On subdivisions of four blocks cycles with two non-consecutive blocks of length one in digraphs with large chromatic number

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Abstract

A cycle with four blocks $C(k_1, k_2, k_3, k_4)$ is an oriented cycle formed of four blocks of lengths k_1, k_2, k_3 and k_4 respectively. Recently, Cohen et al. conjectured that for every positive integers k_1, k_2, k_3, k_4, there is an integer $g(k_1, k_2, k_3, k_4)$ such that every strongly connected digraph D containing no subdivisions of $C(k_1, k_2, k_3, k_4)$ has a chromatic number at most $g(k_1, k_2, k_3, k_4)$. This conjecture is confirmed by Cohen et al. for the case of $C(1, 1, 1, 1)$ and by Al-Mniny for the case of $C(k_1, 1, 1, 1)$. In this paper, we affirm Cohen et al.'s conjecture for the case where $k_2 = k_4 = 1$, namely $g(k_1, 1, k_3, 1) = O((k_1 + k_3)^2)$. Moreover, we show that if in addition D is Hamiltonian, then the chromatic number of D is at most $6k$, with $k = \max\{k_1, k_3\}$.

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1 Introduction

Throughout this paper, all graphs are considered to be simple, that is, there are no loops and no multiple edges. By giving an orientation to each edge of a graph G, the obtained

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oriented graph is called a digraph. Reciprocally, the graph obtained from a digraph D by ignoring the directions of its arcs is called the underlying graph of D, and denoted by $G(D)$ (a circuit of length 2 in D correspond to one edge in $G(D)$). The chromatic number of a digraph D, denoted by $\chi(D)$, is the chromatic number of its underlying graph. A digraph D is said to be k-chromatic if $\chi(D) = k$.

An oriented path (resp. oriented cycle) is an orientation of a path (resp. cycle). The length of a path (resp. cycle) is the number of its edges. The order of a path (resp. cycle) is the number of its vertices. An oriented path (resp. oriented cycle) is said to be directed if all its arcs have the same orientation. More formally, an oriented path P whose vertex-set is $V(P) = \{x_1, x_2, \ldots, x_n\}$ and edge-set is $E(P) = \{(x_i, x_{i+1}); 1 \leqslant i \leqslant n-1\}$ is called a directed path. In this case, we write $P = x_1, x_2, \ldots, x_n$. Given an oriented path P (resp. oriented cycle C), a block is a maximal directed subpath of P (resp. of C). We denote by $P(k_1, k_2, \ldots, k_n)$ (resp. $C(k_1, k_2, \ldots, k_n)$) the oriented path (resp. oriented cycle) formed of n blocks of lengths $k_1, k_2, \ldots, k_{n-1}$ and k_n respectively. Since any two consecutive blocks must have opposite directions, one may easily see that an oriented cycle cannot have an odd number of blocks. Hence, n must be even for any oriented cycle $C(k_1, k_2, \ldots, k_n)$.

Given a digraph D, a directed path (resp. a directed cycle) in D is said to be Hamiltonian if it passes through all the vertices of D. If D has a Hamiltonian directed cycle, then D is called a Hamiltonian digraph. Moreover, D is said to be strongly connected if for any two vertices x and y there is a directed path from x to y. However, D is said to be acyclic if it contains no directed cycles. Given a digraph H, a subdivision of H, denoted by S-H, is a digraph H' obtained from H by replacing each arc (x, y) by an xy-dipath of length at least 1, all new paths being internally disjoint. If a digraph D does not contain a subdivision of H as a subdigraph, then D is said to be H-subdivision-free.

An important question to be asked is the following:

Problem 1. Which are the graphs G such that every graph with sufficiently high chromatic number contains G as a subgraph?

In this context, Erdős and Hajnal [10] proved that every graph with chromatic number at least k contains an odd cycle of length at least k. A counterpart of this theorem for even length was settled by Mihok and Schiermeyer [16]: Every graph with chromatic number at least k contains an even cycle of length at least k. Further results on graphs with prescribed lengths of cycles have been obtained [12, 13, 15, 16, 19].

In their article, Cohen et al. [8] investigated a generalization of Problem 1 by considering the analogous problem for directed graphs:

Problem 2. Which are the digraphs D such that every k-chromatic digraph contains D as a subdigraph?

A famous theorem by Erdős [9] states that there exist digraphs with arbitrarily large chromatic number and arbitrarily high girth. This implies that if D is a digraph containing an oriented cycle, there exist digraphs with arbitrarily high chromatic number with no subdigraph isomorphic to D. Thus the only possible candidates to answer Problem 2 are the oriented trees. Burr [7] conjectured that every $(2k-2)$-chromatic digraph contains every oriented tree T of order k, and he was able to prove that every $(k-1)^2$-chromatic digraph contains a copy of any oriented tree T of order k. The best known bound, due to Addario-Berry et al. [3], is in $(k/2)^2$. For special oriented trees, better bounds on the chromatic number are known. The most famous one, known as Gallai-Roy theorem, deals with directed paths:

Theorem 3. *(Gallai [11], Roy [17]) Every k-chromatic digraph contains a directed path of length $k-1$.*

However, for paths with two blocks, the best possible upper bound has been determined by Addario-Berry et al. as follows:

Theorem 4. *(Addario-Berry et al. [2]) Let k_1 and k_2 be positive integers such that $k_1 + k_2 \geqslant 3$. Every $(k_1 + k_2 + 1)$-chromatic digraph D contains any two-blocks path $P(k_1, k_2)$.*

The following famous theorem of Bondy shows that the story does not stop here:

Theorem 5. *(Bondy [6]) Every strong digraph D contains a directed cycle of length at least $\chi(D)$.*

The strong connectivity assumption is indeed necessary, because there exist acyclic digraphs (transitive tournaments) with large chromatic number and no directed cycle. Since any directed cycle of length at least k can be seen as a subdivision of the directed cycle C_k of length k, Cohen et al. conjectured that Bondy's theorem can be extended to all oriented cycles:

Conjecture 6. (Cohen et al. [8]) For every positive integers $k_1, k_2, \dots, k_n$, there exists a constant $g(k_1, k_2, \dots, k_n)$ such that every strongly connected digraph containing no subdivisions of the oriented cycle $C(k_1, k_2, \dots, k_n)$ has a chromatic number at most $g(k_1, k_2, \dots, k_n)$.

Cohen et al. [8] noticed that the strongly connected connectivity assumption is also necessary in Conjecture 6. This follows from proving the existence of acyclic digraphs with large chromatic number and no subdivisions of C for any oriented cycle C:

Theorem 7. *(Cohen et al. [8]) For any positive integers b, c, there exists an acyclic digraph D with $\chi(D) \geqslant c$ in which all oriented cycles have more than b blocks.*

In their article, Cohen et al. [8] proved Conjecture 6 for the case of two-blocks cycles. More precisely, they showed that the chromatic number of strong digraphs with no subdivisions of a two-blocks cycle $C(k_1, k_2)$ is bounded from above by $O((k_1 + k_2)^4)$:

Theorem 8. *(Cohen et al. [8]) Let k_1 and k_2 be positive integers such that $k_1 \geqslant k_2 \geqslant 2$ and $k_1 \geqslant 3$. If D is a strong digraph having no subdivisions of $C(k_1, k_2)$, then the chromatic* *number of D is at most $(k_1+k_2-2)(k_1+k_2-3)(2k_2+2)(k_1+k_2+1)$.*

More recently, this bound was improved by Kim et al. as follows:

Theorem 9. *(Kim et al. [14]) Let k_1 and k_2 be positive integers such that $k_1 \geqslant k_2 \geqslant 1$ and $k_1 \geqslant 2$. If D is a strong digraph having no subdivisions of $C(k_1,k_2)$, then the chromatic number of D is at most $2(2k_1-3)(k_1+2k_2-1)$.*

In [2], Addario et al. asked if the upper bound of the chromatic number of strongly connected digraphs having no subdivisions of $C(k_1,k_2)$ can be improved to $O(k_1+k_2)$, which remains an open problem. More recently, Al-Mniny et al. [5] introduced the notion of secant edges and provided a positive answer to Addario et al.'s question for the class of digraphs having a Hamiltonian directed path.

