, 0\}$. Then*

$$\pi_2[\mathcal{D}, L] \propto_T \pi_3[\mathcal{D}, a, b].$$

We let $\overrightarrow{P_k}$ be the root-connected directed path of length k with the root being one of the leaves and $\overrightarrow{S_k}$ be the root-connected directed star with k edges emanating from the root. The proof of the following proposition is analogous to that of Proposition 28 with $\overrightarrow{P_k}$ and $\overrightarrow{S_k}$ playing the roles of P_k and S_k. We have $T(\overrightarrow{P_k}; x, y) = 1 + \sum_{i=1}^{k}(x-1)^i y^{i-1}$ and $T(\overrightarrow{S_k}; x, y) = x^k$, so $\overrightarrow{P_k}$ and $\overrightarrow{S_k}$ have the same Tutte polynomials as P_k and S_k, respectively. See Example 15.

Proposition 35. *Let L be the line $y = -1$. For $a \notin \{\frac{1}{2}, 1\}$ we have*

$$\pi_2[\mathcal{D}, L] \propto_T \pi_3[\mathcal{D}, a, -1].$$

Next we classify the complexity of $\pi_3[\mathcal{D}, 1, b]$ for $b \notin \{0, 1\}$. Suppose we have a root-connected digraph D and generate a random subgraph (D, p) of D by deleting each edge with probability p independently of all the other edges. Let $g(D; p)$ denote the probability that (D, p) is root-connected and let g_j be the number of subsets A of $E(D)$ with size j so that $D|A$ is root-connected. Notice that g_j is equal to the number of subsets A of E with $|A| = j$ and $\rho(A) = \rho(E)$. Then

$$g(D; p) = \sum_{j=0}^{|E(D)|} g_j p^{|E(D)|-j}(1-p)^j.$$

Provan and Ball [35] showed that the following problem is #P-hard for each rational p with $0 < p < 1$, and computable in polynomial time when $p = 0$ or $p = 1$.

#Connectedness Reliability
Input: A digraph D.
Output: $g(D; p)$.

It is straightforward to restrict the input to digraphs in $\mathcal{D}$.

Corollary 36. *The following problem is #P-hard for each rational p with $0 < p < 1$.*

#Bipartite Connectedness Reliability
Input: $D \in \mathcal{D}$.
Output: $g(D; p)$.

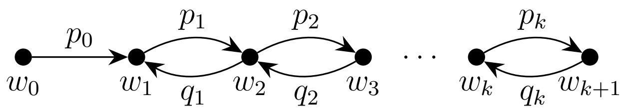

Figure 2: A tailed k-digon.

Proof. We reduce #CONNECTEDNESS RELIABILITY to #BIPARTITE CONNECTEDNESS RELIABILITY. Let D be an input to #CONNECTEDNESS RELIABILITY. If D is not root-connected then just return 0. Otherwise, form D' from D by replacing each directed edge uv by a path of length two directed from u to v. Then D' is bipartite and can be constructed from D in polynomial time. For each directed edge uv of D, the new vertex w between u and v has in-degree one, so the directed edge uw must be present in any root-connected subgraph of D'. Moreover, there is a bijection between root-connected subgraphs of D and root-connected subgraphs of D' that, for each edge uv of D, maps subgraphs of D containing uv to subgraphs of D' containing both edges of the path in D' replacing uv. Thus $g(D';p) = \frac{1}{p^{|E(D)|}} g(D;p)$, so $g(D;p)$ may be computed from $g(D';p)$ in polynomial time, as required. □

We now use this result to classify the complexity of points along the line $x = 1$.

Proposition 37. *The computational problem $\pi_3[\mathcal{D}, 1, b]$ is #P-hard for $b > 1$.*

Proof. Let D be a root-connected, bipartite digraph with edge set E and rank function ρ. Then for $0 < p < 1$ we have

$$g(D;p) = \sum_{\substack{A \subseteq E(D):\\ \rho(A)=\rho(D)}} p^{|E(D)|-|A|}(1-p)^{|A|} = p^{|E(D)|-\rho(D)}(1-p)^{\rho(D)} \sum_{\substack{A \subseteq E(D):\\ \rho(A)=\rho(D)}} \left(\frac{1-p}{p}\right)^{|A|-\rho(A)}$$
$$= p^{|E(D)|-\rho(D)}(1-p)^{\rho(D)} T\left(D; 1, \frac{1}{p}\right).$$

Evaluating $g(D;p)$ is therefore Turing-reducible to evaluating $T(D; 1, \frac{1}{p})$ for $0 < p < 1$. Therefore, $\pi_3[\mathcal{D}, 1, b]$ is #P-hard for $b > 1$. □

In order to determine the complexity of the point $\pi_3[\mathcal{D}, 1, -1]$, we introduce a new operation on root-connected digraphs which we call the *k-digon-stretch*. We define a *tailed k-digon* from u to v to be the digraph defined as follows. The vertex set is $\{w_0 = u, w_1, \ldots, w_k, w_{k+1} = v\}$. There is an edge w_0w_1 and a directed cycle of length 2 on w_i and w_{i+1} for each i with $1 \leqslant i \leqslant k$. An example of a tailed k-digon is shown in Figure 2. (The labelling of the edges will be needed later.) For a root-connected digraph D, the k-digon-stretch of D is constructed by replacing every directed edge uv in D by a tailed k-digon from u to v. We denote the k-digon-stretch of D by D_k.

Theorem 38. *Let D be a root-connected digraph. Then*

$$T(D_k; 1, y) = (k+1)^{|E(D)|-\rho(D)} y^{k|E(D)|} T\left(D; 1, \frac{k+y}{k+1}\right).$$

Proof. Let S be a subset of edges of a tailed k-digon from u to v. If S contains all the edges on the unique directed uv-path through the k-tailed digon, then S is said to *admit a uv-dipath*. Let A be a subset of $E(D_k)$ and $P(A)$ be the set of edges uv in D for which A admits a uv-dipath.

We have $\rho(A) = \rho(D_k)$ if and only if

(i) for each directed edge uv of D and each vertex w of the corresponding tailed k-digon from u to v in D_k, A includes the edges of a path in the k-tailed digon from either u or v to w, and

(ii) $\rho(P(A)) = \rho(D)$.

Note that $\rho(D_k) = k|E(D)| + \rho(D)$. We can write A as the disjoint union $A = \bigcup_{e \in E(D)} A_e$ where A_e is the set of edges of A belonging to the tailed k-digon corresponding to e. The Tutte polynomial of D_k along the line $x = 1$ is given by

$$T(D_k; 1, y) = \sum_{\substack{A \subseteq E(D_k): \\ \rho(A) = \rho(D_k)}} (y-1)^{|A| - \rho(D_k)} = \sum_{\substack{B \subseteq E(D): \\ \rho(B) = \rho(D)}} \sum_{\substack{A \subseteq E(D_k): \\ \rho(A) = \rho(D_k) \\ P(A) = B}} (y-1)^{|A| - \rho(D_k)}$$

$$= \sum_{\substack{B \subseteq E(D): \\ \rho(B) = \rho(D)}} \sum_{\substack{A \subseteq E(D_k): \\ \rho(A) = \rho(D_k), \\ P(A) = B}} \underbrace{\left(\prod_{\substack{e \in E(D): \\ e \notin P(A)}} (y-1)^{|A_e| - k} \right)}_{(1)} \underbrace{\left(\prod_{\substack{e \in E(D): \\ e \in P(A)}} (y-1)^{|A_e| - (k+1)} \right)}_{(2)} (y-1)^{|P(A)| - \rho(D)}. \tag{5}$$

Consider a tailed k-digon from u to v with vertex set labelled as described just before the statement of the theorem. For $0 \leqslant i \leqslant k$, let p_i denote the edge $w_i w_{i+1}$; for $1 \leqslant i \leqslant k$, let q_i denote the edge $w_{i+1} w_i$.

