# Rainbow Triangles
# Sharing One Common Vertex or Edge

Xiaozheng Chen[a] Bo Ning[b]

Submitted: Jul 1, 2023; Accepted: Jul 7, 2025; Published: Aug 22, 2025



**Abstract**

Let $G$ be an edge-colored graph on $n$ vertices. For a vertex $v$, the *color degree* of $v$ in $G$, denoted by $d^c(v)$, is the number of colors appearing on the edges incident with $v$. Denote by $\delta^c(G) = \min\{d^c(v) : v \in V(G)\}$. By a theorem of H. Li, an $n$-vertex edge-colored graph $G$ contains a rainbow triangle if $\delta^c(G) \geqslant \frac{n+1}{2}$. Inspired by this result, we consider two related questions concerning edge-colored books and friendship subgraphs of edge-colored graphs. Let $k \geqslant 2$ be a positive integer. We prove that if $\delta^c(G) \geqslant \frac{n+k-1}{2}$ where $n \geqslant 3k-2$, then $G$ contains $k$ rainbow triangles sharing one common edge; and if $\delta^c(G) \geqslant \frac{n+2k-3}{2}$ where $n \geqslant 2k+9$, then $G$ contains $k$ rainbow triangles sharing one common vertex. The special case $k = 2$ of both results improves H. Li's theorem. The primary novelty in our proof of the first result lies in the integration of the recent technique for identifying rainbow cycles, which was developed by Czygrinow, Molla, Nagle, and Oursler, with certain counting methods from Li, Ning, Shi, and Zhang [J. Graph Theory, 107(4), 2024]. The proof of the second result is facilitated by the implicit use of the machinery underlying the work on Turán numbers for matchings, as established by Erdős and Gallai.

**Mathematics Subject Classifications:** 05C15, 05C38

## 1 Introduction

In 1907, Mantel [25] proved that every triangle-free graph on $n$ vertices has size at most $\lfloor \frac{n^2}{4} \rfloor$. Rademacher (see [11, pp.91]) showed that there is not just one triangle, but indeed at least $\lfloor \frac{n}{2} \rfloor$ triangles in a graph $G$ on $n$ vertices and at least $\frac{n^2}{4}+1$ edges. The $k$-fan (usually called *friendship graph*), denoted by $F_k$, is a graph which consists of $k$ triangles sharing a common vertex. The *book* $B_k$ is a graph which consists of $k$ triangles sharing a common

[a]School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450000, China (cxz@zzu.edu.cn).
[b]College of Computer Science, Nankai University, Tianjin 300350, China (bo.ning@nankai.edu.cn). (B. Ning, corresponding author).

 https://doi.org/10.37236/12170

edge. Erdős [10] extended Mantel's theorem and conjectured that there is a $B_{\lceil \frac{n}{6} \rceil}$ in $G$ if $e(G) > \frac{n^2}{4}$, which was later confirmed by Edwards in an unpublished manuscript [8], and independently by Khadžiivanov and Nikiforov [20]. Erdős, Füredi, Gould, and Gunderson [11] also studied Turán numbers of $F_k$, and proved that $ex(n, F_k) = \lfloor \frac{n^2}{4} \rfloor + k^2 - k$ if $k$ is odd; and $ex(n, F_k) = \lfloor \frac{n^2}{4} \rfloor + k^2 - \frac{3k}{2}$ if $k$ is even, for $n \geqslant 50k^2$. These results immediately imply the fact that every graph on $n$ vertices with minimum degree at least $\frac{n+1}{2}$ contains a $B_k$ for $n \geqslant 6k$ and also a $F_k$ for $n \geqslant 50k^2$. In this paper, we consider edge-colored versions of these extremal problems.

A subgraph of an edge-colored graph is *properly colored* (*rainbow*) if every two adjacent edges (all edges) have pairwise different colors. The rainbow and properly-colored subgraphs have been shown closely related to many graph properties and other topics, such as classical stability results on Turán functions [27], Bermond-Thomassen Conjecture [13], and Caccetta-Häggkvist Conjecture [1], etc. For more rainbow and properly-colored subgraphs under Dirac-type color degree conditions, we refer to [14, 15, 9, 5, 7].

The study of rainbow triangles has a rich history, and there are many classical open problems on them. In some classical problems, the host graph is complete. One conjecture due to Erdős and Sós [12] asserts that the maximum number of rainbow triangles in a 3-edge-coloring of the complete graph $K_n$, denoted by $F(n)$, satisfies $F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd$, where $a+b+c+d = n$ and $a, b, c, d$ are as equal as possible. By using flag algebras, Balogh, Hu, Lidický, Pfender, Volec, and Young [2] confirmed this conjecture for $n$ sufficiently large and $n = 4^k$ for any $k \geqslant 1$. Another example is a recent conjecture by Aharoni (see [1]), which would imply Caccetta-Häggkvist Conjecture [4], a fundamental open problem in the theory of digraphs. Aharoni's conjecture says that given any positive integer $r$, if $G$ is an $n$-vertex edge-colored graph with $n$ color classes and each of size at least $n/r$, then $G$ contains a rainbow cycle of length at most $r$. For more recent developments on Aharoni's conjecture, we refer to the work [6, 17, 18] and more references therein. A special case of Aharoni's conjecture is about rainbow triangles. The relationship between directed triangles and rainbow triangles has been extensively used before (see [24, 21, 22]).

To illustrate this connection, we introduce a construction by Li [21]. Suppose that $D$ is an $n$-vertex digraph such that the out-degree of every vertex is at least $n/3$. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$. We construct an edge-colored graph $G$ such that: $V(G) = V(D)$; for each arc $\overrightarrow{v_i v_j} \in A(D)$, we color the edge $v_i v_j$ with the color $j$. In this way, the number of colors appearing on edges incident with $v_i$ different from $i$ equals to $d^+_D(v_i)$. Thus, finding a directed triangle in $D$ is equivalent to finding a rainbow triangle in such a corresponding edge-colored graph. More importantly, the idea of constructing an auxiliary digraph will also play a crucial role in the proofs presented in this paper.

Our theme of this paper is closely related to the color degree conditions for rainbow triangles. Let $G$ be an edge-colored graph. For every vertex $v \in V(G)$, the *color degree* of $v$, denoted by $d^c_G(v)$ (or in short, $d^c(v)$), is the number of distinct colors appearing on the edges which are incident to $v$. The *minimum color degree* of $G$, denoted by $\delta^c(G)$ (or in short, $\delta^c$), equals to $\min\{d^c(v) : v \in V(G)\}$. It is an easy observation that every graph on $n$ vertices contains a triangle if the minimum degree is at least $\frac{n+1}{2}$. H. Li and

Wang [24] considered a rainbow version and conjectured that the minimum color degree condition $\delta^c(G) \geqslant \frac{n+1}{2}$ ensures the existence of a rainbow triangle in $G$. This conjecture was confirmed by H. Li [21] and also independently by B. Li-Ning-Xu-Zhang in [22].

**Theorem 1** (H. Li [21], B. Li-Ning-Xu-Zhang [22])**.** *Let $G$ be an edge-colored graph on $n$ vertices. If $\delta^c(G) \geqslant \frac{n+1}{2}$ then $G$ contains a rainbow triangle.*

Indeed, B. Li, Ning, Xu and Zhang [22] proved a weaker condition $\sum_{v\in V(G)} d^c(v) \geqslant \frac{n(n+1)}{2}$ suffices for the existence of rainbow triangles, and moreover, characterized the exceptional graphs under the condition $\delta^c(G) \geqslant \frac{n}{2}$. Very recently, X. Li, Ning, Shi, and Zhang [23] established a counting version of Theorem 1. Specifically, they demonstrated that in any edge-colored graph $G$, there exist at least $\frac{1}{6}\delta^c(G)(2\delta^c(G) - n)n$ rainbow triangles, where this bound is best possible.