On the other hand, for the case of four-blocks cycles, Conjecture 6 is still unresolved unless for some cases. For every positive integers k_1, k_2, k_3, k_4, a cycle with four blocks $C(k_1,k_2,k_3,k_4)$ is an oriented cycle formed of four blocks of lengths k_1, k_2, k_3 and k_4 respectively. The order of the blocks is shown in Figure 1.

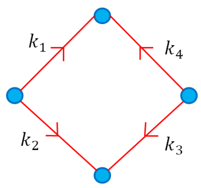

Figure 1: A four blocks cycle $C(k_1,k_2,k_3,k_4)$

In fact, the restriction of Conjecture 6 on four-blocks cycles was confirmed by Cohen et al. [8] for the case where $k_1=k_2=k_3=k_4=1$ and by Al-Mniny [4] for the case where k_1 is arbitrary and $k_2=k_3=k_4=1$ as follows:

Theorem 10. *(Cohen et al. [8]) Let D be a strongly connected digraph with no subdivisions of $C(1,1,1,1)$, then the chromatic number of D is at most 24.*

Theorem 11. *(Al-Mniny [4]) Let k_1 be a positive integer and let D be a strongly connected digraph with no subdivisions of $C(k_1,1,1,1)$, then the chromatic number of D is at most $8^3 \cdot k_1$.*

In this paper, we confirm Conjecture 6 for the four-blocks cycles $C(k_1,1,k_3,1)$ as follows:

Theorem 12. *Let D be a strongly connected digraph having no subdivisions of $C(k_1,1,k_3,1)$ and let $k=\max\{k_1,k_3\}$, then the chromatic number of D is at most $36\cdot(2k)\cdot(4k+2)$.*

Moreover, we provide a linear bound for the chromatic number of Hamiltonian digraphs having no subdivisions of $C(k_1,1,k_3,1)$. More precisely, we prove the following:

Theorem 13. *Let D be a Hamiltonian digraph having no subdivisions of $C(k_1, 1, k_3, 1)$ and let $k = max\{k_1, k_3\}$. Then D is $(6k-1)$-degenerate and thus $\chi(D) \leqslant 6 \cdot k$.*

The paper is organized as follows: In Section 2, we introduce some terminologies and notations that will be used throughout the coming sections. In Section 3, we prove Theorem 12 by using the simple notion of a maximal-tree and the technique of digraphs decomposing. Then in Section 4, we prove Theorem 13 that reduces the chromatic number obtained in Theorem 12 for the class of Hamiltonian digraphs having no subdivisions of $C(k_1, 1, k_3, 1)$.

2 Preliminaries

In this section, we introduce some basic definitions and terminologies that will be elementary for the coming sections.

In what follows, we denote by $[l] := \{1, 2, \ldots, l\}$ for every positive integer l. A graph G is said to be d-degenerate, if any subgraph of G contains a vertex having at most d neighbors. Using an inductive argument, one may easily see the following statement:

Lemma 14. *If G is d-degenerate graph, then G is $(d+1)$-colorable.*

Given two digraphs D_1 and D_2, $D_1 \cup D_2$ is defined to be the digraph whose vertex-set is $V(D_1) \cup V(D_2)$ and whose arc-set is $A(D_1) \cup A(D_2)$. The next lemma will be useful for the coming proofs:

Lemma 15. $\chi(D_1 \cup D_2) \leqslant \chi(D_1) \times \chi(D_2)$ *for any two digraphs D_1 and D_2.*

Proof. For $i \in \{1, 2\}$, let $\phi_i : V(D_i) \longrightarrow \{1, 2, \ldots, \chi(D_i)\}$ be a proper $\chi(D_i)$-coloring of D_i. Define ψ, the coloring of $V(D_1 \cup D_2)$, as follows:

$$\psi(x) = \begin{cases} (\phi_1(x), 1) & x \in V(D_1) \setminus V(D_2); \\ (\phi_1(x), \phi_2(x)) & x \in V(D_1) \cap V(D_2); \\ (1, \phi_2(x)) & x \in V(D_2) \setminus V(D_1). \end{cases}$$

We may easily verify that ψ is a proper coloring of $D_1 \cup D_2$ with color-set

$$\{1, 2, \ldots, \chi(D_1)\} \times \{1, 2, \ldots, \chi(D_2)\}.$$

Consequently, it follows that $\chi(D_1 \cup D_2) \leqslant \chi(D_1) \times \chi(D_2)$. □

A consequence of the previous lemma is that, if we partition the arc-set of a digraph D into $A_1, A_2, \ldots, A_l$, then bounding the chromatic number of all spanning subdigraphs D_i of D with arc-set A_i gives an upper bound for the chromatic number of D.

Let D be a digraph. For a dipath or a directed cycle H of D and for any two vertices u, v of H, we denote by $H[u, v]$ the subdipath of H with initial vertex u and terminal vertex

v. Also, we denote by $H[u,v[, H]u,v]$ and $H]u,v[$ the dipaths $H[u,v]-v, H[u,v]-u$ and $H[u,v]-\{u,v\}$, respectively. Given an oriented cycle C in D, a vertex u of C is said to be a source if the two neighbors of u in C are both out-neighbors. If u is a vertex of D, we denote by ${N_D}^+(u)$ (resp. $N_D^-(u)$) the set of vertices v such that (u,v) (resp. (v,u)) is an arc of D. The out-degree (resp. in-degree) of u, denoted by $d^+(u)$ (resp. $d^-(u)$), is the cardinality of $N^+(u)$ (resp. $N^-(u)$). The maximum out-degree of D is defined by $\Delta^+(D) = \max_{u\in V(D)} d^+(u)$. For a vertex u of a graph G, we denote by $N_G(u)$ the set of all neighbors of u in G, by $d_G(u)$ the cardinality of $N_G(u)$ and by $\delta(G) = \min_{u\in V(G)} d_G(u)$.

