# $s$-Stable Kneser Graph are Hamiltonian

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**Abstract**

The Kneser Graph $K(n,k)$ has as vertices all $k$-subsets of $\{1,\dots,n\}$ and edges connecting two vertices if they are disjoint. The $s$-stable Kneser Graph $K_{s-\mathrm{stab}}(n,k)$ is obtained from the Kneser graph by deleting vertices with elements at cyclic distance less than $s$. In this article, we show that connected $s$-Stable Kneser graphs are Hamiltonian.

**Mathematics Subject Classifications:** 05C45, 05C75, 05C85

## 1 Introduction

Let $[n]=\{1,\dots,n\}$. For each $n \geqslant 2k$, $n,k \in \{1,2,3,\dots\}$, the **Kneser Graph** $K(n,k)$ has as vertices the set $\binom{[n]}{k}=\{A \subseteq [n]:\ |A|=k\}$, the $k$-subsets of $[n]$, where two vertices are adjacent if they are disjoint. Kneser graphs have been widely studied throughout the literature. Two of the most significant problems in the study of Kneser graphs are determining their chromatic number, which Lovász [8] proved to be $n-k+2$, and investigating their Hamiltonicity. It was long conjectured that, except for $K(5,2)$, all connected Kneser graphs are Hamiltonian. Many articles studied this problem (see [6, 11, 13, 15]) until recently (2023), when Merino, Mütze, and Namrata showed in [9] that the conjecture is true.

In [14], Schrijver introduced a family of subgraphs of Kneser graphs that are vertex critical in terms of their chromatic number, which means that the removal of any vertex results in a lower chromatic number. These graphs received the name of Schrijver graphs, are denoted by $SG(n,k)$, and are obtained from the Kneser graph $K(n,k)$ by deleting vertices containing consecutive elements modulo $n$. Schrijver showed that the chromatic number of $SG(n,k)$ is $n-k+2$, and that deleting any vertex from $SG(n,k)$ reduces its chromatic number. This family of graphs was generalized in [2], where Alon, Drewnoski, and Luczak introduced the concept of $s$-stable Kneser graphs.

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\{1,3,5\}$ | x | | x | | x | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

Figure 1: Grid of the vertex $\{1,3,5\}$ in $K_{2-\text{stab}}(9,3)$.

The ***s*-Stable Kneser Graph**, $K_{s-\text{stab}}(n,k)$, is the graph that has as vertex set $\binom{[n]}{k}_s = \left\{A \in \binom{[n]}{k} : s \leqslant\mid i-j \mid\leqslant n-s, \text{ for every pair } i,j \in A\right\}$, and edges between disjoint vertices. Notice that $K_{s-\text{stab}}(n,k)$ is an induced subgraph of $K(n,k)$, and that $K_{2-\text{stab}}(n,k)$ and $SG(n,k)$ are the same graph. Since the family of $s$-stable Kneser graphs was introduced, both it in general, and more specifically the family of Schrijver graphs, started receiving much attention, partly due to their applications to topology [1]. Thus, several properties of this family have been studied, such as its automorphism group [4, 16], its chromatic number [10, 17], its diameter [7], hom-idempotence [17], independence complexes [5], and neighborhood complexes [3].

The problem of Hamiltonicity of $s$-Stable Kneser graphs began by the authors of this manuscript in 2019, and earlier results were presented at the annual meeting of Unión Matemática Argentina (Argentinian Mathematical Union) in 2019 and 2021. The main result of this article is the following.

**Theorem 1.** *Let $k \geqslant 1$ and $s \geqslant 3$. The graph $K_{s-\text{stab}}(n,k)$ is Hamiltonian if and only if $n \geqslant sk$. The graph $K_{2-\text{stab}}(n,k)$ is Hamiltonian if and only if $n \geqslant 2k+1$.*

For a conference talk (in Spanish) presenting these results, we direct the reader to https://www.youtube.com/watch?v=9ZJ9-ruKmG0. Similar results were obtained independently by Mütze and Namrata in [12].

The rest of the article is organized as follows. In Section 2 we introduce some notation and provide some insight into the behavior of the vertices of $K_{s-\text{stab}}(n,k)$. Then, in Section 3 we introduce some auxiliary graphs defined by the orbits of the vertices of $K_{s-\text{stab}}(n,k)$ under element rotation and use them to prove our main result.

## 2 Preliminaries

**Definition 2.** The ***s*-Stable Kneser Graph**, $K_{s-\text{stab}}(n,k)$, has vertex set $\binom{[n]}{k}_s = \left\{A \in \binom{[n]}{k} : s \leqslant\mid i-j \mid\leqslant n-s, \text{ for every pair } i,j \in A\right\}$, and edges between disjoint vertices.