Hu, Li and Yang [19] proposed the following conjecture: Let $G$ be an edge-colored graph on $n \geqslant 3k$ vertices. If $\delta^c(G) \geqslant \frac{n+k}{2}$ then $G$ contains $k$ vertex-disjoint rainbow triangles. Besides the work on Turán numbers of books and $k$-fans mentioned before, our other motivation is to study the converse of Hu-Li-Yang's conjecture, i.e., rainbow triangles sharing vertices or edges. We shall study the existence of rainbow triangles sharing one common vertex or an edge under color degree conditions.

Our original result is the following one which improves Theorem 1. In fact, we can go farther.

**Theorem 2.** *Let $G$ be an edge-colored graph on $n$ vertices with $\delta^c(G) \geqslant \frac{n+1}{2}$.*
*(i) If $n \geqslant 4$ then $G$ contains two rainbow triangles sharing one common edge.*
*(ii) If $n \geqslant 13$ then $G$ contains two rainbow triangles sharing one common vertex.*

Our main results are given as follows.

**Theorem 3.** *Let $k \geqslant 2$ be a positive integer and $G$ be an edge-colored graph on $n \geqslant 3k-2$ vertices. If $\delta^c(G) \geqslant \frac{n+k-1}{2}$ then $G$ contains $k$ rainbow triangles sharing one edge.*

**Theorem 4.** *Let $k \geqslant 2$ be a positive integer and $G$ be an edge-colored connected graph on $n \geqslant 2k + 9$ vertices. If $\delta^c(G) \geqslant \frac{n+2k-3}{2}$ then $G$ contains $k$ rainbow triangles only sharing one common vertex.*

Setting $\delta^c(G) = \frac{n+k-1}{2}$ in Theorem 3, the following example shows that the bound "$n \geqslant 3k-2$" is sharp. Furthermore, it follows from Example 1 that the tight color degree should be at least $\delta^c \geqslant \frac{n+k}{2}$ when $n \leqslant 3k - 3$.

**Example 1.** Let $G$ be a properly-colored balanced complete 3-partite graph $G[V_1, V_2, V_3]$ with $|V(G)| = 3k - 3$ and $|V_1| = |V_2| = |V_3| = k - 1$, where $k \geqslant 1$ is a positive integer. Then for each vertex $v \in V(G)$, $d^c(v) = d(v) = 2k - 2 = \frac{n+k-1}{2}$ while $G$ contains no $B_k$ and $F_k$ (see Figure 1).

The main novelty of our proof of Theorem 3 is a combination of the recent new technique for finding rainbow cycles due to Czygrinow, Molla, Nagle, and Oursler [5] and some recent counting technique from [23]. In particular, Czygrinow et al. [5] extended

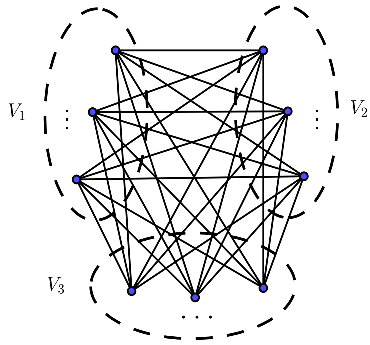


Figure 1: An extremal graph for Theorem 3

Theorem 1 by proving that for every integer $\ell$, every edge-colored graph $G$ on $n \geqslant 432\ell$ many vertices satisfying $\delta^c(G) \geqslant \frac{n+1}{2}$ admits a rainbow $\ell$-cycle $C_\ell$. One novel concept introduced in [5] is the notion of a restriction color, which will be utilized in our proof.

Meantime, both Theorem 3 and Theorem 4 improve Theorem 1 as they imply Theorem 2 by setting $k = 2$. On the other hand, Theorem 4 slightly improves Theorem 9 in [23] by a different method.

This paper is organized as follows. In Section 2, we introduce some necessary notations and terminology. In Section 3, we prove some lemmas and general versions of Theorem 2, i.e., Theorems 3 and 4. We conclude this paper with some more discussions on the sharpness of our results, together some propositions on $F_k$ and $B_k$ on uncolored graphs.

## 2 Notations

In this paper, all graphs considered are simple and connected graphs. Let $G$ be an edge-colored graph. Let $\mathcal{C}(G)$ be the set of colors appearing on $E(G)$. For two disjoint subsets $V_1, V_2 \subseteq V(G)$, the set of colors appearing on the edges between $V_1$ and $V_2$ in $G$ is denoted by $\mathcal{C}(V_1, V_2)$. Some of our notations come from [5, 23]. For a vertex $v \in V(G)$, let $\mathcal{C}(v)$ be the set of colors appearing on the edges incident with $v$. For a color $\alpha \in \mathcal{C}(G)$ and a vertex $v \in V(G)$, define the *$\alpha$-neighborhood* of $v$ as

$$N_\alpha(v) = \{u \in N(v) \mid c(uv) = \alpha\},$$

and *$\alpha$-neighborhood in $X$* of $v$ as

$$N_\alpha(v, X) = \{u \in N(v) \cap X \mid c(uv) = \alpha\},$$

where $X \subseteq V(G)$, and $N(v)$ is the neighborhood of $v$ in $G$. Define

$$N_!(v) = \bigcup_{\alpha \in \mathcal{C}(G)} \{N_\alpha(v) \mid |N_\alpha(v)| = 1\}.$$

For the sake of simplicity, let $d_\alpha(v) = |N_\alpha(v)|$ and $d_\alpha(v, X) = |N_\alpha(v, X)|$. Moreover, let $N[v]$ be the *closed neighborhood* of $v$ in $G$, that is, $N[v] = N(v) \cup \{v\}$. The *monochromatic degree of* $v$, denoted by $d^{mon}(v)$, is the maximum number of edges incident to $v$ colored with a same color (i.e., $d^{mon}(v) = \max\{d_\alpha(v) \mid \alpha \in \mathcal{C}(G)\}$.) For a graph $G$, we denote the *monochromatic degree* of $G$ by $\Delta^{mon}(G) = \max\{d^{mon}(v) \mid v \in V(G)\}$.

The following concept of restriction was first introduced in [5, Section 3].

**Definition 5** (Restriction color [5])**.** Let $G$ be an edge-colored graph. Fix $v \in V(G)$ and $X \subseteq N(v)$. For $y \in V(G) \setminus \{v\}$ and $\alpha \in \mathcal{C}(E)$, we say $(v, X)$ restricts color $\alpha$ for $y$ if there is some $x \in N(y) \cap X$ such that $\alpha = c(xy) \neq c(vx)$ and $\alpha \notin \mathcal{C}(\{y\}, N(y) \setminus X)$. Denote by $\sigma_{v,X}(y)$ the number of colors $\alpha \in \mathcal{C}(E)$ restricted for $y$ by $(v, X)$.

Denote by $rt(v)$ the number of rainbow triangles containing $v$; by $rt(v, x)$ the number of rainbow triangles containing both $v$ and $x$ (i.e., containing the edge $vx$); and $rt(v, X) = \sum_{x \in X} rt(v, x)$.

## 3 Edge-colored books and friendship graphs

According to the definition of restriction color, we have the following proposition.

