A Characterization of 4-Connected Graphs with no $K_{3,3} + v$-Minor

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Abstract

Let $K_{3,3} + v$ be the graph obtained by adding a new vertex v to $K_{3,3}$ and joining v to the four vertices of a 4-cycle. In this paper, we characterize all 4-connected graphs that do not contain $K_{3,3} + v$ as a minor.

Mathematics Subject Classifications: 05C83

1 Introduction

All graphs in this article are simple. Define the *contraction* of an edge e as identifying the two ends of e and then deleting all but one edge from each resulting parallel family. Given two graphs G and H, we say that H is a minor of G, if there is a subgraph of G to which we can apply a sequence of edge contractions and deletions to obtain a graph isomorphic to H. We call G H*-free* if H is not a minor of G. The Robertson-Seymour Graph Minors project has shown that minor-closed classes of graphs can be characterized by finitely many forbidden minors. We can get some graph classes which have many interesting properties in the process of excluding small minors. In addition, many important problems in graph theory can be formulated in terms of H-free graphs. For instance, Tutte's 4-flow conjecture asserts that every bridgeless Petersen-free graph admits a 4-flow.

Ding and Liu [3] surveyed all H-free graphs for 3-connected H with at most 11 edges. For graphs with 12 edges, there are 51 3-connected graphs. Let G be a 3-connected graph. We define G to be *internally* 4*-connected* if the order of G is at least five, and for every 3-separation $\{G_1, G_2\}$ of G, exactly one of G_1, G_2 is isomorphic to $K_{1,3}$. In addition, there are only three internally 4-connected graphs with 12 edges, the cube, the octahedron (or *Oct* for short), and the Wagner graph V_8. Maharry [7, 8] characterized all cube-free graphs and 4-connected *Oct*-free graphs. Ding [1] characterized all *Oct*-free graphs, and Maharry

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and Robertson [9] characterized internally 4-connected V_8-free graphs. For graphs with 13 edges, there are again only three internally 4-connected ones, which are described below. Let Oct^+ be the graph obtained from the octahedron by adding an edge. And Maharry [6] characterized all 4-connected Oct^+-free graphs. Let cube+e denote the graph obtained by adding a long diagonal to the cube. Let $K_{3,3} + v$ be the graph obtained by adding a new vertex v to $K_{3,3}$ and joining v to the four vertices of a 4-cycle. Ding [2] pointed out that cube+e-free and $K_{3,3} + v$-free graphs remain uncharacterized. In this paper, we characterize all 4-connected $K_{3,3} + v$-free graphs.

To state our main result we need to define a few classes of graphs and symbols. We use G/e to denote the graph obtained from G by contracting e, and $G\backslash e$ to denote the graph obained from G by deleting e. For each integer $n \geqslant 3$, let DW_n denote a *double-wheel*, which is the graph on $n + 2$ vertices obtained from a cycle C_n by adding two adjacent vertices and connecting them to all vertices on the cycle. Let $\mathcal{DW} = \{DW_n : n \geqslant 3\}$. For each integer $n \geqslant 5$, let C_n^2 be the graph obtained from a cycle C_n by joining all pairs of vertices of distance two on the cycle. Let $\mathcal{C}_0 = \{C_{2n}^2 : n \geqslant 3\}$, $\mathcal{C}_1 = \{C_{2n+1}^2 : n \geqslant 2\}$ and $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$. The graph $L(G)$ is called the *line graph* of G if $V(L(G)) = E(G)$, and for any two vertices e, f in $V(L(G))$, e and f are adjacent in $L(G)$ if and only if they are adjacent edges in G. Our main result is the following.

Theorem 1. *A 4-connected graph G is $K_{3,3}+v$-free if and only if either G is planar or G belongs to $\mathcal{DW} \cup \mathcal{C}_1 \cup \{L(K_{3,3}), K_6, K_6\backslash e, \Gamma_1, \Gamma_2\}$, where Γ_1 and Γ_2 are the last two graphs shown below.*

