Bounded Fractional Intersecting Families are Linear in Size

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Abstract

Using the sunflower method, we show that if $\theta \in (0,1) \cap \mathbb{Q}$ and $\mathcal{F}$ is a $O(n^{1/3})$-bounded θ-intersecting family over $[n]$, then $|\mathcal{F}| = O(n)$, and that if $\mathcal{F}$ is $o(n^{1/3})$-bounded, then $|\mathcal{F}| \leqslant (\frac{3}{2}+o(1))n$. This partially solves a conjecture of Balachandran, Mathew and Mishra that any θ-intersecting family over $[n]$ has size at most linear in n, in the regime where we have no very large sets.

Mathematics Subject Classifications: 05D05, 03E05

1 Introduction

The study of intersecting families of set systems has a long and storied history in extremal combinatorics, with the prototypical problem taking the form: how large can a family of subsets of $[n]$ be under the constraint that the sets satisfy some intersection properties? One of the classic theorems in this direction is Fisher's Inequality [11], which states that if $1 \leqslant \lambda \leqslant n$, and $\mathcal{F} \subseteq 2^{[n]}$ is a set family with $|F \cap G| = \lambda$ for all distinct $F, G \in \mathcal{F}$, then $|\mathcal{F}| \leqslant n$. With its applications to experimental design, this fundamental theorem is one of the cornerstones of design theory, where decades of work have culminated in a series of stunning existence results; by Wilson [16] for strength-2 designs, and independently by Keevash [14] and by Glock, Kühn, Lo, and Osthus [13] for the general case. In the context of extremal set theory, various generalisations of Fisher's Inequality continue to provide fertile ground for research and theory-building; for instance, one can allow various different intersection sizes, place restrictions on the sizes of sets in the family, require all pairwise intersections to be the same, or consider intersections of multiple sets. Some of the highlights along these lines include the de Bruijn–Erdős Theorem [4],

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the Erdős–Ko–Rado Theorem [9], the Ray-Chaudhuri–Wilson Inequality [15], the Frankl–Wilson Inequality [12], the Alon–Babai–Suzuki Inequality [1], the Erdős–Rado Sunflower Lemma [10] and the recent improvement by Alweiss, Lovett, Wu, and Zhang [2]. For a more complete survey of restricted intersection theorems, we refer the reader to the monograph of Babai and Frankl [3].

Returning to Fisher's Inequality, observe that it greatly restricts the set family — we can only take a linear number of subsets from the exponentially many possibilities. However, this is perhaps not so surprising, as the condition imposed is very strong. Indeed, if λ is large, then $\mathcal{F}$ can only consist of large sets, since we trivially require $|F| \geqslant \lambda$ for each $F \in \mathcal{F}$, while if λ is small, then the large sets in our family must be essentially disjoint, and so we do not have space to pack many of them.

It is therefore natural to wonder what happens if we allow the intersections to scale with the sizes of the subsets, which leads us to the notion of fractional intersecting families, introduced by Balachandran, Mathew, and Mishra [6]. Given $\theta \in (0,1) \in \mathbb{Q}$, a *(fractional) θ-intersecting family* $\mathcal{F}$ over $[n]$ is a collection of nonempty subsets of $[n]$ such that for all $A, B \in \mathcal{F}$ with $A \neq B$, $|A \cap B| \in \{\theta|A|, \theta|B|\}$.[1] In [6], the following upper bound is proved for the size of any θ-intersecting family over $[n]$.

Theorem 1 (Balachandran–Mathew–Mishra [6], 2019)**.** *Let $\theta = \frac{a}{b} \in (0,1) \cap \mathbb{Q}$, and let $\mathcal{F}$ be a θ-intersecting family over $[n]$. Then, $|\mathcal{F}| = O\left(\frac{n \log^2 n}{\log \log n}\right)$, where the implicit constant depends on b.*

On the other hand, the best-known constructions give θ-intersecting families over $[n]$ of size only linear in n.

Example 2. The *sunflower family* $\mathcal{F}_s$ over $[n]$ is defined as follows:

$$\mathcal{F}_s = \begin{cases} \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-1)n\}, & n \equiv 0 \pmod 2; \\ \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-2)(n-1)\}, & n \equiv 1 \pmod 2. \end{cases}$$

This is easily seen to be a $\frac{1}{2}$-intersecting family, also called a *bisection closed* family. Note that $|\mathcal{F}_s| = \lfloor 3n/2 \rfloor - 2$.