**Proposition 6.** *Let $v$ be a vertex of $G$, $x \in N(v)$, $X = N_{c(vx)}(v)$, and $Y = N(v) \setminus X$. Then $rt(v, x) \geqslant \sigma_{v,Y}(x)$ and $rt(v, X) \geqslant \sum_{x \in X} \sigma_{v,Y}(x)$.*

**Proof of Proposition 6.** For $y \in Y \cap N(x)$, we have $c(vx) \neq c(vy)$. If $(v, Y)$ restricts $c(xy)$ for $x$, then $c(xy) \neq c(vy)$ and $c(xy) \notin \mathcal{C}(\{x\}, N(x) \setminus Y)$, which also implies that $c(xy) \neq c(vx)$. Thus, $vxyv$ is a rainbow triangle. It follows $rt(v, x) \geqslant \sigma_{v,Y}(x)$ and $rt(v, X) \geqslant \sum_{x \in N_{c(vx)}(v)} \sigma_{v,Y}(x)$.

*Remark* 7. We say that an edge-colored graph $G$ is edge-minimal if $\delta^c(G - e) < \delta^c(G)$ for every $e \in E(G)$. In the proofs of Theorem 3 and Theorem 4, it is assumed that $G$ is a counterexample with $e(G)$ as small as possible with respect to $n$ and $\delta^c$, which implies that $G - e$ must be a graph with $\delta^c(G - e) < \delta^c$ for every $e \in E(G)$, otherwise $G - e$ is a counterexample with a smaller size respect to $n$ and $\delta^c$ than $G$. Moreover,such edge-minimal graphs $G$ do not contain monochromatic paths of length 3 or monochromatic triangles.

The form of the following lemma is motivated by [23], but its proof is a mixture of techniques from [5, 23].

**Lemma 8.** *Let $G$ be an edge-colored graph which is edge-minimal. Then for any $v \in V(G)$ and $\alpha \in \mathcal{C}(v)$, we have,*

$$\begin{aligned} rt(v, N_\alpha(v)) \geqslant & \sum_{x \in N_\alpha(v)} \Big(d^c(x) + d^c(v) - n\Big) \\ & + d_\alpha(v) \sum_{\beta \in \mathcal{C}(v)} \Big(d_\beta(v) - 1\Big) - d_\alpha(v)\big(d_\alpha(v) - 1\big) - \sum_{y \in N_!(v)} d_{c(vy)}(y, N_\alpha(v)). \end{aligned}$$

*Proof of Lemma 8.* Fix a vertex $v \in V(G)$ and $\alpha \in \mathcal{C}(v)$. For convenience, let $X = N_\alpha(v)$, $Y = N(v) \setminus X$. Define a directed graph $D$ on $V(D) = X \cup Y$ as follows: the arc $\overrightarrow{yx}$ exists if and only if $c(xy) = c(vy)$ for $x \in X$ and $y \in Y$. Since $G$ is edge-minimal, the existence of $\overleftarrow{xy}$ gives $d_{c(vy)}(v) = 1$ and then $y \in N_!(v)$. (Indeed, since the arc $\overleftarrow{xy}$ exists, we have $c(xy) = c(vy)$. If $d_{c(vy)}(v) \geqslant 2$, there exists a monochromatic path of length 3, a contradiction!) Evidently, $d_D^+(y) = d_{c(vy)}(y, X)$. Thus,

$$\sum_{x \in X} d_D^-(x) = \sum_{y \in Y} d_D^+(y) = \sum_{y \in N_!(v)} d_D^+(y) = \sum_{y \in N_!(v)} d_{c(vy)}(y, X). \tag{1}$$

In addition, as for $x \in X$, there are $\sigma_{v,Y}(x)$ colors that are restricted by $(v, Y)$ for $x$. Hence, there are at most $d_D^-(x) + \sigma_{v,Y}(x)$ colors that appear on edges from $x$ to $Y$ but no edges from $x$ to $N(x) \setminus Y$. Then there are at least $d^c(x) - d_D^-(x) - \sigma_{v,Y}(x)$ vertices in $V(G) \setminus (Y \cup \{x\})$. Therefore,

$$n - |Y| - 1 \geqslant d^c(x) - d_D^-(x) - \sigma_{v,Y}(x)$$

$$\Rightarrow \quad \sigma_{v,Y}(x) \geqslant d^c(x) + |Y| + 1 - d_D^-(x) - n \tag{2}$$

Note that $|Y| = d(v) - d_\alpha(v) = \sum_{\beta \in \mathcal{C}(v)} d_\beta(v) - d_\alpha(v) = d^c(v) + \sum_{\beta \in \mathcal{C}(v)} (d_\beta(v) - 1) - d_\alpha(v)$. According to Proposition 6 and combining Ineqs (1) and (2), we can get that

$$\begin{aligned}
& rt(v, X) \\
\geqslant & \sum_{x \in X} \sigma_{v,Y}(x) \\
\geqslant & \sum_{x \in X} \Big(d^c(x) + |Y| + 1 - n\Big) - \sum_{y \in N_!(v)} d_{c(vy)}(y, X) \\
= & \sum_{x \in X} \Big(d^c(x) + d^c(v) + \sum_{\beta \in \mathcal{C}(v)} (d_\beta(v) - 1) - d_\alpha(v) + 1 - n\Big) - \sum_{y \in N_!(v)} d_{c(vy)}(y, X) \\
\geqslant & \sum_{x \in X} \Big(d^c(x) + d^c(v) - n\Big) + d_\alpha(v)\Big(\sum_{\beta \in \mathcal{C}(v)} (d_\beta(v) - 1)\Big) - d_\alpha(v)\big(d_\alpha(v) - 1\big) - \sum_{y \in N_!(v)} d_{c(vy)}(y, X).
\end{aligned}$$

The proof is complete. □

Since $rt(v) = \frac{1}{2} \sum_{\alpha \in \mathcal{C}(v)} rt(v, N_\alpha(v))$, we have the following corollary.

**Corollary 9.** *Let $G$ be an edge-colored graph which is edge-minimal. Then we have,*

$$\begin{aligned}
rt(v) \geqslant \frac{1}{2} \Big( & \sum_{x \in N(v)} \big(d^c(x) + d^c(v) - n\big) \\
& + d(v)\big(\sum_{\alpha \in \mathcal{C}(v)} (d_\alpha(v) - 1)\big) - \sum_{\alpha \in \mathcal{C}(v)} d_\alpha(v)(d_\alpha(v) - 1) - \sum_{y \in N_!(v)} d_{c(vy)}(y, N(v))\Big).
\end{aligned}$$

For each vertex $v \in V(G)$ and $\alpha \in \mathcal{C}(v)$, let

$$B_\alpha(v) = d_\alpha(v)\Big(\sum_{\beta\in\mathcal{C}(v)} \big(d_\beta(v)-1\big)\Big) - d_\alpha(v)\big(d_\alpha(v)-1\big) - \sum_{y\in N_!(v)} d_{c(vy)}(y, N_\alpha(v)). \tag{3}$$

Then by Lemma 8,

$$rt(v, N_\alpha(v)) \geqslant \sum_{x\in N_\alpha(v)} \Big(d^c(x)+d^c(v)-n\Big) + B_\alpha(v). \tag{4}$$

Moveover, let $\alpha_0$ be the color such that $d_{\alpha_0}(v) = d^{mon}(v)$. Set

$$\begin{aligned} &B(v)\\ &= \sum_{\alpha\in\mathcal{C}(v)} B_\alpha(v)\\ &= d(v)\Big(\sum_{\alpha\in\mathcal{C}(v)}\big(d_\alpha(v)-1\big)\Big) - \sum_{\alpha\in\mathcal{C}(v)} d_\alpha(v)\big(d_\alpha(v)-1\big) - \sum_{y\in N_!(v)} d_{c(vy)}(y, N(v)).\\ &= \underbrace{\big(d(v)-d^{mon}(v)\big)\big(d^{mon}(v)-1\big) - \sum_{y\in N_!(v)} d_{c(vy)}(y,N(v))}_{①} + \underbrace{\sum_{\alpha\in\mathcal{C}(v)\setminus\{\alpha_0\}} \big(d(v)-d_\alpha(v)\big)\big(d_\alpha(v)-1\big)}_{②}. \end{aligned} \tag{5}$$

Then by Corollary 9,

$$rt(v) \geqslant \frac{1}{2}\Big(\sum_{x\in N(v)} \big(d^c(x)+d^c(v)-n\big) + B(v)\Big). \tag{6}$$

We say an edge $xy \in E(G)$ is a *rainbow triangle edge* of $v$ if $vxyv$ is a rainbow triangle. Denote by $\Phi(v)$ the edge set of rainbow triangle edges of $v$.