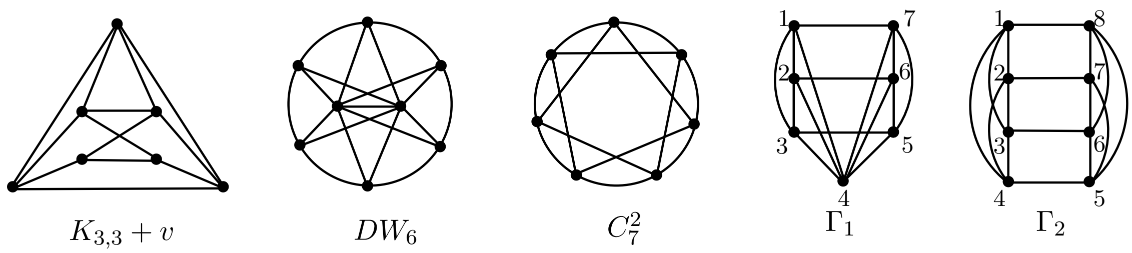

Figure 1: Some graphs in Theorem 1 .

Let G be a 3-connected graph. We call G *weakly 4-connected* if for every 3-separation $\{G_1, G_2\}$ of G, one of G_1 or G_2 contains at most four edges. For each internally 4-connected graph H, Ding has described in [2] how 3-connected H-free graphs can be constructed from weakly 4-connected H-free graphs. This result indicates that to determine all $K_{3,3}+v$-free graphs, it is sufficient to determine all weakly 4-connected $K_{3,3}+v$-free graphs. However, this is a challenging problem. Instead, we focus on determining all 4-connected $K_{3,3} + v$-free graphs, which provides a significant step towards a complete characterization of all weakly 4-connected graphs with no $K_{3,3} + v$-minor.

2 Preliminaries

In this section, we introduce some definitions and known results to prove Theorem 1.

Let G be a graph. For a vertex v in G, let $N_G(v)$ denote the set of vertices of G that are adjacent to the vertex v, and simply write $N(v)$ when there is no ambiguity. If G is 4-connected, then a 4-*split* of v produces a new graph G' as follows. Given two sets $A, B \subseteq N_G(v)$ with $A \cup B = N_G(v)$ and $\min\{|A|, |B|\} \geqslant 3$, the graph G' is obtained by adding to $G - v$ two adjacent vertices a and b such that $N_{G'}(a) = A \cup \{b\}$ and $N_{G'}(b) = B \cup \{a\}$. We also call G' a *split* of G. Note that G' is also 4-connected and $G'/ab = G$.

A sequence of 4-connected graphs $G_0, G_1, \ldots, G_n$ is called a (G_0, G_n)-*chain* if each G_i $(i < n)$ has an edge e_i such that $G_i/e_i = G_{i+1}$. The following theorem due to Qin and Ding [10] is an important tool to generate all 4-connected graphs. Let $\mathcal{L} = \{L(G) : G$ is an internally 4-connected cubic graph$\}$. For convenience, we abbreviate $K_{3,3} + v$ as K^v in the rest of this paper.

Theorem 2 ([10]). *Let G be a 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$. If G is planar, then there exists a (G, C_6^2)-chain; if G is non-planar, then there exists a (G, K_5)-chain.*

The following results are necessary to prove Theorem 1.1.

Lemma 3 ([5]). *If $G \in \mathcal{L}$ is 4-connected and K^v-free, then G is planar or $G = L(K_{3,3})$.*

Lemma 4 ([5]). *Graphs in $\mathcal{C}$ are all 4-connected K^v-free graphs.*

Lemma 5 ([5]). *If a 4-connected graph G is K^v-free, then it is planar, C_{2k+1}^2 $(k \geqslant 2)$, $L(K_{3,3})$ or it is obtained from C_5^2 by repeatedly 4-splitting vertices.*

Lemma 6 ([4]). *The only 4-splits of C_5^2 are K_6, $K_6\backslash e$, DW_4.*

Thus, we next characterize K^v-free graphs obtained from K_6, $K_6\backslash e$ and DW_4 by repeatedly 4-splitting vertices.

3 Proof of Theorem 1

In this section, we prove the following lemma from which the Theorem 1.1 follows.

Lemma 7. *Let G be a 4-connected graph obtained from K_6, $K_6\backslash e$ and DW_4 by repeatedly 4-splitting vertices. Then G is K^v-free if and only if $G \in \mathcal{DW} \cup \{\Gamma_1, \Gamma_2\}$.*

Next we divide the proof of Lemma 7 into a sequence of lemmas.