In the first product above we are considering edges $e = uv$ for which $e \notin P(A)$. Thus A_e does not contain all of $p_0, \ldots, p_k$. Let j be the smallest integer such that $p_j \notin A_e$. As we are only interested in sets A with $\rho(A) = \rho(D_k)$, each of $q_{j+1}, \ldots, q_k$ belongs to A_e. Thus $|A_e| \geqslant k$. Moreover each of $p_{j+1}, \ldots, p_k$ and $q_1, \ldots, q_j$ may or may not belong to A_e. As there are $k+1$ possibilities for j, summing

$$\prod_{\substack{e \in E(D): \\ e \notin P(A)}} (y-1)^{|A_e| - k}$$

over all possible choices of A_e for $e \notin P(A)$ gives $\big((k+1)y^k\big)^{|E(D)| - |P(A)|}$.

In the second product above we are considering edges $e = uv$ for which $e \in P(A)$. Thus A_e contains all of $p_0, \ldots, p_k$. So $|A_e| \geqslant k+1$. Moreover each of $q_1, \ldots, q_k$ may or may not belong to A_e. Summing

$$\prod_{\substack{e \in E(D): \\ e \in P(A)}} (y-1)^{|A_e| - (k+1)}$$

over all possible choices of A_e for $e \in P(A)$ gives $y^{k|P(A)|}$.

Thus the right side of Equation 5 becomes

$$\begin{aligned}
&\sum_{\substack{B\subseteq E(D):\\ \rho(B)=\rho(D)}} y^{k|B|}\left((k+1)y^k\right)^{|E(D)|-|B|}(y-1)^{|B|-\rho(D)}\\
&\quad = y^{k|E(D)|}\sum_{\substack{B\subseteq E(D):\\ \rho(B)=\rho(D)}}(k+1)^{\rho(B)-|B|+|E(D)|-\rho(D)}(y-1)^{|B|-\rho(B)}\\
&\quad = y^{k|E(D)|}(k+1)^{|E(D)|-\rho(D)}T\left(D;1,\frac{y+k}{k+1}\right).
\end{aligned}$$

□

We now complete the classification of complexity for points on the line H_0^x.

Proposition 39. *The problem $\pi_3[\mathcal{D},1,b]$ is #P-hard for $b \notin \{0,1\}$.*

Proof. For $b \notin \{-1,0,1\}$ the result follows immediately from Propositions 34 and 37. If D is root-connected and bipartite, then D_3 is root-connected and bipartite and by Theorem 38,

$$T(D_3;1,-1) = (-1)^{|E(D)|}4^{|E(D)|-\rho(D)}T\left(D;1,\frac{1}{2}\right).$$

As D_3 can be constructed from D in polynomial time, $\pi_3(\mathcal{D},1,\frac{1}{2}) \propto \pi_3(\mathcal{D},1,-1)$, so $\pi_3(\mathcal{D},1,-1)$ is #P-hard. □

We now show that evaluating the Tutte polynomial of a root-connected, bipartite digraph at most points on the hyperbola H_α for $\alpha \neq 0$ is at least as hard as evaluating it at the point $(1+\alpha,2)$.

Proposition 40. *Let α be in $\mathbb{Q}-\{0\}$ and (a,b) be a point on H_α with $b \notin \{-1,0\}$, then*

$$\pi_3[\mathcal{D},1+\alpha,2] \propto_T \pi_3[\mathcal{D},a,b].$$

Proof. For α in $\mathbb{Q}-\{0\}$, the hyperbola H_α crosses the line $y=2$ at the point $(1+\alpha,2)$. By Proposition 34, we know that for any point (a,b) on H_α with $b \notin \{-1,0\}$ we have $\pi_3[\mathcal{D},1+\alpha,2] \propto_T \pi_2[\mathcal{D},H_\alpha] \propto_T \pi_3[\mathcal{D},a,b]$. □

We will now show that evaluating the Tutte polynomial of a root-connected, bipartite digraph at most of the points on the line $y=2$ is #P-hard. This will enable us to classify the complexity of most points lying on the hyperbola H_α for all $\alpha \in \mathbb{Q}-\{0\}$.

Proposition 41. *The problem $\pi_3[\mathcal{D},a,2]$ is #P-hard for $a \neq 2$.*

Proof. We begin by proving that when L is the line $y = 2$ we have

$$\pi_2[\mathcal{D}, L] \propto_T \pi_3[\mathcal{D}, a, 2]$$

for $a \notin \{1, 2\}$. Let D be a root-connected, bipartite digraph and let $z = x - 1$. Along L the Tutte polynomial of D has the form

$$T(D; x, 2) = \sum_{A \subseteq E(D)} z^{\rho(D)-\rho(A)} = \sum_{i=0}^{\rho(D)} t_i z^i$$

for some $t_0, t_1, \ldots, t_{\rho(D)}$. We will now show that for most values of a, we may determine all of the coefficients t_i in polynomial time from $T(D \sim \overrightarrow{S_k}; a, 2)$ for $k = 0, 1, \ldots, \rho(D)$. For each such k, $D \sim \overrightarrow{S_k}$ is root-connected and bipartite, and can be constructed in polynomial time. By Theorem 20, we have

$$T(D \sim \overrightarrow{S_k}; a, 2) = a^{k\rho(D)} T\left(D; \frac{2^k(a-1)^{k+1}}{a^k} + 1, 2\right).$$

Therefore we may compute $T\left(D; \frac{2^k(a-1)^{k+1}}{a^k} + 1, 2\right)$ from $T(D \sim \overrightarrow{S_k}; a, 2)$ when $a \neq 0$. For $a \notin \{0, \frac{2}{3}, 1, 2\}$ the values of $\left(\frac{2^k(a-1)^{k+1}}{a^k} + 1, 2\right)$ are pairwise distinct for $k = 0, 1, \ldots, \rho(D)$. Therefore by evaluating $T(D \sim \overrightarrow{S_k}; a, 2)$ for $k = 0, 1, \ldots, \rho(D)$ where $a \notin \{0, \frac{2}{3}, 1, 2\}$, we obtain $\sum_{i=0}^{\rho(D)} t_i z^i$ for $\rho(D) + 1$ distinct values of z. This gives us $\rho(D) + 1$ linear equations for the coefficients t_i, and so by Lemma 14, they may be recovered in polynomial time. Hence evaluating the Tutte polynomial of a root-connected, bipartite digraph along the line $y = 2$ is Turing-reducible to evaluating it at the point $(a, 2)$ for $a \notin \{0, \frac{2}{3}, 1, 2\}$.

