A note on directed analogues of the Sidorenko and forcing conjectures

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Submitted: Nov 14, 2022; Accepted: Apr 9, 2025; Published: Sep 5, 2025

Abstract

We study analogues of Sidorenko's conjecture and the forcing conjecture in oriented graphs, showing that natural variants of these conjectures in directed graphs are equivalent to the asymmetric, undirected analogues of the conjectures.

Mathematics Subject Classifications: 05C20, 05D40

1 Introduction

Estimating the minimum possible number of copies of a graph A in another graph H with a given number of vertices and edges is a central problem in extremal graph theory. One of the most important open problems in this area is *Sidorenko's conjecture.* Informally, Sidorenko's conjecture states that every dense graph G has asymptotically at least as many copies of any fixed bipartite graph as is expected in a random graph with the same number of vertices and the same edge density as H. This conjecture was independently posed by Erdős and Simonovits [Sim84] and Sidorenko [Sid93] and often appears in the language of *graph homomorphisms*, vertex maps of graphs that send edges to edges. We use the standard notation for a graph H, that its vertex set is denoted $V(H)$, its edge set $E(H)$, its number of vertices is $v(H) = |V(H)|$, and its number of edges is $e(H) = |E(H)|$.

Definition 1. Given undirected graphs A, H, we call $f : V(A) \to V(H)$ a *homomorphism* if for $x, y \in V(A)$, $f(x)$ is adjacent to $f(y)$ whenever x is adjacent to y. Let $h(A, H)$ be the number of homomorphisms $V(A) \to V(H)$. The *homomorphism density* of A in H is the fraction of vertex maps $A \to H$ that are homomorphisms, given by $t(A, H) := h(A, H)/v(H)^{v(A)}$.

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By a slight abuse of notation, we will often refer to a homomorphism $f : V(A) \to V(H)$ as a homomorphism between graphs $f : A \to H$.

Conjecture 2 (Sidorenko's Conjecture)**.** For every bipartite graph A and graph H, we have

$$t(A, H) \geqslant t(K_2, H)^{e(A)}.$$

Sidorenko [Sid93] showed that this conjecture holds for several types of graphs, including complete bipartite graphs, trees, and even cycles. While Sidorenko's conjecture remains open, a variety of special cases have been resolved. Conlon, Fox, and Sudakov [CFS10] showed that the conjecture holds for bipartite graphs with one vertex complete to the other side. In several other works, including [LS11, KLL16, CKLL18, Sze14], further progress has been made to show the conjecture holds for larger subfamilies of the collection of bipartite graphs. The Möbius ladder on 10 vertices, the undirected graph obtained by deleting the edges of a 10-cycle from $K_{5,5}$, is a notorious small open case of Sidorenko's conjecture.

Sidorenko's conjecture has a wide variety of applications to the study of random matrix theory, Markov chains, and to the study of *quasirandomness*. Quasirandom graphs were first studied by Thomason [Tho87] and Chung, Graham, and Wilson [CGW89]; they observed that a large number of properties that Erdős-Renyi random graphs satisfy are actually equivalent. This motivated the characterization of deterministic graph families that satisfy these equivalent properties. For $p \in [0, 1]$, we say that a sequence of distinct undirected graphs $\{H_n\}_{n=1}^{\infty}$ is *p-quasirandom* if $t(K_2, H_n) \to p$, and for all fixed undirected graphs A, $t(A, H_n) \to p^{e(A)}$ as $n \to \infty$.

There is a strengthening of Sidorenko's conjecture on characterizing quasirandom graph properties. A graph A is *p-forcing* if for all families of graphs $\{H_n\}_{n=1}^{\infty}$, the family $\{H_n\}_{n=1}^{\infty}$ is p-quasirandom if and only if the density of H_n is asymptotically p, and the number of copies of A in H_n is asymptotically the number expected in the Erdős-Renyi graphs $G(v(H_n), p)$. We call A *forcing* if it is p-forcing for all $p \in [0, 1]$. The *forcing conjecture*, initially posed by Skokan and Thoma [ST04] states that subgraphs are forcing if and only if they are bipartite and contain a cycle (showing these conditions are necessary is straightforward).

While there has been an extensive effort over the past decades to resolve parts of Sidorenko's conjecture and related problems, the analogues of these problems for oriented graphs remain relatively poorly understood. The first discussion of directed quasirandomness appeared in the context of tournaments in [CG91]. Recently, substantial progress has been made in understanding oriented subgraph counts and quasirandomness in tournaments and more general oriented graphs as in [Gri13, CPS19, CR17, BLSS21, HKK+23, FHMZ24].