Lemma 8. *Every graph in $\mathcal{DW}$ is K^v-free.*

Proof. Observe that every $G \in \mathcal{DW}$ has a set S of at most two vertices such that the maximum degree of $G - S$ is at most two. This is a property preserved by all minor of G, but K^v does not posses it. Thus G is K^v-free. $\square$

Lemma 9. *All graphs in* $\{K_6, K_6\backslash e, \Gamma_1, \Gamma_2\}$ *are* K^v*-free.*

Proof. Since $K^v - v$ is bipartite, K^v does not contain two disjoint triangles. Note that $|E(\Gamma_1)| = 15 = |E(K^v)| + 2$. If Γ_1 contains a K^v-minor, then K^v is obtained from Γ_1 by deleting two edges incident with the vertex 4. If e, f are edges of Γ_1 that are incident with 4, then $\Gamma_1\backslash\{e, f\}$ contains two disjoint triangles, 123 and 567, so this graph is not isomorphic to K^v.

Observe that Γ_2 has eight vertices. If Γ_2 contains a K^v-minor, we may assume that K^v is obtained from Γ_2 by contracting one edge. Up to symmetry, there are exactly two such constructions, $\Gamma_2/12$ and $\Gamma_2/45$. Since $|E(\Gamma_2/12)| = 13$, no edges of $\Gamma_2/12$ can be deleted. Note that $\Gamma_2/12$ has two disjoint triangles, so it is not isomorphic to K^v. Additionally, $\Gamma_2/45$ is isomorphic to Γ_1, which is K^v-free.

Furthermore, K_6 and $K_6\backslash e$ are K^v-free, since $|V(K_6)| = |V(K_6\backslash e)| < |V(K^v)|$. □

Lemma 10. *Every graph obtained by adding an edge to* Γ_1 *or* Γ_2 *contains* K^v *as a minor.*

Proof. By symmetry, there is only one way to add an edge to each of Γ_1 and Γ_2, respectively. It is straightforward to verify that both $\Gamma_1 + 16$ and $\Gamma_2 + 16$ contain K^v as a minor (see Figure 2). □

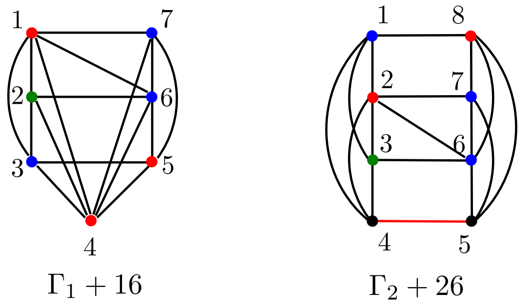

Figure 2: Graphs $\Gamma_1 + 16$ and $\Gamma_2 + 26$ (The vertices of color red and blue belong to the two color classes of $K_{3,3}$ respectively, and the vertex of color green corresponds to v).

Lemma 11. *Every graph obtained by adding an edge to* DW_n $(n \geqslant 5)$ *contains* K^v *as a minor.*

Proof. First, we consider the case when $n = 5$. Up to symmetry, there is a unique way to add an edge. The resulting graph, denoted by DW_5^e, contains K^v as a minor (see Figure 3). Next, we consider $n \geqslant 6$. Let $\mathcal{F} = \{F : F$ is the graph obtained by adding an edge to $DW_n\}$. Note that all the graphs in $\mathcal{F}$ will always contain DW_5^e as a minor by contracting some edges on the cycle $C = 1, 2, \ldots, n$. Therefore, every graph obtained by adding an edge to DW_n $(n \geqslant 6)$ contains K^v as a minor. □

Lemma 12. *The only* K^v*-free splits of* DW_4 *are* Γ_1 *and* DW_5*.*

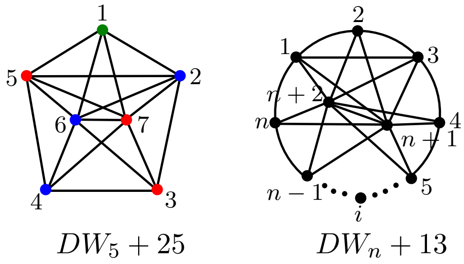

Figure 3: Graphs $DW_5 + 25$ and $DW_n + 13$.