Let $K_{s-\text{stab}}(n,k)$, and let $V$ be a vertex of the graph. For a better understanding of vertices and edges, we represent the elements of $[n]$ as a grid of $n$ squares and a vertex $V$ as marking $k$ of the positions of the grid with an "$x$". The remaining positions are referred to as empty. Then, for instance, the vertex $\{1,3,5\}$ of $K_{2-\text{stab}}(9,3)$ is a grid in which positions 1, 3 and 5 are "$x$" (see Figure 1). Notice, in particular, that when $n = sk$, $K_{s-\text{stab}}(n,k)$ is isomorphic to $K_s$.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| X | | X | | | X | | | |
| | X | | X | | | X | | |
| | | X | | X | | | X | |
| | | | X | | X | | | X |
| X | | | | X | | X | | |
| | X | | | | X | | X | |
| | | X | | | | X | | X |
| X | | | X | | | | X | |
| | X | | | X | | | | X |

(a) Class $(1, 2, 3)$.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| X | | X | | | | X | | |
| | X | | X | | | | X | |
| | | X | | X | | | | X |
| X | | | X | | X | | | |
| | X | | | X | | X | | |
| | | X | | | X | | X | |
| | | | X | | | X | | X |
| X | | | | X | | | X | |
| | X | | | | X | | | X |

(b) Class $(1, 3, 2)$.

Figure 2: Grid of vertices in $K_{2-\text{stab}}(9, 3)$.

It is also interesting to think of a vertex as a list of the spaces between the "$x$". We define the class of a vertex as follows.

**Definition 3.** The class of a vertex $V \in \binom{[n]}{k}_s$ is defined as the orbit of $V$ under the cyclic rotation of the elements of $[n]$.

As this rotation defines an automorphism of $K_{s-\text{stab}}(n, k)$, it will be helpful to study when a vertex in a class is adjacent to a vertex in a different class. For this purpose, we represent the class of $V$ with the list of spaces between the elements of $V$, although there is some ambiguity in this as other vertices in the same class may present a rotation of the list of spaces. As an example, in Figure 2 we present all the elements of the classes $(1, 2, 3)$ and $(1, 3, 2)$ of vertices in $K_{2-\text{stab}}(9, 3)$. As we can notice, those classes are different. Nevertheless, $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$ are the same class.

Let $V = \{a_1, a_2, \ldots, a_k\}$ be a set, we define $V + 1 = \{a_1+1, a_2+1, \ldots, a_k+1\}$ (addition modulo $n$) as the rotation of $V$. If $V$ is a vertex in the graph $K_{s-\text{stab}}(n, k)$, then, since there are at least $s-1$ positions between any two elements of $V$, rotating all the elements by one position guarantees that there is no intersection between $V$ and its rotation. As we can visualize the elements of a vertex as "$x$" marks on a grid, the rotation of a vertex corresponds to shifting all the "$x$" marks one position to the right (cyclically). Let $\mathcal{A}$ be a class of vertices in $K_{s-\text{stab}}(n, k)$, we define its order $|\mathcal{A}|$ to be the number of vertices in the class. Then, given a vertex $V \in \mathcal{A}$, we can form a cycle spanning all vertices in $\mathcal{A}$ by successive rotations.

**Claim 4.** *For each class $\mathcal{A}$ and each vertex $V \in \mathcal{A}$,*

$$V, V + 1, V + 2, \ldots, V + |\mathcal{A}| - 1, V$$

*is a cycle spanning the vertices of $\mathcal{A}$.*

*Proof.* In the previous discussion, we have seen that $V$ and $V + 1$ are adjacent; in the same way, we can observe adjacency between $V + i$ and $V + i + 1$ for $1 \leqslant i \leqslant |\mathcal{A}| - 1$. As in particular, $V + |\mathcal{A}| = V$, we get the desired cycle. □

We denote by $(a_1, a_2, \ldots, a_i)^j$ the concatenation of $j$ copies of $a_1, a_2, \ldots, a_i$. In other words, $(a_1, a_2, a_3, a_4)^3 = (a_1, a_2, a_3, a_4, a_1, a_2, a_3, a_4, a_1, a_2, a_3, a_4)$. Using this notation, we can represent the order of a class as follows.

**Claim 5.** $|\mathcal{A}| = \dfrac{n}{d}$ *if and only if $d$ is the maximum number such that* $\mathcal{A} = (a_1, a_2, \ldots, a_{\frac{k}{d}})^d$.