In this article, we investigate natural analogues of the Sidorenko and forcing properties in oriented graphs. Throughout, we work with *oriented graphs*, which are formed from (simple) undirected graphs by orienting each edge. We can similarly define homomorphism density for directed graphs as in definition 1, where $f : V(B) \to V(G)$ is a homomorphism if for every edge $(x, y) \in E(B)$ we have $(f(x), f(y)) \in E(G)$. An oriented graph B has the *directed Sidorenko property* if for any oriented graph G with edge density p, the number of copies of B in G is at least as many as expected if we orient each edge of an Erdős-Renyi random graph with $v(G)$ vertices and density p uniformly at random. We formalize this notion as a definition.

Definition 3 (Directed Sidorenko)**.** An oriented graph B is said to have the *directed Sidorenko property* if for every oriented graph G,

$$t(B,G) \geqslant t(\vec{K}_2, G)^{e(B)}, \tag{1.1}$$

where $\vec{K}_2$ refers to the oriented complete graph on two vertices.

For a bipartite oriented graph to have the directed Sidorenko property, it must have a homomorphism to an edge (see observation 20). We conjecture that this necessary property is in fact sufficient.

Conjecture 4. If B is a bipartite oriented graph with a homomorphism $B \to \vec{K}_2$, then B has the directed Sidorenko property.

We prove that conjecture 4 is equivalent to a previously studied conjecture. To do so, we first need to define some additional notation. Throughout this work, when we say A is a bipartite graph, it also fixes a bipartition $V(A) = A_1 \sqcup A_2$ of its vertices, where we refer to A_1 as the first part and A_2 the second part. Here, $t_{\text{bip}}(A,H)$ is the fraction of vertex maps respecting the corresponding bipartitions, i.e. sending vertices in the A_1 to U_1 and A_2 to U_2, that are homomorphisms.

We say a bipartite undirected graph A has the *asymmetric Sidorenko property* if $t_{\text{bip}}(A,H) \geqslant p^{e(A)}$ for all bipartite undirected graphs $H = (U_1 \sqcup U_2, F)$ with bipartite density $p = \frac{|F|}{|U_1||U_2|}$ (definition 10).

Throughout this work, we let $\overline{B}$ denote the *underlying undirected graph* of oriented graph B, the undirected graph obtained by forgetting the directions of all of the edges in B.

Theorem 5. *If B is an oriented graph with a homomorphism $B \to \vec{K}_2$, then $\overline{B}$ has the asymmetric Sidorenko property if and only if B has the directed Sidorenko property.*

We also exhibit two necessary conditions for an oriented graph to have the directed forcing property (defined precisely in definition 19).

Theorem 6. *If an oriented graph $B = (V,E)$ has the directed forcing property, then it has a homomorphism to $\vec{K}_2$ and $\overline{B}$ has a cycle.*

We conjecture the two necessary conditions above characterize oriented graphs with the directed forcing property.

Conjecture 7. If B is an oriented graph with a homomorphism to an edge and whose underlying undirected graph has a cycle, then B has the directed forcing property.

We also obtain an analogous result to our reduction from the asymmetric Sidorenko conjecture for oriented graphs with the directed forcing property.

Theorem 8. *Let B be any oriented graph with a homomorphism $B \to \vec{K}_2$. Then $\overline{B}$ has the asymmetric forcing property if and only if B has the directed forcing property.*

The asymmetric Sidorenko property is a stronger notion, as it implies the Sidorenko property. This can be seen by considering an auxiliary bipartite graph H' constructed from some undirected graph H by making two copies of the vertex set of H and connecting pairs of vertices in H' between the two parts if they correspond to edges in H. Then, for any undirected A with the asymmetric Sidorenko property, by counting homomorphic copies of A in H' that send vertices to corresponding parts, we can find the desired Sidorenko lower bound on the number of homomorphisms of A in H.

It is unknown if the asymmetric and classical Sidorenko properties are equivalent (although this is conjectured to be the case [CFS10]). However, although there are no known black box reductions from the standard Sidorenko conjecture to the asymmetric Sidorenko conjecture, essentially all graphs known to have the Sidorenko property are also known to satisfy the asymmetric Sidorenko property (for example, in [CFS10] it was observed that any bipartite graph with a vertex complete to one part has the asymmetric Sidorenko property).