A tree is a connected graph containing no cycles. An oriented tree is an orientation of a tree. An out-tree is an oriented tree in which all vertices have in-degree at most 1. This implies that an out-tree has exactly one vertex of in-degree 0, called the source. Given a digraph D having a spanning out-tree T with source r, the level of a vertex x with respect to T, denoted by $l_T(x)$, is the order of the unique rx-directed path in T. For a positive integer i, we define $L_i(T) := \{x \in V(T) | l_T(x) = i\}$. For a vertex x of D, the ancestors of x are the vertices that belong to $T[r,x]$. If y is an ancestor of x with respect to T, we write $y \leqslant_T x$. Denoting by $S(x)$ the set of the vertices y of D such that x is an ancestor of y, T_x is defined to be the subtree of T rooted at x and induced by $S(x)$. For two vertices x_1 and x_2 of D, the least common ancestor z of x_1 and x_2, abbreviated by l.c.a$\{x_1, x_2\}$, is the common ancestor of x_1 and x_2 having the highest level in T. Note that the latter notion is well-defined since r is a common ancestor of all vertices. For two vertices x and y, we define $\min_T\{x,y\} := \{x\}$ if $l_T(x) < l_T(y)$ and $\min_T\{x,y\} := \{y\}$ if $l_T(y) < l_T(x)$. An arc (x,y) of D is said to be forward with respect to T if $l_T(x) < l_T(y)$. Otherwise, (x,y) is called a backward arc. If for every backward arc (x,y) of D $y \leqslant_T x$, then T is called a final out-tree of D. In such case, one may easily see that $D[L_i(T)]$ is an empty digraph for all $i \geqslant 1$.

The next proposition shows an interesting structural property on digraphs having a spanning out-tree:

Proposition 16. *Given a digraph D having a spanning out-tree T, then D contains a final out-tree.*

Proof. Initially, set $T_0 := T$. If T_0 is final, there is nothing to do. Otherwise, there is an arc (x,y) of D which is backward with respect to T_0 such that y is not ancestor of x. Let T_1 be the out-tree obtained from T_0 by adding (x,y) to T_0, and deleting the arc of head y in T_0. We can easily see that the level of each vertex in T_1 is at least its level in T_0, and there exists a vertex (y) whose level has strictly increased. Since the level of a vertex cannot increase infinitely, we can see that after a finite number of repeating the above process we reach an out-tree which is final. □

3 The existence of S-$C(k_1, 1, k_3, 1)$ in strong digraphs

From now on, we consider k_1 and k_3 to be two positive integers and $k = \max\{k_1, k_3\}$. The aim of this section is to bound from above the chromatic number of strongly connected digraphs having no subdivisions of $C(k, 1, k, 1)$. To this end, we consider D to be a digraph having a final spanning out-tree T rooted at r without subdivisions of $C(k, 1, k, 1)$. Then we partition the vertex-set of D into subsets $V_1, V_2, \ldots, V_{2k}$, where $V_i := \cup_{\alpha \geqslant 0} L_{i+\alpha(2k)}(T)$ for all $1 \leqslant i \leqslant 2k$. After that, denoting by D_i the subdigraph of D induced by V_i, we partition the arc-set of D_i as follows:

$$A_1 := \{(x, y) | l_T(x) < l_T(y) \text{ and } x \leqslant_T y\};$$

$$A_2 := \{(x, y) | l_T(x) > l_T(y) \text{ and } y \leqslant_T x\};$$

$$A_3 := A(D_i) \setminus (A_1 \cup A_2).$$

In the coming sections, we denote by D_i^j the spanning subdigraph of D_i whose arc-set is A_j, for $1 \leqslant i \leqslant 2k$ and $j = 1, 2, 3$.

3.1 Coloring D_i^1

The main goal of this section is to prove that $\chi(D_i^1) \leqslant 6$. To this end, we are going to prove that D_i^1 is a 5-wheel-free digraph. For any integer $k \geqslant 3$, a k-wheel is a graph formed by a cycle C and a vertex u not in $V(C)$, called the center, such that u has at least k neighbors in C. A wheel with a cycle C and a center u is denoted by (C, u). A graph G is said to be k-wheel-free graph if it does not contain a k-wheel as a subgraph.

Theorem 17. *(G.E. Turner [18]) For any integer $k \geqslant 4$, if G is a k-wheel-free graph, then G contains a vertex of degree at most k.*

Note that the result of Turner in [18] is slightly weaker than Theorem 17, but the proof of Turner proves exactly Theorem 17 (see [1]). Due to an inductive argument, Theorem 17 easily implies the following result:

Corollary 18. *For any integer $k \geqslant 4$, if G is a k-wheel-free graph, then G is $(k+1)$-colorable.*

Before going into details, we would like to outline the way we follow to prove that D_i^1 is a 5-wheel-free digraph. The plan is first to reduce the question about the existence of a 5-wheel with a cycle C in D_i^1 to the existence of a 5-wheel with a cycle C in a well-defined family $\mathcal{C}$ of cycles (this part will be done in Subsection 3.1.2 in which we describe the structure of cycles expected to exist in D_i^1 according to the number and length of blocks, and according to the position of the vertices of the cycle with respect to $\leqslant_T$). To this end, we prove in Subsection 3.1.1 a very useful lemma that describes the possible positions of the vertices of any three internally disjoint directed paths of D_i^1 with respect to $\leqslant_T$. Finally, we prove in Subsection 3.1.3 that D_i^1 is a 5-wheel-free digraph by considering all the possible positions for the center of the wheel and its neighbors in each expected cycle in D_i^1, that is, in each cycle in $\mathcal{C}$.

3.1.1 Properties of internally disjoint directed paths of D_i^1

In the following, we study the structural properties of any three internally disjoint directed paths of D_i^1. For this purpose, we prove a very useful lemma that our proofs heavily rely on (see Figure 2):

Lemma 19. *Let $R_1 = u_1, \ldots, u_n$, $R_2 = r_1, \ldots, r_s$ and $R_3 = v_1, \ldots, v_f$ be vertex-disjoint directed paths in D_i^1 of length at least 1, except possibly $u_n = r_s$ or $u_1 = r_1$. Then non of the following occurs:*

1. *$V(R_1)$ and $V(R_2)$ are ancestors, $v_1 \leqslant_T r_1 \leqslant_T u_1 \leqslant_T v_f$ ($r_1 \neq u_1$), and one of the below holds:*

   a. $u_n \in T_{v_f}$ and $r_s \in T_{v_f}$;

   b. For all $1 < j \leqslant f$ with $r_1 \leqslant_T v_j$, neither u_n and v_j are ancestors nor r_s and v_j are ancestors.

2. *$V(R_1), V(R_2)$ and $V(R_3)$ are ancestors, $u_n \neq r_s$, u_1 and r_1 are ancestors of v_1, v_1 is an ancestor of r_s and u_n, and r_s and u_n are ancestors of v_f.*

3. *$l(R_j) = 1$ for $j = 1, 2, 3$, with $r_1 \leqslant_T v_1 \leqslant_T u_1 \leqslant_T v_2 \leqslant_T u_2 \leqslant_T r_2$, $u_2 \neq r_2$, and $u_1 \neq r_1$.*

4. *$u_1 \leqslant_T v_1$, u_n and v_f are not ancestors, $\alpha \notin R_1 \cup R_3$ with $\alpha = l.c.a\{u_n, v_f\}$, and $l(T[\alpha, u_i]) \geqslant k$ for all $u_i \in T_\alpha$.*

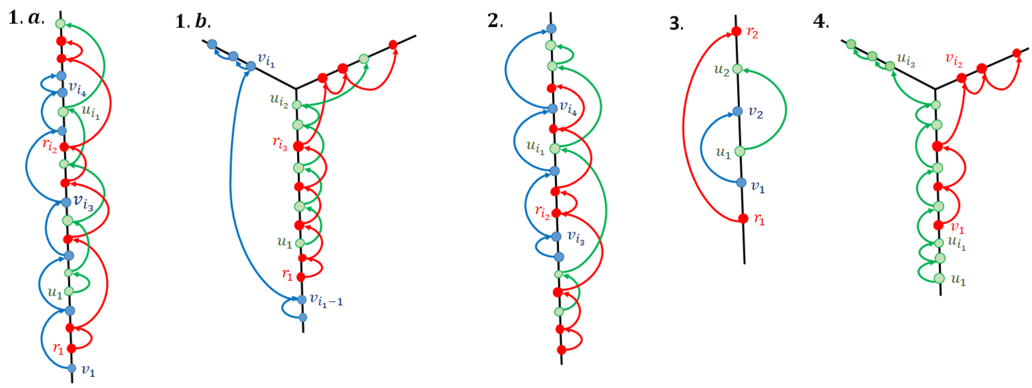

Figure 2: Illustration of Lemma 19.