First we prove a result on a vertex with maximum monochromatic degree, i.e., a vertex $v \in V(G)$ with $d^{mon}(v) = \Delta^{mon}(G)$.

**Lemma 10.** *Let $G$ be an edge-colored graph which is edge-minimal. Then for a vertex $v \in V(G)$ with $d^{mon}(v) = \Delta^{mon}(G)$ we have $B(v) \geqslant 0$. Let $\alpha_0$ be the color such that $d_{\alpha_0}(v) = d^{mon}(v)$. If $\Delta^{mon}(G) \geqslant 2$ and $B(v) = 0$ then there hold:*
*(a) $N_!(v) = N(v) \setminus N_{\alpha_0}(v)$ and $d_\alpha(v) = 1$ for $\alpha \in \mathcal{C}(v) \setminus \{\alpha_0\}$;*
*(b) $d_{c(vy)}(y) = d^{mon}(y) = \Delta^{mon}(G)$ for all $y \in N_!(v)$; and*
*(c) if $B_{\alpha_0}(v) = 0$, then $E[N_{\alpha_0}(v), N_!(v)] \subseteq \Phi(v)$.*

*Proof of Lemma 10.* If $\Delta^{mon}(G) = 1$, then obviously $B(v) = 0$. Suppose that $\Delta^{mon}(G) \geqslant 2$. Since $d_{c(vy)}(y, N(v)) \leqslant \Delta^{mon}(G) - 1 = d^{mon}(v) - 1$ for all $y \in N_!(v)$, we have part ① in Eq (5) is more than or equal to 0. It guarantees that $B(v) \geqslant 0$.

Moreover suppose $B(v) = 0$ and $\Delta^{mon}(G) \geqslant 2$. Then the following two equalities hold.

$$\big(d(v)-d^{mon}(v)\big)\big(d^{mon}(v)-1\big) - \sum_{y\in N_!(v)} d_{c(vy)}(y, N(v)) = 0, \tag{7}$$

$$\sum_{\alpha\in\mathcal{C}(v)\setminus\{\alpha_0\}}\big(d(v)-d_\alpha(v)\big)\big(d_\alpha(v)-1\big)=0. \tag{8}$$

If there exists a vertex $y\in N_!(v)$ such that $d_{c(vy)}(y,N(v))<d^{mon}(v)-1$, then $\big(d(v)-d^{mon}(v)\big)\big(d^{mon}(v)-1\big)>\sum_{y\in N_!(v)}d_{c(vy)}(y,N(v))$, a contradiction to Eq (7). Hence for all $y\in N_!(v)$ we have $d_{c(vy)}(y,N(v))=d^{mon}(v)-1$, which implies that (b) holds. Then

$$0=B(v)\geqslant\big(d(v)-d^{mon}(v)-|N_!(v)|\big)\big(d^{mon}(v)-1\big)+\sum_{\alpha\in\mathcal{C}(v)\setminus\{\alpha_0\}}\big(d(v)-d_\alpha(v)\big)\big(d_\alpha(v)-1\big)\geqslant 0.$$

Since $\Delta^{mon}(G)=d^{mon}(v)\geqslant 2$, we have $d(v)-d^{mon}(v)-|N_!(v)|=0$ and $d_\alpha(v)=1$ for $\alpha\in\mathcal{C}(v)\setminus\{\alpha_0\}$. Therefore, (a) holds.

We prove (c) in the following. Since $d(v)-d^c(v)=d^{mon}(v)-1$ and $d_\alpha(v)-1=0$ for $\alpha\in\mathcal{C}(v)\setminus\{\alpha_0\}$ by (a), we have $B_{\alpha_0}(v)=-\sum_{y\in N_!(v)}d_{c(vy)}(y,N_{\alpha_0}(v))$ by Eq (3). Suppose that $B_{\alpha_0}(v)=0$. Then $d_{c(vy)}(y,N_{\alpha_0}(v))=0$ for all $y\in N_!(v)$. Since $G$ is edge-minimal and $d^{mon}(v)\geqslant 2$, we have $\alpha_0\notin\mathcal{C}(N_{\alpha_0}(v),N_!(v))$; indeed, if $\alpha_0\in\mathcal{C}(N_{\alpha_0}(v),N_!(v))$, then there exists a monochromatic path of length 3. Furthermore, for any edge $xy\in E[N_{\alpha_0}(v),N_!(v)]$, $c(xy)\notin\{c(vx),c(vy)\}$. Thus $xy$ is a rainbow triangle edge of $v$. Hence $E[N_{\alpha_0}(v),N_!(v)]\subseteq\Phi(v)$. This proves (c). The proof is complete. □

### 3.1 Edge-colored books

We first give a lemma on $B_k$ in an uncolored graph $G$ with restriction on the standard minimum degree $\delta(G)$.

**Lemma 11.** *Let $k\geqslant 2$ be a positive integer and $G$ be a graph on $n\geqslant 3k-2$ vertices. If $\delta(G)\geqslant\frac{n+k-1}{2}$ then $G$ contains a $B_k$.*

*Proof of Lemma 11.* Suppose to the contrary that there is no $B_k$ in $G$. If there exists a vertex $v\in V(G)$ with $d(v)\geqslant\frac{n+k}{2}$, then $|N(v)\cap N(u)|\geqslant\lceil\frac{n+k}{2}\rceil+\lceil\frac{n+k-1}{2}\rceil-n\geqslant k$ where $u\in N(v)$. In this case, there is a $B_k$. Suppose $d(v)=\frac{n+k-1}{2}$ for all $v\in V(G)$. (If $\frac{n+k-1}{2}$ is not an integer, $\delta(G)\geqslant\frac{n+k-1}{2}$ guarantees that $\delta(G)\geqslant\frac{n+k}{2}$.)

**Claim.**

*Proof.* As there is no $B_k$ in $G$, we have $|N(v)\cap N(u)|\leqslant k-1$. Thus,

$$\begin{aligned}k-1\geqslant|N(v)\cap N(u)|&\geqslant d(v)+d(u)-|N(v)\cup N(u)|\\&\geqslant\frac{n+k-1}{2}+\frac{n+k-1}{2}-n\\&\geqslant k-1.\end{aligned}$$

Hence $|N(v)\cap N(u)|=k-1\geqslant 1$ and $N(u)\cup N(v)=n$. It follows that $V(G)\setminus N(v)\subseteq N(u)$ and $V(G)\setminus N(u)\subseteq N(v)$. □

Let $uv \in E(G)$ and $w \in N(v) \cap N(u)$. According to Claim, we have $V(G) \setminus N(v) \subseteq N(w)$. Hence, $V(G) \setminus N(v) \subseteq N(u) \cap N(w)$. Since each vertex in $V(G)$ is of degree $\frac{n+k-1}{2}$, we have $|V(G) \setminus N(v)| = \frac{n-k+1}{2}$. Then $|N(u) \cap N(w)| \geqslant \lceil \frac{n-k+1}{2} \rceil \geqslant k$, a contradiction. This proves Lemma 11. □

Now we are ready to prove Theorem 3.