We begin with some preliminaries on the directed and asymmetric Sidorenko properties along with graphons in section 2. Using the language of graphons, we prove theorem 5 in section 3. We subsequently study the directed forcing property in section 4. We conclude with some open problems and further directions in section 5.

2 Preliminaries

Throughout, we let oriented graph $G = (V, E)$ have vertex set V of size $v(G)$ and edge set E of size $e(G)$ with density $p = e(G)/\binom{v(G)}{2}$ and underlying undirected graph $\overline{G}$.

Definition 9. Given an oriented graph G, for vertex $v \in V$, its *out-neighborhood* is denoted by $N^+(v) = \{w : (v, w) \in E\}$ and its *out-degree* is $d^+(v) = |N^+(v)|$. We analogously define the *in-neighborhood* $N^-(v)$ and associate to v its *in-degree* $d^-(v)$.

2.1 Directed and asymmetric Sidorenko

Recall from definition 3 that oriented graph B has the directed Sidorenko property if it is systematically over-represented in oriented graphs.

Remark 2.1. We can alternatively define B to satisfy the directed Sidorenko property if for every oriented graph G on n vertices,

$$N_L(B, G) \geqslant \left(\left(\frac{e(G)}{v(G)^2} \right)^{e(B)} - o(1) \right) v(G)^{v(B)}, \tag{2.1}$$

where $N_L(B, G)$ denotes the number of labeled copies of B in G (i.e. the number of homomorphisms $f : B \to G$ that are injective) and $o(1)$ is a quantity that goes to 0 as $v(G) \to \infty$. It is not hard to show that this alternative definition is equivalent to that given by definition 3.

Remark 2.2. If G is a random oriented graph, where each pair of vertices appears as an edge with constant probability p oriented uniformly at random (independently), an injective map $V(B) \mapsto V(G)$ yields a copy of B in G with probability $(p/2)^{e(B)}$. Thus, Equation (2.1) says that the number of copies of B in G is asymptotically at least a natural random bound.

We will reduce the directed Sidorenko property to an undirected, bipartite Sidorenko property.

Definition 10. Let $A = (V_1 \sqcup V_2, E)$ and $H = (U_1 \sqcup U_2, F)$ be bipartite undirected graphs. Let $t_{\text{bip}}(A, H)$ be the density of maps $f : V(A) \to V(H)$ where $f(V_i) \subseteq U_i$ for $i = 1, 2$ that are homomorphisms. We say that A has the *asymmetric Sidorenko property* if $t_{\text{bip}}(A, H) \geqslant t_{\text{bip}}(K_2, H)^{e(A)}$ holds for all bipartite undirected graphs H.

Note that in the above definition, H has bipartite edge density $t_{\text{bip}}(K_2, H) = \frac{|F|}{|U_1||U_2|}$.

2.2 Graphons

The relationship between asymmetric Sidorenko/forcing properties and their oriented analogues is most naturally seen by leveraging the language of graphons, as in [Lov12]. The language of graphons will also give us a clean framework to understand quasirandomness and forcing in oriented and bipartite graphs. The study of quasirandomness in oriented graphs began with Chung and Graham [CG91], who studied quasirandom tournaments. Recently, several researchers have studied more general quasirandom oriented graphs [Gri13, AGH11]. For conciseness, below we define quasirandomness and forcing *only* in the setting of graphons; however, the notions we define below specify to the classical notions of quasirandomness in oriented and undirected bipartite graphs, and to forcing in bipartite undirected graphs.

Definition 11. A *graphon* is a measurable function $W : [0, 1]^2 \to [0, 1]$. Given a directed graph $G = ([n], E)$, we define its associated *directed graphon* to be the function $W_G : [0, 1]^2 \to [0, 1]$ obtained by equipartitioning $[0, 1] = I_1 \sqcup \cdots \sqcup I_n$ into equal length intervals, and letting $W_G(x, y) = \mathbf{1}\{(i, j) \in E\}$ for $x \in I_i, y \in I_j$ for $1 \leqslant i, j \leqslant n$ (here $\mathbf{1}$ denotes the indicator functions that $(i, j) \in E$).

