Combinatorics of Castelnuovo–Mumford Regularity of Binomial Edge Ideals

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Abstract

Since the introduction of binomial edge ideals by Herzog et al. and independently Ohtani, there has been significant interest in relating algebraic invariants of the binomial edge ideal with combinatorial invariants of the underlying graph. Here, we take up a question considered by Herzog and Rinaldo regarding Castelnuovo–Mumford regularity of block graphs. To this end, we introduce a new invariant $\nu(G)$ associated to any simple graph G, defined as the maximal total length of a certain collection of induced paths within G subject to conditions on the induced subgraph. We prove that for any graph G, $\nu(G) \leqslant \operatorname{reg}(J_G) - 1$, and that the length of a longest induced path of G is less than or equal to $\nu(G)$; this refines an inequality of Matsuda and Murai. We then investigate the question: when is $\nu(G) = \operatorname{reg}(J_G) - 1$? We prove that equality holds for closed graphs, and for bipartite graphs G such that J_G is Cohen-Macaulay. For block graphs, we prove that $\nu(G)$ admits a combinatorial characterization independent of any auxiliary choices, and we prove that $\nu(G) = \operatorname{reg}(J_G) - 1$. This gives $\operatorname{reg}(J_G)$ a combinatorial interpretation for block graphs, and thus answers the question of Herzog and Rinaldo.

Mathematics Subject Classifications: 13C70, 05E40, 13F65

1 Introduction

Castelnuovo–Mumford regularity, introduced by David Mumford in the 1960s [MB66], is a fundamental invariant in commutative algebra and algebraic geometry that roughly measures how *complicated* a module or sheaf is. It is an interesting and difficult question to provide a combinatorial interpretation of the Castelnuovo–Mumford regularity for families of ideals possessing an underlying combinatorial structure. One such family of ideals, studied extensively over the past decade, is the class of binomial edge ideals J_G associated to a graph G. These were introduced by Herzog et al. [HHH$^+$10] and independently Ohtani [Oht11]; see Section 2 for precise definitions. There are elegant combinatorial upper bounds for $\operatorname{reg}(J_G)$, the Castelnuovo–Mumford regularity of the binomial edge ideal, in

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terms of: maximum number of clique disjoint edges of the graph [MMK21] and number of vertices of the graph [KSM16]. On the other hand, Matsuda and Murai proved that the length of a longest induced path of G gives a lower bound for $\operatorname{reg}(J_G)-1$ [MM13, Corollary 2.3]. More recently, [ASS24] and [JVS24] have investigated the question of giving a lower bound on $\operatorname{reg}(J_G) - 1$ via the v-domination number of binomial edge ideals. Inspired by Matsuda and Murai's lower bound, we ask the question:

Question 1. When is it the case that $\sum_{i=1}^{\ell}|E(P_i)| \leqslant \operatorname{reg}(J_G) - 1$ for vertex-disjoint induced paths $P_1, \dots, P_\ell$ of G?

Notice that the case of $\ell = 1$ is the Matsuda–Murai lower bound. In general, an arbitrary choice of vertex-disjoint induced paths does not realize a lower bound for $\operatorname{reg}(J_G)-1$. For instance, the complete graph on $n \geqslant 2$ vertices has $\operatorname{reg}(J_G) = 2$ yet $\lfloor \frac{n}{2} \rfloor$ vertex-disjoint induced edges. Our observations are that (1) it is possible to label the vertices of G so that each path P_i corresponds to a monomial in the generating set of $\operatorname{in}_{\text{lex}}(J_G)$ via Herzog et al's characterization of $\operatorname{in}_{\text{lex}}(J_G)$; and (2) that with restrictions on the edges appearing between the P_i's the corresponding monomials realize a regular sequence whose free resolution is a subcomplex of the free resolution of $\operatorname{in}_{\text{lex}}(J_G)$. Thus, this provides a lower bound on $\operatorname{reg}(\operatorname{in}_{\text{lex}}(J_G))$ in terms of the total number of edges appearing in the P_i. From this lower bound on the Castelnuovo–Mumford regularity of the initial ideal, we obtain a lower bound on the Castelnuovo–Mumford regularity for J_G via Conca–Varbaro's theorem on the preservation of extremal Betti numbers for ideals with squarefree initial ideal. In Section 3, we define the invariant $\nu(G)$ along these lines, and we prove the inequality

$$\nu(G) \leqslant \operatorname{reg}(J_G) - 1,$$

(Theorem 30).

In the remainder of this paper, we take up the question of equality of $\nu(G)$ and $\operatorname{reg}(J_G) - 1$. Various authors have considered the question of describing $\operatorname{reg}(J_G)$, a purely algebraic invariant, in terms of combinatorial properties of G. Ene and Zarojanu showed that $\operatorname{reg}(J_G) - 1$ agrees with the length of the longest induced path of the graph when G is a closed graph [EZ15]. Jayanthan and Kumar gave a combinatorial interpretation of $\operatorname{reg}(J_G)$ when G is bipartite and J_G is Cohen–Macaulay [JK19]. The graphs having $\operatorname{reg}(J_G) \leqslant 3$ have been classified by Kiani and Saeedi Madani in [SMK12] and [SMK18]. For the computation of $\operatorname{reg}(J_G)$ in further cases, see the survey article [MD22]. In Section 4, we show that $\nu(G)$ agrees with $\operatorname{reg}(J_G) - 1$ when G is a closed graph (Corollary 33), and when G is bipartite and J_G is Cohen-Macaulay (Corollary 37).

In Sections 5, 6, and 7, we take up the question of understanding $\operatorname{reg}(J_G)$ in the case when G is a block graph. Previously, various authors have considered the case when G is a tree (a special case of a block graph); see, for instance, Jayanthan et al. [JNR19] and the references therein. In [HR18], Herzog and Rinaldo studied the extremal Betti numbers of J_G when G is a block graph and obtained a combinatorial characterization for $\operatorname{reg}(J_G)$ for a subclass of all block graphs. The question of providing a combinatorial description of $\operatorname{reg}(J_G)$ when G is a block graph has remained open and was singled out by Herzog and Rinaldo as an important open question in the theory of binomial edge ideals [HR18].

In Section 7, we prove the main result of this paper:

Theorem 2. *For a block graph G,*

1. *$\nu(G)$ admits a combinatorial characterization solely in terms of vertex-disjoint induced paths of G that do not admit certain induced subgraphs (Theorem 51),*
2. *$\nu(G) = \mathrm{reg}(J_G) - 1$ (Theorem 64).*

Theorem 51 does not depend on *any* choice of labeling of the induced paths nor on a choice of labeling of the block graph G. We prove Theorem 64 by adapting the theory of Malayeri–Saeedi Madani–Kiani developed in [MMK21], where they provide a method to check whether a function is an upper bound for $\mathrm{reg}(J_G) - 1$ for every graph G. This answers the question of Herzog and Rinaldo. Moreover, this work shows that $\nu(G)$ gives a uniform computation for $\mathrm{reg}(J_G) - 1$ across many of the families of graphs considered thus far in the literature.

2 Background

2.1 Graphs

A (multi)graph G is a pair (V, E) where V is a set and the elements are called vertices and E is a multiset of pairs of vertices $\{a, b\}$, where we possibly allow repetition of the vertices appearing in an edge. When we wish to emphasize the vertex set (respectively edge set) of G, we write $V(G)$ (respectively $E(G)$). An element appearing in E multiple times is called a multi-edge, and an element of E of the form $\{v, v\}$ for some $v \in V$ is called a loop. A graph having no loops nor multi-edges is called a simple graph, whereas a graph potentially having loops or multi-edges is called a multigraph. In this paper, when we write ‘graph’ without the adjective ‘simple’ or ‘multigraph,’ we implicitly mean a simple graph. When we wish to consider a multigraph, we explicitly state that the graph is a multigraph. By a labeling of a set of vertices $W \subseteq V$, we mean a choice of an injective map $\phi : W \to S$ where S is a set of labels. When the vertices of G have been labeled by a set possessing a total order, we utilize the notation $v < w$ for vertices v and w to mean that $\phi(v) < \phi(w)$, where ϕ is the choice of labeling. By $[n]$, we denote the set of integers from 1 to n inclusive. By a graph G on $[n]$, we mean that $|V(G)| = n$, and there is a labeling of the vertices of G by $[n]$.

For a vertex $v \in V(G)$, we define the **neighbors** of v in G as the set:

$$N_G(v) := \{w \mid \{v, w\} \in E(G)\},$$

and we define the **degree** of v in G by

$$\deg_G(v) := |N_G(v)|.$$

We recall the following graph-theoretic constructions. For further information on the terminology introduced here, we refer the reader to [Wes96]. For a graph G and

$W \subseteq V(G)$, we define the **induced subgraph of G on W**, which we denote by $\mathrm{Ind}_G(W)$, as follows:

$$V(\mathrm{Ind}_G(W)) := W$$
$$E(\mathrm{Ind}_G(W)) := \{\{a,b\} \mid \{a,b\} \in E(G), a \in W, b \in W\}.$$

Given a subgraph H of G, we say that H is an **induced subgraph** of G if $\mathrm{Ind}_G(H) = H$. For a graph G and $W \subseteq V(G)$, we define the graph $G \smallsetminus W$ as follows:

$$V(G \smallsetminus W) := V(G) \smallsetminus W$$
$$E(G \smallsetminus W) := \{\{a,b\} \in E(G) \mid a \notin W, b \notin W\}.$$

For a connected graph G and $v \in V(G)$, we say that v is a **cut vertex** of G if $G \smallsetminus \{v\}$ has strictly more connected components than G.

We recall the well-known result that being an induced subgraph is transitive.

Lemma 3. *Let K be an induced subgraph of H, and H be an induced subgraph of G. Then, K is an induced subgraph of G.*

For a graph G, we define a **path** of G to be either: (i) a singleton vertex v having no edges, or (ii) a sequence of vertices and edges $v_1, e_1, v_2, e_2, \ldots, v_{n-1}, e_{n-1}, v_n$ for some $n \geqslant 1$ satisfying:

1. $v_i \in V(G)$,
2. $e_i \in E(G)$,
3. $v_i \neq v_j$ for all $i \neq j$, and
4. $e_j = \{v_j, v_{j+1}\}$ for all $1 \leqslant j \leqslant n-1$.

We denote such a path by $[v_1, \ldots, v_n]$. For a path P of G and $v \in V(P)$, we say that v is a **terminal vertex** of P if $\deg_P(v) = 1$. We say that a vertex $v \in V(P)$ is an **internal vertex** if $\deg_P(v) = 2$. We denote by ∂P the set of terminal vertices of P, and we denote by P° the set of internal vertices of P.

Definition 4. Let P_i be an induced path of G for $1 \leqslant i \leqslant \ell$, and suppose that $V(P_i) \cap V(P_j) = \varnothing$ for all $1 \leqslant i < j \leqslant \ell$. We call an edge $e = \{a,b\} \in E(G)$ an **induced edge** with respect to $P_1, \ldots, P_\ell$ if $a \in P_i$ and $b \in P_j$ for some $1 \leqslant i \neq j \leqslant \ell$. We denote by P_{Ind} the induced subgraph of G on the vertices $\bigcup_{i=1}^{\ell} V(P_i)$. We call a vertex $v \in V(P_{\mathrm{Ind}})$ an **internal vertex** (respectively, **terminal vertex**) of P_{Ind} whenever v in an internal vertex (respectively, terminal vertex) of P_i for some $1 \leqslant i \leqslant \ell$.

We recall the definition of a directed graph and a key lemma about directed acyclic graphs.

Definition 5. A **directed graph** G consists of a set of vertices V and a set of directed edges (or arcs) A. We denote a **directed edge** of a graph G from the vertex v to the vertex w (with $v \neq w$) by (v, w). Given a directed edge (v, w), we say that v and w are the **initial vertex** and **terminal vertex**, respectively. We use the notation $A(G)$ to emphasize the set of arcs associated to G.

We write **directed multigraph** to indicate that repetitions of directed edges are allowed and that directed edges having the same initial and terminal vertex are allowed. We explicitly state when an object is a directed multigraph; otherwise, by directed graph, we always assume that there are no loops or repetitions of directed edges.

We say that a directed (multi)graph G is **directed acyclic** if G does not contain any directed cycle or any loop. A **topological ordering** or **topological sorting** of a directed (multi)graph G is an integer labeling of the vertices of G such that whenever (i, j) is a directed edge of G, then $j < i$.

Lemma 6 ([TS92, Theorem 5.13, p.118])**.** *Let G be a directed graph. G is directed acyclic if and only if G admits a topological sorting.*

2.2 Binomial Edge Ideals

The main object of study in this paper are binomial edge ideals, introduced by Herzog et al. [HHH$^+$10] and Ohtani [Oht11], which associate to any simple graph a binomial ideal as follows. For a survey of binomial edge ideals, the reader is referred to [SM16] or [MD22].

Definition 7 ([HHH$^+$10])**.** Let $G = (V, E)$ be a finite simple graph with vertex set V labeled by $\{1, \dots, n\}$ and edge set E. Fix a field $\mathbb{K}$. Consider the polynomial ring $S := \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$, and for each edge $\{i, j\} \in E$ with $i < j$ define $f_{ij} := x_i y_j - x_j y_i \in S$. Define the binomial edge ideal of G, denoted J_G, to be the ideal

$$J_G := (\{f_{ij} \mid \{i, j\} \in E\}). \tag{1}$$

In [HHH$^+$10], the authors provided a combinatorial description for a Gröbner basis of J_G with respect to the lexicographic term order on S induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. Throughout this paper, we only consider this term order on S. We recall their result below.

Definition 8 ([HHH$^+$10, p.6])**.** Let G be a simple graph on $[n]$, and let i and j be two vertices of G with $i < j$. A path on the vertices $i_0, i_1, \dots, i_r$ of G with $i = i_0$ and $i_r = j$ is called **admissible** if:

1. $i_k \neq i_l$, for all $1 \leqslant k \neq l \leqslant r$,
2. for each $k = 1, \dots, r - 1$ one has either $i_k < i$ or $i_k > j$,
3. for any proper subset $\{j_1, \dots, j_s\}$ of $\{i_1, \dots, i_{r-1}\}$ the sequence $i, j_1, \dots, j_s, j$ is not a path.