Proof. Assume the contrary is true. First, assume that (1.a) holds. Let i_1 and i_2 be maximal satisfying $u_{i_1} \leqslant_T v_f$ and $r_{i_2} \leqslant_T v_f$. Note that the existence of u_{i_1} and r_{i_2} is guaranteed by the fact that $r_1 \leqslant_T u_1 \leqslant_T v_f$. Assume without loss of generality that

$r_{i_2} \leqslant_T u_{i_1}$. Let i_3 be maximal satisfying $v_{i_3} \leqslant_T r_{i_2}$ and let i_4 be minimal satisfying $u_{i_1} \leqslant_T v_{i_4}$. Possibly, $v_{i_3} = v_1$ and $v_{i_4} = v_f$. This implies that $T[v_{i_3}, r_{i_2}] \cap R_3 = \{v_{i_3}\}$ and $T[u_{i_1}, v_{i_4}[\cap(R_1 \cup R_2 \cup R_3) = \{u_{i_1}\}$. If $r_s = u_n$, then the union of $T[v_{i_3}, r_{i_2}] \cup R_2[r_{i_2}, r_s]$, $R_3[v_{i_3}, v_{i_4}]$, $T[u_{i_1}, v_{i_4}]$ and $R_1[u_{i_1}, u_n]$ is a S-$C(k,1,k,1)$ in D, a contradiction. Else, assume without loss of generality that $r_s \leqslant_T u_n$, and let i_5 be chosen to be minimal such that $r_s \leqslant_T u_{i_5}$. Then the union of $T[v_{i_3}, r_{i_2}] \cup R_2[r_{i_2}, r_s] \cup T[r_s, u_{i_5}]$, $R_3[v_{i_3}, v_{i_4}]$, $T[u_{i_1}, v_{i_4}]$ and $R_1[u_{i_1}, u_{i_5}]$ is a S-$C(k,1,k,1)$ in D, a contradiction. Now assume that $(1.b)$ holds. Since $r_1 \leqslant_T v_f$, it follows that r_s and v_f are not ancestors, and u_n and v_f are not ancestors. Consequently, l.c.a$\{u_n, v_f\} \notin R_3$. Let i_1 be minimal satisfying $r_1 \leqslant_T v_{i_1}$. Possibly, $v_{i_1} = v_f$. Then v_{i_1} and u_n are not ancestors, and v_{i_1} and r_s are not ancestors. Let i_2 and i_3 be maximal satisfying $u_{i_2} \leqslant_T v_{i_1}$ and $r_{i_3} \leqslant_T v_{i_1}$. Assume without loss of generality that $r_{i_3} \leqslant_T u_{i_2}$. This implies that $T[v_{i_1-1}, r_{i_3}] \cap R_3 = \{v_{i_1-1}\}$ and $T[u_{i_2}, v_{i_1}[\cap(R_1 \cup R_2 \cup R_3) = \{u_{i_2}\}$. If $r_s = u_n$, then the union of $T[v_{i_1-1}, r_{i_3}] \cup R_2[r_{i_3}, r_s]$, (v_{i_1-1}, v_{i_1}), $T[u_{i_2}, v_{i_1}]$ and $R_1[u_{i_2}, u_n]$ is a S-$C(k,1,k,1)$ in D, a contradiction. Else, assume without loss of generality that $r_s \leqslant_T u_n$, and let i_4 be chosen to be minimal such that $r_s \leqslant_T u_{i_4}$. Then the union of $T[v_{i_1-1}, r_{i_3}] \cup R_2[r_{i_3}, r_s] \cup T[r_s, u_{i_4}]$, (v_{i_1-1}, v_{i_1}), $T[u_{i_2}, v_{i_1}]$ and $R_1[u_{i_2}, u_{i_4}]$ is a S-$C(k,1,k,1)$ in D, a contradiction. Assume now that (2) holds. Let i_1 and i_2 be minimal satisfying $v_1 \leqslant_T u_{i_1}$ and $v_1 \leqslant_T r_{i_2}$. Assume without loss of generality that $r_{i_2} \leqslant_T u_{i_1}$. Let i_3 be maximal satisfying $v_{i_3} \leqslant_T r_{i_2}$, and let i_4 be minimal satisfying $u_{i_1} \leqslant_T v_{i_4}$. Possibly, $v_{i_3} = v_1$ and $v_{i_4} = v_f$. This implies that $T]v_{i_3}, r_{i_2}[\cap(R_1 \cup R_2 \cup R_3) = \phi$ and $T]u_{i_1}, v_{i_4}[\cap(R_1 \cup R_3) = \phi$. If $r_1 = u_1$, then the union of $R_1[u_1, u_{i_1}] \cup T[u_{i_1}, v_{i_4}]$, $R_2[r_1, r_{i_2}]$, $T[v_{i_3}, r_{i_2}]$ and $R_3[v_{i_3}, v_{i_4}]$ is a S-$C(k,1,k,1)$ in D, a contradiction. Else, assume without loss of generality that $u_{i_1-1} \leqslant_T r_{i_2-1}$. Hence, the union of $R_1[u_{i_1-1}, u_{i_1}] \cup T[u_{i_1}, v_{i_4}]$, $T[u_{i_1-1}, r_{i_2-1}] \cup R_2[r_{i_2-1}, r_{i_2}]$, $T[v_{i_3}, r_{i_2}]$ and $R_3[v_{i_3}, v_{i_4}]$ is a S-$C(k,1,k,1)$ in D, a contradiction. Let' assume now that (3) holds, then the union of $T[r_1, v_1] \cup (v_1, v_2)$, R_2, $R_1 \cup T[u_2, r_2]$ and $T[u_1, v_2]$ is a S-$C(k,1,k,1)$ in D, a contradiction. Finally if (4) holds, let i_1 be maximal satisfying $u_{i_1} \leqslant_T v_1$ and let i_2, i_3 be minimal satisfying $\alpha \leqslant_T v_{i_2}$ and $\alpha \leqslant_T u_{i_3}$. Possibly, $u_{i_1} = u_1, v_{i_2} = v_f$ and $u_{i_3} = u_n$. This implies that $T[u_{i_1}, v_1] \cap R_1 = \{u_{i_1}\}$ and $T[\alpha, v_{i_2}] \cap R_3 = \{v_{i_2}\}$. Then the union of $T[u_{i_1}, v_1] \cup R_3[v_1, v_{i_2}]$, $R_1[u_{i_1}, u_{i_3}]$, $T[\alpha, u_{i_3}]$ and $T[\alpha, v_{i_2}]$ is a S-$C(k,1,k,1)$ in D, a contradiction. This terminates the proof of Lemma 19. ☐

From now on, we say that $[R_1, R_2, R_3]$ satisfies Lemma 19$(1.a)$ (resp. $(1.b), (2), (3)$) if there exist three directed paths R_1, R_2, R_3 satisfying the conditions of Lemma 19$(1.a)$ (resp. $(1.b), (2), (3)$). Also, we say that $[R_1, R_3]$ satisfies Lemma 19(4), if there exist two directed paths R_1 and R_3 satisfying the conditions of Lemma 19(4).