Given a bipartite graph $H = (V \sqcup W, F)$ with $V = \{v_1, \ldots, v_n\}, W = \{w_1, \ldots, w_m\}$, we define its associated *bipartite graphon* $W_H : [0, 1]^2 \to [0, 1]$ as following. We equipartition the interval $[0, 1]$ into equal length intervals $[0, 1] = I_1 \sqcup \cdots \sqcup I_n$ and $[0, 1] = J_1 \sqcup \cdots \sqcup J_m$, and let $W_H(x, y) = \mathbf{1}\{(v_i, w_j) \in F\}$ for $x \in I_i, y \in J_j$ for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$.

Symmetric graphons (which satisfy $W(x, y) = W(y, x)$) are commonly studied, as they correspond to undirected graph limits. Here, we study graphons more generally (in the *asymmetric* setting) as limit objects of sequences of either directed graphs or bipartite undirected graphs with specified parts.

The space of graphons is compact under the cut metric, given by identifying graphons with cut distance zero, as defined below.

Definition 12. Given a measurable $W : [0,1]^2 \to [0,1]$, the *cut norm* of W is given by

$$\|W\|_\square = \sup_{S\times T\subset[0,1]^2} \left| \int_{S\times T} W(x,y)dxdy \right|,$$

taking the supremum over measurable S, T. Given two graphons W, U, we define their *cut distance* as

$$\delta_\square(U,W) = \inf_{\phi:[0,1]\to[0,1]} \|U - W^\phi\|,$$

taking the infimum over measure-preserving maps ϕ, where $W^\phi(x,y) = W(\phi(x),\phi(y))$.

In terms of cut distance, graphons are the limiting objects of graphs. Formally we have the following result (see e.g. [Lov12] for proofs and more detailed exposition about graphons).

Theorem 13 (Corollary 11.15 [Lov12])**.** *Suppose that $W : [0,1]^2 \to [0,1]$ is a measurable function. Then there exists a sequence of bipartite graphs $\{H_n\}_{n=1}^\infty$ such that*

$$\lim_{n\to\infty} \delta_\square(W, W_{H_n}) = 0.$$

If $W(x,y)+W(y,x) \leqslant 1$ for all $x,y \in [0,1]$, there is a sequence of oriented graphs $\{G_n\}_{n=1}^\infty$ with

$$\lim_{n\to\infty} \delta_\square(W, W_{G_n}) = 0.$$

The cut norm also offers a succinct characterization of quasirandomness for graphons.

Definition 14. A sequence of graphons $\{W_n\}_{n=1}^\infty$ is *p-quasirandom* if

$$\lim_{n\to\infty} \|W_n - p\|_\square = 0.$$

The canonical example of a p-quasirandom sequence of graphons is if W_n is the graphon corresponding to Erdős-Renyi random graoh $\mathcal{G}(n,p)$. Quasirandomness of a family of oriented or bipartite graphs is equivalent to quasirandomness of the associated graphon family (c.f. §1.4, Examples 11.37-38 in [Lov12]).

The above definition of quasirandomness appears at first to have little to do with counts of oriented subgraphs. This relationship becomes more apparent via the below characterization (culminating in theorem 18) of graphon quasirandomness via homomorphism density of subgraphs.

Definition 15. Given graphon W and directed graph B, the *B-density in W* is

$$t(B,W) = \int_{[0,1]^{V(B)}} \prod_{(i,j)\in E(B)} W(x_i,x_j) \prod_{i\in V(B)} dx_i$$

Given graphon W and bipartite undirected graph $A = (A_1 \sqcup A_2, E(A))$, the *bipartite A-density in W* is

$$t_{\text{bip}}(A,W) = \int_{[0,1]^{A_1\sqcup A_2}} \prod_{(v_i,w_j)\in E(A)} W(x_i,y_j) \prod_{v_i\in A_1} dx_i \prod_{w_j\in A_2} dy_j.$$

A sequence of graphons is *left convergent* to graphon W if for every (bipartite/directed) graph F, $t(F,W_n) \to t(F,W)$ as $n\to\infty$.

The above definitions have the following consequence.

Observation 16. *Let $\overline{B} = (V_1 \sqcup V_2, E')$ be an undirected bipartite graph, and let $B = (V,E)$ be the directed graph obtained from $\overline{B}$ by directing all edges from V_1 to V_2. Then $t(B,W) = t_{\text{bip}}(\overline{B},W)$ for any graphon $W : [0,1]^2 \to [0,1]$.*

observation 16 follows by examining the expression for the homomorphism densities given in definition 15 and noticing that the integrands are the same because the edges of $B = (V_1 \sqcup V_2, E(B))$ are exactly those of the form (x,y) for $x\in V_1, y\in V_2$ and where $(x,y)\in E(\overline{B})$.