We now consider the cases where $a = 0$ or $a = \frac{2}{3}$. The digraph $D \sim \overrightarrow{P_2}$ is root-connected and bipartite, and may be constructed in polynomial time. By Theorem 20, we have

$$T(D \sim \overrightarrow{P_2}; 0, 2) = 2^{\rho(D)} T\left(D; \frac{(-1)^3 2^2}{2} + 1, 2\right) = 2^{\rho(D)} T(D; -1, 2).$$

Therefore $\pi_3[\mathcal{D}, -1, 2] \propto_T \pi_3[\mathcal{D}, 0, 2]$. Similarly we have

$$T\left(D \sim \overrightarrow{P_2}; \frac{2}{3}, 2\right) = \left(\frac{8}{9}\right)^{\rho(D)} T\left(D; \frac{(-\frac{1}{3})^3 2^2}{\frac{8}{9}} + 1, 2\right) = \left(\frac{8}{9}\right)^{\rho(D)} T\left(D; \frac{5}{6}, 2\right).$$

Therefore $\pi_3[\mathcal{D}, 5/6, 2] \propto_T \pi_3[\mathcal{D}, 2/3, 2]$. Putting all this together we get $\pi_2[\mathcal{D}, L] \propto_T \pi_3[\mathcal{D}, a, 2]$ for all a in $\mathbb{Q} - \{1, 2\}$. Consequently $\pi_3[\mathcal{D}, 1, 2] \propto_T \pi_3[\mathcal{D}, a, 2]$, for all a in $\mathbb{Q} - \{2\}$.

By Proposition 39, we know that $\pi_3[\mathcal{D}, 1, 2]$ is #P-hard. This completes the proof. □

Theorem 42. *Let α be in $\mathbb{Q} - \{0, 1\}$ and (a, b) be a point on H_α with $b \neq 0$. Then $\pi_3[\mathcal{D}, a, b]$ is #P-hard.*

Proof. Suppose first that $b \neq -1$. By Proposition 40, $\pi_3[\mathcal{D}, 1+\alpha, 2] \propto_T \pi_3[\mathcal{D}, a, b]$. As $\alpha \neq 1$, Proposition 41, implies $\pi_3[\mathcal{D}, a, b]$ is #P-hard.

Now suppose that $b = -1$. As $(a, b) \notin H_1$, we have $a \neq \frac{1}{2}$. So by Proposition 35, $\pi_3[\mathcal{D}, 1, -1] \propto_T \pi_3[\mathcal{D}, a, -1]$. By Proposition 39, $\pi_3[\mathcal{D}, 1, -1]$ is #P-hard. Therefore $\pi_3[\mathcal{D}, a, -1]$ is #P-hard. □

The only remaining points we need to classify are those lying on the line $y = 1$. To do this we prove that the problem of evaluating the Tutte polynomial of a root-connected, bipartite digraph at most fixed points along this line is at least as hard as the analogous problem for rooted graphs.

Theorem 43. *The problem $\pi_3[\mathcal{D}, a, 1]$ is #P-hard for a in $\mathbb{Q} - \{1\}$.*

Proof. Let G be a connected, rooted, bipartite graph with root r. Construct a rooted digraph D with root r by replacing every edge of G by a pair of oppositely directed edges. Then D is root-connected and bipartite, and can be constructed from G in polynomial time. We can define a natural map $f : 2^{E(D)} \to 2^{E(G)}$ so that $f(A)$ is the set of edges of G for which at least one corresponding directed edge is included in A.

If $\rho_G(A) = |A|$ then the root component of $G|A$ is a tree and includes all the edges of A. Similarly if $\rho_D(A') = |A'|$ then the root component of $D|A'$ is an arborescence rooted at r and includes all the edges of A'. For every subset A of E with $\rho_G(A) = |A|$, there is precisely one choice of A' with $\rho_D(A') = |A'|$ and $f(A') = A$, obtained by directing all the edges of A away from r. Thus there is a one-to-one correspondence between subsets A of E with $\rho_G(A) = |A|$ and subsets A' of $E(D)$ with $\rho_D(A') = |A'|$, and this correspondence preserves the sizes of the sets. Therefore we have

$$T(D; x, 1) = \sum_{\substack{A' \subseteq E(D): \\ |A'| = \rho_D(A')}} (x-1)^{\rho(D) - |A'|} = \sum_{\substack{A \subseteq E: \\ |A| = \rho_G(A)}} (x-1)^{\rho(G) - |A|} = T(G; x, 1).$$

So $\pi_3[\mathcal{G}, a, 1] \propto_T \pi_3[\mathcal{D}, a, 1]$. So by Proposition 33, we deduce that $\pi_3[\mathcal{D}, a, 1]$ is #P-hard for $a \neq 1$. □

7 Binary Greedoids

In our final section we let $\mathbb{G}$ be the class of binary greedoids. We present analogous results to those in the previous section by finding the computational complexity of evaluating the Tutte polynomial of a binary greedoid at a fixed rational point, eventually proving Theorem 6. As before, it is convenient to think of the input as being a binary matrix rather than its binary greedoid.

We begin by examining the easy points of Theorem 6. Let Γ be a binary greedoid with element set E and rank function ρ. If a point (a, b) lies on the hyperbola H_1 then, following the remarks at the end of Section 3 $T(\Gamma; a, b)$ is easily computed.

We now focus on the hard points. The k-thickening operation will again be crucial. Given a binary matrix M, the k-thickening M^k of M is obtained by replacing each column of M by k copies of the column. We have $\Gamma(M^k) = (\Gamma(M))^k$, so Theorem 17 can be applied to compute the $T(M^k)$ in terms of $T(M)$. Let I_k denote the $k \times k$ identity matrix. Then $\Gamma(I_k) \cong \Gamma(P_k)$, so $T(I_k) = T(P_k) = 1 + \sum_{j=1}^{k} (x-1)^j y^{j-1}$.

The next proposition is analogous to that of Proposition 27. We omit its proof because proof of Proposition 27 and those of Propositions 24–26 on which it depends rely only on the thickening operation which behaves uniformly in the rooted graph and binary cases.

Proposition 44. *Let L be either H_0^x, H_0^y, or H_α for $\alpha \in \mathbb{Q} - \{0\}$. Let (a, b) be a point on L such that $(a, b) \neq (1, 1)$ and $b \notin \{-1, 0\}$. Then*

$$\pi_2[\mathcal{B}, L] \propto_T \pi_3[\mathcal{B}, a, b].$$

A binary matroid is a matroid that can be represented over the finite field $\mathbb{Z}_2$. Every graphic matroid is also binary, so Theorem 3 and Lemma 12 imply that $\pi_2[\mathcal{B}, 1, b]$ is #P-hard providing $b \neq 1$. This immediately gives the following.

Proposition 45. *The problem $\pi_3[\mathcal{B}, 1, b]$ is #P-hard for all b in $\mathbb{Q} - \{1\}$.*

The following result has been announced by Vertigan in [9] and slightly later in [42], but up until now no written proof has been published. For completeness, we provide a proof in Appendix A.