3.1.2 The landscape of cycles in D_i^1

This subsection is devoted to reduce the question about the existence of a 5-wheel with a cycle C in D_i^1 to the question about the existence of a 5-wheel with a cycle C in $\mathcal{C}$ for a crucial family $\mathcal{C}$ of cycles to be defined below.

We first define a special class of cycles $\mathcal{C}$ on at most 8 blocks in D_i^1 by $\mathcal{C} := C_2 \cup C_4 \cup C_6 \cup C_8$, where $C_2 = \{C \in D_i^1; C \text{ is a 2-blocks cycle}\}$ and C_i is the set of cycles in D_i^1 with i blocks defined below, for $i = 4, 6, 8$ (see Figure 3). Now we are going to define the class C_i of cycles with i blocks for $i = 4, 6, 8$. To this end, we need to define eight internally disjoint directed paths in D_i^1 as follows: $P_1 = n_1, \dots, n_t$; $P_2 = m_1, \dots, m_l$; $Q_1 = x_1, \dots, x_{t_1}$; $Q_2 = y_1, \dots, y_{l_1}$; $Q_3 = z_1, \dots, z_m$; $Q_4 = w_1, \dots, w_r$; $Q_5 = c_1, \dots, c_{\alpha_1}$; $Q_6 = d_1, \dots, d_{\alpha_2}$, with $t, l, t_1, l_1, m, r, \alpha_1, \alpha_2 \geqslant 2$. We advise here the reader to skip the definitions of C_i exposed below and move directly to Lemma 20. While reading the proof of Lemma 20, one can check each cycle and go back to its definition in $\mathcal{C}$.

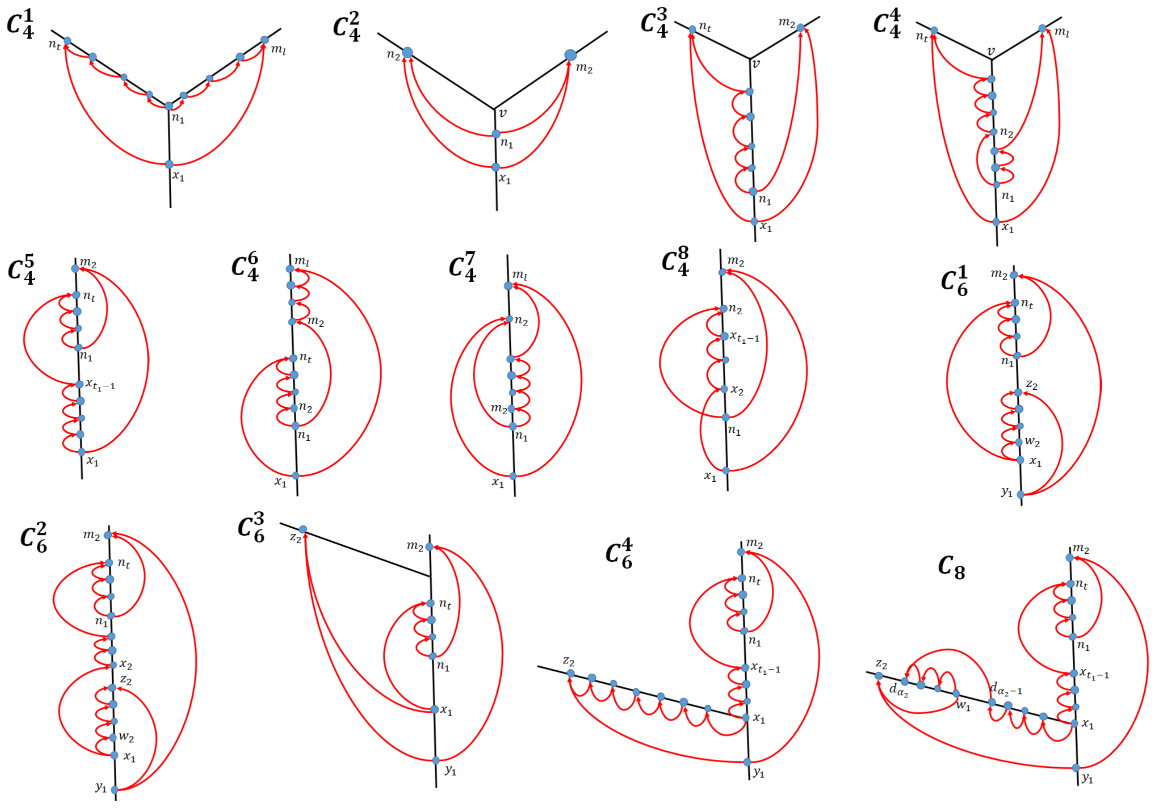

Figure 3: $\mathcal{C} := C_2 \cup C_4 \cup C_6 \cup C_8$, with $C_4 = \bigcup_{i=1}^{8} C_4^j$ and $C_6 = \bigcup_{i=1}^{4} C_6^j$. Blocks with a length of at least 2 do not necessarily have the length drawn in the figure. All the cycles are drawn according to their location in T.

Let C be a cycle of D_i^1 with at most 8 blocks. First, we will define $C_4 = \bigcup_{i=1}^{8} C_4^j$, with C_4^j is a class of cycles on 4 blocks for $j = 1, \dots, 8$, and containing cycles with the form $P_1 \cup P_2 \cup Q_1 \cup Q_2$. In this case $x_1 = y_1$, $n_t = x_{t_1}$, $n_1 = m_1$, and $m_l = y_{l_1}$:

- $C_4^1 := \{C; C$ is a 4-blocks cycle such that $l(Q_1) = l(Q_2) = 1$, n_t and m_l are not ancestors with $x_1 \leqslant_T n_1 = \text{l.c.a}\{n_t, m_l\}$, and $l(P_j) \geqslant 1$ for $j = 1, 2\}$,
- $C_4^2 := \{C; C$ is a 4-blocks cycle such that $l(C_4^2) = 4$, n_2 and m_2 are not ancestors with $x_1 \leqslant_T n_1 \leqslant_T \text{l.c.a}\{n_2, m_2\}\}$,