*Proof of Theorem 3.* We prove the theorem by contradiction. Let $G$ be a counterexample such that $e(G)$ is as small as possible respect to $n$ and $\delta^c$. By Lemma 11, we can assume $\Delta^{mon}(G) \geqslant 2$. By Remark 1, $G$ is edge-minimal.

**Claim 1.** For any vertex $v \in V(G)$ with $d^{mon}(v) = \Delta^{mon}(G)$, let $\alpha_0$ be the color such that $d_{\alpha_0}(v) = d^{mon}(v)$. Then $B(v) = B_\alpha(v) = 0$ for $\alpha \in \mathcal{C}(v)$ and $rt(v, x) = k - 1$ for $x \in N(v)$.

*Proof.* We use $d_\alpha$ instead of $d_\alpha(v)$ in the following. If there exists a color $\alpha \in \mathcal{C}(v)$ such that $rt(v, N_\alpha(v)) \geqslant (k-1)d_\alpha + 1$, then there exists a vertex $x_0 \in N_\alpha(v)$ satisfying $rt(v, x_0) \geqslant k$, a contradiction. Thus, for all $\alpha \in \mathcal{C}(v)$, we have

$$rt(v, N_\alpha(v)) \leqslant (k-1)d_\alpha \leqslant \sum_{x \in N_\alpha(v)} (d^c(x) + d^c(v) - n),$$

where the second inequality holds as the condition that $d^c(x) + d^c(v) \geqslant n + k - 1$ due to our assumption on $\delta^c(G)$ in the theorem statement. It follows that $B_\alpha(v) \leqslant 0$ for all $\alpha \in \mathcal{C}(v)$ by Ineq (4). Then by Lemma 10, $0 \leqslant B(v) = \sum_{\alpha \in \mathcal{C}(v)} B_\alpha(v) \leqslant 0$. That is, $B(v) = B_\alpha(v) = 0$ for $\alpha \in \mathcal{C}(v)$. Then we have $rt(v, N_\alpha(v)) = (k-1)d_\alpha$ for all $\alpha \in \mathcal{C}(v)$ and $d_\alpha = 1$ for $\alpha \in \mathcal{C}(v) \setminus \{\alpha_0\}$ by (a) of Lemma 9. Since $G$ contains no $k$ rainbow triangles sharing one common edge, $rt(v, x) = k - 1$ for all $x \in N(v)$. □

Let $v$ be a vertex with $d^{mon}(v) = \Delta^{mon}(G)$ and $\Gamma$ be the subgraph of $G$ induced by $N[v]$. Then we have the following claim.

**Claim 2.** $d^c_\Gamma(x) \leqslant k$ for $x \in N_!(v)$.

*Proof.* For any $x \in N_!(v)$, since $B(v) = 0$, $d_{c(vx)}(x) = \Delta^{mon}(G)$ by b) of Lemma 10. Thus, $B(x) = B_{c(vx)}(x) = 0$ by Claim 1 and $N(x) = N_!(x) \cup N_{c(vx)}(x)$ by (a) of Lemma 9. For any vertex $u \in N(v)$, if $u \in N_!(v)$ then $uv \in \Phi(x)$ as $v \in N_{c(vx)}(x)$. Thus $uvxu$ is a rainbow triangle, which means that $xu \in \Phi(v)$. Hence for any $u \in N(v) \cap N(x)$, $u \in N_{c(vx)}(x)$ or $xu \in \Phi(v)$. Therefore,

$$d^c_\Gamma(x) \leqslant rt(v, x) + 1 = k. \qquad \square$$

Then, we infer for $x \in N_{!}(v)$,

$$\begin{aligned}\frac{n+k-1}{2} \leqslant d^c(x) &\leqslant d^c_\Gamma(x) + |G - \Gamma| \\ &\leqslant d^c_\Gamma(x) + n - \big(\Delta^{mon}(G) + d^c(v)\big) \\ &\leqslant k + n - \Delta^{mon}(G) - d^c(v) \\ &\leqslant \frac{n+k+1}{2} - \Delta^{mon}(G),\end{aligned}$$

that is, $\Delta^{mon}(G) = 1$, a contradiction. □

### 3.2 Edge-colored friendship graphs

The aim of this subsection is to prove Theorem 4.

A *covering* of a graph $G$ is a subset $K$ of $V(G)$ such that every edge of $G$ has at least one end vertex in $K$. A covering $K^*$ is a *minimum covering* if $G$ has no covering $K$ with $|K| < |K^*|$. The number of vertices in a minimum covering of $G$ is called the *covering number* of $G$, and is denoted by $\tau(G)$.

**Lemma 12.** *For a graph $G$ on $n$ vertices, we have $\tau(G) \leqslant n-1$. Furthermore, $G$ is complete if and only if $\tau(G) = n-1$.*

*Proof of Lemma 12.* The sufficiency of condition is clear. Now we establish its necessity. Suppose that $G$ is not complete. Then there exist two vertices $u, v \in V(G)$ such that $uv \notin E(G)$. Hence $V(G) \setminus \{u, v\}$ is a covering of $G$. Thus, $\tau(G) \leqslant n-2$. □

A *matching* in a graph is a set of pairwise disjoint edges. The *matching number* of a graph $G$ is the maximum number of pairwise disjoint edges in $G$, denoted by $\nu(G)$. The Tutte-Berge Formula [26, 3] is very useful when we study the problem related to matching number. A *partition* of $G$ is a family of pairwise disjoint subsets $V_1, V_2, \ldots, V_p$ of $V(G)$ satisfying $\bigcup_{1 \leqslant i \leqslant p} V_i = V(G)$.

**Definition 13.** ($A_0$-partition) Let $A_0$ be a subset of $V(G)$. We say $(A_0, A_1, \ldots, A_p)$ is a $A_0$-partition of $G$ if $(A_0, A_1, \ldots, A_p)$ is a partition of $G$ such that $G[A_i]$ is a component of $G - A_0$ for $1 \leqslant i \leqslant p$.

**Theorem 14** (Tutte-Berge Formula)**.** *A graph $G$ satisfies $\nu(G) = s$ if and only if $G$ contains a $A_0$-partition $(A_0, A_1, \ldots, A_p)$ with $|A_i|$ odd for $1 \leqslant i \leqslant p$, and*

$$s = |A_0| + \sum_{i=1}^{p} \frac{|A_i| - 1}{2}.$$

Now we will get a useful lemma which shows the relationship between the covering number and the matching number of a graph.

**Lemma 15.** *Let $G$ be a connected graph on $n \geqslant 2\nu(G)+2$ vertices. Then $G$ contains a $A_0$-partition $(A_0, A_1, \ldots, A_p)$ with $|A_i|$ odd for $1 \leqslant i \leqslant p$, $A_0 \neq \emptyset$, and*

$$\tau(G) \leqslant 2\nu(G) - |A_0| \leqslant 2\nu(G) - 1.$$

*Furthermore, if $\tau(G) = 2\nu(G) - 1$, then*

*(1) $G[A_i]$ is complete for $1 \leqslant i \leqslant p$;*

*(2) $\tau(G) = n - p$ and there exists a minimum covering $K$ of $G$ satisfying $A_0 \subseteq K$ and $|K \cap A_i| = |A_i| - 1$ for all $1 \leqslant i \leqslant p$.*

*Proof of Lemma 15.* By Tutte-Berge Formula, $G$ contains a $A_0$-partition $(A_0, A_1, \ldots, A_p)$ with $|A_i|$ odd for $1 \leqslant i \leqslant p$. By Lemma 12, we have $\tau(G[A_i]) \leqslant |A_i| - 1$ for $1 \leqslant i \leqslant p$.