Example 3. The *Hadamard family* $\mathcal{F}_H$ over $[2m]$ is constructed from an $m \times m$ normalized Hadamard matrix H as follows. View the rows $A_1, \ldots, A_{3m}$ of the following block matrix as the $\{\pm 1\}$-incidence vectors of subsets of $[2m]$, where J denotes the $m \times m$ all-ones matrix:

$$\begin{bmatrix} H & H \\ H & -H \\ H & -J \end{bmatrix}.$$

Then, $\mathcal{F}_H = \{A_i : i \in [3m] \setminus \{1, m+1\}\}$. One can show using the orthogonality of the rows of H that $\mathcal{F}_H$ is a bisection closed family over $[2m]$. Writing $2m = n$, we see that $|\mathcal{F}_H| = 3n/2 - 2$.

[1] More generally, given a set L of proper fractions, a *(fractional) L-intersecting family* $\mathcal{F}$ over $[n]$ is a collection of subsets of $[n]$ such that for all $A, B \in \mathcal{F}$ with $A \neq B$, $|A \cap B| \in \{\theta|A|, \theta|B|\}$ for some $\theta \in L$.

It was conjectured in [6] that *any* θ-intersecting family over $[n]$ is at most linear in size.[2]

Conjecture 4 (Balachandran–Mathew–Mishra [6], 2019)**.** For $\theta \in (0,1) \cap \mathbb{Q}$, there is a constant $c > 0$ such that for any θ-intersecting family $\mathcal{F}$ over $[n]$, $|\mathcal{F}| \leqslant cn$.

Moreover, the fact that two very different constructions give rise to maximal bisection closed families over $[n]$ of the same size raises the question whether, for $\theta = 1/2$, we have $|\mathcal{F}| \leqslant \lfloor 3n/2 \rfloor - 2$ for any bisection closed family $\mathcal{F}$ over $[n]$. In [7], there are constructions of bisection closed families over $[n]$ for $n \leqslant 15$ which have size greater than $\lfloor 3n/2 \rfloor - 2$, so the constructions in Examples 2 and 3 are possibly extremal only for large n.

In this note, we make some progress towards resolving the conjecture by proving the following result. We say that a family of sets is *w-bounded*, for a positive real w, if every set in the family has size at most w.

Theorem 5. *Let $\theta \in (0,1) \cap \mathbb{Q}$ and $w = O(n^{1/3})$ be a positive real. There is a constant $C > 0$ such that the following holds: for all sufficiently large n, if $\mathcal{F}$ is a w-bounded θ-intersecting family over $[n]$, then $|\mathcal{F}| \leqslant Cn$.*

In fact, with the slightly stronger assumption that the family is $o(n^{1/3})$-bounded, we can give an explicit constant that is often tight, and characterise families attaining the bound. To describe the asymptotically optimal families, we need to introduce a bit of notation related to our constructions.

First, recall that a *sunflower* is a set family where all pairwise intersections are the same; that is, for all distinct $F, F' \in \mathcal{F}$, we have $F \cap F' = \bigcap_{F'' \in \mathcal{F}} F''$. The common intersection $C = \bigcap_{F'' \in \mathcal{F}} F''$ is called the *core*, while the (pairwise disjoint) remainders of the sets $F \setminus C$ are called *petals*. Observe that the family $\mathcal{F}_s$ from Example 2 is the union of 2- and 4-uniform sunflowers, whose cores are nested; the definition below generalises this notion of neatly-arranged sunflowers. In what follows, given a set family $\mathcal{F}$ and a uniformity k, we denote by $\mathcal{F}(k)$ the collection of all sets in $\mathcal{F}$ of size k.

Definition 6. Let $\mathcal{F}$ be a family over $[n]$, and let $k_1 < \cdots < k_t$ be the sizes of sets in $\mathcal{F}$. We say that $\mathcal{F}$ is a *bouquet* if

1. each $\mathcal{F}(k_j)$ is a sunflower with at least two petals;
2. $C_{k_1} \subsetneq C_{k_2} \subsetneq \cdots \subsetneq C_{k_t}$, where C_{k_j} denotes the core of $\mathcal{F}(k_j)$;
3. for any $F \in \mathcal{F}$ we have $F \cap C_{k_t} = C_{|F|}$.