- $C_4^3 := \{C; C$ is a 4-blocks cycle such that $l(Q_1) = l(Q_2) = l(P_2) = 1$, $l(P_1) \geqslant 2$, n_t and m_2 are not ancestors with $x_1 \leqslant_T n_1 \leqslant_T n_{t-1} \leqslant_T \text{l.c.a}\{n_t, m_2\}\}$,
- $C_4^4 := \{C; C$ is a 4-blocks cycle such that $l(Q_1) = l(Q_2) = 1$, $l(P_j) \geqslant 2$ for $j = 1, 2$, n_t and m_l are not ancestors with $x_1 \leqslant_T n_1 \leqslant_T m_{l-1} \leqslant_T n_2 \leqslant_T n_{t-1} \leqslant_T \text{l.c.a}\{n_t, m_l\}\}$,
- $C_4^5 := \{C; C$ is a 4-blocks cycle such that $l(P_2) = l(Q_2) = 1$, $l(P_1) \geqslant 1$, $l(Q_1) \geqslant 2$, and $x_1 \leqslant_T x_{t_1-1} \leqslant_T n_1 \leqslant_T n_t \leqslant_T m_2\}$,
- $C_4^6 := \{C; C$ is a 4-blocks cycle such that $l(Q_1) = l(Q_2) = 1$, $l(P_j) \geqslant 1$ for $j = 1, 2$, and $x_1 \leqslant_T n_1 \leqslant_T n_t \leqslant_T m_2\}$,
- $C_4^7 := \{C; C$ is a 4-blocks cycle such that $l(P_1) = l(Q_1) = l(Q_2) = 1$, $l(P_2) \geqslant 2$, $x_1 \leqslant_T n_1 \leqslant_T m_{l-1} \leqslant_T n_2 \leqslant_T m_l\}$,
- $C_4^8 := \{C; C$ is a 4-blocks cycle such that $l(P_1) = l(P_2) = l(Q_2) = 1$, $l(Q_1) \geqslant 2$, and $x_1 \leqslant_T n_1 \leqslant_T x_2 \leqslant_T n_2 \leqslant_T m_2\}$.

Now we will define $C_6 = \bigcup_{i=1}^4 C_6^j$, with C_6^j is a class of cycles on 6 blocks, for $j = 1, \ldots, 4$, and containing cycles with the form $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4$. In this case $n_1 = m_1$, $y_{l_1} = m_l$, $z_1 = y_1$, $z_m = w_r$, $w_1 = x_1$, and $x_{t_1} = n_t$:

- $C_6^1 := \{C; C$ is a 6-blocks cycle such that $l(P_2) = l(Q_1) = l(Q_2) = l(Q_3) = 1$, $l(P_1) \geqslant 1$, $l(Q_4) \geqslant 1$, and $y_1 \leqslant_T x_1 \leqslant_T z_2 \leqslant_T n_1 \leqslant_T n_t \leqslant_T m_2\}$,
- $C_6^2 := \{C; C$ is a 6-blocks cycle such that $l(P_2) = l(Q_2) = l(Q_3) = 1$, $l(Q_1) \geqslant 2$, $l(P_1) \geqslant 1$, $l(Q_4) \geqslant 1$, $y_1 \leqslant_T x_1 \leqslant_T z_2 \leqslant_T x_2 \leqslant_T x_{t_1-1} \leqslant_T n_1 \leqslant_T n_t \leqslant_T m_2\}$,
- $C_6^3 := \{C; C$ is a 6-blocks cycle such that $l(P_2) = l(Q_1) = l(Q_2) = l(Q_3) = l(Q_4) = 1$, $l(P_1) \geqslant 1$, z_2 and m_2 are not ancestors with $y_1 \leqslant_T x_1 \leqslant_T n_1 \leqslant_T n_t \leqslant_T \text{l.c.a}\{z_2, m_2\}\}$,
- $C_6^4 := \{C; C$ is a 6-blocks cycle such that $l(P_2) = l(Q_2) = l(Q_3) = 1$; $l(P_1), l(Q_1), l(Q_4) \geqslant 1$, z_2 and m_2 are not ancestors with $x_1 = \text{l.c.a}\{z_2, m_2\}$, $y_1 \leqslant_T x_1 \leqslant_T x_{t_1-1} \leqslant_T n_1 \leqslant_T n_t \leqslant_T m_2$, and $y_1 \leqslant_T x_1 \leqslant_T z_2\}$.

Now we will define C_8, a class of cycles on 8 blocks, and containing cycles with the form $P_1 \cup P_2 \cup (\bigcup_{j=1}^6 Q_j)$. In this case $n_1 = m_1$, $y_{l_1} = m_l$, $z_1 = y_1$, $z_m = w_r$, $w_1 = c_1$, $c_{\alpha_1} = d_{\alpha_2}$, $d_1 = x_1$, and $x_{t_1} = n_t$.

- $C_8 := \{C; C$ is an 8-blocks cycle such that $C = P_1 \cup P_2 \cup (\bigcup_{j=1}^6 Q_j)$, $l(P_2) = l(Q_2) = l(Q_3) = l(Q_4) = 1$, $l(P_1), l(Q_1), l(Q_5), l(Q_6) \geqslant 1$, z_2 and m_2 are not ancestors with $x_1 = \text{l.c.a}\{z_2, m_2\}$, $y_1 \leqslant_T x_1 \leqslant_T x_{t_1-1} \leqslant_T n_1 \leqslant_T n_t \leqslant_T m_2$, and $x_1 \leqslant_T d_{\alpha_2-1} \leqslant_T w_1 \leqslant_T d_{\alpha_2} \leqslant_T z_2\}$.

The following lemma describes the structure of all cycles expected to exist in D_i^1, and reduces the question about the existence of a 5-wheel with a cycle C in D_i^1 to the question about the existence of a 5-wheel with a cycle C in $\mathcal{C}$:

Lemma 20. *Let C be a cycle in D_i^1, then $C \in \mathcal{C}$.*

Proof. If C is a 2-blocks cycle, then $C \in C_2$ and so $C \in \mathcal{C}$. Now assume that C is a cycle with at least 4 blocks. Let n_1 be a source of C with maximal level with respect to $\leqslant_T$. Let $P_1 = n_1, \ldots, n_t$, $P_2 = m_1, \ldots, m_l$, $Q_1 = x_1, \ldots, x_{t_1}$, and $Q_2 = y_1, \ldots, y_{l_1}$ be blocks of C, with $n_1 = m_1$, $x_{t_1} = n_t$, $y_{l_1} = m_l$, and $t, l, t_1, l_1 \geqslant 2$. Clearly, x_1 and y_1 are sources of C. Moreover, $x_1 \leqslant_T n_1$ and $y_1 \leqslant_T n_1$, due to the definition of D_i^1 and the maximality of n_1.

Assertion 21. If n_t and m_l are not ancestors, then $C \in \bigcup_{j=1}^{4} C_4^j$.

Proof of Assertion 21. Let $v = \text{l.c.a}\{n_t, m_l\}$.