*Proof.* Let $\mathcal{A} = (a_1, a_2, \ldots, a_k)$ such that $|\mathcal{A}| = \dfrac{n}{d}$. Let $V \in \mathcal{A}$, it verifies that $V + \dfrac{n}{d} = V$, hence the number of "$x$"s between the first $\dfrac{n}{d}$ positions on the grid is $\dfrac{k}{d}$. As $\dfrac{n}{d}$ is the smallest number such that $V + \dfrac{n}{d} = V$, then $d$ is the maximum number to divide the grid into equal sections. Finally, $\mathcal{A} = (a_1, a_2, \ldots, a_{\frac{k}{d}}, \ldots, a_1, a_2, \ldots, a_{\frac{k}{d}}) = (a_1, a_2, \ldots, a_{\frac{k}{d}})^d$.

Let $\mathcal{A}$ be a class of vertices in $K_{s-\text{stab}}(n, k)$, and $d$ be the maximum number such that $\mathcal{A} = (a_1, a_2, \ldots, a_{\frac{k}{d}})^d$. As $\displaystyle\sum_{i=1}^{k} a_i = n - k$, then $\displaystyle\sum_{i=1}^{\frac{k}{d}} a_i = \frac{n-k}{d}$. Given $V \in \mathcal{A}$, if we rotate $V$ the sum of spaces $\displaystyle\sum_{i=1}^{\frac{k}{d}} a_i$ plus the number of "$x$"s between those spaces we obtain $V$, that is $V + \frac{n-k}{d} + \frac{k}{d} = V + \frac{n}{d} = V$. Hence $|\mathcal{A}| \leqslant \dfrac{n}{d}$, and due to the conditions on $d$ we obtain $|\mathcal{A}| = \dfrac{n}{d}$. □

Notice that, to have a class of order $n/d$, $d$ must divide both $n$ and $k$. Consequently, it must also divide the number of blank places, $n - k$. As an example, let us consider the grid of the graph $K_{3-\text{stab}}(36, 6)$. There are $36 - 6 = 30$ blank places, which must be divided into 6 blocks, with each block containing at least 2 blank places. We know that $d$ is a common divisor between 36 and 6, if and only if $d$ is a common divisor between 30 and 6. As $1, 2, 3$ and 6 are the common divisors between 36 and 6, there are classes of order $36, 18, 12$ and 6. Table 1 shows some of those classes.

| Class | Order |
|---|---|
| (2,2,2,2,11,11) | 36 |
| $(2, 2, 11)^2 = (2, 2, 11, 2, 2, 11)$ | 36/2=18 |
| $(2, 8)^3 = (2, 8, 2, 8, 2, 8)$ | 36/3=12 |
| $(6)^6 = (6, 6, 6, 6, 6, 6)$ | 36/6=6 |

Table 1: Some classes of the graph $K_{3-\text{stab}}(36, 6)$.

Although the proof of the following lemma is elementary, we include it here for the sake of completeness.

**Lemma 6.** *Let $A$ and $B$ be vertices in different classes of $K_{s-\text{stab}}(n, k)$. Then, $A$ and $B$ are adjacent if and only if $A + 1$ and $B + 1$ are adjacent.*

*Proof.* Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$. If $A$ and $B$ are adjacent, then $A \cap B = \emptyset$.

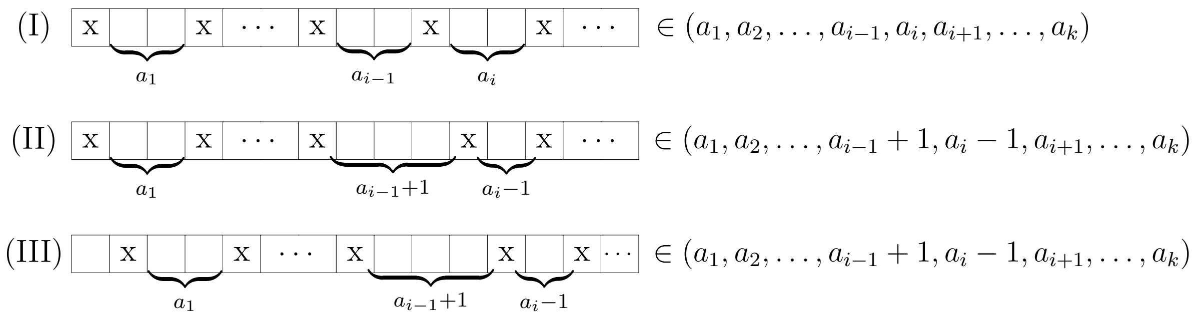
Suppose that $c \in (A+1 \cap B+1)$. Then, $c = a_i + 1$ and $c = b_j + 1$, which implies that $a_i = b_j$. Then $A+1$ and $B+1$ are adjacent. □

## 3 Class Graph and proof of the main theorem

**Definition 7.** Let $K_{s-\mathrm{stab}}(n,k)$ with $n \geqslant sk+1$, we assign every vertex to its class as we have shown before. The **Class Graph** of the $s$-Stable Kneser Graph $K_{s-\mathrm{stab}}(n,k)$, denoted by $CK_{s-\mathrm{stab}}(n,k)$, has as vertices the classes of the vertices of $K_{s-\mathrm{stab}}(n,k)$, and edges as follows: if two vertices are neighbors in $K_{s-\mathrm{stab}}(n,k)$, then their classes are neighbors in $CK_{s-\mathrm{stab}}(n,k)$.