In our main result, we provide an explicit upper bound, and show that any family that attains it must essentially be a bouquet with specific set sizes.

[2] The conjecture is implicit in [6], and explicitly stated for the case when $\theta = 1/2$.

Theorem 7. *Let $a, b \in \mathbb{N}$ such that $1 \leqslant a < b$ and $\gcd(a, b) = 1$. Let $\theta = a/b$, and let $\mathcal{F}$ be a $o(n^{1/3})$-bounded θ-intersecting family over $[n]$. Then $|\mathcal{F}| \leqslant (C_\theta + o(1))n$, where $C_\theta = \frac{1}{b-a}\sum_{i=1}^{\lfloor b/a \rfloor} \frac{1}{i}$, and this constant is best possible for $\theta \in \{1/3\} \cup [1/2, 1)$.*

Furthermore, if we have equality, then there is some bouquet $\mathcal{F}^* \subseteq \mathcal{F}$ such that $|\mathcal{F} \setminus \mathcal{F}^*| = o(n)$, and almost all elements of $[n]$ are contained in sets of size ib for all $1 \leqslant i \leqslant \lfloor b/a \rfloor$.

2 Main results

Our proofs consist of two parts. First, we show that one can remove a small number of sets from a bounded θ-intersecting family to obtain a bouquet.

Proposition 8. *Let $\theta \in (0, 1) \cap \mathbb{Q}$ and $w > 1$. Let $\mathcal{F}$ be a w-bounded θ-intersecting family over $[n]$. Then $\mathcal{F}$ contains a bouquet $\mathcal{F}^*$ with $|\mathcal{F} \setminus \mathcal{F}^*| \leqslant w^3$.*

We then utilise the structure of bouquets to bound their size.

Proposition 9. *Let $a, b \in \mathbb{N}$ such that $1 \leqslant a < b$ and $\gcd(a, b) = 1$. Let $\theta = a/b$, and let $\mathcal{F}^*$ be a θ-intersecting bouquet over $[n]$. Then $|\mathcal{F}^*| \leqslant C_\theta n$, where $C_\theta = \frac{1}{b-a}\sum_{i=1}^{\lfloor b/a \rfloor} \frac{1}{i}$.*

After proving these propositions, we shall combine them to deduce Theorems 5 and 7.

Proof of Proposition 8

We will require the following result of Deza [8] that implies that a large uniform θ-intersecting family must be a sunflower.

Theorem 10 (Deza [8], 1974)**.** *Let $\mathcal{F}$ be a w-bounded family of subsets of $[n]$ such that all pairwise intersections have the same cardinality. If $|\mathcal{F}| \geqslant w^2 - w + 2$, then $\mathcal{F}$ is a sunflower.*

Since $\mathcal{F}$ is w-bounded, if a level $\mathcal{F}(k)$ is non-empty, then $k \leqslant w$. Call a level $\mathcal{F}(k)$ *small* if $|\mathcal{F}(k)| \leqslant w^2$. Note that $|\mathcal{F}(1)| \leqslant 1 < w^2$: the θ-intersecting property requires that any two distinct singleton sets in $\mathcal{F}$ have intersection of size equal to $\theta \in (0, 1)$, which is impossible. Thus, we can bound the number of sets in small levels by

$$\sum_{k\,:\,|\mathcal{F}(k)| \leqslant w^2} |\mathcal{F}(k)| = |\mathcal{F}(1)| + \sum_{k>1\,:\,|\mathcal{F}(k)| \leqslant w^2} |\mathcal{F}(k)| \leqslant 1 + (w-1)w^2 < w^3. \tag{1}$$

We remove these sets from $\mathcal{F}$, and shall show that what remains must be a bouquet (after removing at most one more set, if needed). Let $k_1 < k_2 < \cdots < k_t$ be the remaining nonempty levels.