Claim 22. *C is a 4-blocks cycle and $(Q_1 \cup Q_2) \cap T]x_1, n_1[= \phi$.*

Proof of Claim 22. Assume by contradiction that this is not the case. Let s_1 and s_2 be maximal such that $x_{s_1} \leqslant_T n_1$ and $y_{s_2} \leqslant_T n_1$. Note that in case C is not a 4-blocks cycle, then $x_1 \neq y_1$ and possibly $x_{s_1} = x_1$ or $y_{s_2} = y_1$. Otherwise, according to our assumption, we may have either $x_{s_1} = x_1$ or $y_{s_2} = y_1 = x_1$ but not both. Thus, $x_{s_1} \neq y_{s_2}$. Assume without loss of generality that $x_{s_1} \leqslant_T y_{s_2}$. According to the choice of s_1, it follows that $x_{s_1+1} \in T]n_1, n_t]$. Consequently, $[P_2, Q_2[y_{s_2}, y_{l_1}], (x_{s_1}, x_{s_1+1})]$ satisfies Lemma 19(1.a) or Lemma 19(1.b), a contradiction. This confirms Claim 22. ⧫

Therefore, according to Claim 22, we have $C \cap T[r, n_1[= \{x_1\} = \{y_1\}$. Moreover, observe that if $v = n_1$ then $l(Q_1) = 1$, since otherwise the union of $(x_1, x_2) \cup T[x_2, n_j]$, (x_1, y_2), $T[n_1, y_2]$ and $P_1[n_1, n_j]$ is a S-$C(k, 1, k, 1)$ in D, where j is minimal such that $x_2 \leqslant_T n_j$. By symmetry, if $v = n_1$ then $l(Q_2) = 1$ and so $C \in C_4^1$. Assume now that $v \neq n_1$. Observe that $v \notin V(D_i^1)$. In fact, if $v \in D_i^1 \backslash (Q_1 \cup P_2)$, then $[Q_1, P_2]$ satisfies Lemma 19(4), a contradiction. Else if $v \in Q_1 \cup P_2$, then $v \in D_i^1 \backslash (Q_2 \cup P_1)$ and so $[Q_2, P_1]$ satisfies Lemma 19(4), a contradiction. Hence, $v \notin V(D_i^1)$.

Claim 23. $l(Q_1) = l(Q_2) = 1$.

Proof of Claim 23. Assume first that $Q_1 \cap T[n_1, v] \neq \phi$, and let y_j be the vertex of Q_2 satisfying $n_1 \leqslant_T x_2 \leqslant_T y_j$. If y_j and n_t are ancestors, then $[P_1, Q_1[x_2, n_t], Q_2[y_1, y_j]]$ satisfies Lemma 19(1.a), a contradiction. This means that y_j and n_t are not ancestors and so $[P_1, Q_1[x_2, n_t], Q_2[y_1, y_j]]$ satisfies Lemma 19(1.b), a contradiction. This proves that $Q_1 \cap T[n_1, v] = \phi$. By symmetry, we have $Q_2 \cap T[n_1, v] = \phi$. Now assume that $Q_1 \cap T]v, n_t[\neq \phi$. Hence, $x_2 \neq n_t$ and so the union of $(x_1, x_2) \cup T[x_2, n_{j_1}]$, (y_1, y_2), $T[n_{j_2}, y_2]$ and $P_1[n_{j_2}, n_{j_1}]$ is a S-$C(k, 1, k, 1)$ in D, where j_1 is minimal such that $x_2 \leqslant_T n_{j_1}$ and j_2 is maximal such that $n_{j_2} \leqslant_T v$. Thus, $Q_1 \cap T]v, n_t[= \phi$ and by symmetry $Q_2 \cap T]v, m_l[= \phi$. As a result, $Q_1 = (x_1, n_t)$ and $Q_2 = (y_1, m_l)$. This yields the desired claim. ⧫

Notice that $l(T[v, n_t]) < k$, since else $[(x_1, n_t), P_2]$ satisfies Lemma 19(4), a contradiction. By symmetry, $l(T[v, m_l]) < k$. Hence, $P_1 \cap T[v, n_t[= \phi$ and $P_2 \cap T[v, m_l[= \phi$. If $l(P_1) = l(P_2) = 1$, then $C = C_4^2$. Thus let us consider the opposite and assume without loss of generality that $m_{l-1} \leqslant_T n_{t-1}$. If $m_{l-1} = n_1$, then $l(P_2) = 1$ and so $C \in C_4^3$. Now assume that $m_{l-1} \neq n_1$. This implies that $l(P_1) > 1$ and $l(P_2) > 1$. Observe that for all f in

$P_1 \cap T]m_{l-1}, n_{t-1}]$, there is no w in $T]n_1, m_{l-1}[$ such that $(w, f) \in A(P_1)$, since otherwise $[(w, f), Q_1, (m_{j-1}, m_j)]$ satisfies Lemma 19(3), where j is minimal such that $w \leqslant_T m_j$. Hence, $m_{l-1} \leqslant_T n_2$ and so $C \in C_4^4$. This confirms Assertion 21. $\lozenge$

Assertion 24. Let $R = \bigcup_{j=1}^{4} R_j$ be a 4-blocks path in D_i^1, where $R_1 = r_1, \ldots, r_s$, $R_2 = u_1, \ldots, u_n$, $R_3 = g_1, \ldots, g_\kappa$, and $R_4 = v_1, \ldots, v_h$ are the 4 blocks of R, with $r_s = u_n$, $g_1 = u_1$, $g_\kappa = v_h$, $r_1 \neq v_1$, $r_1 \leqslant_T u_1$, $v_1 \leqslant_T u_1$, and $u_n \leqslant_T g_\kappa$. Then $l(R_3) = 1$, $r_{s-1} \leqslant_T u_1$, and $v_{h-1} \leqslant_T r_1$.

Proof of Assertion 24. We are going to prove first that $v_{h-1} \leqslant_T r_1$. Indeed, $R_4 \cap T]r_1, u_1[= \phi$, since otherwise $[R_3, R_4[v_j, v_h], R_1]$ satisfies Lemma 19(1.a), with j is minimal such that $r_1 \leqslant_T v_j \leqslant_T u_1$, a contradiction. Moreover, $R_4 \cap T]u_1, g_\kappa[= \phi$, since otherwise $[R_1, R_4[v_1, v_{h-1}], R_3]$ satisfies Lemma 19(2), a contradiction. This gives that $v_{h-1} \leqslant_T r_1$. Now we want to show that $r_{s-1} \leqslant_T u_1$. In fact, $R_1 \cap T]u_1, r_s[= \phi$. If not, let j be minimal such that $u_1 \leqslant_T r_j \leqslant_T r_s$ and let i be maximal such that $u_i \leqslant_T r_j$. According to our assumption together with the previous observation, we get that $[(v_{h-1}, v_h), (r_{j-1}, r_j), (u_i, u_{i+1})]$ satisfies Lemma 19(3), a contradiction. This proves that $r_{s-1} \leqslant_T u_1$. To end the proof, it remains to prove that $l(R_3) = 1$. Assume otherwise and consider the possible positions of g_2. If $g_2 \leqslant_T r_s$, let j be maximal satisfying $u_j \leqslant_T g_2$. Then the union of $T[u_j, g_2] \cup R_3[g_2, g_\kappa]$, $R_2[u_j, u_n]$, $T[v_{h-1}, r_1] \cup R_1$ and (v_{h-1}, v_h) is a S-$C(k, 1, k, 1)$ in D, a contradiction. Thus $r_{s-1} \leqslant_T g_1 \leqslant_T r_s \leqslant_T g_2 \leqslant_T v_h$ and so $[(v_{h-1}, v_h), (r_{s-1}, r_s), (g_1, g_2)]$ satisfies Lemma 19(3), a contradiction. This implies that $g_2 = g_\kappa$ and so $l(R_3) = 1$. $\lozenge$

Assertion 25. If n_t and m_l are ancestors and C is a 4-blocks cycle, then $C \in \bigcup_{j=5}^{8} C_4^j$.