The above notions of homomorphism density are consistent with the analogous definitions in graphs in the following sense. For any directed graphs B and G, $t(B,G) = t(B,W_G)$ where W_G is the directed graphon associated to G; for any bipartite graphs A and H, $t_{\text{bip}}(A,H) = t_{\text{bip}}(A,W_H)$ with W_H the bipartite graphon associated to H.

Proposition 17 (§11, [Lov12])**.** *A sequence of bipartite graphs $\{H_n\}_{n=1}^\infty$ is p-quasirandom if and only if*

$$\lim_{n\to\infty} \|W_{H_n} - p\|_\square = 0.$$

A sequence of directed graphs $\{G_n\}_{n=1}^\infty$ is p-quasirandom if and only if

$$\lim_{n\to\infty} \|W_{G_n} - p\|_\square = 0.$$

By identifying graphons that differ on a measure 0 set, the space of graphons becomes a compact space with the cut distance metric (c.f. Theorem 9.23, [Lov12]). As shown in [BCL+08], left convergence is equivalent to convergence in cut distance; a sequence of graphons $\{W_n\}_{n=1}^\infty$ left converges to W if and only if $\delta_\square(W_n,W)\to 0$ as $n\to\infty$. Quantitatively, we have the following counting lemma (which establishes one direction of the equivalence).

Theorem 18 (Lemma 10.23, [Lov12]). *Let U, W be graphons. If B is an oriented graph, then*

$$|t(B,W) - t(B,U)| \leqslant e(B)\delta_{\square}(U,W).$$

If A is a bipartite undirected graph, we similarly have

$$|t_{\text{bip}}(A,W) - t_{\text{bip}}(A,U)| \leqslant e(A)\delta_{\square}(U,W).$$

This equivalence establishes a homomorphism density version of quasirandomness as an immediate consequence; it also allows us to give a simple definition of the directed forcing property in the language of graphons.

Definition 19. Oriented graph B has the *directed forcing property* if the following condition holds for all $p \in [0,1]$. Take any sequence of oriented graphs $\{G_n\}_{n=1}^{\infty}$ such that both $t(\vec{K}_2, G_n) \to p/2$ and $t(B, W_{G_n}) \to \left(\frac{p}{2}\right)^{e(B)}$ as $n \to \infty$. Then $\{G_n\}_{n=1}^{\infty}$ is p-quasirandom.

Bipartite undirected graph A has the *asymmetric forcing property* if the following condition holds for all $p \in [0,1]$. Take any sequence of bipartite undirected graphs, $\{H_n\}_{n=1}^{\infty}$ with bipartite density tending to p as $n \to \infty$ and $t_{\text{bip}}(A, W_{H_n}) \to p^{e(B)}$. Then, $\{H_n\}_{n=1}^{\infty}$ is p-quasirandom.

Colloquially, an oriented graph B has the directed forcing property, if counting the number of copies of B in an oriented graph sequence $\{G_n\}$ is *sufficient* to certify quasirandomness of $\{G_n\}$.

3 Directed Sidorenko property

Which oriented graphs have the directed Sidorenko property? We reduce this question to the analogous conjecture in the asymmetric, undirected setting.

Let $G = \vec{K}_{n,n} = (V_1 \sqcup V_2, E)$ be a complete balanced bipartite oriented graph on $2n$ vertices where all edges $e = (v_1, v_2) \in E$ are oriented so that $v_1 \in V_1$ and $v_2 \in V_2$. Notice that if oriented graph B does not have a homomorphism to an edge, there are zero copies of B in G. Thus, in order to have the directed Sidorenko property, an oriented graph B must have a homomorphism to an edge, giving the following.

Observation 20. *If oriented graph B has the directed Sidorenko property, then it has a homomorphism $B \to \vec{K}_2$.*

Therefore, we only need to analyze directed graphs B obtained from a bipartite graph $\overline{B}$ by orienting all edges from the first part to the second. We aim to show theorem 5, which says that B is directed Sidorenko if and only if $\overline{B}$ is asymmetric Sidorenko. We first give the following characterization of the Sidorenko properties using graphons.