1. By Theorem 10, each $\mathcal{F}(k_j)$ is a sunflower with at least two sets.

2. Let C_{k_j} be the core of $\mathcal{F}(k_j)$. Since $\mathcal{F}$ is θ-intersecting, $|C_{k_j}| = \theta k_j$ for all $1 \leqslant j \leqslant t$. Now, let $1 \leqslant j < j' \leqslant t$, and suppose $F' \in \mathcal{F}(k_{j'})$. Then $|F' \cap F| \geqslant \theta k_j = |C_{k_j}|$ for every $F \in \mathcal{F}(k_j)$, since $\mathcal{F}$ is θ-intersecting. If $C_{k_j} \nsubseteq F'$, then F' must intersect every petal in $\mathcal{F}(k_j)$. But then $|F'| \geqslant |\mathcal{F}(k_j)| > w^2$, which is not possible since $\mathcal{F}$ is w-bounded. Thus, $C_{k_j} \subseteq F'$ for every $F' \in \mathcal{F}(k_{j'})$, which implies that $C_{k_j} \subseteq C_{k_{j'}}$.

3. Let $j < t$ and $F \in \mathcal{F}(k_j)$. If $F \cap (C_{k_t} \setminus C_{k_j}) \neq \emptyset$, then for any $G \in \mathcal{F}(k_t)$ we have $|F \cap G| > |C_{k_j}| = \theta k_j$. Thus, necessarily, $|F \cap G| = \theta k_t$. Again, F is not large enough to meet every petal of $\mathcal{F}(k_t)$, and so we have $C_{k_t} \subseteq F$. Now, for any $j' < t$, if there was another such set $F' \in \mathcal{F}(k_{j'})$, then we would have $|F \cap F'| \geqslant |C_{k_t}| = \theta k_t \notin \{\theta|F|, \theta|F'|\}$, contradicting that $\mathcal{F}$ is θ-intersecting. Hence, there is at most one such set; if so, we remove it, and the remaining family satisfies $F \cap C_{k_t} = C_{|F|}$.

Then, having removed at most w^3 sets, we are left with a bouquet $\mathcal{F}^*$. □

Proof of Proposition 9

Let $k_1 < \cdots < k_t$ be the nonempty levels in the bouquet $\mathcal{F}^*$ over $[n]$, and set $Y = [n] \setminus C_{k_t}$. Note that for each $F \in \mathcal{F}^*$ we have $|F \cap Y| = (1-\theta)|F|$. Moreover, for each j, the sets in $\mathcal{F}^*(k_j)$ are pairwise disjoint over Y. Thus, we have

$$|\mathcal{F}^*| = \sum_{F \in \mathcal{F}^*} 1 = \sum_{F \in \mathcal{F}^*} \sum_{y \in F \cap Y} \frac{1}{(1-\theta)|F|} = \sum_{y \in Y} \sum_{\substack{F \in \mathcal{F}^*: \\ y \in F}} \frac{1}{(1-\theta)|F|}.$$

For each $y \in Y$, let $S_y = \{|F| : F \in \mathcal{F}^*, y \in F\}$. Then we have

$$|\mathcal{F}^*| = \frac{1}{1-\theta} \sum_{y \in Y} \sum_{s \in S_y} \frac{1}{s}. \tag{2}$$

Now observe that if $F, F' \in \mathcal{F}^*$ and $|F| < \theta|F'|$, then we must have $|F \cap F'| = \theta|F|$. However, $F \cap F' \cap C_{i_t} = C_{|F|}$, which is of size $\theta|F|$, and so $F \cap F' \cap Y = \emptyset$. This means that for every $y \in Y$, we have $\max S_y \leqslant \frac{1}{\theta} \min S_y$. Moreover, since $\mathcal{F}^*$ is θ-intersecting, b must divide $|F|$ for every $F \in \mathcal{F}^*$. Thus, for every $y \in Y$, we have some $m_y \in \mathbb{N}$ such that $S_y \subseteq \{bm_y, b(m_y+1), \ldots, b\lfloor m_y/\theta \rfloor\}$, and

$$\sum_{s \in S_y} \frac{1}{s} \leqslant \sum_{i=m_y}^{\lfloor m_y/\theta \rfloor} \frac{1}{bi} = \frac{1}{b} \sum_{i=m_y}^{\lfloor m_y/\theta \rfloor} \frac{1}{i}.$$

Hence we have

$$|\mathcal{F}^*| = \frac{1}{1-\theta} \sum_{y \in Y} \sum_{s \in S_y} \frac{1}{s} \leqslant \frac{1}{(1-\theta)b} \sum_{y} \sum_{i=m_y}^{\lfloor m_y/\theta \rfloor} \frac{1}{i}.$$