Proof. We first consider splitting a degree-4 vertex of DW_4. Suppose both of the two new vertices have degree four. Up to symmetry, there are exactly three such splits, denoted by G_1, G_2 and G_3, which are shown in Figure 4. The first two splits, G_1 and G_2, contain K^v as a minor. The third split, G_3 is isomorphic to DW_5, which is K^v-free by Lemma 8.

Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by G, is obtained from G_1, G_2, or G_3 by adding edges. If G contains G_1 or G_2, then G contains a K^v-minor. Therefore, we assume that G is obtained from DW_5 by adding edges, which contains a K^v-minor by Lemma 11.

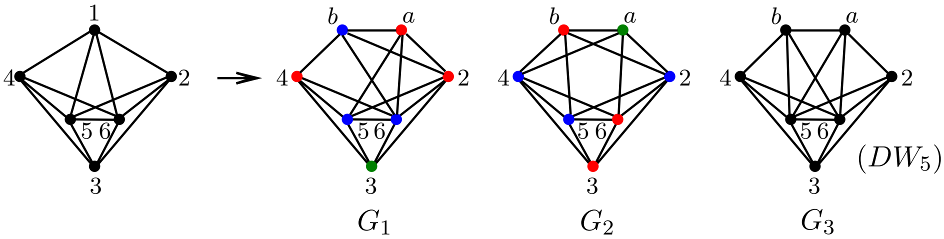

Figure 4: Three splits G_1, G_2, G_3 of DW_4.

Next we consider splitting a degree-5 vertex. Suppose both of the two new vertices, a and b, have degree four. Up to symmetry, there are exactly four such splits, denoted by H_1, H_2, H_3 and H_4, which are shown in Figure 5. The last three splits, H_2, H_3 and H_4, contain K^v as a minor. The first split, H_1 is isomorphic to Γ_1, which is K^v-free by Lemma 9.

Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by H, is obtained from H_1, H_2, H_3, or H_4 by adding edges. If H contains H_2, H_3, or H_4 then H contains a K^v-minor. So we assume that H is obtained from Γ_1 by adding edges, which contains a K^v-minor by Lemma 10.

In summary, the only K^v-free splits of DW_4 are Γ_1 and DW_5. □

Lemma 13. *The only K^v-free splits of Γ_1 is Γ_2.*

Proof. We first claim that splitting a degree-4 vertex of Γ_1 must result in a K^v-minor. Let $\{1, 2, 3, 4, 5, 6, 7\}$ be the vertices of Γ_1. Up to symmetry, we consider splitting the vertex 1. Suppose both of the two new vertices, a and b, have degree four. By symmetry,

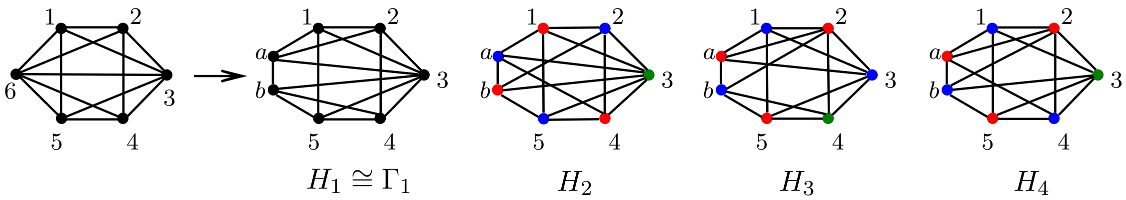

Figure 5: Another four splits H_1, H_2, H_3, H_4 of DW_4

there are four such splits, denoted by G_1, G_2, G_3, and G_4, each of which contains K^v as a minor, as shown in Figure 6.

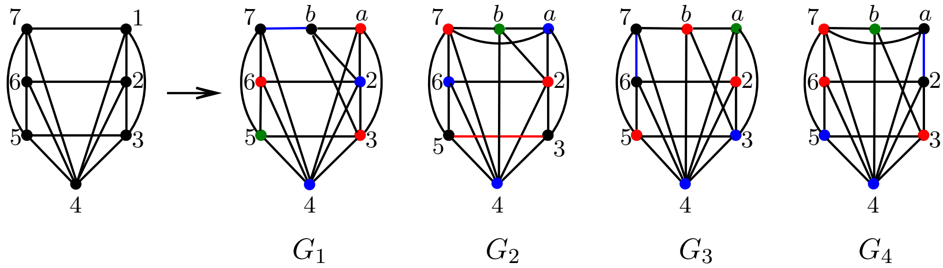

Figure 6: Four splits G_1, G_2, G_3, G_4 of Γ_1 (The colored edges mean to be contracted and the resulted new vertices belong to the same color classes of $K_{3,3}$).