Given such an admissible path, we define the monomial

$$u_\pi = \left(\prod_{i_k>j} x_{i_k}\right)\left(\prod_{i_l<i} y_{i_l}\right),$$

and we denote by m_π the monomial $x_i y_j u_\pi$.

Remark 9. Item 1 and Item 3 of Definition 8 establish that π is an induced path of G.

Theorem 10 ([HHH+10, Theorem 2.1])**.** *Let G be a simple graph on $[n]$. Let $<$ be the lexicographic order on $S = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$ induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. Then, the set of binomials*

$$\mathscr{G} := \bigcup_{i<j} \{u_\pi f_{ij} \mid \pi \text{ is an addmissible path from } i \text{ to } j\}$$

is a reduced Gröbner basis of J_G.

Consequently, J_G is a radical ideal [HHH+10, Corollary 2.2].

2.3 Castelnuovo–Mumford Regularity

We recall the definition of Castelnuovo–Mumford regularity of a finitely generated graded R-module, where R is a polynomial ring. Let $R := \mathbb{K}[z_1, \dots, z_m]$ be standard graded. Given a finitely generated graded R-module M, let

$$F_\bullet : 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0$$

be a minimal graded free R-resolution of M where $F_i = \bigoplus_{j\in\mathbb{Z}} R(-j)^{b_{ij}}$. The b_{ij} are the **graded Betti numbers** of M, non-negative integers, and for each i, only finitely many of the b_{ij} are non-zero. The **Castelnuovo–Mumford regularity** of M is defined as follows:

$$\operatorname{reg}(M) := \max\{j - i \mid b_{ij} \neq 0\}. \tag{2}$$

The reader is referred to [Pee11] or [BCV21] for further information regarding Castelnuovo-Mumford regularity.

The next result of Conca and Varbaro [CV20] shows that under the assumption that a homogeneous ideal has a squarefree initial ideal (with respect to some term order), then the extremal Betti numbers of the ideal and of its initial ideal coincide; in particular, their regularities coincide.

Theorem 11 ([CV20, Corollary 2.7])**.** *Let $I \subseteq R := \mathbb{K}[z_1, \dots, z_m]$ be a homogeneous ideal such that $\operatorname{in}(I)$ is square-free with respect to some term order (not necessarily lexicographic order). Then, the extremal Betti numbers of R/I and those of $R/\operatorname{in}(I)$ coincide (positions and values). In particular, $\operatorname{reg}(R/I) = \operatorname{reg}(R/\operatorname{in}(I))$.*

3 The Invariant $\nu(G)$

3.1 Definition and Motivation

It is a result of Matsuda and Murai that:

Theorem 12 ([MM13, Corollary 2.2])**.** *If H is an induced subgraph of G, then*

$$\operatorname{reg}(S/J_H) \leqslant \operatorname{reg}(S/J_G).$$

Theorem 12 is most often used in the form of the following corollary.

Corollary 13 ([MM13, Corollary 2.3])**.** *Let G be a graph, then*

$$\ell(G) \leqslant \operatorname{reg}(S/J_G)$$

where $\ell(G)$ is the length of a longest induced path within G.

However, Theorem 12 also implies the slightly stronger result that if $P_1, \ldots, P_\ell$ are vertex-disjoint induced paths of G having no induced edges (Definition 4), then

$$\sum_{i=1}^{\ell} |E(P_i)| \leqslant \operatorname{reg}(S/J_G).$$

It is perhaps natural to ask:

Question 14. What induced edges can we allow between vertex-disjoint induced paths $P_1, \ldots, P_\ell$ of G while retaining the lower bound

$$\sum_{i=1}^{\ell} |E(P_i)| \leqslant \operatorname{reg}(S/J_G)? \tag{3}$$

For example, it can be checked with Macaulay 2 [GS] that the graph G in Figure 1, consisting of the induced paths $[4, 6, 5]$ and $[1, 3, 2]$ together with the induced edge $\{3, 4\}$, satisfies $\operatorname{reg}(S/J_G) = 4$.

3.2 Directed Oriented Induced Paths

We introduce the following definition, which provides a sufficient condition for vertex-disjoint induced paths to satisfy equation (3).

Definition 15. Let P be an induced path of a graph G. We say that P together with a surjection $\phi_P : \{1, 2\} \to \partial P$ is an **oriented induced path**. We say that ϕ_P is an **orientation**.

Remark 16. When an oriented path P has exactly one vertex, $\phi_P(1) = \phi_P(2)$. Otherwise, ϕ_P is a bijection. In the latter case, we think of ϕ as specifying a start and an end vertex for P_i. This distinction between the terminal vertices of P_i is necessary due to the asymmetry between the terminal vertices of admissible paths in Definition 8.

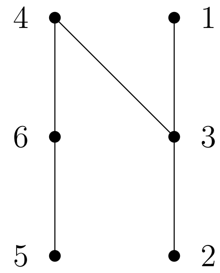


Figure 1: DOIP Paths

Definition 17. Let $\underline{P} := P_1, \ldots, P_\ell$ be vertex-disjoint induced paths of G. Let P_{Ind} denote the induced subgraph of G on $\bigcup_{i=1}^{\ell} V(P_i)$. For a choice of

1. σ a permutation on the set $\{1, \ldots, \ell\}$, and
2. orientations $\phi_i : \{1, 2\} \to \partial P_i$ for $1 \leqslant i \leqslant \ell$,

we say that $(\underline{P}, \sigma, \underline{\phi_i})$ are **directed oriented induced paths (DOIP)** if whenever $\sigma(i) \leqslant \sigma(j)$ and Q is an induced path of P_{Ind} having terminal vertices $\phi_{\sigma(i)}(1)$ and $\phi_{\sigma(j)}(2)$, then Q contains P_k as subgraph for some $1 \leqslant k \leqslant \ell$. We say that $\underline{P}$ is DOIP if there exists a choice of σ and orientations $\underline{\phi_i}$ such that $(\underline{P}, \sigma, \underline{\phi_i})$ is DOIP.

Example 18. In Figure 1, we define the paths $P_1 = [1, 3, 2]$ and $P_2 = [4, 6, 5]$. Then, P_1 and P_2 are DOIP. Indeed, we let $\sigma = \text{id}_{\{1,2\}}$, and $\phi_1(1) = 1$, $\phi_1(2) = 2$, $\phi_2(1) = 4$, and $\phi_2(2) = 5$. It is now clear that any induced path of P_{Ind} from vertex 1 to either vertex 2 or vertex 4 contains either P_1 or P_2. Likewise, for any induced path from vertex 4 to vertex 5.

Remark 19. If in Example 18, we were to change σ from the identity permutation to the transposition $(2\,1)$ while keeping P_1, P_2, ϕ_1, and ϕ_2 as in Example 18, then $[4, 3, 2]$ is an induced path of P_{Ind} from $\phi_{\sigma(1)}(1)$ to $\phi_{\sigma(2)}(2)$ which does not contain P_1 or P_2.

If in Example 18, we were to keep P_1, P_2, and ϕ_1 unchanged, and we were to change σ from the identity permutation to the transposition $(2\,1)$ and ϕ_2 to $\phi_2(1) = 5$ and $\phi_2(2) = 4$, then P_1 and P_2 are DOIP with respect to these choices.

In the next example, we demonstrate vertex-disjoint induced paths which are not DOIP.

Example 20. We consider Figure 2. Consider the induced paths $P_1 = [1, 3, 2]$ and $P_2 = [4, 6, 5]$ in each of the three graphs depicted in this figure. Then, P_1 and P_2 are not DOIP. For each of these graphs, any pair of terminal vertices from P_1 and P_2 can be connected by an induced path not containing P_1 or P_2. The existence of such paths is an obstruction to the paths P_1 and P_2 being DOIP.

Furthermore, for the center and rightmost graphs, there is an induced path connecting the terminal vertices of P_1, which does not contain P_1 nor P_2. The existence of such a path is also an obstruction to the DOIP property.

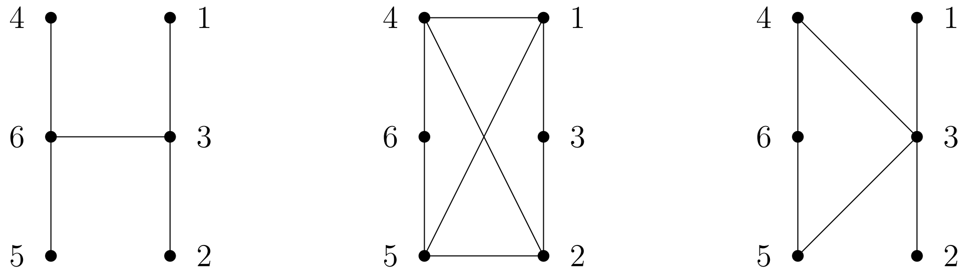


Figure 2: Non-DOIP paths

We observe that we can find subpaths in these graphs which are DOIP; the paths $P_1 = [4, 6]$ and $P_2 = [1, 3, 2]$ are DOIP for all of these graphs.

The notion of $P_1, \ldots, P_\ell$ being DOIP captures the idea that any induced path Q of P_{Ind} with $\partial Q \subseteq \bigcup_{i=1}^{\ell} \partial P_i$, such that Q does not contain some P_k, *travels* from top to bottom and from left to right. We make this idea precise using the notion of directed acyclic graphs.

Definition 21. Let $\underline{P} := P_1, \ldots, P_\ell$ be vertex-disjoint induced paths of G, let P_{Ind} be the induced subgraph of G on $\bigcup_{i=1}^{\ell} V(P_i)$, and let $\phi_i : \{1, 2\} \to \partial P_i$ be orientations for $1 \leqslant i \leqslant \ell$. Then, we define $K_{P_{\text{Ind}}}$ to be the directed multigraph with vertex set $[\ell]$, and with a multiarc (i, j) for each induced path Q from $\phi_i(1)$ to $\phi_j(2)$ whenever $1 \leqslant i, j \leqslant \ell$ and Q does not contain P_k for every $1 \leqslant k \leqslant \ell$.

Remark 22. Up to isomorphism of multigraphs, $K_{P_{\text{Ind}}}$ does not depend on the choice of labeling σ of the paths $P_1, \ldots, P_\ell$. However, as Example 23 illustrates, $K_{P_{\text{Ind}}}$ does depend on the choice of orientations ϕ_i.

Example 23. In Figure 3, let $P_1 = [1, 2]$ and $P_2 = [3, 5, 4]$. Let $\phi_1(1) = 1$, $\phi_1(2) = 2$, $\phi_2(1) = 3$, and $\phi_2(2) = 4$. Then, $K_{P_{\text{Ind}}}$ is the directed multigraph on the vertex set $\{1, 2\}$ with multiarcs: $(2, 1)$ corresponding to the induced path $[3, 5, 2]$.

Now, let us suppose that $\phi_1(1) = 2$, $\phi_1(2) = 1$, $\phi_2(1) = 3$, and $\phi_2(2) = 4$. Then, $K_{P_{\text{Ind}}}$ is the directed multigraph on the vertex set $\{1, 2\}$ with multiarcs: $(2, 1)$ corresponding to the induced path $[3, 1]$, and $(1, 2)$ corresponding to the induced path $[2, 5, 4]$.

Theorem 24. *Let $\underline{P} := P_1, \ldots, P_\ell$ be vertex-disjoint induced paths of G. Then, $\underline{P}$ is DOIP if and only if there exist orientations $\phi_i : \{1, 2\} \to \partial P_i$ for $1 \leqslant i \leqslant \ell$ such that the multigraph $K_{P_{\text{Ind}}}$ corresponding to these orientations is directed acyclic.*

Proof. ($\Longrightarrow$) Immediate consequence of Definition 17.

($\Longleftarrow$) By Lemma 6, there exists an ordering of the vertices of $K_{P_{\text{Ind}}}$ under which $K_{P_{\text{Ind}}}$ admits a topological sorting. Let σ be the bijection which realizes this ordering of the vertices of $K_{P_{\text{Ind}}}$. □

Corollary 25. *An induced path P_1 of G is DOIP.*

Proof. Follows immediately from Theorem 24, since $K_{P_{\text{Ind}}}$ is a singleton vertex possessing no multiarcs. □

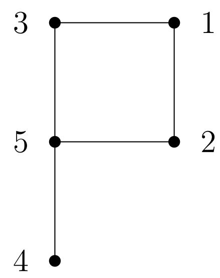


Figure 3: Dependence of $K_{P_{\mathrm{Ind}}}$ on Orientations

3.3 DOIP and Regularity

In this subsection, we establish that paths which are DOIP satisfy equation (3).

Definition 26. Let $m \in S$ be a monomial. We define the **support** of m to be the subset of variables of S which divide m. We denote by $\mathrm{Supp}(m)$ the support of m. For a monomial ideal $I \subseteq S$ and $W \subseteq S$ any set of monomials, we define

$$I_W := (\{m \in I \mid m \text{ is a monomial and } \mathrm{Supp}(m) \subseteq W\}).$$

Lemma 27. *Let $\underline{P} := P_1, \ldots, P_\ell$ be vertex-disjoint induced paths of a graph G which are DOIP. Then, there exists a labeling of the vertices of G such that P_i with respect to this labeling is an admissible path in the sense of Definition 8 for $1 \leqslant i \leqslant \ell$, and that*

$$(\mathrm{in}(J_G))_W = (m_1, \ldots, m_\ell). \tag{4}$$

Where we denote the monomial associated to P_i in Definition 8 by m_i, and we define $W := \bigcup_{i=1}^{\ell} \mathrm{Supp}(m_i)$.

Proof. We may suppose that the paths $P_1, \ldots, P_\ell$ are labeled such that $(\underline{P}, \mathrm{id}_{[\ell]}, \underline{\phi_i})$ is DOIP. For $1 \leqslant i \leqslant \ell$, label vertex $\phi_i(1)$ by the integer $2i-1$, and label vertex $\phi_i(2)$ by the integer $2i$. Label the remaining vertices of G by distinct consecutive integers larger than 2ℓ. With respect to this labeling, the P_i are admissible for each $1 \leqslant i \leqslant \ell$. Moreover, by Theorem 10, we have that

$$y_j \in W := \bigcup_{i=1}^{\ell} \mathrm{Supp}(m_i)$$

if and only if $j = 2i$ for $1 \leqslant i \leqslant \ell$.