Our goal is to find a spanning tree of the graph $CK_{s-\mathrm{stab}}(n,k)$ with certain properties; therefore, we focus on a specific subset of edges. Given a class $\mathcal{A} = (a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k)$, every class of the form $\mathcal{B} = (a_1, a_2, \ldots, a_{i-1}+1, a_i - 1, a_{i+1}, \ldots, a_k)$ is called a friend class of $\mathcal{A}$. In other words, $\mathcal{B}$ is a friend class of $\mathcal{A}$ if it can be obtained from it by adding 1 to $a_{i-1}$ and subtracting 1 from $a_i$ for some $1 \leqslant i \leqslant k$, where $a_0 = a_k$. Notice that for $\mathcal{B}$ to be a class, $a_i$ must be at least $s$. Thus, each class has at most $k$ friends.

Notice that if we start with a vertex from the class $(a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k) \in CK_{s-\mathrm{stab}}(n,k)$, as shown in (I) below. Then we add 1 to "$a_{i-1}$" and subtract one from "$a_i$", obtained in (II) by moving the "$x$" between those positions, one place to the right.




After that, we move every "$x$" in (II) one position to the right, getting (III), still in the same class. We conclude that (I) is not a neighbor of (II), but (I) is a neighbor of (III). Then, $(a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k)$ is adjacent to $(a_1, a_2, \ldots, a_{i-1}+1, a_i - 1, a_{i+1}, \ldots, a_k)$ in $CK_{s-\mathrm{stab}}(n,k)$. Thus, we have the following.

**Claim 8.** *Friend classes are adjacent in $CK_{s-\mathrm{stab}}(n,k)$.*

*Proof.* Suppose $\mathcal{A} = (a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k)$ and $\mathcal{B} = (a_1, a_2, \ldots, a_{i-1}+1, a_i - 1, a_{i+1}, \ldots, a_k)$ are friend classes. In particular, this implies $a_i - 1 \geqslant 1$. For $1 \leqslant j \leqslant k$, let

$b_j = \sum_{\ell=1}^{j} a_\ell$ and consider vertices

$$A = \{1, b_1+2, b_2+3, \ldots, b_{i-2}+i-1, b_{i-1}+i, b_i+i+1, \ldots, b_k+k+1\},$$
$$B = \{2, b_1+3, b_2+4, \ldots, b_{i-2}+i, b_{i-1}+i+2, b_i+i+2, \ldots, b_k+k+2\}.$$

Notice that $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Further, we have

$$\{2, b_1+3, b_2+4, \ldots, b_{i-2}+i\} = \{1, b_1+2, b_2+3, \ldots, b_{i-2}+i-1\}+1$$
$$\text{and}$$
$$\{b_i+i+2, \ldots, b_k+k+2\} = \{b_i+i+1, \ldots, b_k+k+1\}+1.$$

Thus,

$$\{2, b_1+3, b_2+4, \ldots, b_{i-2}+i, b_{i-1}+i+2, b_i+i+2, \ldots, b_k+k+2\} \subset A+1,$$

which does not intersect $A$. Finally,

$$b_{i-1}+i < b_{i-1}+i+2 < b_i+i+1,$$

where the second inequality follows as

$$b_i+i+1-(b_{i-1}+i+2) = a_i - 1 \geqslant 1.$$

Thus, $A \cap B = \emptyset$. Therefore, friend classes are adjacent in $CK_{s-\mathrm{stab}}(n,k)$. □

**Definition 9.** We define $SCK_{s-\mathrm{stab}}(n,k)$ as the spanning subgraph of $CK_{s-\mathrm{stab}}(n,k)$ induced by the edges between friend classes.