Now, suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by G, is obtained from the four initial graphs by adding edges. Then G contains a K^v-minor.

Next, we consider splitting a degree-6 vertex. Suppose both of the two new vertices, a and b, have degree four. Up to symmetry, there are exactly three such splits, denoted by H_1, H_2, and H_3, which are shown in Figure 7. The first two splits, H_1 and H_2, contain K^v as a minor. The third split, H_3 is isomorphic to Γ_2, which is K^v-free by Lemma 9.

Now, suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by H, is obtained from H_1, H_2, or H_3 by adding edges. If H contains H_1 or H_2, then H contains a K^v-minor. Thus we assume that H is obtained from Γ_2 by adding edges, which contains a K^v-minor by Lemma 8. □

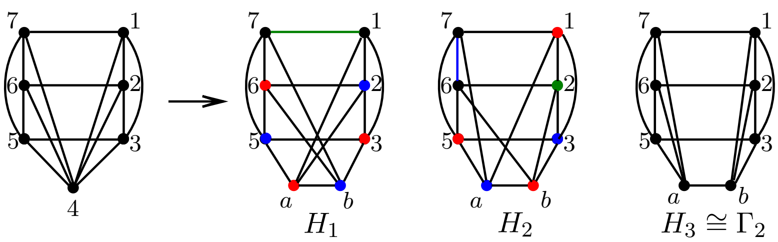

Figure 7: Another three splits H_1, H_2, H_3 of Γ_1.

Lemma 14. *Every 4-split of Γ_2 contains a K^v-minor.*

Proof. By symmetry, we only need to consider splitting the vertex 1. Suppose both of the two new vertices, a and b, have degree four. Up to symmetry, there are exactly two such splits, denoted by H_1, H_2, which are shown in Figure 8. In addition, both of H_1 and H_2 contain K^v as a minor. Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by H, is obtained from H_1 or H_2 by adding edges. Consequently, H contains a K^v-minor. □

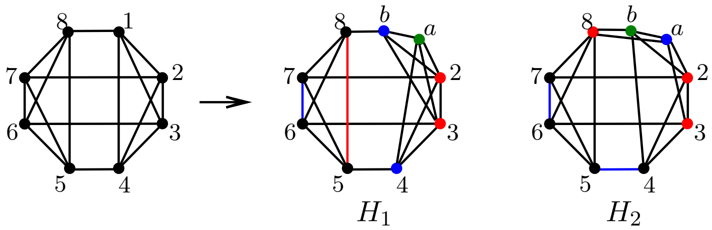

Figure 8: Two splits G_1, G_2 of Γ_2.

Lemma 15. *For each $n \geqslant 5$, the only K^v-free split of DW_n is DW_{n+1}.*

Proof. Suppose the vertices on the rim are labeled $1, \dots, n$, in the order they appear on the cycle. Let G be a split of DW_n. We first consider the case where G is obtained by splitting a vertex of degree $n+1$. Let a and b be the two new adjacent vertices.

For $n \geqslant 6$, without loss of generality, we can assume that a has degree exceeding four while b has degree four. Let i $(1 \leqslant i \leqslant n)$ be a neighbor of a. Note that $|N_G(a) \cup N_G(b)| \geqslant 9$, we can choose i such that $i \notin N_G(b)$. Then $G_1 = G/ij$ $(j = i+1)$ is a split of DW_{n-1}, since $G_1/ab = DW_{n-1}$, and $d_{G_1}(a), d_{G_1}(b) \geqslant 4$. Therefore, by repeating this process, we conclude that G contains a minor that is obtained by splitting a vertex of degree 6 of DW_5. Up to symmetry, there are two splits such that both of the two new vertices have degree four. Since both splits contain K^v-minor (as illustrated in Figure 9), it follows that G contains a K^v-minor.