Theorem 46 (Vertigan)**.** *Evaluating the Tutte polynomial of a binary matroid is #P-hard at the point $(1, 1)$.*

Using this result, we are able to fill in the missing point $(1, 1)$ from the previous result and also establish hardness along the line $y = 1$.

Proposition 47. *The problem $\pi_3[\mathcal{B}, a, 1]$ is #P-hard for all a.*

Proof. By Proposition 44 we have $\pi_2[\mathcal{B}, H_0^y] \propto_T \pi_3[\mathcal{B}, a, 1]$ for $a \neq 1$. The result now follows from Theorem 46. □

Proposition 48. *Let Γ be a binary greedoid and let $\Gamma' = \Gamma(I_k)$. Then*

$$T(\Gamma \approx \Gamma'; x, y) = T(\Gamma; x, y)(x-1)^k y^k + T(\Gamma; 1, y)\Big(1 + \sum_{j=1}^{k} (x-1)^j y^{j-1} - (x-1)^k y^k\Big).$$

Proof. The proof follows immediately from Theorem 21. □

We now classify the complexity of $\pi_3[\mathcal{B}, a, b]$ when $b = 0$ or $b = -1$.

Proposition 49. *The problem $\pi_3[\mathcal{B}, a, 0]$ is #P-hard for all $a \neq 0$.*

Proof. Let M be a binary matrix with linearly independent rows. Then from Proposition 48, we have $T(M \approx I_1; a, 0) = aT(M; 1, 0)$. Therefore when $a \neq 0$ we have $\pi_3[\mathcal{B}, 1, 0] \propto_T \pi_3[\mathcal{B}, a, 0]$. The result now follows from Proposition 45. □

Proposition 50. *The problem $\pi_3[\mathcal{B}, a, -1]$ is #P-hard for all $a \neq \frac{1}{2}$.*

Proof. Let M be a binary matrix with linearly independent rows. We have

$$(2a-1)T(M;1,-1) = T(M \approx I_1; a, -1) + (a-1)T(M; a, -1).$$

Thus, $\pi_3[\mathcal{B}, 1, -1] \propto_T \pi_3[\mathcal{B}, a, -1]$. By using Proposition 45, we deduce that $\pi_0[\mathcal{B}, a, -1]$ is #P-hard. □

Our final result, together with Propositions 45, 47 and 50, completes the proof of Theorem 6.

Theorem 51. *Let (a, b) be a point in H_α for $\alpha \in \mathbb{Q} - \{0, 1\}$ with $b \neq -1$. Then $\pi_3[\mathcal{B}, a, b]$ is #P-hard.*

Proof. For $\alpha \in \mathbb{Q} - \{0, 1\}$, the hyperbola H_α crosses the x-axis at the point $(1 - \alpha, 0)$. By Proposition 44 since $b \neq -1$ and $(a, b) \neq (1, 1)$ we have $\pi_3[\mathcal{B}, 1 - \alpha, 0] \propto_T \pi_3[\mathcal{B}, a, b]$. The result now follows from Proposition 49. □

A Counting bases in a represented matroid

In this appendix, we present a proof that counting the number of bases of a represented matroid is #P-complete. More precisely, we consider the following family of counting problems. Let $\mathbb{F}$ be a field.

Counting Bases of $\mathbb{F}$-Represented Matroids
Input: A $(0, 1)$-matrix A.
Output: The number of bases of $M(A)$, the matroid represented by A over the field $\mathbb{F}$.

Theorem 52. *For every field $\mathbb{F}$,* Counting Bases of $\mathbb{F}$-Represented Matroids *is #P-complete.*

A proof of this result was announced nearly 30 years ago by Dirk Vertigan — it first seems to have been referred to in [9] and slightly later in [42], where it is described as an unpublished manuscript — but no written proof has been circulated. Sketches of the proof have been presented by Vertigan in talks, for example, at the Conference for James Oxley in 2019 [40]. The second author was present at this meeting and the material in this section has been produced from his incomplete recollection of the talk. All the key ideas are due to Vertigan but the details including any errors, omissions or unnecessary complications are due to the authors. As pointed out to us by Dillon Mayhew [31], Vertigan's proof presented in [40] introduced an intermediate step involving weighted bases; our proof does not require this intermediate step but this comes at the cost of introducing a larger matrix in the reduction. We provide the proof, partly as a service to the community because we know of several colleagues who have tried to recreate it and partly because a referee has pointed out the undesirability of relying on an unpublished result. Although our original aim was only to establish the special case of Theorem 52 relevant for our work, it turns out that little extra effort is required to prove Theorem 52 in full generality.

We require very little matroid theory other than basic notions such as rank, circuits and the closure operator. As we work exclusively with matroids having representations drawn from a specific family of matrices considered over different fields, the claims we make about the associated matroids can easily be checked by considering the representing matrices. For background on matroids see [33].

To prove hardness, we give a reduction from counting perfect matchings in a graph, a problem which is well-known to be #P-complete [39]. Clearly, it makes no difference to the complexity of counting perfect matchings if we restrict ourselves to loopless graphs having an even number of vertices and no isolated vertices. Given such a graph G with n vertices, we construct a family of matrices $\{A_k : 1 \leqslant k \leqslant n/2+1\}$ with entries in $\{0,1\}$. By considering these matrices as being defined over different fields, we obtain two corresponding families of matroids. Which family arises depends on whether the field has characteristic two. Thus the proof of Theorem 52 splits into two parts depending on whether the characteristic of the underlying field is two.

We shall generally think of matrices as coming with sets indexing their rows and columns. If A is a matrix with sets X and Y indexing its rows and columns respectively, then we say that A is an $X \times Y$ matrix. For non-empty subsets X' and Y' of X and Y, respectively, $A[X',Y']$ is the submatrix of A obtained by deleting the rows indexed by elements of $X - X'$ and the columns indexed by elements of $Y - Y'$.

Throughout this section we shall take G to be a graph with an even number of vertices, having no isolated vertices. We suppose that the vertex set of G is $V = \{v_1, \dots, v_n\}$ and its edge set is $E = \{e_1, \dots, e_m\}$. Let k be a strictly positive integer and let

$$X = \{v_1, \dots, v_n\} \cup \{f_{i,j} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant k\}$$

and

$$Y = \{v_1, \dots, v_n, e_1, \dots, e_m\} \cup \{w_{i,j}, x_{i,j}, y_{i,j}, z_{i,j} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant k\}.$$

Here X includes all the vertices of G, Y includes all the vertices and edges of G, and both include several new elements. The matrix A_k is an $X \times Y$ matrix. To specify its entries suppose that e_i has endvertices v_a and v_b with $a < b$. Then for each j with $1 \leqslant j \leqslant k$, taking $X' = \{v_a, v_b, f_{i,j}\}$ and $Y' = \{v_a, v_b, e_i, w_{i,j}, x_{i,j}, y_{i,j}, z_{i,j}\}$, we let

$$A_k[X',Y'] = \begin{array}{c} \\ v_a \\ v_b \\ f_{i,j} \end{array}\begin{array}{c} \begin{array}{ccccccc} v_a & v_b & e_i & w_{i,j} & x_{i,j} & y_{i,j} & z_{i,j} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

We complete the definition of A_k by setting every as yet unspecified entry to zero. Overall, A_k has the following form.