Proposition 21. *An oriented graph B satisfies the directed Sidorenko property if and only if for all measurable functions $W : [0,1]^2 \to [0,1]$, we have*

$$t(B,W) \geqslant \left| \int_{[0,1]^2} W(x,y)\, dxdy \right|^{e(B)} = t(\vec{K}_2, W)^{e(B)}.$$

Proof. First, if B satisfies $t(B,W) \geqslant t(\vec{K}_2,W)^{e(B)}$ for all W, then for any oriented graph G, we take $W = W_G$ and derive that $t(B,G) \geqslant (e(G)/v(G)^2)^{e(B)}$. This means that B satisfies the directed Sidorenko property.

On the other hand, suppose that B satisfies the directed Sidorenko property. As a consequence, for any $W : [0,1]^2 \to [0,1]$, by theorem 13, there exists a sequence of directed graphs $\{G_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \delta_\square(W/2, W_{G_n}) = 0$. Since B satisfies the directed Sidorenko property, we have

$$t(B, W_{G_n}) = t(F, G_n) \geqslant t(\vec{K}_2, W_{G_n})^{e(B)}.$$

Then by theorem 18, we have $t(B,W/2) \geqslant t(B,W_{G_n}) - e(B)\delta_\square(W/2,W_{G_n})$ and $t(\vec{K}_2,W_{G_n}) \geqslant t(\vec{K}_2,W/2) - \delta_\square(W/2,W_{G_n})$. Hence we have

$$t(B,W/2) \geqslant \left| \int_{[0,1]^2} \frac{1}{2} W(x,y)\, dxdy - \delta_\square(W/2,W_{G_n}) \right|^{e(B)} - e(B)\delta_\square(W/2,W_{G_n}).$$

Since this is true for all n, as $n \to \infty$, we get $t(B,W/2) \geqslant t(\vec{K}_2,W/2)^{e(B)}$. Multiplying both sides by $2^{e(B)}$, we have the desired inequality, thereby establishing the equivalence. ∎

Using the same argument, we can prove the following analogue for bipartite graphs.

Proposition 22. *An undirected bipartite graph A satisfies the asymmetric Sidorenko property if and only if for all measurable functions $W : [0,1]^2 \to [0,1]$, we have*

$$t_{\mathrm{bip}}(A,W) \geqslant \left| \int_{[0,1]^2} W(x,y)\, dxdy \right|^{e(A)}.$$

Now we are ready to prove theorem 5.

Proof of theorem 5. Let $\overline{B}$ be an undirected bipartite graph, and B be the directed graph obtained from $\overline{B}$ by orienting all edges from the first part to the second. We aim to show that $\overline{B}$ is asymmetric Sidorenko if and only if B is directed Sidorenko. By proposition 22, $\overline{B}$ is asymmetric Sidorenko if and only if for all measurable $W : [0,1]^2 \to [0,1]$ we have

$$t_{\mathrm{bip}}(\overline{B},W) \geqslant \left| \int_{[0,1]^2} W(x,y)\, dxdy \right|^{e(\overline{B})}.$$

By observation 16, we have $t_{\mathrm{bip}}(\overline{B},W) = t(B,W)$. As $e(\overline{B}) = e(B)$, we have $\overline{B}$ is asymmetric Sidorenko if and only if

$$t(B,W) \geqslant \left| \int_{[0,1]^2} W(x,y)\, dxdy \right|^{e(B)}$$

for all measurable $W : [0,1]^2 \to [0,1]$. This is, by proposition 21, equivalent to B being directed Sidorenko. Hence we conclude that the two conditions are indeed equivalent. ∎

4 Directed forcing property

We first show that, in order to have the directed forcing property, an oriented graph must have a homomorphism to $\vec{K}_2$.

Proposition 23. *If an oriented graph B satisfies the directed forcing property, then it has a homomorphism $B \to \vec{K}_2$.*

Proof. We may assume that B has no isolated vertices (by removing any isolated vertices that exist). For $\lambda \in [0,1]$, we define graphon $W^{(\lambda)}$ as below and in fig. 1

$$W^{(\lambda)}(x,y) = \begin{cases} 1-\lambda & \text{if } x \in [1/4,1/2], y \in [0,1/4], \\ \lambda/4 & \text{if } x \in [1/2,1], y \in [1/2,1], \\ 0 & \text{otherwise.} \end{cases}$$

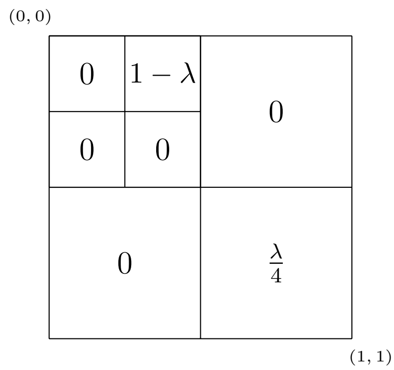

Figure 1. Visual depiction of the graphon $W^{(\lambda)}$.