Let $K$ be a subset of $V(G)$ consisting of $A_0$ and $|A_i|-1$ vertices in $A_i$ for all $1 \leqslant i \leqslant p$. Apparently, $K$ is a covering of $G$ and $|K| = |A_0| + \sum_{1\leqslant i\leqslant p}(|A_i| - 1) = n - p$. Since $G$ is a connected graph and $G[A_i]$ is a component of $G - A_0$, we have $E[A_0, A_i] \neq \emptyset$ for $1 \leqslant i \leqslant p$. Then,

$$\tau(G) \leqslant |K| = |A_0| + \sum_{1\leqslant i\leqslant p} \tau(G[A_i]) \leqslant |A_0| + \sum_{1\leqslant i\leqslant p} (|A_i| - 1) = n - p = 2\nu(G) - |A_0|, \quad (9)$$

where the last equality holds by Theorem 14.

If $A_0 = \emptyset$, then $\nu(G) = \frac{n-1}{2}$ as $G$ is connected by Theorem 14. Then $n = 2\nu(G)+1$, a contradiction. Thus $A_0 \neq \emptyset$. Since $|A_0| \geqslant 1$, if $\tau(G) = 2\nu(G) - 1$, then all inequalities become equalities in Ineq (9). Hence $\tau(G[A_i]) = |A_i| - 1$ for all $1 \leqslant i \leqslant p$. From Lemma 12, $G[A_i]$ is complete. Since $\tau(G) = |K| = |A_0| + \sum_{1\leqslant i\leqslant p}(|A_i| - 1)$, $K$ is a minimum covering of $G$. □

Now we are ready to prove Theorem 4.

*Proof of Theorem 4.* We prove the theorem by contradiction. Let $G$ be a counterexample with $e(G)$ is as small as possible respect to $n$ and $\delta^c$. Choose $v \in V(G)$ such that $d^{mon}(v) = \Delta^{mon}(G)$. Recall $\Phi(v)$ is the edge set of rainbow triangle edges of $v$. Let $\Gamma$ denote the subgraph induced by the edge set $\Phi(v)$. Then $\nu(\Gamma) \leqslant k-1$. By Lemma 10, $B(v) \geqslant 0$ as $d^{mon}(v) = \Delta^{mon}(G)$. Since $d(v) \geqslant 2k+3$ from the facts $n \geqslant 2k+9$ and $\delta^c \geqslant \frac{n+2k-3}{2}$, we have $rt(v) \geqslant \frac{1}{2}\sum_{x\in N(v)}(d^c(v) + d^c(x) - n) \geqslant \frac{4k^2-9}{2}$ by Eq (6). Thus, $e(\Gamma) = rt(v) > \binom{2k-1}{2}$, which gives that $|V(\Gamma)| \geqslant 2k \geqslant 2\nu(\Gamma)+2$ and $\Gamma$ is not complete. Then by Lemma 15

$$\tau(\Gamma) \leqslant 2\nu(\Gamma) - 1 \leqslant 2k - 3. \quad (10)$$

Let $\varsigma$ be a color satisfying $d^{mon}(v) = d_\varsigma(v)$. Let $Y = N_\varsigma(v)$ if $\Delta^{mon}(G) \geqslant 2$, otherwise $Y = \emptyset$.

**Claim 1.** For any minimum covering $K$ of $\Gamma$, let $X$ be a maximum rainbow neighborhood of $v$ in $N(v) \setminus (K \cup Y)$, i.e., all vertices in $X$ have a unique color to $v$. Set $U = X \cup Y$ and $T = K \cup U$. Then we have

(1) $\tau(\Gamma) = 2k - 3$;

(2) For all $u \in U$, $K$ is a rainbow neighbor set of $u$ and $c(vu) \notin \mathcal{C}(u, K)$; and

(3) $d_{c(vu)}(u) = d^{mon}(u) = \Delta^{mon}(G)$ for $u \in X$ and $E[X, Y] = \emptyset$.

*Proof.* We have $|X| \geqslant d^c(v) - \tau(\Gamma) - 1$, and

$$|T| \geqslant d^{mon}(v) + d^c(v) - 1. \tag{11}$$

For $u \in U$, define an index $q(u)$ which refers to the number of colors which appear on the edges between $u$ and $K$ other than appear on the edges between $u$ and $U \cup \{v\}$. That is, $q(u) := |\mathcal{C}(u, K) \setminus \mathcal{C}(u, U \cup \{v\})|$.

For all $u \in U$, we have

$$\begin{aligned} d^c(u) &\leqslant d^c_{T\cup\{v\}}(u) + n - \left|T\right| - 1 \\ &\leqslant d^c_{T\cup\{v\}}(u) + n - d^c(v) - \Delta^{mon}(G). \end{aligned} \tag{12}$$

Since each rainbow triangle edge of $v$ is incident to at least one vertex in $K$, and $X$ is a rainbow neighbor set of $v$, for $u_1, u_2 \in X$ or $u_1 \in X$ and $u_2 \in Y$, if $u_1u_2 \in E(G)$ we have

$$c(u_1u_2) \in \{c(vu_1), c(vu_2)\}. \tag{13}$$

Now we distinguish two cases to show that (1)-(3) hold respectively.

**Case 1.** $\Delta^{mon}(G) = 1$.

In this case, $G$ is properly-colored and $Y = \emptyset$. By (13), $G[U]$ consists of isolated vertices. Thus for $u \in U$, we have $d^c_{T\cup\{v\}}(u) = |c(uv) \cup \mathcal{C}(\{u\}, K)| \leqslant 1 + q(u)$. Hence by Ineq (12), we have

$$\begin{aligned} d^c(u) &\leqslant 1 + q(u) + n - d^c(v) - 1 \\ &\leqslant q(u) + n - d^c(v). \end{aligned}$$

Thus, $2k - 3 \leqslant d^c(u) + d^c(v) - n \leqslant q(u) \leqslant |K| \leqslant 2k - 3$ by Ineq (10) (following from our bound on $\delta^c(G)$), which gives that $q(u) = \left|K\right| = 2k - 3$ for all $u \in U$. Therefore, (1) holds in this case. According to the definition of $q(u)$, (2) also holds in this case. Since $G[U]$ consists of isolated vertices, (3) is implied straightly.

**Case 2.** $\Delta^{mon}(G) \geqslant 2$.

We can obtain an oriented graph $D[X]$ on $X$ as follows: orient each edge $x_1x_2$ in $G[X]$ by $\overrightarrow{x_1x_2}$ if $c(x_1x_2) = c(vx_2)$. Then for $x \in X$, all out-arcs from $x$ are assigned pairwise distinct colors which are different from $c(vx)$, we have

$$|\mathcal{C}(x, X) \setminus \{c(vx)\}| = d^+_{D[X]}(x).$$

Since all in-arcs for $x$ are assigned the color $c(vx)$, we have

$$d^{\text{non}}_{G[X]}(x) = \begin{cases} d^-_{D[X]}(x) & \text{if } d^-_{D[X]}(x) \geqslant 1 \\ 1 & \text{if } d^-_{D[X]}(x) = 0 \text{ and } d^+_{D[X]}(x) \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

It guarantees that $d^-_{D[X]}(x) \leqslant \Delta^{mon}(G) - 1$ for any $x \in X$.