$$A_k = \begin{array}{c} \\ V \\ f_{i,j} \end{array}\begin{array}{c} \begin{array}{cccccc} V & E & w_{i,j} & x_{i,j} & y_{i,j} & z_{i,j} \end{array} \\ \begin{bmatrix} I_n & M(G) & 0 & X & Y & kM(G) \\ 0 & 0 & I_{km} & I_{km} & I_{km} & I_{km} \end{bmatrix} \end{array}.$$

Here the columns of A_k have been ordered so that those labelled by elements of V come before those labelled by elements of E which in turn come before all the $k|E|$ columns labelled $w_{i,j}$ followed by all the $k|E|$ columns labelled $x_{i,j}$, then the $y_{i,j}$ columns and the $z_{i,j}$ columns. Within blocks of columns, we place the column labelled $w_{i,j}$ before the column labelled $w_{i',j'}$ if $i < i'$, or $i = i'$ and $j < j'$, and similarly for the columns labelled $x_{i,j}$, $y_{i,j}$ or $z_{i,j}$, and the rows labelled $f_{i,j}$. With this convention, many of the blocks of the matrix become identity matrices of appropriate sizes. The block $A_k[V, E]$ becomes the vertex–edge incidence matrix of G with the non-zero entries in each column indicating the endvertices of the corresponding edge. The block $A_k[V, \{z_{i,j} : 1 \leqslant i \leqslant |E|, 1 \leqslant j \leqslant k\}]$ is obtained from $A_k[V, E]$ by repeating each column k times, so that repeated columns are ordered consecutively. Finally the matrix labelled X has a single one in each column. If edge $e = v_a v_b$ with $a < b$, then the single one in each column $x_{e,j}$ occurs in row v_a. The matrix labelled Y is similar but the single one in each column $y_{e,j}$ occurs in row v_b.

Fix $\mathbb{F}$ and let $N_k = M(A_k)$, that is, the matroid with element set Y represented by A_k considered over $\mathbb{F}$. Taking Y' as in the previous paragraph, if $\mathbb{F}$ has characteristic two, then $N_k|Y'$ is isomorphic to the Fano matroid F_7 and otherwise $N_k|Y'$ is isomorphic to the non-Fano matroid F_7^- obtained from F_7 by relaxing the circuit-hyperplane $\{e_i, x_{i,j}, y_{i,j}\}$. Now let $M_k = N_k \setminus (V \cup E)$. Note that for each vertex v and edge e of G, N_k contains elements e and v, but M_k contains neither. Clearly $r(N_k) = |V| + |E|k$ and because G has no isolated vertices, $r(M_k) = r(N_k)$.

We shall show that for each k, every basis of M_k corresponds to what we call a *feasible template* of G, that is, a subgraph of G in which some edges are directed (possibly in both directions) and some are labelled, satisfying certain properties which we describe below. In particular, we will see that the bidirected edges in a feasible template form a matching in G. Furthermore, the number of bases of M_k corresponding to each feasible template depends only on k and the numbers of edges directed and labelled in each possible way, and is easily computed. By varying k and counting the number of bases of M_k, we can recover the number of feasible templates with each possible number of bidirected edges. The number of feasible templates with $n/2$ bidirected edges is equal to the number of perfect matchings of G.

A *template* of G is a spanning subgraph of G in which edges may be bidirected, that is, two arrows are affixed one pointing to each endvertex, (uni)directed or undirected, and are labelled according to the following rules.

- Every bidirected edge is unlabelled.
- A (uni)directed edge $e = v_a v_b$ with $a < b$ is labelled either wx or yz if e is directed towards a and is labelled either wy or xz if e is directed towards b.
- An undirected edge is labelled either wz or xy.

Even though the matroid M_k itself depends on whether $\mathbb{F}$ has characteristic two, the proofs of the two cases have a great deal in common. To prevent repetition we describe the common material here, before finishing the two cases separately. For $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$, let $F_{i,j} = \{w_{i,j}, x_{i,j}, y_{i,j}, z_{i,j}\}$ and for $1 \leqslant i \leqslant m$, let $F_i = \bigcup_{1 \leqslant j \leqslant k} F_{i,j}$. For all

i and j, the set $F_{i,j}$ is a circuit and $r(M_k \setminus F_{i,j}) < r(M_k)$. Let B be a basis of M_k. Then $1 \leqslant |B \cap F_{i,j}| \leqslant 3$. Moreover, for all i, $r(F_i) = k+2$ and $r(M_k \setminus F_i) \leqslant r(M_k) - k$, so $k \leqslant |B \cap F_i| \leqslant k+2$. By combining these observations, we get the following.

1. If $|B \cap F_i| = k$, then $|B \cap F_{i,j}| = 1$ for $1 \leqslant j \leqslant k$.

2. If $|B \cap F_i| = k+1$, then $|B \cap F_{i,j}| = 2$ for some j with $1 \leqslant j \leqslant k$ and $|B \cap F_{i,j}| = 1$, otherwise.

3. If $|B \cap F_i| = k+2$, then either $|B \cap F_{i,j}| = 3$ for some j with $1 \leqslant j \leqslant k$ and $|B \cap F_{i,j}| = 1$, otherwise, or $|B \cap F_{i,j}| = |B \cap F_{i,j'}| = 2$ for distinct j, j' with $1 \leqslant j, j' \leqslant k$ and $|B \cap F_{i,j}| = 1$, otherwise.

The main idea in the proof is to use templates to classify each basis B of M_k according to $|B \cap F_i|$ for $i = 1, \ldots, m$ and additionally when $|B \cap F_i| = k+1$ according to $B \cap F_{i,j}$ for the unique value j with $|B \cap F_{i,j}| = 2$.

Suppose edge e_i joins vertices v_a and v_b in G and $a < b$. If $|B \cap F_i| = k$, then $\mathrm{cl}_{N_k}(B \cap F_i) - E(M_k) = \emptyset$. If $|B \cap F_i| = k+1$, then $|B \cap F_{i,j}| = 2$ for precisely one value j^* of j and $\mathrm{cl}_{N_k}(B \cap F_i) - E(M_k)$ depends on $B \cap F_{i,j^*}$.

- If $B \cap F_{i,j^*}$ is $\{w, x\}$ or $\{y, z\}$, then $\mathrm{cl}_{N_k}(B \cap F_i) - E(M_k) = \{v_a\}$.
- If $B \cap F_{i,j^*}$ is $\{w, y\}$ or $\{x, z\}$, then $\mathrm{cl}_{N_k}(B \cap F_i) - E(M_k) = \{v_b\}$.
- If $B \cap F_{i,j^*}$ is $\{w, z\}$, then $\mathrm{cl}_{N_k}(B \cap F_i) - E(M_k) = \{e_i\}$.
- If $B \cap F_{i,j^*}$ is $\{x, y\}$, then $\mathrm{cl}_{N_k}(B \cap F_i) - E(M_k)$ is $\{e_i\}$ when $\mathbb{F}$ has characteristic two and is empty otherwise.