We next establish equation (4). As the reverse inclusion is clear, it suffices to prove the forward inclusion. Let $m \in \mathrm{in}(J_G)$ be a monomial such that $\mathrm{Supp}(m) \subseteq W$. Then, there exist monomials $u \in S$ and $m' \in \mathrm{in}(J_G)$, a minimal generator, such that $m = u \cdot m'$. As Theorem 10 gives a reduced Gröbner basis of J_G, we have that $m' = m_Q$ for some admissible path Q of G. As $V(Q) \subseteq W$, it follows, in particular, that Q is an induced path of P_{Ind}. In order for $\text{Supp}(m_Q) \subseteq W$, it is necessary that one of the terminal vertices of Q is $\phi_i(2)$ for some $1 \leqslant i \leqslant \ell$. In order for Q to be admissible, it is necessary that the other terminal vertex of Q is $\phi_j(1)$ for some $1 \leqslant j \leqslant i$ (because all the other vertices of G are labeled by integers larger than $2i$). Now, as $\underline{P}$ is DOIP, Definition 17 implies that P_k is a subgraph of Q for some $1 \leqslant k \leqslant \ell$. We observe that:

1. Q does not contain P_r as a subgraph for $1 \leqslant r < j$. Otherwise, $y_{2r-1} \mid m_Q$, but $y_{2r-1} \notin W$ (because $x_{2r-1} \in W$).

2. Q does not contain P_r as a subgraph for $j < r < i$. Otherwise, Q contains the vertex $2r-1$. But $2r-1$ is strictly between the terminal vertices of Q, which are $2j-1$ and $2i$, contradicting Q being admissible.

3. Q does not contain P_r as a subgraph for $i < r \leqslant \ell$. Otherwise, $x_{2r} \mid m_Q$, but $x_{2r} \notin W$ (because $y_{2r} \in W$).

It follows from these observations that $j = k = i$. In order for Q to:

1. contain P_k as a subgraph,

2. be admissible, and

3. have terminal vertices $2j-1$ and $2i$,

it must be the case that $Q = P_k$. Consequently, $m \in (m_1, \ldots, m_\ell)$, which completes the proof. □

Example 28. We illustrate Lemma 27 in the context of the graph in Figure 1.

Let $P_1 = [1,3,2]$, and $P_2 = [4,6,5]$. We observe that $m_1 = x_1x_3y_2$ and that $m_2 = x_4x_6y_5$. Hence, $W = \{x_1, x_3, x_4, x_6, y_2, y_5\}$. It can be checked that

$$\text{in}(J_G) = (x_5y_6, x_4y_6, x_4x_6y_5, x_3y_4, x_2y_3, x_1y_3, x_1x_3y_2).$$

We see that

$$\text{in}(J_G)_W = (x_4x_6y_5, x_1x_3y_2).$$

Definition 29. Let G be a graph, we define the invariant

$$\nu(G) := \max\left\{\sum_{i=1}^{\ell} |E(P_i)| \mid \underline{P} \text{ is DOIP}\right\}.$$

Theorem 30. *Let G be a graph, then*

$$\nu(G) \leqslant \text{reg}(S/J_G).$$

Proof. It suffices to show that if $P_1, \ldots, P_\ell$ is DOIP, then

$$\sum_{i=1}^{\ell} |E(P_i)| \leqslant \operatorname{reg}(S/J_G).$$

Label the vertices of G as in Lemma 27. Since $\operatorname{in}(J_G)$ is a squarefree monomial ideal, we have by Theorem 11 that

$$\operatorname{reg}(S/\operatorname{in}(J_G)) = \operatorname{reg}(S/J_G).$$

It is well known that

$$\operatorname{reg}(S/\operatorname{in}(J_G)_W) \leqslant \operatorname{reg}(S/\operatorname{in}(J_G)).$$

(See, for example, [Pee11].) Lemma 27 implies that $\operatorname{in}(J_G)_W = (m_1, \ldots, m_\ell)$ is a complete intersection. Hence,

$$\begin{aligned} \operatorname{reg}(S/(m_1, \ldots, m_\ell)) &= \sum_{i=1}^{\ell} (\deg(m_i) - 1) \\ &= \sum_{i=1}^{\ell} |E(P_i)|, \end{aligned}$$

which completes the proof. □

We recover Matsuda and Murai's lower bound for Castelnuovo–Mumford regularity as a corollary.

Corollary 31 ([MM13, Corollary 2.2]). *For a graph G,*

$$\ell(G) \leqslant \nu(G) \leqslant \operatorname{reg}(S/J_G).$$

Proof. Follows immediately from Corollary 25 and Theorem 30. □

4 Equality of $\nu(G)$ and $\operatorname{reg}(S/J_G)$

In this section, we recall two families of graphs for which a combinatorial description of the Castelnuovo–Mumford regularity of the binomial edge ideal is known. We show that for these two families, their Castelnuovo–Mumford regularity coincide with the invariant $\nu(G)$.

4.1 Closed Graphs

In [HHH$^+$10], closed graphs were introduced. A graph G is said to be **closed** if there exists a labeling of the vertices such that the set

$$\{x_iy_j - x_jy_i \mid \{i,j\} \in E(G)\}$$

is a quadratic Gröbner basis of J_G ([HHH$^+$10, Theorem 1.1]). Crupi and Rinaldo showed that closed graphs are interval graphs [CR14, Theorem 2.4]. In [EZ15], Ene and Zarojanu computed the Castelnuovo–Mumford regularity for closed graphs.

Theorem 32 ([EZ15, Theorem 2.2])**.** *Let G be a closed graph, then*

$$\ell(G) = \operatorname{reg}(S/J_G).$$

Corollary 33. *Let G be a closed graph, then*

$$\ell(G) = \nu(G) = \operatorname{reg}(S/J_G).$$

Proof. Corollary 25 and Theorem 30 imply that

$$\ell(G) \leqslant \nu(G) \leqslant \operatorname{reg}(R/J_G).$$

Equality throughout now follows from Theorem 32. □

4.2 Cohen–Macaulay Bipartite Graphs

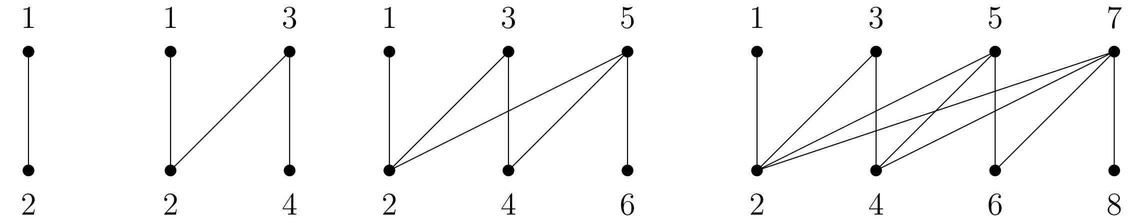
In [BMS18], the authors study when the binomial edge ideal of a bipartite graph is unmixed, and they give a combinatorial characterization of when the binomial edge ideal of a bipartite graph is Cohen–Macaulay. Using this characterization of Cohen–Macaulay binomial edge ideals of bipartite graphs, Jayanthan and Kumar computed the regularity for this family of graphs [JK19, Theorem 4.7]. We now recall this characterization of Cohen–Macaulay binomial edge ideals of bipartite graphs (we use the notation from both [BMS18] and [JK19]).

Definition 34 ([BMS18, p.2])**.** For every $m \geqslant 1$, let F_m be the graph on the vertex set $[2m]$ and with edge set

$$E(F_m) := \{(2i, 2j-1)\} \mid i = 1, \dots, m, j = i, \dots, m\}.$$

The operation $*$: For $i = 1, 2$, let G_i be a graph having at least one vertex f_i of degree one. We define $(G_1, f_1) * (G_2, f_2)$ to be the graph obtained by identifying the vertices f_1 and f_2.

The operation $\circ$: For $i = 1, 2$, let G_i be a graph with at least one vertex f_i of degree one, and let v_i be its neighbor, and we assume that $\deg_{G_i}(v_i) \geqslant 3$. We define $(G_1, f_1) \circ (G_2, f_2)$ to be the graph obtained from G_1 and G_2 by deleting the vertices f_1, f_2 and identifying the vertices v_1 and v_2.



(a) F_1 (b) F_2 (c) F_3 (d) F_4

Figure 4: F_m, $m \leqslant 4$

Figure 4 depicts F_i for $1 \leqslant i \leqslant 4$. Example 38 and Figure 5 includes a concrete description and illustration of these operations $*$ and $\circ$ for a particular bipartite graph G.

Theorem 35 ([BMS18, Theorem 6.1])**.** *Let G be a connected bipartite graph. The following properties are equivalent:*

1. *J_G is Cohen–Macaulay,*
2. *There exists $s \geqslant 1$ such that $G = G_1 * \cdots * G_s$, where $G_i = F_{n_i}$ or $G_i = F_{m_{i,1}} \circ \cdots \circ F_{m_{i,t_i}}$, for some $n_i \geqslant 1$ and $m_{i,j} \geqslant 3$ for each $j = 1, \ldots, t_i$.*

With the decomposition of G as in Theorem 35, define the following:

$$A = \{i \in [s] \mid G_i = F_{n_i}, n_i \geqslant 2\},$$
$$B = \{i \in [s] \mid G_i = F_{n_i}, n_i = 1\}, \text{ and}$$
$$C = \{i \in [s] \mid G_i = F_{m_{i,1}} \circ \cdots F_{m_{i,t_i}}, t_i \geqslant 2\}.$$

For each $i \in C$, let

$$C_i = \{j \in \{2, \ldots, t_i - 1\} \mid m_{i,j} \geqslant 4\} \cup \{1, t_i\}, \text{ and}$$
$$C_i' = \{j \in \{2, \ldots, t_i - 1\} \mid m_{i,j} = 3\}.$$

Set $\alpha = |A| + \sum_{i \in C} |C_i|$ and $\beta = |B| + \sum_{i \in C} |C_i'|$.

Theorem 36 ([JK19, Theorem 4.7])**.** *Let $G = G_1 * \cdots * G_s$ be a Cohen–Macaulay connected bipartite graph. Let α and β be defined as above, then* $\operatorname{reg}(S/J_G) = 3\alpha + \beta$.

Corollary 37. *Let $G = G_1 * \cdots * G_s$ be a Cohen–Macaulay connected bipartite graph. Then, $\nu(G) = \operatorname{reg}(S/J_G)$.*

Proof. For convenience of the proof we assume that the vertices of each G_i are labeled by the integers 1 through $|V(G_i)|$. For $1 \leqslant i \leqslant s$, we construct a path P_i inside G_i as follows:

1. If $i \in A$, let P_i be the path $[1, 2, 3, 4]$ on G_i,
2. If $i \in B$, let P_i be the path $[1, 2]$ on G_i,

3. Suppose $i \in C$ and that $G_i = F_{m_{i,1}} \circ \cdots F_{m_{i,t_i}}, t_i \geqslant 2$. For $1 \leqslant j \leqslant m_{i,t_i}$ construct a path $P_{i,j}$ inside $F_{m_{i,j}}$ as follows:

   (a) If $j = 1$, let $P_{i,j}$ be the path $[1, 2, 3, 4]$ on $F_{m_{i,1}}$.

   (b) If $j = t_i$, let $P_{i,j}$ be the path $[2 \cdot m_{i,t_i} - 3, 2 \cdot m_{i,t_i} - 2, 2 \cdot m_{i,t_i} - 1, 2 \cdot m_{i,t_i}]$ on $F_{m_{i,t_i}}$.

   (c) If $j \in C_i \setminus \{1, t_i\}$, let $P_{i,j}$ be the path $[3, 4, 5, 6]$ on $F_{m_{i,j}}$.

   (d) If $j \in C'_i$, let $P_{i,j}$ be the path $[3, 4]$ on $F_{m_{i,j}}$.

   Put $P_i = \bigcup_{j=1}^{m_{i,t_i}} P_{i,j}$.