Moreover for $x \in X$ and $y \in Y$, $c(xy) = c(vx)$ if $xy \in E(G)$ by (13) (otherwise, there is a monochromatic $P_3$ as $\Delta^{mon}(G) \geqslant 2$, a contradiction to the fact that $G$ is edge-minimal).

Thus for $x \in X$, we have

$$\begin{aligned} d^c_{T\cup\{v\}}(x) &= |\mathcal{C}(x,X) \setminus \{c(vx)\}| + |\mathcal{C}(x,K) \setminus \mathcal{C}(x, X\cup\{v\})| + 1 \\ &\leqslant d^+_{D[X]}(x) + 1 + q(x). \end{aligned}$$

By Ineq (12), we have $d^c(x) \leqslant d^+_{D[X]}(x) + 1 + q(x) + n - d^c(v) - \Delta^{mon}(G)$. Hence

$$\begin{aligned} q(x) &\geqslant d^c(x) + d^c(v) - n - d^+_{D[X]}(x) + \Delta^{mon}(G) - 1 \\ &\geqslant 2k - 3 - d^+_{D[X]}(x) + \Delta^{mon}(G) - 1. \end{aligned}$$

Since $q(x) \leqslant |K| \leqslant 2k-3$, we have $d^+_{D[X]}(x) \geqslant \Delta^{mon}(G) - 1$ for $x \in X$. It is clear that

$$d^+_{D[X]}(x) = d^-_{D[X]}(x) = \Delta^{mon}(G) - 1,$$

as $d^-_{D[X]}(x) \leqslant \Delta^{mon}(G) - 1$ for $x \in X$. Then $q(x) = 2k-3$ for all $x \in X$.

By Ineq (12), for $y \in Y$,

$$\begin{aligned} \frac{n+2k-3}{2} \leqslant d^c(y) &\leqslant |Y| - 1 + |\mathcal{C}(y,K) \setminus \mathcal{C}(y, U\cup\{v\})| + 1 + n - \Delta^{mon}(G) - d^c(v) \\ &\leqslant \Delta^{mon}(G) + q(y) + n - \Delta^{mon}(G) - d^c(v) \\ &\leqslant \frac{n-2k+3}{2} + q(y). \end{aligned} \tag{14}$$

Then $q(y) = 2k-3$ for all $y \in Y$.

To sum up, $q(u) = \big|K\big| = 2k-3$ for all $u \in U$. Note that $|K| = \tau(G)$ as seen in Lemma 15. Therefore, (1) holds in this case.

According to the definition of $q(u)$, $K$ is a rainbow neighbor set of $u$ and $c(vu) \notin \mathcal{C}(u,K)$ for all $u \in U$. Then (2) holds for $u \in U$ in this case.

Meanwhile $d^-_{D[X]}(x) = \Delta^{mon}(G) - 1$ implies that $d_{c(vx)}(x) = d^{mon}(x) = \Delta^{mon}(G)$ and $E[X,Y] = \emptyset$ as $c(xy) = c(vx)$ if $xy \in E(G)$. Hence (3) holds for this case. □

**Claim 2.** $\Gamma$ contains a $A_0$-partition $(A_0, A_1, \ldots, A_p)$ with $\nu(\Gamma) = k-1$, $|A_0| = 1$ and $p \geqslant 3$.

*Proof.* Since $V(\Gamma) \geqslant 2\nu(\Gamma)+2$, $A_0$ is not empty by Lemma 15. Since $2k-3 = 2(k-1)-1 = \tau(\Gamma) \leqslant 2\nu(\Gamma) - |A_0|$ by (1) of Claim 1 and Lemma 15, we have $\nu(\Gamma) = k-1$ and $|A_0| = 1$. Thus $\tau(\Gamma) = 2\nu(\Gamma) - 1$, which gives that $\tau(\Gamma) = |V(\Gamma)| - p$ by (2) of Lemma 15. Then we have $p \geqslant 3$. □

W.l.o.g., we assume that $K_0$ is a minimum covering satisfying (2) of Lemma 15. Let $X$ be a maximum rainbow neighborhood of $v$ in $N(v) \setminus (K_0 \cup Y)$. Set $U = X \cup Y$ and $T = K_0 \cup U$. Now we distinguish two cases.

**Case 1.** $K_0 \setminus A_0 \neq \emptyset$. That is $K_0 \cap A_i \neq \emptyset$ for some $i \in [p]$.

For such $w \in K_0 \cap A_i$, there exists at most one vertex $u' \in A_i \setminus K_0$, such that $wu' \in E(\Gamma) = \Phi(v)$ as $|A_i \setminus K_0| = 1$. Since $|X| \geqslant d^c(v) - |K_0| - 1 \geqslant \frac{n+2|K_0|-3}{2} - (2k-3) - 1 \geqslant 3$, there must exist one vertex $u \in X \setminus A_i$ such that $c(vu) \neq c(vw)$. If $w, v \in V(\Gamma)$, since $w$ and $v$ are in different components of $\Gamma - B_0$, we have $wu \notin E(\Gamma)$. Then $c(wu) \in \{c(vw), c(vu)\}$. By (2) of Claim 1, $c(wu) = c(vw)$. Thus, $c(vw) \notin \mathcal{C}(v, U)$ as $G$ is edge-minimal. Thus $d^{mon}(w) \geqslant d_{c(vw)}(w) \geqslant |U \cup \{v\}| - |\{u'\}|$. Since $|U| = |T| - |K_0| \geqslant d^c(v) + \Delta^{mon}(G) - 1 - (2k-3)$ by (1) of Claim 1, we have the following inequality:

$$\begin{aligned} d^{mon}(w) &\geqslant \frac{n+2k-3}{2} + \Delta^{mon}(G) - (2k-3) - 1 \\ &\geqslant \Delta^{mon}(G) + 1 \end{aligned}$$

as $n \geqslant 2k+1$. This is a contradiction.

**Case 2.** $K_0 = A_0$.

Since $|A_0| = |\{w\}| = 1$ and $|K_0| = 2k-3$ by (1) of Claim 1, we have $k = 2$. By Claim 2, $p \geqslant 3$ implies that there exist at least three vertices $u_1, u_2$ and $u_3$ such that $wu_i \in \Phi(v)$ for $1 \leqslant i \leqslant 3$. If $u_i \in X$, then $u_i$ is also a vertex with $d^{mon}(u) = \Delta^{mon}(G)$ by (3) of Claim 1, we have $|\Phi(u_i)| \geqslant 2$. Since $vwu_iv$ are rainbow triangles for $i = 1, 2, 3$. If there exists a rainbow triangle containing $vu_i$ or $wu_i$, there exist 2 rainbow triangles sharing one common edge. Therefore there exist two rainbow triangles with exactly one common vertex $u_i$, a contradiction. Thus, $u_i \in Y$ for $1 \leqslant i \leqslant 3$. Since $E[X, Y] = \emptyset$ and $|X| \geqslant d^c(v) - 2 \geqslant \frac{n-3}{2}$, $Y$ is properly colored as $\delta^c(G) \geqslant \frac{n+1}{2}$. Thus, $u_1u_2u_3u_1$ is a rainbow triangle. Therefore there exist two rainbow triangles with exactly one common vertex $u_i$, a contradiction. □

## 4 Concluding remarks

One may wonder the sharpness of Theorems 3 and 4. For Theorem 3, by Example 1, we know when $k \geqslant \frac{n}{3} + 1$ (in this case, the subgraph $B_k$ is with order $\Theta(n)$), the color degree guaranteeing a properly colored $B_k$ should be larger than $\frac{n+k-1}{2}$.

For uncolored friendship subgraphs, we first prove the following result.