For any λ, we know that $t(\vec{K}_2, W^{(\lambda)}) = \int_{[0,1]^2} W^{(\lambda)}(x,y)\, dxdy = 1/16$. If B does not have a homomorphism to $\vec{K}_2$, we cannot have a homomorphic copy of B in $W^{(\lambda)}$ where the images of the vertices of B, $x_1, \ldots, x_{v(B)}$, lie in $[0,1/2]$. In other words, when $\lambda = 0$, we have $t(B, W^{(0)}) = 0$. On the other hand, when $\lambda = 1$, we see that the integrand for $t(B, W^{(1)})$ is nonzero only when all of x_i lie in $[1/2,1]$, and $W^{(1)}(x_i, x_j) = 1/4$ in this case. Hence

$$t(B, W^{(1)}) = \left(\frac{1}{2}\right)^{v(B)} \left(\frac{1}{4}\right)^{e(B)} \geqslant \left(\frac{1}{16}\right)^{e(B)}.$$

In the last inequality, we note that B has no isolated vertices, so $v(B) \leqslant 2e(B)$. Since $t(B, W^{(0)}) \leqslant \left(\frac{1}{16}\right)^{e(B)} \leqslant t(B, W^{(1)})$, by continuity of $t(B, W^{(\lambda)})$ there exists $\lambda_0 \in [0,1]$ such that

$$t(B, W^{(\lambda_0)}) = \left(\frac{1}{16}\right)^{e(B)} = t\left(\vec{K}_2, W^{(\lambda_0)}\right)^{e(B)}.$$

However, $W^{(\lambda_0)}$ is not the constant function up to a measure zero set, so B is not forcing. ■

Finally, we obtain an analogous result to theorem 5 for oriented graphs which have the directed forcing property. We show that if B has a homomorphism to an edge, then $\overline{B}$ has the asymmetric forcing property if and only if B has the directed forcing property.

Proof of theorem 8. First, we show that if B has the directed forcing property, then $\overline{B}$ has the asymmetric forcing property. Suppose that $\{H_n\}_{n=1}^{\infty}$ is a sequence of bipartite graphs such that

$$\lim_{n\to\infty} t_{\text{bip}}(\overline{B}, H_n) = p^{e(B)}.$$

For each $n \geqslant 1$, we set $W_n = \frac{1}{2} W_{H_n}$. Then,

$$t_{\text{bip}}(\overline{B}, W_n) = 2^{-e(B)} t_{\text{bip}}(\overline{B}, W_{H_n}) = 2^{-e(B)} t_{\text{bip}}(\overline{B}, H_n).$$

Because W_n satisfies that $W_n(x, y) + W_n(y, x) \leqslant 1/2 + 1/2 = 1$ for all $x, y \in [0, 1]$, by theorem 13, there exists a directed graph G_n such that $\delta_\square(W_{G_n}, W_n) \leqslant 1/n$. Also note that by observation 16, $t_{\text{bip}}(\overline{B}, W_n) = t(B, W_n)$. Hence,

$$t(B, W_n) = t_{\text{bip}}(\overline{B}, W_n) = 2^{-e(B)} t_{\text{bip}}(\overline{B}, H_n).$$

By theorem 18, we have

$$\begin{aligned} \left|t(B, G_n) - (p/2)^{e(B)}\right| &\leqslant |t(B, W_{G_n}) - t(B, W_n)| + \left|2^{-e(B)} t_{\text{bip}}(\overline{B}, H_n) - 2^{-e(B)} p^{e(B)}\right| \\ &\leqslant \frac{e(B)}{n} + 2^{-e(B)} \left|t_{\text{bip}}(\overline{B}, H_n) - p^{e(B)}\right|. \end{aligned}$$