We define $\underline{P}$ as $\bigcup_{i=1}^{s} P_i$ and P_{Ind} as the induced subgraph of G on $V(\underline{P})$. The construction of the P_i yields that for $1 \leqslant i \leqslant s-1$, either P_i and P_{i+1} share a terminal vertex or there is no induced edge between P_i and P_{i+1}. The construction of G via Definition 34 implies that there are no edges between $V(G_i)$ and $V(G_j)$ whenever $|i - j| > 1$. Hence, in particular, P_i and P_j have no induced edge whenever $|i - j| > 1$. It follows that P_{Ind} is a disjoint union of induced paths. Hence, in particular, $\underline{P}$ is DOIP. Thus,

$$\begin{aligned}
3\alpha + \beta &= \sum_{i=1}^{s} |E(P_i)| && \text{(by construction of the } P_i) \\
&\leqslant \nu(G) && (\underline{P} \text{ is DOIP}) \\
&\leqslant \operatorname{reg}(S/J_G) && \text{(Theorem 30)} \\
&= 3\alpha + \beta. && \text{(Theorem 36)}
\end{aligned}$$

□

Example 38. We consider the Cohen–Macaulay bipartite graph $G = G_1 * G_2 * G_3$ where $G_1 = (F_3 \circ F_4 \circ F_3 \circ F_4)$, $G_2 = F_1$, and $G_3 = F_4$. We illustrate the construction of $\underline{P}$ in the proof of Corollary 37 via Figure 5. The graph G_1 is constructed by identifying:

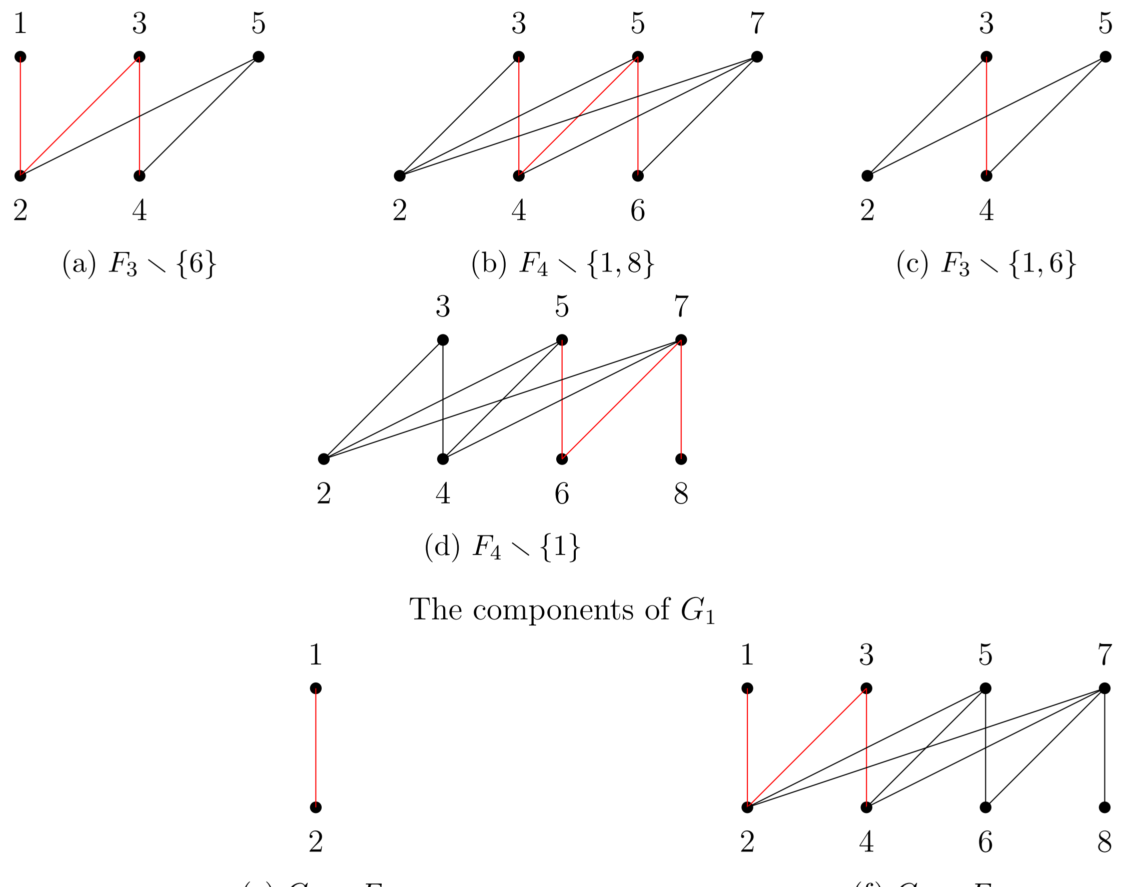
1. vertex 5 of subfigure 5a with vertex 2 of subfigure 5b,
2. vertex 7 of subfigure 5b with vertex 2 of subfigure 5c,
3. vertex 5 of subfigure 5c with vertex 2 of subfigure 5d.

The graph G is constructed from G_1, G_2, and G_3 by identifying:

1. vertex 8 of subfigure 5d with vertex 1 of Figure 5e,
2. vertex 2 of Figure 5e with vertex 1 of Figure 5f.

The proof of Corollary 37 yields that:

1. for G_1, we have that $P_{1,1} = [1, 2, 3, 4]$, $P_{1,2} = [3, 4, 5, 6]$, $P_{1,3} = [3, 4]$, and $P_{1,4} = [5, 6, 7, 8]$. Then, P_1 is the disjoint union of these paths in G_1.
2. for G_2, we have that $P_2 = [1, 2]$, and



(a) $F_3 \setminus \{6\}$ (b) $F_4 \setminus \{1, 8\}$ (c) $F_3 \setminus \{1, 6\}$

(d) $F_4 \setminus \{1\}$

The components of G_1

(e) $G_2 = F_1$ (f) $G_3 = F_4$

Figure 5: Construction of G from G_1, G_2, and G_3

3. for G_3, we have that $P_3 = [1, 2, 3, 4]$.

It follows that $\underline{P} := \bigcup_{i=1}^{\ell} P_i$ consists of four vertex-disjoint induced paths. (In the gluing step, terminal vertices of $P_{1,4}$, P_2, and P_3 coincide. These paths concatenate in the construction of G.) As mentioned in the proof, all four of these vertex-disjoint induced paths contain no induced edges between themselves.

5 Preliminaries on Block Graphs

In this section, we recall the definition and some elementary properties of block graphs. All of the results in this section are well-known to experts. However, lacking a reference for these results, we present their proofs for completeness.

Recall the definition of a block graph.

Definition 39 (Block Graph)**.** A graph G is **biconnected** if G is connected and $G \setminus v$ is connected for every $v \in V(G)$. G is a **clique** or a **complete graph** if for every pair of distinct vertices v and w in $V(G)$, $\{v, w\} \in E(G)$. A subgraph B of G is a **block** of G

if B is a maximal biconnected component of G with respect to inclusion. A graph G is a **block graph** if every block of G is a complete graph. Block graphs are also referred to in the literature as a **tree of cliques**.

We recall the following well-known properties of block graphs.

Lemma 40. *Let G be a block graph. If H is an induced subgraph of G, then H is a block graph.*

Lemma 41. *Let G be a connected block graph. Let v and w be distinct vertices of G. There is a unique shortest path in G connecting v and w, and this shortest path is an induced path in G.*

Lemma 42. *Let G be a block graph. Let Q be an induced path in G. Then, every cycle of G contains no more than one edge of Q.*

Proof. Suppose by contradiction that there exists a cycle, C of G, which contains two or more edges of Q. Since G is a block graph, C induces a complete subgraph of G which would contradict Q being induced. □

Lemma 43. *Let $Q = [v_1, \dots, v_r]$ and $R = [w_1, \dots, w_s]$ be induced paths of a block graph which intersect non-trivially. Then, $Q \cap R$ is connected. Moreover, there is a unique decomposition $Q = Q_1 * \gamma * Q_2$ and $R = R_1 * \gamma * R_2$ where Q_i (respectively R_i) is possibly a singleton vertex of Q (respectively R) for $i = 1, 2$, and $\gamma = Q \cap R$.*

Proof. Let v and w be vertices of $Q \cap R$. Let $[v, w] \mid_Q$ and $[v, w] \mid_R$ be the subpaths of Q and R from v to w. These are induced subpaths of Q and R, respectively, and hence are induced paths of G, by Lemma 3. Lemma 41 implies that these are the same path. Thus, $Q \cap R$ is connected. □

Lemma 44. *Let $R_1 = [w_1, \dots, w_s]$ and $R_2 = [w_1^{'}, \dots, w_t^{'}]$ be disjoint induced paths of a block graph, and $Q = [v_1, \dots, v_r]$ be an induced path of a block graph. Suppose that $v_1 = w_1$ and $v_r = w_t^{'}$. Then, there is a unique decomposition of Q as $Q_1 * \gamma * Q_2$ where Q_i is a subpath of R_i for $i = 1, 2$, and $\gamma \cap R_i$ is a vertex for $i = 1, 2$.*

Proof. Apply Lemma 43 to R_1 and Q. Because $R_1 \cap Q$ contains v_1, $Q = \gamma^{'} * Q_2^{'}$, where $\gamma^{'} := R_1 \cap Q$ and $Q_2^{'}$ intersects R_1 at a vertex. Because R_1 and R_2 are disjoint, $Q_2^{'}$ contains $w_t^{'}$. Apply Lemma 43 to R_2 and $Q_2^{'}$. Then, $Q_2^{'} = Q_1^{''} * \gamma^{''}$ where $\gamma^{''} := R_2 \cap Q_2^{'}$ and $Q_1^{''}$ intersects R_2 at a vertex. Put $Q_1 := \gamma^{'}$, $\gamma := Q_1^{''}$, and $Q_2 := \gamma^{''}$. Then, $Q = Q_1 * \gamma * Q_2$. □

6 Combinatorial Characterization of DOIP Paths in Block Graphs

In this section, we give several equivalent combinatorial formulations for vertex-disjoint paths of a block graph G to be DOIP, which are in terms of forbidden subgraphs of P_{Ind}.

6.1 Forbidden Subgraphs

Let G be a block graph, and let $\underline{P} := P_1, \ldots, P_\ell$ be vertex-disjoint induced oriented paths of G.

Definition 45. Let Q be an oriented induced path of P_{Ind} with orientation ϕ_Q. We denote by $Q_0 := \phi_Q(1)$ and $Q_r := \phi_Q(2)$. We say that Q is a **strand** of P_{Ind} from P_i to P_j, with $i \neq j$, if:

1. $Q_0 \neq \phi_i(2)$,
2. $Q_r \neq \phi_j(1)$,
3. $V(Q) \cap V(P_i) = \{Q_0\}$,
4. $V(Q) \cap V(P_j) = \{Q_r\}$,
5. Q does not contain P_k for any $1 \leqslant k \leqslant \ell$,

We say that Q is an **internal strand** if Q is a strand from P_i to P_j and, in addition, $Q_0 \in P_i^\circ$ and $Q_r \in P_j^\circ$.

In Definition 45, conditions (1) and (2) are automatically satisfied whenever H is an internal strand. The motivation for Definition 45 is to relate strands and arcs of $K_{P_{\text{Ind}}}$.

Definition 46. Let i and j be distinct integers belonging to $[\ell]$, a a vertex belonging to $V(P_i) \smallsetminus \{\phi_i(2)\}$, and b and c distinct vertices belonging to $V(P_j)$. Let Q be an oriented induced path with orientation ϕ_Q satisfying:

1. $\phi_Q(1) = a$,
2. $V(Q) \cap V(P_i) = \{a\}$,
3. $V(Q) \cap V(P_j) = \varnothing$,
4. $\phi_Q(2)$ is adjacent to b and c in P_{Ind},
5. Q does not contain P_k for any $1 \leqslant k \leqslant \ell$.

We define the subgraph H of P_{Ind} as follows:

$$V(H) := V(Q) \cup \{b, c\}$$
$$E(H) := E(Q) \cup \{\{\phi_Q(2), b\}, \{\phi_Q(2), c\}\}.$$

We say that H is a **fork** of P_{Ind} from P_i to P_j. We say that H is an **internal fork** of P_{Ind} from P_i to P_j if, in addition, $a \in P_i^\circ$.

For a vertex $v \in H$, we say that v is a **terminal vertex** of H if $v \in \{a, b, c\}$; otherwise, we say that v is an **internal vertex**.

In Definition 46, the requirement that $a \in V(P_i) \setminus \{\phi_i(2)\}$ is automatically satisfied whenever H is an internal fork. Furthermore, in this setting, Lemma 42 implies that $\{b, c\} \in E(P_{\text{Ind}})$.

Definition 47. Let i and j be distinct integers belonging to $[\ell]$, a and b distinct vertices of $V(P_i)$, and c and d distinct vertices belonging to $V(P_j)$. Let Q be an oriented induced path with orientation ϕ_Q satisfying:

1. $\phi_Q(1)$ is adjacent to a and b in P_{Ind},
2. $\phi_Q(2)$ is adjacent to c and d in P_{Ind},
3. $V(Q) \cap V(P_i) = \varnothing$,
4. $V(Q) \cap V(P_j) = \varnothing$,
5. Q does not contain P_k for any $1 \leqslant k \leqslant \ell$.

We define the subgraph H of P_{Ind} as follows:

$$\begin{aligned} V(H) &:= V(Q) \cup \{a, b, c, d\} \\ E(H) &:= E(Q) \cup \{\{\phi_Q(1), a\}, \{\phi_Q(1), b\}, \{\phi_Q(2), c\}, \{\phi_Q(2), d\}\}. \end{aligned}$$

We say that H is a **double fork** of P_{Ind} from P_i to P_j.

For a vertex $v \in H$, we say that v is a **terminal vertex** of H if $v \in \{a, b, c, d\}$; otherwise, we say that v is an **internal vertex**.

Definition 48. Let i and j be distinct integers belonging to $[\ell]$, a and b distinct vertices of $V(P_i)$, and c and d distinct vertices belonging to $V(P_j)$. Let H denote the induced subgraph on $\{a, b, c, d\}$. We say that H is a **complete ladder** if H is a complete graph.

Remark 49. We observe that in Definition 48, a complete ladder is K_4, the complete graph on 4 vertices. However, not every K_4 in P_{Ind} is a complete ladder. For example, a K_4 whose vertices are terminal vertices of distinct P_i would not be a complete ladder, nor would it realize an internal strand, an internal fork, or a double fork.

Example 50. Let G be the graph in Figure 6. We observe that G is a block graph. We consider the vertex-disjoint induced paths

$$\begin{array}{llll} P_1 := [1, 2, 3] & P_2 := [4, 5, 6] & P_3 := [7, 8] & \\ P_4 := [9, 10, 11] & P_5 := [12, 13] & P_6 := [14, 15, 16] & P_7 := [17, 18] \\ P_8 := [19, 20] & P_9 := [21, 22, 23] & P_{10} := [24, 25, 26] & P_{11} := [27, 28, 29]. \end{array}$$

For $1 \leqslant i \leqslant 11$, we define the orientation ϕ_i such that $\phi_i(1) < \phi_i(2)$. We present some subgraphs of G that illustrate Definitions 45, 46, 47, and 48.

1. (Internal) Strand:

(a) $Q = [2, 5]$ is an (internal) strand from P_1 to P_2, $\phi_Q(1) = 2$, and $\phi_Q(2) = 5$,

(b) $[5, 10, 11, 13, 15]$ is an (internal) strand from P_2 to P_6.

(c) $[5, 2]$ is a strand from P_2 to P_1,

(d) $[17, 20]$ is a strand from P_7 to P_8,

(e) $[2, 5, 10, 11, 13, 15, 18, 19, 21, 25, 28]$ is an internal strand from P_1 to P_{11}.

2. (Internal) Fork: (we just list the vertices of the fork)

(a) $\{7, 8, 9\}$ is a fork from P_4 to P_3, $Q = [9]$ is the singleton path,

(b) $\{7, 8, 9, 10, 5\}$ is an internal fork from P_2 to P_3, $Q = [5, 10, 9]$, $\phi_Q(1) = 5$, $\phi_Q(2) = 9$,

(c) $\{17, 18, 19, 21\}$ is a fork from P_9 to P_7,

(d) $\{22, 25, 27, 28\}$ is an internal fork from P_9 to P_{11}.