**Proposition 16.** *Let $k \geqslant 2$ be a positive integer and $G$ be a graph on $n \geqslant 3k-1$ vertices. If $\delta(G) \geqslant \frac{n+k-1}{2}$ then $G$ contains a $F_k$.*

*Proof of Proposition 16.* We proceed the proof by induction on $k$. The basic case $k = 2$ is easily derived from Theorem 4. Let $k \geqslant 3$ and suppose the result holds for $k-1$. Suppose to the contrary that there is no $F_k$ in $G$. Let $v$ be the center of a $F_{k-1}$ and $S = V(F_k - v)$. Since $\delta(G) \geqslant \frac{n+k-1}{2}$, each edge is contained in at least $k-1$ triangles. As $n \geqslant 3k-1$, there exist $k$ vertices in $N(v) \setminus S$. According to pigeonhole principle, there exists a $F_k$ in $N[v]$. □

As we have already mentioned in the introduction, as corollaries of the result of Erdős et al. [11] on $k$-fans and Erdős' conjecture on books (see [10, 8, 20]), respectively, the following results hold.

**Proposition 17.** *Let $k \geqslant 2$ be a positive integer and $G$ be a graph on $n \geqslant 50k^2$ vertices. If $\delta(G) \geqslant \frac{n+1}{2}$ then $G$ contains a $F_k$.*

**Proposition 18.** *Let $k \geqslant 2$ be a positive integer and $G$ be a graph on $n \geqslant 6k$ vertices. If $\delta(G) \geqslant \frac{n+1}{2}$ then $G$ contains a $B_k$.*

So by reasoning the above results, Theorems 3 and 4 should be at least asymptotically tight when $k = o(n)$.

For Theorem 4, the corresponding color degree condition may be acceptable when one considers a nearly spanning $F_k$. One evidence is that, if we consider the color degree condition for the spanning $F_k$ (that is, $k = \frac{n-1}{2}$), then the sufficient condition is that $\delta^c \geqslant n - 1$ which equals to $\frac{n+2k}{2} + O(1)$ (see the following).

**Fact 19.** *Let $G$ be an edge-colored graph on $n$ vertices where $n$ is odd. If $\delta^c(G) \geqslant n - 1$ then $G$ contains a properly-colored $F_{\frac{n-1}{2}}$.*

### Acknowledgments

The authors are very grateful to the anonymous referee for his/her very carefully reading of the manuscript and useful comments. The results presented here were reported by the first author in a workshop to congratulate the birthday of Prof. Jinjiang Yuan. The authors are also very grateful to Prof. Jinjiang Yuan for his encouragement and comments. X. Chen was supported by NSFC (No. 12301457) and B. Ning was partially supported by NSFC (No. 12371350) Fundamental Research Funds for the Central Universities, Nankai University (No. 63243151).

## References

[1] R. Aharoni, M. DeVos, and R. Holzman. Rainbow triangles and the Caccette-Häggkvist conjecture. *J. Graph Theory*, 92(4): 347–360, 2019.

[2] J. Balogh, P. Hu, B. Lidický, F. Pfender, J. Volec, and M. Young. Rainbow triangles in three-colored graphs. *J. Combin. Theory. Ser. B*, 126: 83–113, 2017.

[3] C. Berge. Sur le couplage maximum d'un graphe (French). *C. R. Acad. Sci. Paris*, 247:258–259, 1958.

[4] L. Caccetta and R. Häggkvist. On minimal digraphs with given girth. Proc. Ninth Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Boca Raton, FLA, 181–187, 1978.

[5] A. Czygrinow, T. Molla, B. Nagle, and R. Oursler. On odd rainbow cycles in edge-colored graphs. *European J. Combin.*, 94: Paper No. 103316, 10 pp, 2021.

[6] M. Devos, M. Drescher, D. Funk, S. de la Maza, K. Guo, T. Huynh, B. Mohar, and A. Montejano. Short rainbow cycles in graphs and matroids. *J. Graph Theory*, 96(2): 192–202, 2020.

[7] L. Ding, J. Hu, G. Wang, and D. Yang. Properly colored short cycles in edge-colored graphs. *European J. Combin.*, 100, Paper No. 103436, 12 pp, 2022.

[8] C.S. Edwards. A lower bound for the largest number of triangles with a common edge. (unpublished manuscript), 1977.

[9] S. Ehard and E. Mohr. Rainbow triangles and cliques in edge-colored graphs. *European J. Combin.*, 84, Paper No. 103037, 12pp, 2020.

[10] P. Erdős. On a theorem of Rademacher-Turán. *Illinois J. Math.*, 6: 122–127, 1962.

[11] P. Erdős, Z. Füredi, R.J. Gould, and D.S. Gunderson. Extremal graphs for intersecting triangles. *J. Combin. Theory. Ser. B*, 64(1): 89–100, 1995.

[12] P. Erdős and A. Hajnal. On Ramsey like theorems. Problems and results. Combinatorics, Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972, Math. Inst. Appl., Oxford, 1972, pp. 123–140, Inst. Math. Appl., Southend, 1972.

[13] S. Fujita, R. Li, and G. Wang. Decomposing edge-coloured graphs under colour degree constraints. *Combin. Probab. Comput.*, 28(5): 755–767, 2019.

[14] S. Fujita, R. Li, and S. Zhang. Color degree and monochromatic degree conditions for short properly colored cycles in edge-colored graphs. *J. Graph Theory*, 87(3): 362–373, 2018.

[15] S. Fujita, B. Ning, C. Xu, and S. Zhang. On sufficient conditions for rainbow cycles in edge-colored graphs. *Discrete Math.*, 342(7): 1956–1965, 2019.

[16] T. Gallai. On factorisation of graphs. *Acta Math. Acad. Sci. Hung.*, 1: 133–153, 1950.

[17] P. Hompe, P. Pelikánová, A. Pokorná, and S. Spirkl. On Aharoni's rainbow generalization of the Cacceta-Hăggkvist conjecture. *Discrete Math.*, 344: 112319, pp 4, 2021.

[18] P. Hompe and S. Spirkl. Further approximations for Aharoni's rainbow generalization of the Caccetta-Hăggkvist conjecture. *Electronic J. Combin.*, 29(1), #P1.55, 2022.

[19] J. Hu, H. Li, and D. Yang. Vertex-disjoint rainbow triangles in edge-colored graphs. *Discrete Math.*, 343(12), 112117, 5 pp, 2020.

[20] N. Khadžiivanov and V. Nikiforov. Solution of a problem of P. Erdős about the maximum number of triangles with a common edge in a graph (in Russian). *C. R. Acad. Bulgare Sci.*, 32: 1315–1318, 1979.

[21] H. Li. Rainbow $C_3$'s and $C_4$'s in edge-colored graphs. *Discrete Math.*, 313(19): 1893–1896, 2013.

[22] B. Li, B. Ning, C. Xu, and S. Zhang. Rainbow triangles in edge-colored graphs. *European J. Combin.*, 36: 453–459, 2014.

[23] X. Li, B. Ning, Y. Shi, and S. Zhang. Counting rainbow triangles in edge-colored graphs. *J. Graph Theory*, 107(4): 742–758, 2024.

[24] H. Li and G. Wang. Color degree and heterochromatic cycles in edge-colored graphs. *European J. Combin.*, 33(8): 1958–1964, 2012.

[25] W. Mantel. Problem 28. soln. by H. Gouvental, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff. *Wiskundige Opgaven*, 10: 60–61, 1907.

[26] W. T. Tutte. The factorization of linear graphs. *J. London Math. Soc.*, 22: 107–111, 1947.

[27] L. Yuan. Anti-Ramsey numbers for paths. arXiv:2102.00807, 2021.