As n tends to infinity, both terms on the right hand side tend to zero. Therefore we have

$$\lim_{n\to\infty} t(B, G_n) - (p/2)^{e(B)} = 0.$$

By the definition of directed forcing property, we conclude that the sequence of directed graphs $\{G_n\}_{n=1}^{\infty}$ is $(p/2)$-quasirandom. By proposition 17, we know that

$$\lim_{n\to\infty} \|W_{G_n} - p/2\|_\square = 0.$$

From our construction of G_n, we know that $\delta_\square(W_{H_n}/2, W_{G_n}) \leqslant 1/n$. By the triangle inequality for the cut-distance, we have $\delta_\square(W_{H_n}/2, p/2) \leqslant 1/n + \|W_{G_n} - p/2\|_\square$. Note that the constant function is invariant under measure-preserving maps, so we can equivalently rewrite this as

$$\|W_{H_n} - p\|_\square \leqslant 2/n + 2\|W_{G_n} - p/2\|_\square.$$

As n tends to infinity, both terms on the right hand side tend to zero. Therefore, we conclude that

$$\lim_{n\to\infty} \|W_{H_n} - p\| = 0,$$

which establishes p-quasirandomness via proposition 17.

The other direction is very similar. Suppose that $\overline{B}$ has the asymmetric forcing property. We aim to show that B has the directed forcing property. Suppose that $\{G_n\}_{n=1}^{\infty}$ is a sequence of directed graphs satisfying that $t(B, G_n) = p^{e(B)}$. Then for W_{G_n}, by observation 16, we have $t_{\text{bip}}(\overline{B}, W_{G_n} = t(B, W_{G_n}) = t(B, G_n)$. Note that W_{G_n} is a measurable function taking values in $[0, 1]$, so by theorem 13 there exists a bipartite graph H_n satisfying that $\delta_\square(W_{H_n}, W_{G_n}) \leqslant 1/n$. Using a similar argument as above, we have by theorem 18 that

$$\left|t_{\text{bip}}(\overline{B}, H_n) - p^{e(B)}\right| \leqslant \frac{e(B)}{n} + \left|t(B, G_n) - p^{e(B)}\right|.$$

Letting n tend to infinity, we have $\lim_{n\to\infty} t_{\text{bip}}(\overline{B}, H_n) = p^{e(B)}$. Hence $\{H_n\}_{n=1}^{\infty}$ is p-quasirandom by the asymmetric forcing property of $\overline{B}$. This shows that $\lim_{n\to\infty} \|W_{H_n} - p\|_\square = 0$. Also, by the same argument as before, we have the triangle inequality $\|W_{G_n} - p\|_\square \leqslant 1/n + \|W_{H_n} - p\|_\square$ and thus $\lim_{n\to\infty} \|W_{G_n} - p\|_\square = 0$. By proposition 17, we conclude that $\{G_n\}_{n=1}^{\infty}$ is p-quasirandom. ■

5 Concluding remarks

The above directed Sidorenko and forcing properties studied here are very natural notions, but not the only Sidorenko-type properties one might investigate in a directed graph. For example, consider the following, slightly different directed Sidorenko-style property that is natural to define for an oriented graph B. We say B satisfies the *second directed Sidorenko property* if

$$t_B(G) \geqslant \frac{1}{2^{e(B)}} t_{\overline{B}}(\overline{G}) \tag{5.1}$$

holds for all oriented graphs G. Equation (5.1) implies (1.1) when the underlying graph $\overline{B}$ satisfies Sidorenko's conjecture. This fact suggests that the second directed Sidorenko property captures a natural "orientation-focused" analogue of the Sidorenko property. Griffiths [Gri13], studied such an orientation-focused analogue of quasirandomness in general oriented graphs.

We can further focus on which patterns of *directions* are systematically overrepresented or underrepresented by studying counts of directed graphs B in *tournaments*, oriented complete graphs. The behavior of subgraph counts in tournaments are quite different than in general directed graphs. For example, there are a small number of special subgraphs, called *impartial oriented graphs* that have the surprising property that for an impartial B, the number of copies of B in any tournament T on n vertices is the same, i.e. the number of copies of B in a tournament T is $f(n)$, where $n = v(T)$, and is otherwise independent of the orientation of T. Some of the simplest examples of impartial oriented graphs are a vertex, single edge, and the oriented graph B with vertex set $\{a, b, c, d\}$ and edge set $\{(a, b), (c, d), (a, c)\}$ (see [ZZ20]).

Further, there are directed graphs (e.g. directed paths [SSZ23] and directed cycles of length not a multiple of 4 [GKLV23]), that are *tournament anti-Sidorenko*, i.e. systematically underrepresented in all tournaments. This phenomenon does not appear in general directed graphs. The tournament Sidorenko and tournament anti-Sidorenko directed properties are further studied in [FHMZ24].

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