3. Double Fork: (we just list the vertices of the double fork)

(a) $\{21, 22, 25, 27, 28\}$ is a double fork, $Q = [25]$ is a singleton path,

(b) $\{17, 18, 19, 21, 25, 27, 28\}$ is a double fork, $Q = [19, 21, 25]$, $\phi_Q(1) = 19$, $\phi_Q(2) = 25$.

4. Complete Ladder: (we just list the vertices of the double fork)

(a) $\{17, 18, 19, 20\}$ is a complete ladder.

We next present examples of subgraphs of G that do not satisfy the requirements of Definitions 45, 46, 47, and 48.

1. Not strands:

(a) $[8, 9, 10, 5]$ is not a strand from P_3 to P_2 because $\phi_Q(1) = 8 = \phi_3(2)$,

(b) $[7, 9]$ is not a strand from P_3 to P_4 because $\phi_Q(2) = 9 = \phi_4(1)$,

(c) $[8, 9]$ is not a strand from P_3 to P_4 (same reason), but it is a strand from P_4 to P_3,

(d) $[1, 2, 5]$ is not a strand from P_1 to P_2 because Q violates condition (3) of Definition 45,

(e) $[7, 9, 10, 11, 13]$ is not a strand from P_3 to P_5 because Q contains P_4.

2. Not forks: (we just list the vertices under consideration)

(a) $\{12, 13, 15, 18\}$ is not a fork because $\phi_Q(1) = 18 = \phi_7(2)$,

(b) $\{2, 3, 5, 4\}$ is not a fork because it violates condition (2) of Definition 46,

(c) $\{7, 8, 9, 10, 11, 13, 15\}$ is not a fork from P_6 to P_3 because Q contains P_4.

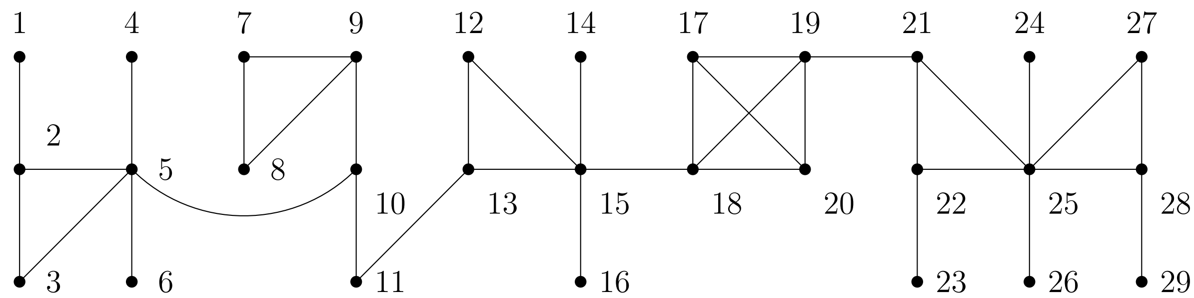


Figure 6: Block Graph

3. Not double forks: (we just list the vertices under consideration)
   (a) $\{7, 8, 9, 10, 11, 13, 15, 18, 19, 20\}$ is not a double fork because $Q = [9, 10, 11, 13, 14, 18]$ contains P_4.

6.2 DOIP Property for Paths and Forbidden Subgraphs

The main result of this subsection is the following theorem, which gives a combinatorial characterization when $\underline{P} = P_1, \ldots, P_\ell$, vertex-disjoint induced paths of a block graph, are DOIP. This characterization does not refer to the labeling of the paths, σ, or to the orientations of the paths P_i. In the subsequent section, we leverage this theorem to prove the equality of $\nu(G)$ and $\operatorname{reg}(S/J_G)$ for block graphs.

Theorem 51. *Let G be a block graph, and let $\underline{P} := P_1, \ldots, P_\ell$ be vertex-disjoint induced paths of G. The following statements are equivalent:*

1. *$\underline{P}$ is DOIP for any choice of orientations ϕ_i, $1 \leqslant i \leqslant \ell$,*
2. *P_{Ind} does not contain an internal strand, an internal fork, a double fork, or a complete ladder as subgraphs.*

The idea for the proof of Theorem 51 is as follows: For (1) implies (2), we show that if P_{Ind} contains an internal strand, an internal fork, a double fork, or a complete ladder as a subgraph, then $K_{P_{\text{Ind}}}$ contains a directed two cycle. For (2) implies (1), it suffices to show that $K_{P_{\text{Ind}}}$ is directed acyclic. We would like to say that if $K_{P_{\text{Ind}}}$ has a directed cycle, then the paths realizing this directed cycle of $K_{P_{\text{Ind}}}$ would realize a *large* cycle in G. This would lead to complications, since the induced subgraph on the vertices of any cycle in a block graph is a complete graph. The difficulty is that the paths realizing the directed cycle of $K_{P_{\text{Ind}}}$ may a priori intersect each other, thwarting this hope. However, our observation is that if these paths intersect, then they introduce an internal strand, an internal fork, a double fork, or a complete ladder as a subgraph in G.

We begin by showing that $K_{P_{\text{Ind}}}$ is a simple multigraph for any block graph.

Proposition 52. *Let $\underline{P} := P_1, \ldots, P_\ell$ be vertex-disjoint induced oriented paths of a block graph G. Then, $K_{P_{\mathrm{Ind}}}$ has no loops or multiarcs, i.e., $K_{P_{\mathrm{Ind}}}$ is a (simple) directed graph.*

Proof. Suppose by contradiction that $K_{P_{\mathrm{Ind}}}$ has a loop or a multiarc. If $K_{P_{\mathrm{Ind}}}$ has a loop, then there is an oriented induced path Q of G such that $\phi_Q(1) = \phi_i(1)$ and $\phi_Q(2) = \phi_i(2)$ for some $1 \leqslant i \leqslant \ell$. If $K_{P_{\mathrm{Ind}}}$ has a multiarc, then there are oriented induced paths Q_1 and Q_2 of G such that $\phi_{Q_1}(j) = \phi_{Q_2}(j)$ for $j = 1, 2$. Lemma 43 implies that the intersection of any two induced paths of a block graph consists of exactly one connected component. The only way that two induced paths can have the same terminal vertices is if they are the same path. Thus, $Q = P_i$ and $Q_1 = Q_2$, a contradiction. □

The following proposition connects arcs of $K_{P_{\mathrm{Ind}}}$ to strands of P_{Ind}.

Proposition 53. *$K_{P_{\mathrm{Ind}}}$ contains the arc (i, j) if and only if P_{Ind} contains a strand from P_i to P_j.*

Proof. ($\Longrightarrow$) There exists an induced path Q from $\phi_i(1)$ to $\phi_j(2)$ realizing the arc (i, j) of $K_{P_{\mathrm{Ind}}}$. Lemma 44 applied to Q, P_i, and P_j implies that $Q = Q_1 * \gamma * Q_2$. Then, γ is a strand from P_i to P_j.

($\Longleftarrow$) Let Q be a strand from P_i to P_j. Let R_i (respectively, R_j) be the subpath of P_i (respectively, from P_j) from $\phi_i(1)$ to $\phi_Q(1)$ (respectively, $\phi_Q(2)$ to $\phi_j(2)$). Let T be the induced path from $\phi_i(1)$ to $\phi_j(2)$. In order to show that T realizes the arc (i, j) of $K_{P_{\mathrm{Ind}}}$, it suffices to show that T does not contain P_k for $1 \leqslant k \leqslant \ell$. We observe that since $R_i * Q * R_j$ is a path from $\phi_i(1)$ to $\phi_j(2)$, we have that

$$\begin{aligned} V(T) &\subseteq R_i * Q * R_j \\ &\subseteq (V(P_i) \smallsetminus \{\phi_i(2)\}) \cup V(Q) \cup (V(P_j) \smallsetminus \{\phi_j(1)\}) \, . \end{aligned}$$

We observe that

1. $V(P_i) \not\subseteq V(T)$ because $\phi_i(2) \notin V(Q)$, as Q is a strand,
2. $V(P_j) \not\subseteq V(T)$ because $\phi_j(1) \notin V(Q)$, as Q is a strand,
3. $V(P_k) \not\subseteq V(T)$ for $k \in [\ell] \smallsetminus \{i, j\}$; otherwise, $V(P_k) \subseteq V(Q)$ (since the P_i are vertex-disjoint), which is impossible because Q is a strand. □

The next result establishes that (1) implies (2) of Theorem 51.

Corollary 54. *Let $\underline{P} := P_1, \ldots, P_\ell$ be vertex-disjoint oriented induced paths of a block graph G. Then, P_{Ind} contains an internal strand, an internal fork, a double fork, or a complete ladder if and only if $K_{P_{\mathrm{Ind}}}$ has a directed cycle of length two.*

Proof. ($\Longrightarrow$) Proposition 53 implies that it suffices to construct a strand from P_i to P_j and a strand from P_j to P_i whenever P_{Ind} contains one of the graphs in question.

If Q is an internal strand from P_i to P_j, then it is clear that Q is a strand from P_i to P_j and vice versa.

Let H be an internal fork from P_i to P_j. Without loss of generality, we may suppose that $b \neq \phi_j(2)$ and that $c \neq \phi_j(1)$. Then, $Q * [\phi_Q(2), c]$ is a strand from P_i to P_j, and $[b, \phi_Q(2)] * Q$ is a strand from P_j to P_i.

Let H be a double fork from P_i to P_j. Without loss of generality, we may suppose that $a \neq \phi_i(2)$, $b \neq \phi_i(1)$, $c \neq \phi_j(2)$, and $d \neq \phi_j(1)$. Then, $[a, \phi_Q(1)] * Q * [\phi_Q(2), d]$ is a strand from P_i to P_j, and $[c, \phi_Q(2)] * Q * [\phi_Q(1), b]$ is a strand from P_j to P_i.

Suppose that H is a complete ladder. Without loss of generality, we may suppose that a (respectively, c) is closer to $\phi_i(1)$ (respectively, $\phi_j(1)$) than b (respectively, d). Then, $[a, d]$ is a strand from P_i to P_j, and $[c, b]$ is a strand from P_j to P_i.

($\Longleftarrow$) Suppose that $K_{P_{\text{Ind}}}$ has a directed cycle of length two; then there exist induced paths Q_1 and Q_2 of P_{Ind} realizing this directed cycle. Without loss of generality, we may suppose that Q_1 (respectively, Q_2) realizes the arc $(1, 2)$ (respectively, $(2, 1)$). Lemma 44 implies that $Q_i = Q_{i,1} * \gamma_i * Q_{i,2}$ for $i = 1, 2$ where $Q_{i,1}$ is a subpath of P_1 and $Q_{i,2}$ is a subpath of P_2. Let a (respectively b) be the terminal vertex of γ_1 (respectively γ_2) contained in P_1. Let c (respectively d) be the terminal vertex of γ_1 (respectively γ_2) contained in P_2. We denote the subpath of P_1 (respectively P_2) having terminal vertices a and b (respectively c and d) by $[a, b]$ (respectively $[c, d]$).

Case 1. Suppose that $V(\gamma_1) \cap V(\gamma_2) = \varnothing$. Then, a, b, c, and d are distinct vertices, and

$$[a, b] * \gamma_2 * [d, c] * \gamma_1$$

is a cycle. Since G is a block graph, the induced subgraph on $\{a, b, c, d\}$ is a complete graph. Thus, P_{Ind} contains a complete ladder.

Case 2. Refer to Figure 7 for this case. Suppose that $V(\gamma_1) \cap V(\gamma_2) \neq \varnothing$. Lemma 44 applied to γ_1 and γ_2 implies that there is a decomposition $\gamma_i = \mu_{i,1} * \omega * \mu_{i,2}$ where $\mu_{i,1}$ intersects P_1 and $\mu_{i,2}$ intersects P_2 for $i = 1, 2$.

When $a = b$ and $c = d$, ω is an internal path. When $a = b$ and $c \neq d$, the subgraph of P_{Ind} having vertices $V(\omega) \cup \{c, d\}$ and edges $E(\omega) \cup E(\mu_{1,2}) \cup E(\mu_{2,2})$ is an internal fork. When $a \neq b$ and $c \neq d$, the subgraph of P_{Ind} having vertices $V(\omega) \cup \{a, b, c, d\}$ and edges $E(\omega) \cup E(\mu_{1,1}) \cup E(\mu_{1,2}) \cup E(\mu_{2,1}) \cup E(\mu_{2,2})$ is a double fork. □

The following lemmas will help us control the intersection of strands.