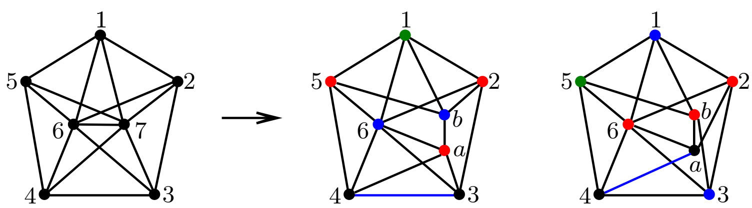

Figure 9: Two minimal splits of DW_5.

Next, suppose that G is obtained by splitting a vertex of degree 4. Suppose both of the two new vertices, a, b, have degree four. Up to symmetry, there are exactly three such splits, denoted by G_1, G_2, and G_3, which are shown in Figure 10. The last two splits,

G_2 and G_3, contain K^v as a minor. The first split G_1 is isomorphic to DW_{n+1}, which is K^v-free by Lemma 8.

Now, suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by G, is obtained from G_1, G_2, or G_3 by adding edges. If G contains G_2 or G_3, then G contains a K^v-minor. So we assume that G is obtained from DW_{n+1} by adding edges, which contains a K^v-minor by Lemma 11. Thus G is either DW_{n+1} or contains a K^v-minor. □

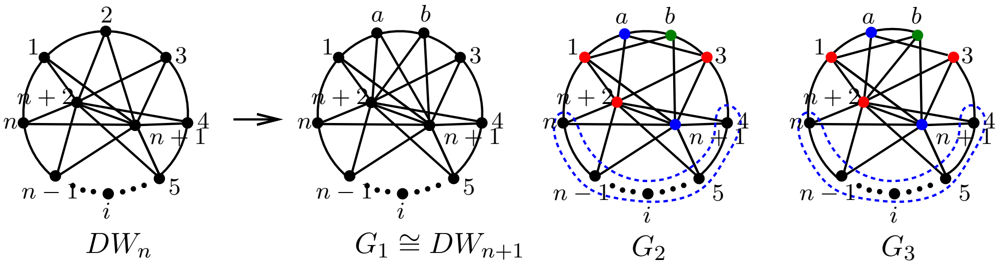

Figure 10: Three splits G_1, G_2, G_3 of DW_n.

Lemma 16. *Every 4-split of $K_6 \backslash e$ or K_6 contains K^v as a minor.*

Proof. According to Lemmas 10 and 11, every graph obtained by adding an edge to Γ_1 or DW_5 contains K^v as a minor. Note that $K_6 \backslash e$ is the graph obtained by adding an edge, say 14, to DW_4. Every graph generated by 4-splitting all vertices except 1 and 4 of $K_6 \backslash e$ is isomorphic to a graph obtained by adding at least one edge to some graph generated by 4-splitting these vertices of DW_4. Thus, these graphs contain K^v-minor by Lemmas 10-12. Next, we consider splitting the vertex 1, up to symmetry.

Suppose both of the two new vertices, a, b, have degree four. Up to symmetry, there are exactly three such splits, denoted by G_1, G_2, and G_3, which are shown in Figure 12. Note that G_1, G_2, and G_3 all contain K^v as a minor. Now, suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by G, is obtained from G_1, G_2, or G_3 by adding edges. Consequently, G contains a K^v-minor. Thus, every 4-split of $K_6 \backslash e$ contains K^v as a minor.

Note that $K_6 \backslash e + 36 \cong K_6$. Based on a similar discussion as above, it suffices to consider splitting the vertex 3. However, the resulting graphs are isomorphic to the graphs obtained by adding 36 to the splits shown in Figure 11, which all contain K^v as a minor. Thus, every 4-split of K_6 contains K^v as a minor. □

Proof of Lemma 7. The necessity follows from Lemmas 8 and 9. For the sufficiency, by Lemma 16, we only need to consider the splits of DW_4. Then Lemma 12 indicates that we only need to consider the splits of Γ_1 and DW_5. Finally, Lemmas 13-15 conclude that all K^v-free splits of Γ_1 and DW_5 belong to $\{\Gamma_2\} \cup \mathcal{DW}$. □