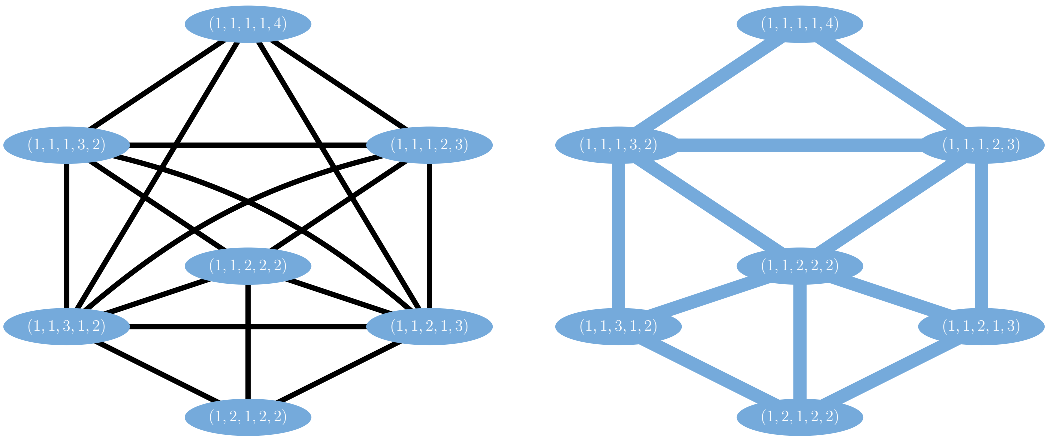
We now proceed to prove that $SCK_{s-\mathrm{stab}}(n,k)$ is connected.

**Lemma 10.** *$SCK_{s-\mathrm{stab}}(n,k)$ is connected, for $n = sk+r$, $r \geqslant 1$, $s \geqslant 2$.*

*Proof.* For any vertex in $K_{s-\mathrm{stab}}(n,k)$ the total amount of spaces is $n-k = sk+r-k = (s-1)k+r$. Consider the vertex $(\underbrace{s-1, s-1, \ldots, s-1}_{k-1}, s-1+r) \in SCK_{s-\mathrm{stab}}(n,k)$.

We would like to prove that this vertex is connected to any vertex through the edges of the graph. Let $(a_1, a_2, a_3, \cdots, a_{k-1}, a_k)$ be a vertex such that $\sum_{i=1}^{k} a_i = n-k$.

As shown in Claim 8, $(s-1, s-1, \ldots, s-1, s-1+r)$ is adjacent to $(s-1, s-1, \ldots, s-1, s, s-2+r)$. Thus, applying the claim successively, we can find a path from $(s-1, s-1, \ldots, s-1, s-1+r)$ to $(s-1, s-1, \ldots, s-1, 2s-2+r-a_k, a_k)$. But continuing with this process with the next coordinate, we can find a path to $(s-1, s-1, \ldots, s-1, 3s-3+r-a_{k-1}-a_k, a_{k-1}, a_k)$. We can keep on going this way and connect $(s-1, s-1, \ldots, s-1, s-1+r)$ to $(a_1, a_2, a_3, \cdots, a_{k-1}, a_k)$. □




(a) Graph $CK_{2-\text{stab}}(13, 5)$. (b) Graph $SCK_{2-\text{stab}}(13, 5)$.

Figure 3: Graphs obtained from $K_{2-\text{stab}}(13, 5)$.

As an example, consider the graph $K_{2-\text{stab}}(13, 5)$ which contains 91 vertices. Since there are $13 - 5 = 8$ spaces to partition into 5 blocks, the classes formed are (1,1,1,1,4), (1,1,1,2,3), (1,1,1,3,2), (1,1,2,1,3), (1,1,3,1,2), (1,1,2,2,2) y (1,2,1,2,2). In Figure 3 (a) we can observe the graph $CK_{2-\text{stab}}(13, 5)$. Then, if we only keep edges between friend classes, we obtain the graph $SCK_{2-\text{stab}}(13, 5)$ illustrated in Figure 3 (b).

Let $G$ be a connected graph, the degree $\deg_G(a)$ of a vertex $a$ is the number of edges incident to it in $G$. A key part of our construction requires that each class $\mathcal{A}$ has at most as many neighbors in $SCK_{s-\text{stab}}(n, k)$ as is has elements. In other words, $\deg_{SCK_{s-\text{stab}}(n,k)}(\mathcal{A}) \leqslant |\mathcal{A}|$ for each $\mathcal{A} \in V(SCK_{s-\text{stab}}(n, k))$.

**Claim 11.** *If $\mathcal{A} \in V(SCK_{s-\text{stab}}(n, k))$, then $\deg_{SCK_{s-\text{stab}}(n,k)}(\mathcal{A}) \leqslant |\mathcal{A}|$, for $n = sk + r$, $r \geqslant 1$, $s \geqslant 2$.*

*Proof.* By Claim 5 $|\mathcal{A}| = \dfrac{n}{d}$, where $d$ is the maximum number such that

$$\mathcal{A} = (a_1, a_2, \ldots, a_{\frac{k}{d}})^d.$$

Notice that when $j - i = p\frac{k}{d}$, with $1 \leqslant p \leqslant d$, the friend class of $\mathcal{A}$ obtained by adding 1 to $a_i$ and subtracting 1 from $a_{i+1}$ is the same as the friend class obtained by adding 1 to $a_j$ and subtracting 1 from $a_{j+1}$. Therefore, $\mathcal{A}$ has at most $\frac{k}{d}$ friend classes. Therefore, by Claim 5, $\deg_{SCK_{s-\text{stab}}(n,k)}(\mathcal{A}) = \frac{k}{d} < \frac{n}{d} = |\mathcal{A}|$. □

In Figure 4 we present a spanning tree of the graph $SCK_{2-\text{stab}}(13, 5)$, rooted in the vertex $(1, 1, 1, 1, 4)$. The vertices of the graph are labeled according to the notation used in the proof of Claim 12.