If $|B \cap F_{i,j}| = 3$, then clearly $\{v_a, v_b, e_i\} \subseteq \mathrm{cl}_{N_k}(B \cap F_{i,j})$. If $|B \cap F_{i,j}| = 2$, then $\mathrm{cl}_{N_k}(B \cap F_{i,j})$ includes precisely one element of $\{e_i, v_a, v_b\}$ except when $B \cap F_{i,j} = \{x, y\}$ and $\mathbb{F}$ has characteristic two. But in any case, if for distinct j, j' we have $|B \cap F_{i,j}| = |B \cap F_{i,j'}| = 2$, then as B is independent, $\mathrm{cl}_{N_k}(B \cap F_{i,j}) \cap \{v_a, v_b, e_i\} \neq \mathrm{cl}_{N_k}(B \cap F_{i,j'}) \cap \{v_a, v_b, e_i\}$. It is now straightforward to check that $\{v_a, v_b, e_i\} \subseteq \mathrm{cl}_{N_k}(B \cap (F_{i,j} \cup F_{i,j'}))$. Summing up, if $|B \cap F_i| = k+2$, then $\mathrm{cl}_{N_k}(B \cap F_i) - E(M_k) = \{v_a, v_b, e_i\}$.

To each subset S of $E(M_k)$, such that for all i, $S \cap F_i$ is independent and for all i and j, $|S \cap F_{i,j}| \geqslant 1$, we associate a template $T(S)$ of G, by starting with an edgeless graph with vertex set $V(G)$ and adding each edge e_i of G such that $|S \cap F_{i,j}| > 1$ for some j, possibly directing or bidirecting it as we now describe. Suppose that $e_i = v_a v_b$ with $a < b$.

- If $\mathrm{cl}_{N_k}(S \cap F_i) - E(M_k) = \{v_a, v_b, e_i\}$, then bidirect e_i.
- If $\mathrm{cl}_{N_k}(S \cap F_i) - E(M_k) = \{v_a\}$, then direct e_i from v_b to v_a.
- If $\mathrm{cl}_{N_k}(S \cap F_i) - E(M_k) = \{v_b\}$, then direct e_i from v_a to v_b.
- If $\mathrm{cl}_{N_k}(S \cap F_i) - E(M_k) \subseteq \{e_i\}$, then do not direct e_i.

In the last three cases above, we also label e_i. To do this let j^* be the unique value of j such that $|S \cap F_{i,j}| = 2$. Then label e_i with the elements of $S \cap F_{i,j^*}$, but with their subscripts omitted. In this way the edge e_i is given two labels from the set $\{w, x, y, z\}$.

A.1 $\mathbb{F}$ has characteristic two

We now focus on the case when $\mathbb{F}$ has characteristic two. The following result is the key step in the proof.

Proposition 53. *A subset B of $E(M_k)$ is a basis of M_k if and only if all of the following conditions hold.*

1. *For all i, $B \cap F_i$ is independent.*
2. *For all i and j, $|B \cap F_{i,j}| \geqslant 1$.*
3. *The subgraph of $T(B)$ induced by its undirected edges is acyclic.*
4. *It is possible to direct the undirected edges of $T(B)$ so that every vertex has indegree one.*

Proof. We first show that the conditions are collectively sufficient. Suppose that B satisfies each of the conditions and that $T(B)$ has b bidirected edges, r (uni)directed edges and u undirected edges. Then the last condition implies that $2b + r + u = n$. Combining this with the first two conditions gives $|B| = km + 2b + r + u = km + n = r(M_k)$. So, it is sufficient to prove that $r(B) = r(M_k)$. We will show that the last two conditions imply that $v_i \in \mathrm{cl}_{N_k}(B)$ for $i = 1, \dots, n$. Then the second condition ensures that $\mathrm{cl}_{N_k}(B) = E(N_k)$ and consequently $r(B) = r(N_k) = r(M_k)$ as required.

Consider a vertex v of G. The last two conditions imply that there is a (possibly empty) path P in $T(B)$ between a vertex v' having indegree one and v, comprising only undirected edges. Suppose that the vertices of P in order are $v_{j_1} = v', v_{j_2}, \dots, v_{j_l} = v$ and that for $1 \leqslant h \leqslant l-1$, the edge joining v_{j_h} and $v_{j_{h+1}}$ in P is e_{i_h}. Then $\{v_{j_1}, e_{i_1}, \dots, e_{i_{l-1}}\} \subseteq \mathrm{cl}_{N_k}(B)$. As $v_{j_h} \in \mathrm{cl}_{N_k}(\{v_{j_{h-1}}, e_{i_{h-1}}\})$ for $h = 2, \dots, l$, we see that $v = v_{j_l} \in \mathrm{cl}_{N_k}(B)$, as required. Thus the conditions are sufficient.

To show that each condition is necessary we suppose that B is a basis of M_k. Clearly the first condition is necessary. We observed earlier that for all i and j, $r(E(M_k) - F_{i,j}) < r(M_k)$, so the second condition is also necessary. Suppose, without loss of generality, that edges $e_1, \dots, e_l$ are undirected and form a cycle in $T(B)$. Then the corresponding elements $e_1, \dots, e_l$ form a circuit in N_k. Because each of $e_1, \dots, e_l$ is undirected, $e_i \in \mathrm{cl}_{N_k}(B \cap F_i)$ for $i = 1, \dots, l$. Thus

$$e_l \in \mathrm{cl}_{N_k}(\{e_1, \dots, e_{l-1}\}) \subseteq \mathrm{cl}_{N_k}\left(\bigcup_{i=1}^{l-1}(B \cap F_i)\right).$$

So there is a circuit of N_k contained in $\{e_l\} \cup (B \cap F_l)$ and another contained in $\{e_l\} \cup \bigcup_{i=1}^{l-1}(B \cap F_i)$. Hence there is a circuit of N_k and consequently of M_k contained in $\bigcup_{i=1}^{l}(B \cap F_i)$, contradicting the fact that B is a basis. Thus the third condition is necessary.

Finally, suppose that $T(B)$ has b bidirected edges, r (uni)directed edges and u undirected edges. Then, as $km + 2b + r + u = |B| = r(M_k) = km + n$, we have $2b + r + u = n$.

Observe that if the undirected edges are assigned a direction, then the sum of the indegrees of the vertices will become n. Suppose that it is impossible to direct the undirected edges of $T(B)$ so that each vertex has indegree one. Then, before directing the undirected edges, there must either be a vertex z with indegree at least two, or two distinct vertices x and y both having indegree at least one and joined by a path P of undirected edges.