Now write $b = a\ell + r$, where $\ell \in \mathbb{N}$ and $0 \leqslant r \leqslant a-1$. Then,

$$\lfloor m_y/\theta \rfloor = \lfloor bm_y/a \rfloor = \ell m_y + \lfloor rm_y/a \rfloor \leqslant \ell m_y + m_y - 1 = (\ell+1)m_y - 1.$$

Thus,

$$\begin{aligned}\sum_{i=m_y}^{\lfloor m_y/\theta\rfloor}\frac{1}{i}&=\sum_{j=1}^{\ell-1}\sum_{i=jm_y}^{(j+1)m_y-1}\frac{1}{i}+\sum_{i=\ell m_y}^{\ell m_y+\lfloor rm_y/a\rfloor}\frac{1}{i}\\&\leqslant\sum_{j=1}^{\ell-1}\sum_{i=jm_y}^{(j+1)m_y-1}\frac{1}{jm_y}+\sum_{i=\ell m_y}^{(\ell+1)m_y-1}\frac{1}{\ell m_y}=\sum_{j=1}^{\ell}\frac{1}{j}.\end{aligned}\tag{3}$$

Noting that $\ell=\lfloor b/a\rfloor$, $\frac{1}{(1-\theta)b}=\frac{1}{b-a}$, and that there are at most n choices for $y\in Y$, we obtain the desired bound. ☐

Theorem 5 immediately follows from Propositions 8 and 9. To establish Theorem 7, we need to characterise the bouquets that attain equality in Proposition 9.

Proof of Theorem 7

Let $\mathcal{F}$ be a w-bounded θ-intersecting family over $[n]$, where $\theta=\frac{a}{b}$. By Proposition 8, we can discard at most $w^3=o(n)$ sets from $\mathcal{F}$ to obtain a bouquet $\mathcal{F}^*\subseteq\mathcal{F}$. By Proposition 9, it follows that $|\mathcal{F}^*|\leqslant C_\theta n$, where $C_\theta=\frac{1}{b-a}\sum_{i=1}^{\lfloor b/a\rfloor}\frac{1}{i}$. Thus, we have $|\mathcal{F}|\leqslant(C_\theta+o(1))n$.

In order to have equality, we must have $|\mathcal{F}^*|\geqslant(C_\theta-o(1))n$. In the proof of Proposition 9, specifically (2), we have $|\mathcal{F}^*|=\frac{1}{1-\theta}\sum_{y\in Y}\sum_{s\in S_y}\frac{1}{s}$, where Y is the set of elements outside the largest core of the bouquet, and S_y is the set of sizes of the sets that contain y. In (3) we then bounded $\frac{1}{1-\theta}\sum_{s\in S_y}\frac{1}{s}$ in terms of $m_y=\min S_y$, showing it achieves its maximum of C_θ when $S_y=\{1,2,\dots,\lfloor b/a\rfloor\}$; we call this latter set S^*.

To prove stability, let $Y=Y_1\cup Y_2\cup Y_3$, where $Y_1=\{y\in Y:S_y=S^*\}$, $Y_2=\{y\in Y:m_y\geqslant 2\}$, and $Y_3=\{y\in Y:m_y=1,S_y\subsetneq S^*\}$.

For $y\in Y_3$, since S_y is missing at least one element from S^*, we have $\frac{1}{1-\theta}\sum_{s\in S_y}\frac{1}{s}\leqslant C_\theta-\frac{1}{(1-\theta)\lfloor b/a\rfloor}$.

For $y\in Y_2$, consider the $j=1$ term in the inequality in (3). Here, in the sum $\sum_{i=m_y}^{2m_y-1}\frac{1}{i}$, we bound the summands from above by $\frac{1}{m_y}$ to show that this sum is at most 1. However, if $m_y\geqslant 2$, then for the terms with $\frac{3}{2}m_y\leqslant i\leqslant 2m_y-1$, of which there are at least $\frac{1}{3}m_y$, the summand is in fact at most $\frac{2}{3m_y}$. Thus, in this case, this sum is at most $\frac{8}{9}$. This shows that for $y\in Y_2$, we have $\frac{1}{1-\theta}\sum_{s\in S_y}\frac{1}{s}\leqslant C_\theta-\frac{1}{9(1-\theta)}$.