The following claim presents the construction we use to obtain a Hamiltonian cycle of $K_{s-\text{stab}}(n, k)$ from a spanning tree of the graph $CK_{s-\text{stab}}(n, k)$.

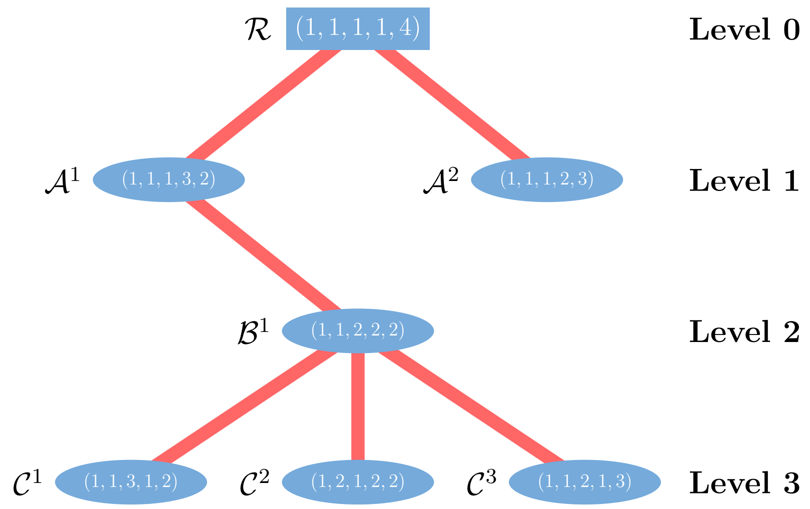



Figure 4: A spanning tree of the graph $SCK_{2-\text{stab}}(13, 5)$, rooted in the vertex $(1, 1, 1, 1, 4)$.

**Claim 12.** *Let $T$ be a spanning tree of $CK_{s-\text{stab}}(n, k)$ with $n \geqslant sk+1$, $s \geqslant 2$. If $\deg_T(\mathcal{A}) \leqslant |\mathcal{A}|$ for every $\mathcal{A} \in V(CK_{s-\text{stab}}(n, k))$, then the $s$-Stable Kneser Graph $K_{s-\text{stab}}(n, k)$ is Hamiltonian.*

*Proof.* Consider $T$ as a rooted tree, and let $\mathcal{R}$ be its root. Partition the vertices in levels according to their distance to $\mathcal{R}$. Thus, vertices in level $\ell$ are at distance $\ell$ from $\mathcal{R}$. In particular, $\mathcal{R}$ is in level 0.

We construct the cycle inductively. Claim 4 assures the existence of a cycle in $K_{s-\text{stab}}(n, k)$ for each class $\mathcal{A}$. For each class $\mathcal{A}$, let $A_1, A_2, \ldots, A_{|\mathcal{A}|}$ be the vertices in the class such that

$$A_1, A_2, \ldots, A_{|\mathcal{A}|}, A_1$$

is a cycle, and such that $A_i$ is adjacent to $B_i$ in $K_{s-\text{stab}}(n, k)$ if $\mathcal{A}$ is adjacent to $\mathcal{B}$ in $T$, $B_i \in \mathcal{B}$, (with $i$ computed modulo $|\mathcal{A}|$ and $|\mathcal{B}|$, respectively).

Let $d = \deg_T(\mathcal{R})$ and let $\mathcal{A}^1, \ldots, \mathcal{A}^d$ be the vertices adjacent to $\mathcal{R}$ in level 1. To connect the cycles

$$\begin{gathered} R_1, R_2, \ldots, R_{|\mathcal{R}|}, R_1 \\ A_1^1, A_2^1, \ldots, A_{|\mathcal{A}^1|}^1, A_1^1 \\ A_1^2, A_2^2, \ldots, A_{|\mathcal{A}^2|}^2, A_2^2 \\ \vdots \\ A_1^d, A_2^d, \ldots, A_{|\mathcal{A}^d|}^d, A_1^d \end{gathered}$$