Lemma 55. *If γ is a strand from P_i to P_k and $V(\gamma) \cap V(P_j) \neq \varnothing$ for some $j \in [\ell] \smallsetminus \{i, k\}$, then $K_{P_{\text{Ind}}}$ contains the arc (i, j) or the arc (j, k).*

Proof. By Lemma 43, we can write γ as $\gamma_1 * \omega * \gamma_2$ where $V(\gamma_n) \cap V(P_j) = \{v_n\}$ for some vertices v_n for $n = 1, 2$, ω is a subpath of P_j, $V(\gamma_1) \cap V(P_i) \neq \varnothing$, and $V(\gamma_2) \cap V(P_k) \neq \varnothing$. If $v_1 \neq \phi_j(1)$, then γ_1 is a strand from P_i to P_j. If $v_2 \neq \phi_j(2)$, then γ_2 is a strand from P_j to P_k. It cannot be the case that both $v_1 = \phi_j(1)$ and $v_2 = \phi_j(2)$, as γ does not contain P_j. □

Proposition 56. *Suppose that P_{Ind} does not contain an internal strand, an internal fork, a double fork, or a complete ladder. Let γ_1 be a strand from P_a to P_b, and γ_2 be a strand from P_b to P_c. If $V(\gamma_1) \cap V(\gamma_2) \neq \varnothing$, then $K_{P_{\text{Ind}}}$ contains the arc (a, c).*

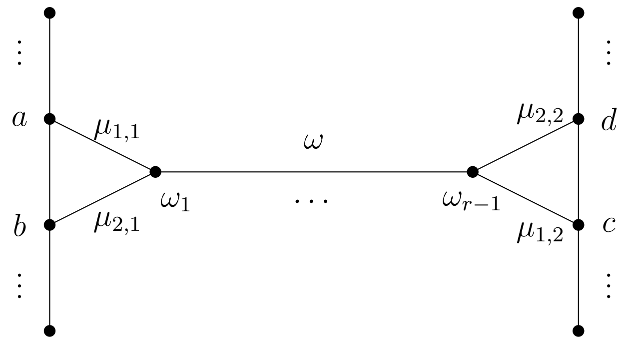


Figure 7: Illustration for Corollary 54

Proof. For this proof, refer to Figure 8. If $V(\gamma_1) \cap V(P_c) \neq \varnothing$, then Lemma 55 implies that (a, c) or (c, b) is an arc of $K_{P_{\text{Ind}}}$. Corollary 54 implies that (c, b) cannot be an arc of $K_{P_{\text{Ind}}}$. Likewise, it is shown that if $V(\gamma_2) \cap V(P_a) \neq \varnothing$, then (a, c) is an arc of $K_{P_{\text{Ind}}}$. Thus, we may assume that $V(\gamma_1) \cap V(P_c) = \varnothing$ and that $V(\gamma_2) \cap V(P_a) = \varnothing$. Lemma 44 implies that there is a decomposition

$$\gamma_i = \mu_{i,1} * \omega * \mu_{i,2},$$

where $\mu_{i,j}$ is determined by the condition of containing the vertex $\phi_{\gamma_i}(j)$ for $1 \leqslant i, j \leqslant 2$. We denote the intersection of $\mu_{2,1}$ (respectively, $\mu_{1,2}$) with P_b by the vertex t (respectively, s). We denote the vertex that is the intersection of $\mu_{2,1}$ and $\mu_{1,2}$ by r, and we denote the vertex which is the intersection of $\mu_{1,1}$ and $\mu_{2,2}$ by v. We denote the vertices belonging to $\mu_{1,1}$ and $\mu_{2,2}$ and adjacent to v by u and w, respectively. We observe that either $r = s = t$ (in which case r is an internal vertex of P_b), or the induced subgraph on $\{r, s, t\}$ is a complete graph (by Lemma 42). We consider the path $Q = \mu_{1,1} * \mu_{2,2}$. We observe that Q is not an induced subpath of P_{Ind} if and only if $\{u, w\} \in E(P_{\text{Ind}})$ (by Lemma 42). Let $\tilde{Q}$ denote the induced subpath of Q. We show that $\tilde{Q}$ does not contain P_k for any $1 \leqslant k \leqslant \ell$. Otherwise, it would follow that $\tilde{Q}$ is a strand from P_a to P_c, and the result would follow from Proposition 53.

Suppose by contradiction that $\tilde{Q}$ contains P_k for some $1 \leqslant k \leqslant \ell$. Since γ_1 does not intersect P_c and γ_2 does not intersect P_a, it follows that $\tilde{Q}$ does not contain P_a or P_c.

Case 1. Suppose that $\tilde{Q} = Q$ is an induced path. Then, v is an internal vertex of P_k; otherwise, P_k would be properly contained in either $\mu_{1,1}$ or in $\mu_{2,2}$, which would contradict γ_1 and γ_2 being strands. When $s \neq t$, the subgraph having vertices $V(\omega) \cup \{s, t\}$ and edges $E(\omega) \cup \{\{r, s\}, \{r, t\}\}$ is an internal fork. When $s = t$, the subgraph having vertices $V(\omega) \cup \{s\}$ and edges $E(\omega) \cup \{\{r, s\}\}$ is an internal strand.

Case 2. Suppose that $\tilde{Q} \neq Q$. Then, $\tilde{Q}$ contains the edge $\{u, w\}$. Moreover, P_k contains the edge $\{u, w\}$, since P_k is not contained in γ_1 or γ_2. It follows that there is a subgraph H with $V(H) \subseteq V(\omega) \cup \{u, w\} \cup \{s, t\}$ and with $E(H) \subseteq E(\omega) \cup \{\{u, v\}, \{r, s\}, \{r, t\}\}$, which is an internal strand, an internal fork, or a double fork. ☐

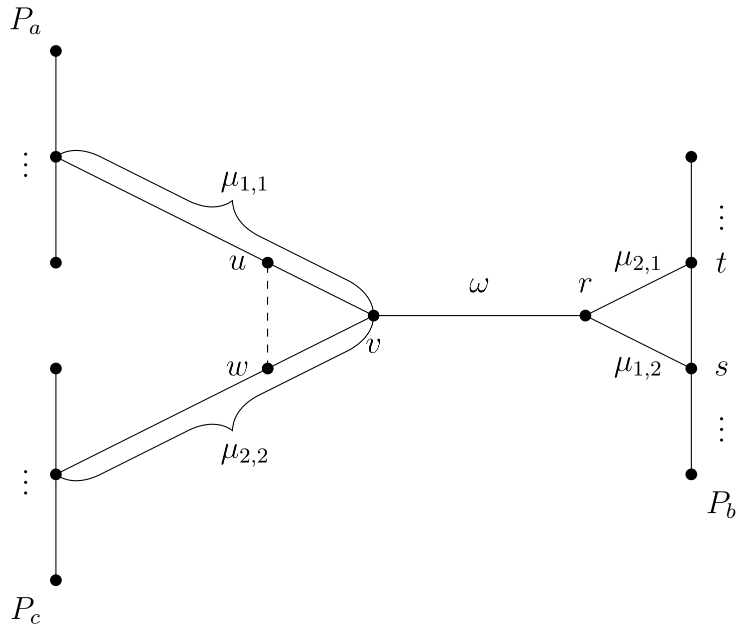


Figure 8: Illustration for Proposition 56

Proposition 57. *Suppose that P_{Ind} does not contain an internal strand, an internal fork, a double fork, or a complete ladder. Let γ_1 be a strand from P_a to P_b, and γ_2 be a strand from P_c to P_d where $a, b, c, d \in [\ell]$ are distinct. If $V(\gamma_1) \cap V(\gamma_2) \neq \varnothing$, then $K_{P_{\text{Ind}}}$ contains at least one of the arcs (a, c), (a, d), (b, d), (c, a), (c, b), (d, a), or (d, b).*

If, in addition $V(\gamma_1) \cap V(P_i) = \varnothing$ for $i \in \{c, d\}$ and $V(\gamma_2) \cap V(P_j) = \varnothing$ for $j \in \{a, b\}$, then $K_{P_{\text{Ind}}}$ contains the arc (a, d) or the arc (c, b).

Proof. For this proof, refer to Figure 9. By Proposition 55, we may assume that $V(\gamma_1) \cap P_i = \varnothing$ for $i \in \{c, d\}$ and $V(\gamma_2) \cap P_i = \varnothing$ for $i \in \{a, b\}$. Lemma 44 implies that there is a decomposition

$$\gamma_i = \mu_{i,1} * \omega * \mu_{i,2},$$

where $\mu_{i,j}$ is determined by the condition of containing the vertex $\phi_{\gamma_i}(j)$ for $1 \leqslant i, j \leqslant 2$.

Case 1. Suppose that $V(\mu_{1,1}) \cap V(\mu_{2,2}) \neq \varnothing$. Then, we have that $V(\mu_{1,1}) \cap V(\mu_{2,2}) = \{v\}$ and that $V(\mu_{1,2}) \cap V(\mu_{2,1}) = r$. (When $|V(\omega)| = 1$, we have that $v = r$.) We denote by u and t the vertices of $\mu_{1,1}$ and $\mu_{2,2}$, respectively, which are adjacent to v. We denote by s and w the vertices of $\mu_{1,2}$ and $\mu_{2,1}$, respectively, which are adjacent to r. We denote by Q_1 and Q_2 the paths $\mu_{1,1} * \mu_{2,2}$ and $\mu_{2,1} * \mu_{1,2}$, respectively. The paths Q_1 and Q_2 are not an induced path if and only if $\{u, t\}$ and $\{w, s\}$ are induced edges of Q_1 and Q_2, respectively. Let $\tilde{Q}_1$ and $\tilde{Q}_2$ denote the induced subpath of P_{Ind} on $V(Q_1)$ and $V(Q_2)$, respectively. We show that it is not possible for both $\tilde{Q}_1$ and $\tilde{Q}_2$ to contain paths P_i and P_j for some $i, j \in [\ell]$ distinct. In which case, it follows that $\tilde{Q}_1$ or $\tilde{Q}_2$ is a strand from P_a

to P_d or a strand from P_c to P_b, respectively. The result then follows from Proposition 53.

Suppose by contradiction that $\tilde{Q}_1$ contains P_i and that $\tilde{Q}_2$ contains P_j, and we consider the following subcases.

Subcase 1.(a). Suppose that $\tilde{Q}_1 = Q_1$ and that $\tilde{Q}_2 = Q_2$. First, we observe that P_i contains v as an internal vertex; otherwise, P_i would belong to γ_1 or γ_2. For similar reasons, P_j contains r as an internal vertex. When $|V(\omega)| = 1$, $r = v$; contradicting P_i and P_j being vertex-disjoint. When $V(\omega) \geqslant 2$, ω would be an internal strand, a contradiction.

Subcase 1.(b). Suppose that $\tilde{Q}_1$ contains the edge $\{u, t\}$ and that $\tilde{Q}_2 = Q_2$. Then, $\{u, t\} \in E(P_i)$, and r is an internal vertex of P_j. It follows that the subgraph having vertices $V(\omega) \cup \{u, t\}$ and edges $E(\omega) \cup \{\{u, v\}, \{t, v\}\}$ is an internal fork, a contradiction.

Subcase 1.(c). Suppose that $\tilde{Q}_1 = Q_1$ and that $\tilde{Q}_2$ contains the edge $\{w, s\}$. This subcase is analogous to subcase 1.(b).

Subcase 1.(d). Suppose that $\tilde{Q}_1$ contains the edge $\{u, t\}$ and that $\tilde{Q}_2$ contains the edge $\{w, s\}$. Then, $\{u, t\} \in E(P_i)$ and $\{w, s\} \in E(P_j)$. Consequently, the subgraph having vertices $V(\omega) \cup \{u, t, w, s\}$ and edges $E(\omega) \cup \{\{u, v\}, \{t, v\}, \{w, r\}, \{s, r\}\}$ is a double fork, a contradiction.

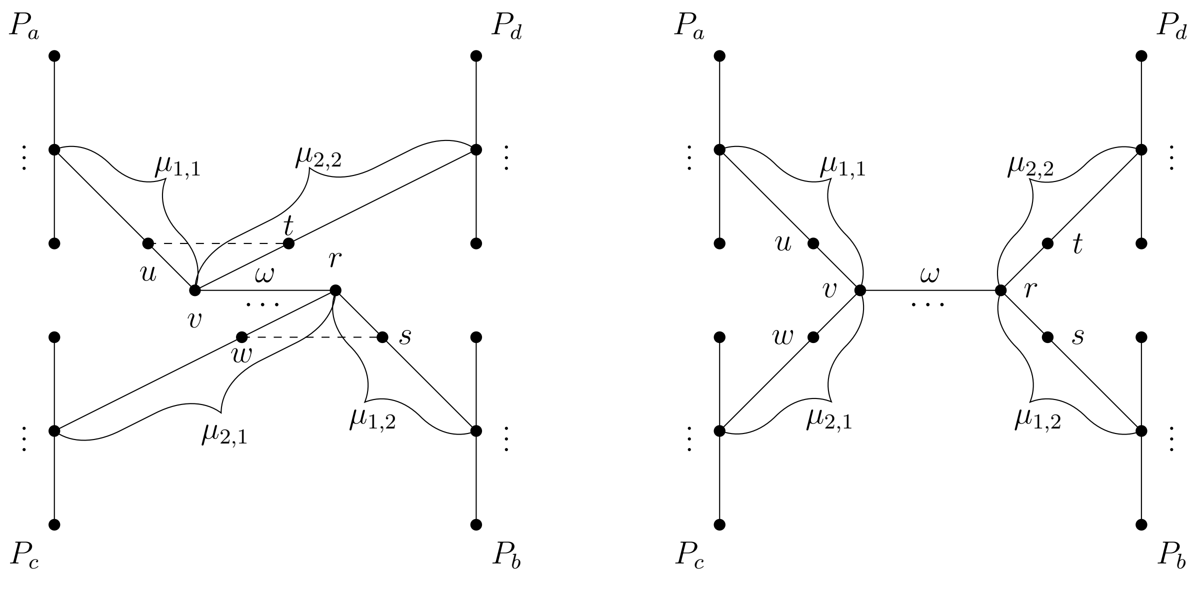
Case 2. Suppose that $V(\mu_{1,1}) \cap V(\mu_{2,1}) \neq \varnothing$. Then, we have that $V(\mu_{1,1}) \cap V(\mu_{2,1}) = \{v\}$ and that $V(\mu_{1,2}) \cap V(\mu_{2,2}) = \{r\}$. (We may suppose that $v \neq r$; otherwise, we would be in Case 1.) We denote by u and w the vertices of $\mu_{1,1}$ and $\mu_{2,1}$, respectively, which are adjacent to v. We denote by s and t the vertices of $\mu_{1,2}$ and $\mu_{2,2}$, respectively, which are adjacent to r. We denote by Q_1 and Q_2 the paths $\mu_{1,1} * \omega * \mu_{2,2}$ and $\mu_{2,1} * \omega * \mu_{1,2}$, respectively. We observe that Q_1 is an induced path of P_{Ind} by Lemma 42 together with the observations that $\mu_{1,1} * \omega$ and $\omega * \mu_{2,2}$ are induced paths of P_{Ind} being subpaths of the induced paths γ_1 and γ_2, respectively. Similarly, Q_2 is an induced path of P_{Ind}. If Q_1 contains the path P_i for some $i \in [\ell]$, then u and t belong to $V(P_i)$; otherwise, P_i would be contained in γ_1 or γ_2. It follows that Q_2 cannot contain any path P_j for $j \in [\ell]$ as such a path would necessarily be vertex-disjoint from P_i and contain ω, which is impossible. □

We are now ready to prove Theorem 51.