Hence, we have

$$|\mathcal{F}^*|=\frac{1}{1-\theta}\sum_{y\in Y}\sum_{s\in S_y}\frac{1}{s}\leqslant C_\theta n-\frac{1}{9(1-\theta)}|Y_2|-\frac{1}{(1-\theta)\lfloor b/a\rfloor}|Y_3|.$$

Thus, to have $|\mathcal{F}^*|\geqslant(C_\theta-o(1))\,n$, we must have $|Y_2|=|Y_3|=o(n)$, which means almost all elements of the ground set are in Y_1, and are thus contained in sets of size $b,2b,\dots,\lfloor b/a\rfloor b$.

Finally, we show this constant C_θ is best possible when $\theta\in\{1/3\}\cup[1/2,1)$ by means of the following constructions of θ-intersecting $o(n^{1/3})$-bounded families over $[n]$.

- For $\theta = a/b \in (1/2, 1)$, let $\mathcal{F}$ be a maximal b-uniform sunflower over $[n]$ with core of size a. Then $\mathcal{F}$ is $\frac{a}{b}$-intersecting and $|\mathcal{F}| = \lfloor \frac{n-a}{b-a} \rfloor = \lfloor C_\theta n - \frac{a}{b-a} \rfloor$.

- For $\theta = 1/2$, the family $\mathcal{F}_s$ has size $\lfloor 3n/2 \rfloor - 2$ over $[n]$, and $C_{1/2} = 3/2$.

- For $\theta = 1/3$, assume $n \equiv 3 \pmod{24}$ for convenience, and consider the family $\mathcal{F} = \mathcal{F}(3) \cup \mathcal{F}(6) \cup \mathcal{F}(9)$, where:
  - $\mathcal{F}(3)$ is a sunflower with core $\{1\}$ and petals $\{\{2i, 2i+1\} : 2 \leqslant i \leqslant (n-1)/2\}$,
  - $\mathcal{F}(6)$ is a sunflower with core $\{1, 2\}$ and petals $\{\{24i + j, 24i + j + 6, 24i + j + 12, 24i + j + 18\} : 0 \leqslant i \leqslant (n-27)/24,\ 4 \leqslant j \leqslant 9\}$, and
  - $\mathcal{F}(9)$ is a sunflower with core $\{1, 2, 3\}$ and petals $\{\{6i-2, 6i-1, 6i, \ldots, 6i+3\} : 1 \leqslant i \leqslant (n-3)/6\}$.

  $\mathcal{F}$ is then $\frac{1}{3}$-intersecting, and $|\mathcal{F}| = (n-3)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right) = \frac{11}{12}(n-3) = C_{1/3} n - \frac{33}{12}$.
  □

3 Concluding remarks

Sharpening the constant. While Theorem 7 establishes the correct constant for certain values of θ, further arguments can be made to improve the constant for other fractions. We briefly illustrate this with the example of $\theta = 1/4$: any $o(n^{1/3})$-bounded $\frac{1}{4}$-intersecting family has size at most $\left(\frac{7}{12} + o(1)\right)n$, rather than the $\left(\frac{25}{36} + o(1)\right)n$ bound that Theorem 7 gives.

For the lower bound, we construct such a family using sets of size $4, 8$ and 16. The sunflowers have nested cores of size $1, 2$ and 4 respectively, and for the petals, we divide the remaining elements into blocks of size 36, arranged in 3×12 rectangles. Each row (of size 12) is the petal of a 16-set, and is partitioned into four petals of size 3 each (for the 4-sets). The 12 columns are paired up to form the petals of the 8-sets, in such a way that they intersect each small petal at most once.

For the upper bound, we first note that if the constant of $\frac{25}{36}$ from the theorem were tight, then almost all elements of the ground set would have to be contained in sets of size 4, 8, 12, and 16. However, it is not hard to show (we omit the details) that sets of size 12 are not compatible with the other set sizes. Given this, one can then show that $\sum_{s \in S_y} \frac{1}{s}$ is maximised when $S_y = \{4, 8, 16\}$, which results in a bound of $\left(\frac{7}{12} + o(1)\right)n$ instead.

It appears a difficult task to see what the correct constant is in general, even for $\theta = 1/b$, and it would be interesting to obtain further results in this direction.