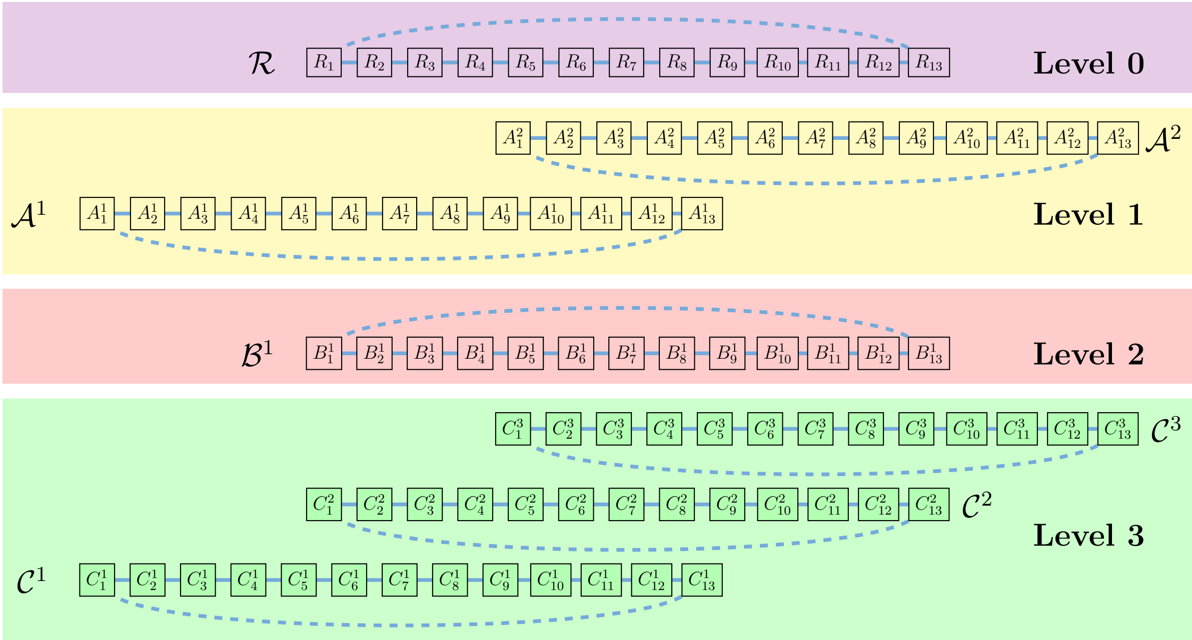



Figure 5: A cycle in each class of the graph $K_{2-\text{stab}}(13, 5)$.

we are going to form a cycle consisting of the following paths

$$\begin{aligned} &R_1, A^1_1, A^1_{|\mathcal{A}^1|}, A^1_{|\mathcal{A}^1|-1}, \dots, A^1_2, R_2\\ &R_2, A^2_2, A^2_1, A^2_{|\mathcal{A}^2|}, \dots, A^2_3, R_3\\ &\vdots\\ &R_d, A^d_d, A^d_{d-1}, A^d_{d-2}, \dots, A^d_{d+1}, R_{d+1}\\ &R_{d+1}, R_{d+2}, R_{d+3}, \dots, R_{|\mathcal{R}|}, R_1. \end{aligned}$$

As $\deg_T(\mathcal{R}) \leqslant |\mathcal{R}|$, we know that the edges $R_iR_{i+1}$ and $R_j, R_{j+1}$ are different if $1 \leqslant i < j \leqslant |\mathcal{R}|$. For $1 \leqslant i \leqslant \deg_T(\mathcal{R})$, change edges $R_iR_{i+1}$ and $A^i_iA^i_{i+1}$ for the edges $R_iA^i_i$ and $R_{i+1}A^i_{i+1}$. This generates the cycle

$$\begin{aligned} &R_1, A^1_1, A^1_{|\mathcal{A}^1|}, A^1_{|\mathcal{A}^1|-1}, \dots, A^1_2, R_2, A^2_2, A^2_1, A^2_{|\mathcal{A}|^2} \dots A^2_3, R_3, \dots,\\ &A^{d-1}_d, R_d, A^d_d, A^d_{d-1}, A^d_{d-2}, \dots, A^d_{d+1}, R_{d+1}, R_{d+2}, \dots, R_{|\mathcal{R}|}, R_1 \end{aligned}$$

containing all vertices in classes from levels 0 and 1.