Lemma 17. *$L(K_{3,3})$ is K^v-free.*

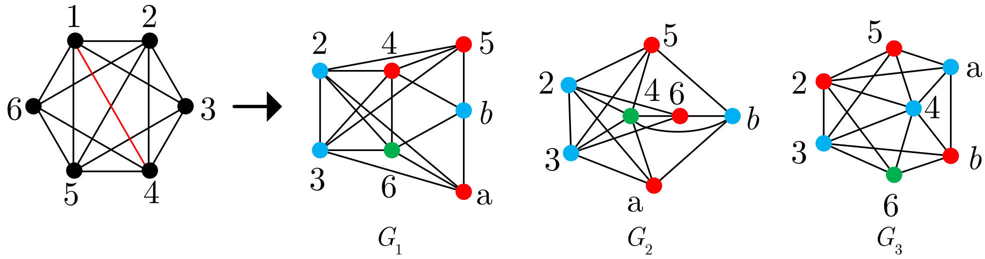

Figure 11: Splits of $K_6 \backslash e$.

Proof. If K^v is a minor of $L(K_{3,3})$, then it can be obtained by contracting two edges e and f in $L(K_{3,3})$ and subsequently deleting some edges. By symmetry, we only need to contract one edge in $L(K_{3,3})$. Let $L(K_{3,3})/e$ denote the graph obtained by contracting an edge e in $L(K_{3,3})$. Then, by symmetry, we can contract one of the edges $\{16, 12, 24, 23, 35, 45, 38\}$ in $L(K_{3,3})/e$. We verify every case in order and up to isomorphism there are six resulting graphs, we denote them by H_i^j, where j is the number of edges in each graph. Note that the graph H_1^{14} has 14 edges. If K^v is a minor of H_1^{14}, then K^v must be obtained by deleting an edge adjacent to the vertex a. However, each of the resulting graphs contains two disjoint triangles, making them K^v-free.

Observe that graphs H_2^{13} and H_3^{13} both have 13 edges. Specifically, the graph H_2^{13} contains two disjoint triangles, while the graph H_3^{13} has a vertex of degree 5, making them K^v-free. The graph H_4^{14} has 14 edges and two cubic vertices, meaning that only the edge between two vertices of degree 5 can be deleted. The resulting graph also contains two disjoint triangles, thus is K^v-free. Note that the graph H_5^{15} is isomorphic to Γ_1, and according to Lemma 9, it is K^v-free. The graph H_6^{14} has a vertex of degree 2 and thus cannot contain a K^v-minor. In summary, $L(K_{3,3})$ is K^v-free. □

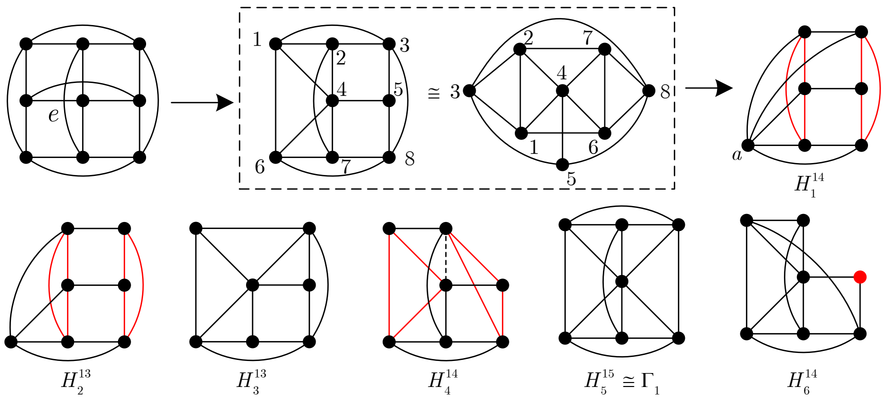

Figure 12: Graphs $L(K_{3,3}), L(K_{3,3})/e$ and H_i^j.

Proof of Theorem 1. By Theorem 2, we only need to consider the graphs in $\mathcal{C} \cup \mathcal{L}$ and the splits of C_5^2. The sufficiency of Theorem 1.1 follows from Lemmas 3-6 and 7. For the necessity, since K^v is non-planar, all planar graphs are K^v-free. Then the result follows from Lemmas 4, 8, 9, and 17. □

Acknowledgments

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