Proof of Theorem 51. (1) $\Longrightarrow$ (2): If $\underline{P}$ is DOIP, then in particular $K_{P_{\text{Ind}}}$ has no directed two cycle. Corollary 54 implies that P_{Ind} does not contain an internal strand, an internal fork, a double fork, or a complete ladder.

(2) $\Longrightarrow$ (1): We assume that P_{Ind} does not contain an internal strand, an internal fork, a double fork, or a complete ladder, and we show that $\underline{P}$ is DOIP. By Theorem 24, this is equivalent to showing that $K_{P_{\text{Ind}}}$ is directed acyclic. Suppose by contradiction that $K_{P_{\text{Ind}}}$ has a minimal cycle of length m, i.e. that $K_{P_{\text{Ind}}}$ has no cycle of size smaller than m. Proposition 52 and Corollary 54 imply that $m \geqslant 3$. Without loss of generality, we may suppose that $(i, i+1)$ are arcs of $K_{P_{\text{Ind}}}$ for $1 \leqslant i \leqslant m-1$ and that $(m, 1)$ is an arc of $K_{P_{\text{Ind}}}$. Proposition 53 implies that there are strands γ_i from P_i to P_{i+1} for $1 \leqslant i \leqslant m-1$ and a strand γ_m from P_m to P_1. Lemma 55 and Propositions 56 and 57 imply for $i, j \in [m]$ that:

1. $V(\gamma_i) \cap V(\gamma_j) = \varnothing$ whenever $|i - j| \geqslant 2$,



(a) Case 1

(b) Case 2

Figure 9: Illustration for Proposition 57

2. $V(\gamma_i) \cap V(\gamma_{i+1}) = \varnothing$, and
3. $V(\gamma_i) \cap V(P_j) = \varnothing$ whenever $j \neq i$ or $j \neq i+1$.

If not, these propositions and lemmas would imply the existence of an arc (i, j) for some $i, j \in [m]$ distinct, which would contradict the assumption that $K_{P_{\text{Ind}}}$ has a minimal cycle of length m. We denote by R_i the unique subpath of P_i having terminal vertices $\phi_{\gamma_{i-1}}(2)$ and $\phi_{\gamma_i}(1)$. It follows that

$$Q := R_1 * \gamma_1 * R_2 * \gamma_2 * \cdots * \gamma_{m-1} * R_m * \gamma_m$$

is a cycle. Since G is a block graph, the induced graph on $V(Q)$ is a complete graph. Thus, at most one of the R_i contains an internal vertex of P_i; otherwise, the edge connecting two distinct internal vertices would be an internal strand. Since $m \geqslant 3$, we may assume without loss of generality that P_2 and P_3 are each a singleton edge. It follows from the definition of strands that $R_2 = P_2$ and that $R_3 = P_3$. Thus, the induced subgraph on $V(P_2) \cup V(P_3)$ is a complete ladder, a contradiction. □

6.3 Another Characterization of DOIP Paths

We take the time to record an equivalent characterization of Thereom 51 in terms of forbidden subgraphs. This reformulation will be particularly useful in the subsequent section as additional constraints are placed upon the forbidden subgraphs.

Definition 58. Let G be a block graph, and let $\underline{P} := P_1, \ldots, P_\ell$ be vertex-disjoint induced paths in G. Let H be an internal strand, an internal fork, or a double fork. We say that H is **edge-disjoint** from $\underline{P}$ if $E(H) \cap E(\underline{P}) = \varnothing$.

Example 59. Let G be the graph depicted in Figure 6, and let $\underline{P}$ be as defined in Example 50. Then, the graph on the vertices $\{2, 3, 5, 10, 9, 8, 7\}$ is a double fork of P_{Ind} which is not edge-disjoint from $\underline{P}$ (because $E(H) \cap E(\underline{P}) = \{\{9, 10\}\}$).

Proposition 60. *Let G be a block graph, and let $\underline{P} := P_1, \dots, P_\ell$ be vertex-disjoint induced paths of G. The following are equivalent:*

1. *P_{Ind} does not contain any of the following as subgraphs:*
   - *(a) an internal strand,*
   - *(b) an internal fork,*
   - *(c) a double fork,*
   - *(d) a complete ladder.*
2. *P_{Ind} does not contain any of the following as subgraphs:*
   - *(a) an internal strand which is edge-disjoint from $\underline{P}$,*
   - *(b) an internal fork which is edge-disjoint from $\underline{P}$,*
   - *(c) a double fork which is edge-disjoint from $\underline{P}$,*
   - *(d) a complete ladder.*

Proof. It is clear that (1) implies (2). That (2) implies (1) follows from Lemmas 61 and 62. □

Lemma 61. *If P_{Ind} contains an internal strand as a subgraph, then P_{Ind} contains an internal strand that is edge-disjoint from $\underline{P}$.*

Proof. Let H be an internal strand of P_{Ind}. We prove by induction on $|E(H)|$ that there exists an internal strand of P_{Ind} that is edge-disjoint from $\underline{P}$. If $|E(H)| = 1$, then by Definition 45, H is edge-disjoint from $\underline{P}$. Suppose that H is an internal strand with $|E(H)| \geqslant 2$. If H is edge-disjoint from $\underline{P}$, there is nothing to prove. Suppose that $E(P_i) \cap E(H) \neq \varnothing$ for some $1 \leqslant i \leqslant \ell$. Lemma 43 implies that $H = H_1 * \gamma * H_2$ where γ is a subpath of P_i with $|E(\gamma)| \geqslant 1$. Because H does not contain P_i, at least one of the terminal vertices of H_1 or H_2 is an internal vertex of P_i. Without loss of generality, suppose that it is H_1. Then, H_1 is an internal strand of P_{Ind} with $|E(H_1)| < |E(H)|$. By the induction hypothesis, applied to H_1, there exists an internal strand of P_{Ind} which is edge-disjoint from $\underline{P}$, which completes the proof. □

Lemma 62. *If P_{Ind} contains an internal fork or a double fork as a subgraph, then P_{Ind} contains an internal strand, an internal fork, or a double fork which is edge-disjoint from $\underline{P}$.*

Proof. Let H be an internal fork or a double fork of P_{Ind}. Let Q be the subpath of H, as defined in Definitions 46 and 47. The proof proceeds by induction on $|E(Q)|$. When $|E(Q)| = 0$, i.e., Q is the path consisting of a singleton vertex, it is clear from the definitions that H is edge-disjoint from $\underline{P}$. When $|E(Q)| \geqslant 1$, the proof proceeds via induction, as in the proof of Lemma 61. We observe that in replicating the proof of Lemma 61 that: if H is an internal fork, then H_1 or H_2 is a strand or a fork with at least one of them being internal; and if H is a double fork, then H_1 and H_2 are forks with at least one of them being internal. □

Example 63. We illustrate Lemma 62 in the context of Example 59. Observe that H_1 and H_2 are the graphs on the vertices $\{2, 3, 5, 10\}$ and $\{9, 8, 7\}$, respectively. Then, H_1 is an internal fork of P_{Ind}.

7 Combinatorial Characterization of reg(S/J_G) for Block Graphs

In this section, we prove the following theorem.

Theorem 64. *Let G be a block graph. Then,*

$$\nu(G) = \operatorname{reg}(S/J_G).$$

It suffices to show that $\operatorname{reg}(S/J_G) \leqslant \nu(G)$, and to do so, we utilize the theory developed in [MMK21], which we now recall.

Definition 65. For a graph G, define $\widehat{G} := G \smallsetminus I_S(G)$ where $I_S(G)$ denotes the set of isolated vertices of G. For $v \in V(G)$, recall that $N_G(v)$ denotes the vertices of G adjacent to v. For $v \in V(G)$, we define the graph G_v as follows:

$$\begin{aligned} V(G_v) &:= V(G) \\ E(G_v) &:= E(G) \cup \{\{u, w\} \mid u, w \in N_G(v)\}. \end{aligned}$$

The graph G_v is referred to as the **completion of G at v**.

Definition 66. We say that a subset $\mathcal{G}$ of all finite graphs is **compatible** if $\mathcal{G}$ satisfies the following conditions:

1. $\bigsqcup_{i=1}^{t} K_{n_i} \in \mathcal{G}$ for all $n_i \in \mathbb{Z}$ with $n_i \geqslant 2$,
2. $\widehat{G} \in \mathcal{G}$ for all $G \in \mathcal{G}$,
3. $G \smallsetminus \{v\} \in \mathcal{G}$ for all $G \in \mathcal{G}$ and $v \in V(G)$,
4. $G_v \in \mathcal{G}$ for all $G \in \mathcal{G}$ and $v \in V(G)$.

Definition 67 ([MMK21, Definition 2.1])**.** Let $\mathcal{G}$ be a subset of all finite graphs. Suppose that $\mathcal{G}$ is compatible. A map $\varphi : \mathcal{G} \to \mathbb{N}_0$ is called **compatible** if it satisfies the following conditions:

1. $\varphi(\widehat{G}) \leqslant \varphi(G)$ for all $G \in \mathcal{G}$,

2. if $G = \bigsqcup_{i=1}^{t} K_{n_i}$, where $n_i \geqslant 2$ for every $1 \leqslant i \leqslant t$, then $\varphi(G) \geqslant t$,

3. if $G \neq \bigsqcup_{i=1}^{t} K_{n_i}$, then there exists $v \in V(G)$ such that

   (a) $\varphi(G \setminus v) \leqslant \varphi(G)$, and

   (b) $\varphi(G_v) < \varphi(G)$.

Theorem 68 ([MMK21, Theorem 2.3])**.** *Let $\mathcal{G}$ be a subset of all finite graphs. Suppose that $\mathcal{G}$ is compatible and that $\varphi : \mathcal{G} \to \mathbb{N}_{\geqslant 0}$ is compatible. Then, for all $G \in \mathcal{G}$,*

$$\operatorname{reg}(S/J_G) \leqslant \varphi(G).$$

Remark 69. Theorem Theorem 68 was originally shown for the set $\mathcal{G}$ of all finite graphs [MMK21]. The proof there has two main steps. First, they show for a graph G that

$$\begin{aligned} \operatorname{reg}(S/J_{G \setminus \{v\}}) &\leqslant \varphi(G \setminus \{v\}) \\ \operatorname{reg}(S/J_{G_v}) &< \varphi(G_v) \end{aligned}$$

Second, they utilize induction on the number of internal vertices of a graph together with the short exact sequence

$$0 \to S/J_G \to S/J_{G_v} \oplus S_v/J_{G \setminus v} \to S_v/J_{G_v \setminus v} \to 0$$

to deduce that $\operatorname{reg}(S/J_G) \leqslant \varphi(G)$. The fact that $\mathcal{G}$ is compatible, i.e., closed under vertex completion and deletion, allows us to apply the induction step in our setting.

Example 70. The class of chordal graphs and the class of block graphs are compatible.

Lemma 71. *Let $\mathcal{G}$ be a compatible subset of finite graphs, and let $\nu : \mathcal{G} \to \mathbb{N}_0$ be defined as in Definition 29. Then, ν satisfies conditions 1, 2, and 3a of Definition 67.*

Proof. The claims follow from the straightforward observations that:

1. If H is an induced subgraph of G, then $\nu(H) \leqslant \nu(G)$,

2. If $G = G_1 \sqcup G_2$, then $\nu(G) = \nu(G_1) + \nu(G_2)$,

3. $\nu(K_n) = 1$ for all $n \geqslant 2$. □

Theorem 68 and Lemma 71 show that to prove Theorem 64 it suffices to show that there exists a vertex c of the block graph G such that $\nu(G_c) < \nu(G)$. We now introduce some notation and prove a few preparatory lemmas.

Definition 72. A vertex v of G is called a **cut vertex** if the number of connected components of $G \setminus \{v\}$ is strictly larger than the number of connected components of G. For a vertex v of G, we define the **clique degree of v**, denoted by $\operatorname{cdeg}(v)$, as the number of maximal distinct cliques of G containing v. The number of maximal cliques in a block graph G is denoted by $c(G)$. We say that a block graph G has the **two-block property** if, for every vertex v of G, $\operatorname{cdeg}(v) \leqslant 2$. We say that a block graph G is a **path of cliques** if every block of G has at most two cut vertices. In a path of cliques, blocks with exactly one cut vertex are called **terminal blocks**.

Example 73. Consider the graph constructed from the complete graph on three vertices by attaching a whisker to each vertex of the complete graph. (This graph is sometimes referred to as the **net**.) This graph has the two-block property but is not a path of cliques.

Lemma 74. *Let G be a block graph that has the two-block property. Let $\mathcal{A}$ denote a collection of edges of G such that no two edges belong to a common clique of G. Let $\underline{P}$ denote the disjoint union of paths obtained from $\mathcal{A}$ after concatenating those edges of $\mathcal{A}$ sharing a terminal vertex. Let P_{Ind} denote the induced subgraph of G on $V(\underline{P})$. Then, P_{Ind} is DOIP.*

Proof. Suppose by contradiction that there exists a block graph G together with a set of edges $\mathcal{A}$ of G satisfying the stated hypotheses such that P_{Ind} is not DOIP. We may suppose that among all such block graphs that G has been chosen to minimize $c(G)$. It is clear that $c(G) \geqslant 3$, as any block graph on two or fewer cliques, together with any choice of edges $\mathcal{A}$, realizes P_{Ind} that is DOIP. By Theorem 51 and Proposition 60, P_{Ind} contains H, an internal strand, an internal fork, or a double fork, which is edge-disjoint from $\underline{P}$ (H is not a complete ladder because no two edges of $\mathcal{A}$ belong to the same clique). By minimality of $c(G)$, we may assume that

1. $V(H) \cap V(B_i) \neq \varnothing$ for all $1 \leqslant i \leqslant c(G)$, and
2. for every block B of G, there exists an edge e of $\mathcal{A}$ contained in B.