In either case we aim to establish a contradiction by showing that there is some vertex v such that $B \cup \{v\}$ contains two distinct circuits in N_k. This would imply that B contains a circuit of N_k and consequently of M_k, giving the required contradiction. In the former case there are distinct edges e_i and e_j directed towards (and possibly away from as well) z in $T(B)$. So $z \in \mathrm{cl}_{N_k}(B \cap F_i) \cap \mathrm{cl}_{N_k}(B \cap F_j)$ implying that $(B \cap F_i) \cup \{z\}$ and $(B \cap F_j) \cup \{z\}$ both contain circuits of N_k including z. But then $(B \cap F_i) \cup (B \cap F_j) = B \cap (F_i \cap F_j)$ contains a circuit of N_k and consequently of M_k, contradicting the fact that B is a basis of M_k. So we may assume that the latter case holds. Suppose that, without loss of generality, the vertices of P in order are $v_1 = x, v_2, \ldots, v_l = y$. Suppose, again without loss of generality, that for $i = 2, \ldots, l$, the edge joining v_{i-1} and v_i in P is e_i, that e_1 is directed towards $x = v_1$ in $T(B)$ and e_{l+1} is directed towards $y = v_l$ in $T(B)$. Then $y \in \mathrm{cl}_{N_k}(B \cap F_{l+1})$ and $x \in \mathrm{cl}_{N_k}(B \cap F_1)$. Furthermore, for each $i = 2, \ldots, l$, $e_i \in \mathrm{cl}_{N_k}(B \cap F_i)$, so $v_i \in \mathrm{cl}_{N_k}\left(\bigcup_{j=1}^{i}(B \cap F_j)\right)$. In particular, $y \in \mathrm{cl}_{N_k}\left(\bigcup_{j=1}^{l}(B \cap F_j)\right)$. So, there is a circuit of N_k contained in $\{y\} \cup \{B \cap F_{l+1}\}$ and another contained in $\{y\} \cup \bigcup_{j=1}^{l}(B \cap F_j)$. Hence there is a circuit of N_k and consequently of M_k contained in $\bigcup_{j=1}^{l+1}(B \cap F_j)$, contradicting the fact that B is a basis. It follows that it possible to direct the undirected edges of each component of $T(B)$ so that every vertex has indegree one, establishing the necessity of the final condition. □

We say that a template T is *feasible* if it satisfies the last two conditions in the previous result, that is, if the subgraph induced by its undirected edges is acyclic and every vertex of the graph obtained from T by contracting the undirected edges has indegree equal to one.

Proposition 54. *Let G be a loopless graph without isolated vertices and let T be a feasible template of G with b bidirected edges. Then the number of bases of M_k with template T is*

$$4^{km}\left(\frac{k}{4}\right)^{n}\left(\frac{4}{k}+12\right)^{b}.$$

Proof. If follows from the definition of feasibility that if a feasible template contains b bidirected edges, then it has $n - 2b$ edges which are either (uni)directed or undirected. Furthermore G has $m - n + b$ edges which are not in T. Suppose that B is a basis with template T. We count the number of choices for B. Suppose that e_i is an edge of G which is not present in T. Then for $j = 1, \ldots, k$, we have $|F_{i,j} \cap B| = 1$, so there are 4^k choices for $B \cap F_i$. Now suppose that e_i is either (uni)directed or undirected in T. Then for all but one choice of j in $1, \ldots, k$, we have $|F_{i,j} \cap B| = 1$ and for the remaining possibility for j, $|F_{i,j} \cap B| = 2$, with the choice of elements of $F_{i,j}$ specified by the labelling of the edge e_i. Thus there are $k \cdot 4^{k-1}$ choices for $B \cap F_i$. Finally suppose that e_i is a bidirected edge. Then there are two subcases to consider. Either $|F_{i,j} \cap B| = 3$ for one value of j and $|F_{i,j} \cap B| = 1$ for all other values of j, or $|F_{i,j} \cap B| = 2$ for two values of j and $|F_{i,j} \cap B| = 1$ for all other values of j. In the former case, for each j, there are four choices of $F_{i,j}$ as there is no restriction on the choice of $F_{i,j}$ beyond its size. Now suppose that $|F_{i,j'} \cap B| = |F_{i,j''} \cap B| = 2$ for $j' \neq j''$. Then we also require that $\mathrm{cl}_{N_k}(B \cap F_{i,j'}) - F_i \neq \mathrm{cl}_{N_k}(B \cap F_{i,j''}) - F_i$. Thus there are $k \cdot 4^k + \binom{k}{2} \cdot 6 \cdot 4 \cdot 4^{k-2}$ choices for $B \cap F_i$.

So the number of bases of M_k with template T is

$$(4^k)^{m-n+b}(k \cdot 4^{k-1})^{n-2b}\left(k \cdot 4^k + \binom{k}{2} \cdot 6 \cdot 4 \cdot 4^{k-2}\right)^b = 4^{km}\left(\frac{k}{4}\right)^n\left(\frac{4}{k} + 12\right)^b.$$

□

Theorem 55. *If $\mathbb{F}$ is a field with characteristic two, then the problem* Counting Bases of $\mathbb{F}$-Represented Matroids *is #P-complete.*

Proof. It is clear that Counting Bases of $\mathbb{F}$-Represented Matroids belongs to #P. To prove hardness, we give a reduction from counting perfect matchings. Let G be a loopless graph with n vertices and m edges. We may assume that G has no isolated vertices and n is even. We can construct representations of the matroids $M_1, \ldots, M_{n/2+1}$ in time polynomial in n and m. For $k = 1, \ldots, n/2+1$, let b_k denote the number of bases of M_k and for $j = 0, \ldots, n/2$, let t_j denote the number of feasible templates of G with j bidirected edges. Then for $k = 1, \ldots, n/2+1$, by Proposition 54, we have

$$b_k = \sum_{j=0}^{n/2} 4^{km}\left(\frac{k}{4}\right)^n\left(\frac{4}{k} + 12\right)^j t_j.$$

Given $b_1, \ldots, b_{n/2+1}$, we may recover $t_0, \ldots, t_{n/2}$ in time polynomial in n and m. In particular, we may recover $t_{n/2}$. But feasible templates with $n/2$ bidirected edges are in one-to-one correspondence with perfect matchings of G. As counting perfect matching is #P-complete by [39], we deduce that when $\mathbb{F}$ has characteristic two, Counting Bases of $\mathbb{F}$-Represented Matroids is #P-complete. □

A.2 $\mathbb{F}$ does not have characteristic two

When $\mathbb{F}$ does not have characteristic two, we can proceed in a similar way, but the proof is a little more complicated as we need to consider more carefully cycles of undirected edges in a template. The following lemma gives us the key property of cycles of undirected edges in the template of a basis.

Lemma 56. *Let G be a loopless graph without isolated vertices. Let C be a cycle of G and Z be a set of $2|C|$ elements of M_k selected as follows. For each i such that e_i is an edge of C, choose j with $1 \leqslant j \leqslant k$ and add either $w_{i,j}$ and $z_{i,j}$, or $x_{i,j}$ and $y_{i,j}$ to Z. To simplify notation we omit the second subscript and for each i denote the elements added to Z by either w_i and z_i, or x_i and y_i. Then both of the following hold.*

1. *If $|\{i : \{w_i, z_i\} \subseteq Z\}|$ is odd then Z is independent in M_k (and N_k) and for each vertex v of C, $v \in \mathrm{cl}_{N_k}(Z)$.*

2. *If $|\{i : \{w_i, z_i\} \subseteq Z\}|$ is even then Z is a circuit in M_k (and N_k).*