Small families. The $o(n)$ error in Theorem 7 is necessary, because of the existence of bisection closed families of size greater than $3n/2$ for $n \leqslant 15$. These are constructed in [7] using the Fano plane. Define the family $\mathcal{F}_{\text{Fano}}$ over $[8]$ as follows:

$$\begin{aligned}\mathcal{F}_{\text{Fano}} &= \mathcal{F}_s \cup \{1357, 1368, 1458, 1467\} \\ &= \{12, 13, 14, 15, 16, 17, 18, 1234, 1256, 1278, 1357, 1368, 1458, 1467\}.\end{aligned}$$

It is easy to check that $\mathcal{F}_{\text{Fano}}$ is a bisection closed family of size 14 over [8], and it arises from the symmetric 2-$(7, 4, 2)$ design. We can similarly modify $\mathcal{F}_s$ using the sets 1357, 1368, 1458, and 1467 to get bisection closed families over $[n]$ of size more than $\lfloor 3n/2 \rfloor - 2$ for $n \leqslant 15$.

Unbounded families. Our constant C_θ in Theorem 7 is strictly smaller than 3/2 when $\theta \neq 1/2$, and the proof of Proposition 9 shows that even when $\theta = 1/2$, we obtain a smaller constant unless almost all elements of $[n]$ are contained in sets of size 2. However, the existence of the Hadamard families $\mathcal{F}_H$ of Example 3 precludes any simple extension of the argument given in this note to try and establish an upper bound of $(\frac{3}{2} + o(1))n$ for the size of an arbitrary bisection closed family, since these are bisection closed families of size $3n/2 - 2$ that do not contain any sets of size 2 (in fact, the set sizes in $\mathcal{F}_H$ are all either $n/2$ or $n/4$).

Families of large sets. The best results known so far in the "large" regime are given in [6]: if all the sets in $\mathcal{F}$ have size at least $\frac{1}{4(1-\theta)}n - \Theta(\sqrt{n})$, then $|\mathcal{F}| = O(n)$.

A linear algebraic reformulation. In [7], the authors consider a related problem of finding bounds on the ranks of certain symmetric matrices. Specifically, large θ-intersecting families induce such matrices of low rank. There the authors construct low rank matrices using bipartite graphs and ask whether any of them arise from θ-intersecting families. Theorem 5 shows that it is not possible for *bounded* bisection closed families to induce such matrices. This explains in a sense why the Fano construction does not seem to extend beyond small values of n to produce larger bisection closed families from $\mathcal{F}_s$.

Hierarchically closed families. Our results also have implications in the setting of *hierarchically* r*-closed* θ-intersecting families, as defined by Balachandran, Bhattacharya, Kher, Mathew, and Sankarnarayanan [5]. Given $r \geqslant 2$, we say $\mathcal{F}$ is hierarchically r-closed θ-intersecting if, for any $2 \leqslant t \leqslant r$ and any t-subset $\{A_1, \ldots, A_t\}$ of $\mathcal{F}$, we have $|\bigcap_{i=1}^t A_i| \in \{\theta|A_i| : i \in [t]\}$. From our previous examples, note that $\mathcal{F}_s$ is hierarchically r-closed for all r, while $\mathcal{F}_H$ is not hierarchically r-closed for any $r \geqslant 3$. Thus, in this sense, the two families are at opposite ends of a spectrum, despite having the same size.[3]

In [5, Theorem 5], it was shown that Conjecture 4 holds for hierarchically closed fractional intersecting families with a constant $c_\theta \leqslant \frac{1}{b-a}(2\log(\theta^{-1}) + 2)$. In the special case of $\theta = 1/2$, the authors improved the constant to the tight $c_{1/2} = 3/2$, and further showed that $\mathcal{F}_s$ from Example 2 is the unique extremal family, up to permutation of the ground set. To prove their results, they showed that a hierarchically closed fractional intersecting family must essentially be a bouquet, and then gave an upper bound on the size of bouquets.

In Proposition 9, we provide sharper estimates on the size of bouquets, and thus we improve the bounds in [5] for hierarchically closed fractional intersecting families, showing

[3]Note that a hierarchically 2-closed family is just a θ-intersecting family as defined in Section 1. So, when we say that a θ-intersecting family $\mathcal{F}$ is *hierarchically closed*, we mean that it is hierarchically r-closed for some $r \geqslant 3$.

that these have size at most $(C_\theta + o(1))n$ as well. Furthermore, since the constructions of bouquets in Theorem 7 are also hierarchically closed, the tightness results carry over as well.

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