(If G did not satisfy these two conditions, then we could produce a smaller counterexample by deleting irrelevant blocks of G.) The first condition implies that G is a path of cliques, since H is necessarily contained in a path of cliques. We may label the cliques of G consecutively, starting from a terminal clique, by $B_1, B_2, B_3, \ldots, B_{c(G)}$. We denote by v_i the vertex common to B_i and B_{i+1} for $1 \leqslant i \leqslant c(G) - 1$. We denote by e_i the edge of $\mathcal{A}$ belonging to B_i. Consequently, H contains the subpath $[v_1, v_2, v_3, \ldots, v_{c(G)-1}]$. In particular, $v_i \in V(P_{\text{Ind}})$. We show that v_2 is not a vertex of e_2 through consideration of two cases below. In which case, we can construct $G^{'}$ by deleting the block B_1 from G, $\mathcal{A}^{'}$ by deleting e_1 from $\mathcal{A}$, and $\underline{P}^{'}$ and $P_{\text{Ind}}^{'}$ coming from $\mathcal{A}^{'}$ and $G^{'}$. If $e_2 = \{a, b\}$, then we construct $H^{'}$ by:

$$V(H^{'}) := (V(H) \smallsetminus \{v_1\}) \cup \{a, b\}$$
$$E(H^{'}) := (E(H \smallsetminus \{v_1\})) \cup \{\{v_2, a\}, \{v_2, b\}\},$$

which is an internal fork or a double fork of $P_{\text{Ind}}^{'}$. This would contradict minimality of $c(G)$.

Claim: e_2 does not contain v_2.

Proof of Claim. We consider the following two cases.

Case 1. Suppose that e_1 does not contain v_1. Then, it must be the case that e_2 contains v_1 (because $V(H) \subseteq V(\underline{P})$). As $e_2 \neq \{v_1, v_2\}$ (because H is edge-disjoint from $\underline{P}$), e_2 does not contain v_2.

Case 2. Suppose that e_1 contains v_1. Then, $E(H) \cap E(B_1) = \varnothing$, since H does contain e_1 and G is a path of cliques having B_1 as a terminal vertex. It follows that v_1 is a terminal vertex of H. Hence, it must be the case that e_2 contains v_1. Since $e_2 \neq \{v_1, v_2\}$, it again follows that e_2 does not contain v_2. □

Corollary 75. *If G is a block graph with the two-block property, then $\nu(G) = \operatorname{reg}(S/J_G) = c(G)$.*

Remark 76. In Corollary 75, the statement that $\operatorname{reg}(S/J_G) = c(G)$ follows from [HR18, Proposition 1.3].

Proof. For this family of graphs, Lemma 74 proves that that $c(G) \leqslant \nu(G)$. For any graph G, we have that $\nu(G) \leqslant \operatorname{reg}(S/J_G) \leqslant c(G)$ by Theorem 30 and [MMK21, Corollary 2.7]. □

Lemma 77. *Let G be a block graph. Suppose that $G = G_1 \cup G_2$ and that $G_1 \cap G_2 = \{c\}$ for some vertex c of G where G_1 has the two-block property and G_2 is a block graph. Let $v_1 \in V(G_1)$. Suppose that $\underline{P}$ is a union of vertex-disjoint induced paths of G that contains the edge $\{v_1, c\}$. If H is an internal strand, an internal fork, or a double fork of P_{Ind} that is edge-disjoint from $\underline{P}$, then*

$$V(H) \cap (V(G_1) \smallsetminus \{c\}) = \varnothing.$$

Proof. Suppose by contradiction that $V(H) \cap (V(G_1) \smallsetminus \{c\}) \neq \varnothing$. Then, c is an internal vertex of H; otherwise, H would be a subgraph of G_1, which would contradict Lemma 74 together with Theorem 51. We denote by a the vertex of $V(H) \cap V(G_1)$ that is adjacent to c. Since $\{v_1, c\} \notin E(H)$, it must be the case that $a \neq v_1$. We define the graph $H^{'}$ by

$$\begin{aligned} V(H^{'}) &:= (V(H) \cap V(G_1)) \cup \{v_1\} \\ E(H^{'}) &:= (E(H) \cap E(G_1)) \cup \{\{a, v_1\}\}. \end{aligned}$$

It follows from Definitions 46, 47 that $H^{'}$ is an internal fork or a double fork of P_{Ind}. This contradicts Lemma 74. □

Lemma 78. *Let G be a block graph, $\underline{P}$ vertex-disjoint induced paths of G, and H an internal strand, an internal fork, or a double fork of P_{Ind} that is edge-disjoint from $\underline{P}$. Suppose that c is an internal vertex of both H and $\underline{P}$. Then, there exists an internal strand or an internal fork of P_{Ind} which is edge-disjoint from $\underline{P}$ and which contains c as a terminal vertex.*

Proof. Since c is an internal vertex of H, there exist subgraphs H_1 and H_2 of H such that

$$\begin{aligned} H &= H_1 \cup H_2 \\ \{c\} &= H_1 \cap H_2. \end{aligned}$$

Now, H_1 is an internal strand or an internal fork of P_{Ind} which is edge-disjoint from $\underline{P}$. □

Proposition 79. *Let G be a block graph. Then, for some cut vertex c of G, we have that $\nu(G_c) < \nu(G)$.*

Proof. If G has the two-block property, then the result follows from Corollary 75. Thus, we may assume that G is a block graph which does not have the two-block property. Pick $c \in \operatorname{Cut}(G)$ such that $\operatorname{cdeg}(c) \geqslant 3$ and $G_1, G_2, \ldots, G_t$ are subgraphs of G, $t \geqslant 3$, satisfying:

1. G_i has the two-block property for $1 \leqslant i \leqslant t-1$,
2. $G = \bigcup_{i=1}^{t} G_i$, and
3. $G_i \cap G_j = \{c\}$ for $1 \leqslant i < j \leqslant t$.

Such a c exists because G does not have the two-block property, and by induction on the number of blocks of G. For $1 \leqslant i \leqslant t$, denote by B_i the block of G_i that contains c. Let B be the block of G_c containing c. Let $\underline{P}^{'}$ be DOIP paths of G_c such that

$$\nu(G_c) = |E(\underline{P}^{'})|.$$

Let $P^{'}_{\text{Ind}}$ be the induced subgraph of G_c on $V(\underline{P}^{'})$. From $\underline{P}^{'}$, we construct a DOIP path $\underline{P}$ of G such that

$$|E(\underline{P}^{'})| < |E(\underline{P})|.$$

From which, it follows that $\nu(G_c) < \nu(G)$. This construction proceeds across several cases.

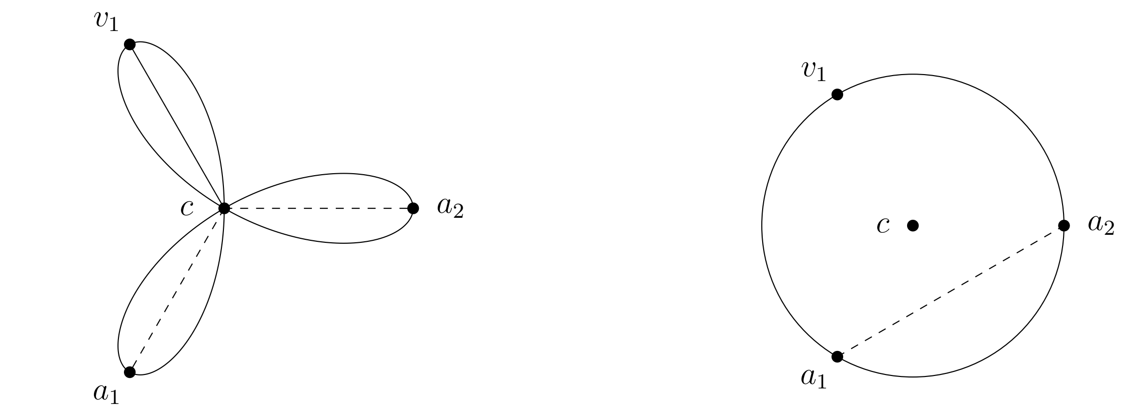
<u>Case 1.</u> For this case, refer to Figure 10. Suppose that $E(B) \cap E(\underline{P}^{'}) = \varnothing$. It follows that $\underline{P}^{'}$ is a subgraph of G. In particular, it follows that $E(B_i) \cap E(\underline{P}^{'}) = \varnothing$ for all $1 \leqslant i \leqslant t$. Pick $v_1 \in B_1 \smallsetminus \{c\}$. We define $\underline{P}$ to be the subgraph of G as follows:

$$\begin{aligned} V(\underline{P}) &:= V(\underline{P}^{'}) \cup \{v_1, c\} \\ E(\underline{P}) &:= E(\underline{P}^{'}) \cup \{\{v_1, c\}\}. \end{aligned}$$

The assumptions that G_1 has the two-block property and that $E(B_1) \cap E(\underline{P}^{'}) = \varnothing$ imply that $\underline{P}$ consists of vertex-disjoint induced paths. Suppose by contradiction that H is an internal strand, an internal fork, or a double fork of P_{Ind} which is edge-disjoint from $E(\underline{P})$. Lemma 77 implies that H does not contain v_1. Hence, H contains c; otherwise, H would be an internal strand, an internal fork, or a double fork of $P^{'}_{\text{Ind}}$, a contradiction. Moreover, c is not a terminal vertex of H, since $\deg_{\underline{P}}(c) = 1$ and $V(H) \cap (V(B_1) \smallsetminus \{c\}) = \varnothing$. Hence, c is an internal vertex of H. We denote the vertices of H adjacent to c by a_1 and a_2, and we define the graph $H^{'}$ of G_c as follows:

$$\begin{aligned} V(H^{'}) &:= V(H) \smallsetminus \{c\} \\ E(H^{'}) &:= (E(H) \smallsetminus \{\{a_1, c\}, \{a_2, c\}\}) \cup \{\{a_1, a_2\}\}. \end{aligned}$$

Because a_1 and a_2 are adjacent to c, $\{a_1, a_2\}$ is indeed an edge of G_c. We observe that the condition $E(B) \cap E(\underline{P}^{'}) = \varnothing$ implies that $\{a_1, a_2\}$ does not contain an edge of $\underline{P}^{'}$.



(a) G, P_{Ind} (solid line), and H (dashed lines)

(b) G_c and $H^{'}$ (dashed line)

Figure 10: Illustration for Case 1

Moreover, this condition implies that if a_i is a terminal vertex of H, then a_i is an internal vertex of $\underline{P}$. It now follows from Definitions 45, 46, 47 that $H^{'}$ is an internal strand, an internal fork, or a double fork of $P^{'}_{\text{Ind}}$, a contradiction.

We next consider the case where $E(B) \cap E(\underline{P}^{'}) \neq \varnothing$. We distinguish between cases based on whether this edge of $E(B) \cap E(\underline{P}^{'})$ contains the vertex c.

<u>Case 2.</u> Suppose that $\{a, c\}$ is an edge of $E(B) \cap E(\underline{P}^{'})$. If necessary, relabel B_1 and G_1 by B_2 and G_2, respectively, so that we may assume that $a \notin V(G_1)$. We pick $v_1 \in V(B_1) \setminus \{c\}$ and construct $\underline{P}$ of G as follows:

$$V(\underline{P}) := V(\underline{P}^{'}) \cup \{v_1\}$$
$$E(\underline{P}) := E(\underline{P}^{'}) \cup \{\{v_1, c\}\}.$$

We observe that $\underline{P}$ consists of vertex-disjoint induced paths of G, since $E(\underline{P}) \cap E(B_1) = \varnothing$ and G_1 has the two-block property. Suppose by contradiction that H is an internal strand, an internal fork, or a double fork of P_{Ind} which is edge-disjoint from $E(\underline{P})$. Lemma 77 implies that H does not contain v_1. Hence, H contains c; otherwise, H would be an internal strand, an internal fork, or a double fork of $P^{'}_{\text{Ind}}$, a contradiction. By Lemma 78, we may assume that c is a terminal vertex of H. Let b denote the vertex of H which is adjacent to c. We construct $H^{'}$ of G_c as follows:

$$V(H^{'}) := V(H) \cup \{b\}$$
$$E(H^{'}) := E(H) \cup \{\{b, a\}\}.$$

Since H is edge-disjoint from $\underline{P}$, $b \neq a$. Hence, $H^{'}$ is an internal fork or a double fork of $P^{'}_{\text{Ind}}$, a contradiction.

<u>Case 3.</u> Suppose that $\{a, b\}$ is an edge of $E(B) \cap E(\underline{P}^{'})$ with $a \neq c$ and $b \neq c$. Let v_1 and v_2 be vertices of $B_1 \setminus \{c\}$ and $B_2 \setminus \{c\}$, respectively. We construct $\underline{P}$, vertex-disjoint induced paths of G from $\underline{P}^{'}$, by deleting the edge $\{a, b\}$ from $\underline{P}^{'}$, removing any isolated

vertices created after deleting this edge, and then adding the edges $\{v_1, c\}$ and $\{v_2, c\}$. Suppose by contradiction that H is an internal strand, an internal fork, or a double fork of P_{Ind} which is edge-disjoint from $\underline{P}$. Lemma 77 allows us to assume that H contains c, and Lemma 78 allows us to assume that c is a terminal vertex of H. We denote by d the vertex of H which is adjacent to c.

Subcase (a). We suppose that $d \neq a$ and that $d \neq b$. Then, we construct $H^{'}$, an internal fork or a double fork of $P^{'}_{\text{Ind}}$, as follows:

$$V(H^{'}) := (V(H) \smallsetminus \{c\}) \cup \{a, b\}$$
$$E(H^{'}) := (E(H) \smallsetminus \{\{c, d\}\}) \cup \{\{a, d\}, \{b, d\}\}.$$

Subcase (b). We suppose without loss of generality that $a = d$. This implies that $a \in V(\underline{P})$. The construction of $\underline{P}$ from $\underline{P}^{'}$ involved deleting the edge $\{a, b\}$ and any isolated vertices. Hence, it must be the case that $\deg_{\underline{P}'}(a) = 2$, i.e., that a is an internal vertex of $\underline{P}^{'}$. We construct the graph $H^{'}$ as follows:

$$V(H^{'}) := V(H) \smallsetminus \{c\}$$
$$E(H^{'}) := E(H) \smallsetminus \{\{a, c\}\}.$$

We observe that H being edge-disjoint from $\underline{P}$ implies that if $a \in V(P_i)$, then $V(H^{'}) \cap V(P_i) = \{a\}$. Hence, $H^{'}$ is an internal strand or an internal fork of $P^{'}_{\text{Ind}}$, a contradiction. $\square$

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