Proof. For an edge e_i of C, we say that e_i is a wz-edge if $\{w_i, z_i\} \subseteq Z$, and otherwise we say that it is an xy-edge. We first prove that Z is either independent or a circuit, depending on the parity of $|\{i : \{w_i, z_i\} \subseteq Z\}|$. Consider the submatrix A of A_k containing just the columns indexed by members of Z and consider the coefficients of a non-trivial linear combination of these columns summing to zero. As each row of A is either zero or contains two non-zero entries, both equal to one, we may assume that the non-zero coefficients are all ± 1. Furthermore, for every wz-edge e_i, the coefficients of w_i and z_i must sum to zero, and similarly for every xy-edge e_i, the coefficients of x_i and y_i must sum to zero. Now consider two adjacent edges e_i and e_j in C, and let v be their common endvertex. As the row indexed by v contains one non-zero entry in a column indexed by an element of $\{w_i, x_i, y_i, z_i\} \cap Z$ and also one in a column indexed by an element of $\{w_j, x_j, y_j, z_j\} \cap Z$, we deduce that the coefficients of $\{w_i, x_i, y_i, z_i\} \cap Z$ are non-zero if and only those of $\{w_j, x_j, y_j, z_j\} \cap Z$ are non-zero. Consequently all the coefficients in a non-trivial linear combination of the columns of A are non-zero. Now imagine traversing C in G and suppose that e_i and e_j are consecutive (not necessarily adjacent) wz-edges. Then it is not difficult to see that the coefficients of w_i and w_j (and of z_i and z_j) have opposite signs. Thus, if there are an odd number of wz-edges, then no non-trivial linear combination of the columns of A sums to zero and Z is independent. Alternatively, if there are an even number of wz-edges, then one can assign coefficients ± 1 to columns indexed by w_i or z_i meeting the necessary conditions we have established, and then it is not difficult to check that non-zero coefficients may be assigned to all the remaining columns in order to give a non-trivial linear combination of the columns of A summing to zero. Thus Z is dependent, and as we have shown that all coefficients of a non-trivial linear combination of the columns of A summing to zero must be non-zero, we deduce that Z is a circuit.

Finally, suppose that there are an odd number of wz-edges and let $V(C)$ denote the vertex set of C. Then all the non-zero entries of the columns of A_k corresponding to $Z \cup V(C)$ lie in at most $2|C|$ rows, so $r_{N_k}(Z \cup V(C)) \leqslant 2|C| = r_{N_k}(Z)$. Hence $r_{N_k}(Z \cup V(C)) = r_{N_k}(Z)$, so for each vertex v of C, $v \in \mathrm{cl}_{N_k}(Z)$. □

We say that a cycle of a template comprising only undirected edges is *good* if it has an odd number of edges labelled wz. The analogue of Proposition 53 is as follows.

Proposition 57. *A subset B of $E(M_k)$ is a basis of M_k if and only if all of the following conditions hold.*

- *For all i, $B \cap F_i$ is independent.*
- *For all i and j, $|B \cap F_{i,j}| \geqslant 1$.*
- *Every cycle of $T(B)$ comprising only undirected edges is good.*

- *It is possible to direct the undirected edges of $T(B)$ so that every vertex has indegree one.*

Proof. Most of the proof follows that of Proposition 53. The main difference concerns cycles of $T(B)$ comprising undirected edges. The argument to prove the sufficiency of the conditions follows that of Proposition 53 except for the part showing that for every vertex v in G, we have $v \in \mathrm{cl}_{N_k}(B)$. To establish this observe that the last two conditions imply that for every vertex v of G, there is a (possibly empty) path P in $T(B)$, comprising only undirected edges, between a vertex v', that either has indegree one or belongs to a good cycle, and v. Using Lemma 56 for the latter case, we see that in either case $v' \in \mathrm{cl}_{N_k}(B)$ and the proof may continue in the same way as that of Proposition 53.

To show that each condition is necessary we suppose that B is a basis of M_k. The necessity of the first two conditions follows in the same way as in the proof of Proposition 53 and the necessity of the third follows from Lemma 56. Suppose for contradiction, that $T(B)$ has an undirected edge ab belonging to two cycles C_1 and C_2 of $T(B)$ comprising only undirected edges. Then by traversing the edges of $C_2 - C_1$ starting from a and stopping when we first reach a vertex c of C_1 other than a, we may construct a path between two vertices (a and c) of C_1 using only edges of $C_2 - C_1$. Together with the two paths between a and c which together form C_1, this gives three internally vertex-disjoint paths between a and c. It is always possible to combine two of these together to give a circuit including an even number of wz edges. This contradicts the necessity of the third condition. So every undirected edge of $T(B)$ belongs to at most one cycle comprising only undirected edges.

The necessity of the final condition follows from a similar argument to that used in the proof of Proposition 53, but there are more cases to consider. By first trying to direct all the edges of $T(B)$ belonging to good cycles and then all the remaining undirected edges, we observe that if it is not possible to direct the undirected edges of $T(B)$ so that each edge has indegree one, then before directing the undirected edges one of the following must occur.

1. There is a vertex z of $T(B)$ with indegree at least two.
2. There is a vertex z of $T(B)$ belonging to two edge-disjoint good cycles.
3. There is a vertex z of $T(B)$ with indegree one which belongs to a good cycle.
4. There are vertices x and y of $T(B)$ not belonging to the same good cycle and joined by a path P comprising undirected edges and so that each of x and y either has indegree one or belongs to a good cycle.

To show that each possibility leads to a contradiction, the aim is again to show that there is a vertex v of B such that $B \cup \{v\}$ contains two distinct circuits of N_k. The first case is the same as in the proof of Proposition 53. The second and third follow similarly with the aid of Lemma 56 and the final one follows in a similar way to the analogous case in Proposition 53, noting first that by Lemma 56, if necessary, there are disjoint subsets B_x

and B_y of B with $x \in \mathrm{cl}_{N_k}(B_x)$ and $y \in \mathrm{cl}_{N_k}(B_y)$ and then deducing that y (and in fact every vertex of P) belongs to $\mathrm{cl}_{N_k}(B_x)$. □

We amend the definition of feasibility to say that a template T is *feasible* if it satisfies the last two conditions in the previous result, that is, if the subgraph induced by its undirected edges contains no cycle including an even number of edges labelled wz and it is possible to direct the undirected edges of $T(B)$ so that every vertex has indegree one.

Proposition 58. *Let G be a loopless graph without isolated vertices and let T be a feasible template of G with b bidirected edges. Then the number of bases of M_k with template T is*

$$4^{km}\left(\frac{k}{4}\right)^n\left(\frac{3}{k}+13\right)^b.$$

Proof. The proof is very similar to that of Proposition 54. The key difference is counting the number of choices for F_i when i is a bidirected edge and $|F_{i,j'} \cap B| = |F_{i,j''} \cap B| = 2$ for $j' \neq j''$. There are now 26 ways to choose $F_{i,j'}$ and $F_{i,j''}$ compared with 24 when $\mathbb{F}$ has characteristic two.

So the number of bases of M_k with template T is

$$(4^k)^{m-n+b}(k \cdot 4^{k-1})^{n-2b}\left(k \cdot 4^k + \binom{k}{2} \cdot 26 \cdot 4^{k-2}\right)^b = 4^{km}\left(\frac{k}{4}\right)^n\left(\frac{3}{k}+13\right)^b. \qquad \square$$

Theorem 59. *If $\mathbb{F}$ is a field with characteristic other than two, then the problem* Counting Bases of $\mathbb{F}$-Represented Matroids *is #P-complete.*

The proof is identical to that of Theorem 55 except that it uses Proposition 58 rather than Proposition 54.

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