In order to connect vertices from level $\ell$ to level $\ell + 1$, let $\mathcal{B}$ be a vertex in level $\ell$, and let $\mathcal{C}^1, \dots, \mathcal{C}^{|\mathcal{B}|-1}$ be the vertices adjacent to $\mathcal{B}$ at level $\ell + 1$. Let $B_iB_{i+1}$ be the edge that was deleted to connect vertices in the class $\mathcal{B}$ to their neighbor at level $\ell - 1$. As $\deg_T(\mathcal{B}^j) \leqslant |\mathcal{B}^j|$, if $1 \leqslant \alpha < \beta \leqslant |\mathcal{B}^j|-1$ then $B_{i+\alpha}B_{i+\alpha+1}$ and $B_{i+\beta}B_{i+\beta+1}$ are two distinct edges in our current graph, where addition is done modulo $|\mathcal{B}|$. Then, for each $1 \leqslant \alpha \leqslant |\mathcal{B}|-1$, exchange edges $B_{i+\alpha}B_{i+\alpha+1}$ and $C^\alpha_{i+\alpha}C^\alpha_{i+\alpha+1}$ for edges $B_{i+\alpha}C^\alpha_{i+\alpha}$ and $B_{i+\alpha+1}C^\alpha_{i+\alpha+1}$, with addition done modulo $|\mathcal{B}|$ or modulo $|\mathcal{C}^\alpha|$, according to the vertex. Therefore, after repeating this process for all levels of the rooted tree, we obtain a Hamiltonian cycle for $K_{s-\mathrm{stab}}(n, k)$. □

In Figure 5 we present the cycle in each class of the graph $K_{2-\mathrm{stab}}(13, 5)$, labeling the vertices following the notation in the proof of Claim 12. In Figure 6 we show the cycle obtained.

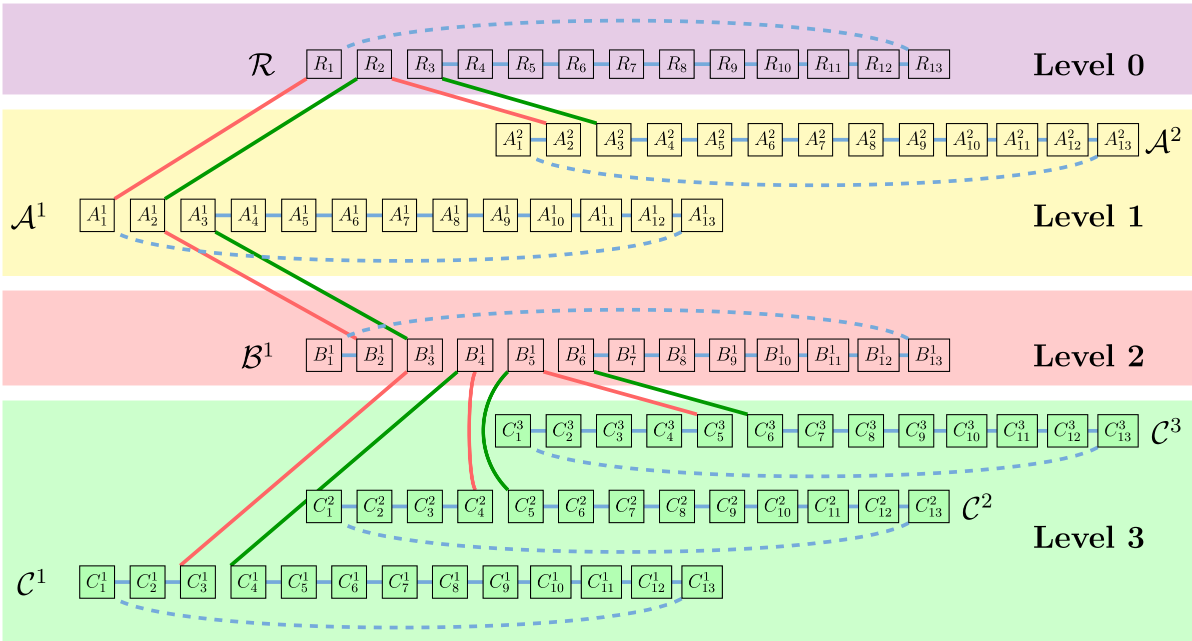



Figure 6: A Hamiltonian cycle of the graph $K_{2-\mathrm{stab}}(13, 5)$.

Our main result follows from Claims 11 and 12.

*Proof of Theorem 1.* When $n = sk$, $K_{s-\mathrm{stab}}(n, k)$ is isomorphic to $K_s$. Thus, if $s \geqslant 3$, it is Hamiltonian.

Assume now that $n \geqslant sk + 1$, and let $T$ be a spanning tree of $SCK_{s-\mathrm{stab}}(n, k)$. By Claim 11, $\deg_T(\mathcal{A}) \leqslant |\mathcal{A}|$. As $V(SCK_{s-\mathrm{stab}}(n, k)) = V(CK_{s-\mathrm{stab}}(n, k))$, $T$ is a spanning tree of $CK_{s-\mathrm{stab}}(n, k)$. Thus, by Claim 12, the $s$-Stable Kneser Graph $K_{s-\mathrm{stab}}(n, k)$ is Hamiltonian. □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