Proof of Assertion 25. Since C is a 4-blocks cycle, then $x_1 = y_1$. Recall that the maximality of n_1 gives that $x_1 \leqslant_T n_1$. Assume without loss of generality that $n_t \leqslant_T m_l$. Note that $Q_2 \cap T]x_1, n_1[= \emptyset$, since otherwise Assertion 24 implies that $y_{l_1-1} \leqslant_T x_1$, a contradiction. If $Q_1 \cap T]x_1, n_1[\neq \phi$, then Assertion 24 together with the previous remark imply that $C \in C_4^5$. Let us assume now that the opposite is true. Hence, $n_1 \leqslant_T x_2$ and $n_1 \leqslant_T y_2$. Clearly, $l(Q_2) = 1$, since else $l_T(n_1) < l_T(y_2) < l_T(m_l)$ and so $[(y_1, y_2), Q_1, P_2]$ satisfies Lemma 19(2), a contradiction. To conclude, we need to prove the following two claims.

Claim 26. *If $P_2 \cap T]n_t, m_l[\neq \phi$, then $C \in C_4^6$.*

Proof of Claim 26. Observe first that $P_2 \cap T]n_1, n_t[= \phi$, since else $[Q_2, (n_{i-1}, n_i), (m_j, m_{j+1})]$ satisfies Lemma 19(3), where j is maximal satisfying $m_j \leqslant_T n_t$ and i is minimal satisfying $m_j \leqslant_T n_i$. Moreover, note that $l(Q_1) = 1$, since else the union of $(x_1, x_2) \cup T[x_2, n_j], Q_2, (n_1, m_2) \cup T[m_2, m_l]$ and $P_1[n_1, n_j]$ is a S-$C(k, 1, k, 1)$ in D, where j is minimal satisfying $x_2 \leqslant_T n_j$. Hence, $C \in C_4^6$. ♦

Claim 27. *If $P_2 \cap T]n_t, m_l[= \phi$, then $C \in \bigcup_{j=6}^{8} C_4^j$.*

Proof of Claim 27. We are going to argue on the possible lengths of P_1. If $l(P_1) = 1$, then either $l(P_2) > 1$ or $l(P_2) = 1$. Suppose first that the former holds. We will prove that $l(Q_1) = 1$. Assume else and consider the possible positions of x_2: If $x_2 \leqslant_T m_{l-1}$, then $[Q_2, (m_{i_2-1}, m_{i_2}), (x_{i_1}, x_{i_1+1})]$ satisfies Lemma 19(3), where i_1 is maximal satisfying $x_{i_1} \leqslant_T m_{l-1}$, and i_2 is minimal satisfying $x_{i_1} \leqslant_T m_{i_2}$. Else if $m_{l-1} \leqslant_T x_2$, then $[P_1, (m_{l-1}, m_l), (x_1, x_2)]$ satisfies Lemma 19(1.a), a contradiction. Hence, $l(Q_1) = 1$ and so $C \in C_4^7$. Now assume that the later holds, i.e. $l(P_2) = 1$. Then either $l(Q_1) = 1$ and so $C \in C_4^6$, or $l(Q_1) > 1$ and so $C \in C_4^8$. Else if $l(P_1) > 1$, we will prove that $l(Q_1) = l(P_2) = 1$. First assume that $l(Q_1) > 1$ and consider the possible positions of x_2: If $x_2 \leqslant_T n_{t-1}$, then $[Q_2, (n_{i_2-1}, n_{i_2}), (x_{i_1}, x_{i_1+1})]$ satisfies Lemma 19(3), where i_1 is maximal satisfying $x_{i_1} \leqslant_T n_{t-1}$, and i_2 is minimal satisfying $x_{i_1} \leqslant_T n_{i_2}$. Else if $n_{t-1} \leqslant_T x_2$, then $[(n_{t-1}, n_t), P_2, (x_1, x_2)]$ satisfies Lemma 19(1.a), a contradiction. Hence, $l(Q_1) = 1$. Now assume that $l(P_2) > 1$ and consider the possible positions of m_2 in $T]n_1, n_t[$: If $m_2 \leqslant_T n_2$, then $[(n_1, n_2), Q_1, P_2[m_2, m_l]]$ satisfies Lemma 19(2), a contradiction. Else if $n_2 \leqslant_T m_2$, then $[Q_2, (n_1, m_2), (n_i, n_{i+1})]$ satisfies Lemma 19(3), where i is maximal satisfying $n_i \leqslant_T m_2$, a contradiction. Hence, $l(P_2) = 1$. As a result, $C \in C_4^6$. This completes the proof of our claim. ♦

In view of what precedes, Assertion 25 is confirmed. ♢

From now on, $V(P_1 \cup P_2 \cup Q_1 \cup Q_2)$ are considered to be ancestors and C is considered to be a cycle with at least six blocks. Let $Q_3 = z_1, \ldots, z_m$ and $Q_4 = w_1, \ldots, w_r$ be two other blocks of C, with $z_1 = y_1$ and $z_m = w_r$. Note that if C is a six-blocks cycle then $w_1 = x_1$. If C is a cycle with at least ten blocks, then consider $Q_5 = c_1, \ldots, c_{\alpha_1}$ and $Q_6 = d_1, \ldots, d_{\alpha_2}$ to be also blocks of C, with $w_1 = c_1$, $c_{\alpha_1} = d_{\alpha_2}$ and $d_1 = x_1$. In what follows, we will assume without loss of generality that $n_t \leqslant_T m_l$. In accordance with Assertion 24, it follows that $l(P_2) = 1$, $y_{l_1-1} \leqslant_T x_1$ and $x_{t_1-1} \leqslant_T n_1$.

The following observation will be very useful for the rest of the proof.

Assertion 28. Let $(p, q) \in A(D_i^1)$ such that one of the following holds:

1. $p \in T[r, y_{l_1-1}[$ and $q \in T_{y_{l_1-1}} - y_{l_1-1}$.
2. $p \in T]x_1, n_1[\backslash Q_1$ and $q \in (T]n_1, m_2[\cup T_{m_2})\backslash(P_1 \cup P_2)$.
3. $p \in T]y_{l_1-1}, x_{t_1-1}[$ and $q \in T]x_{t_1-1}, m_2[\cup T_{m_2}$.

Then $(p, q) \notin A(C)$.

Proof of Assertion 28. Assume else and suppose first that (1) holds. Assume that $q \leqslant_T m_2$. If $q \in T]y_{l_1-1}, x_{t_1-1}]$, then the union of $T[p, y_{l_1-1}] \cup (y_{l_1-1}, m_2)$, $(p, q) \cup T[q, x_{t_1-1}] \cup (x_{t_1-1}, n_t)$, $T[n_1, n_t]$ and P_2 is a S-$C(k, 1, k, 1)$ in D, a contradiction. Else if $q \in T]x_{t_1-1}, n_t[$, then $[(x_{t_1-1}, n_t), (y_{l_1-1}, m_2), (p, q)]$ satisfies Lemma 19(1.a), a contradiction. Else if $q \in T]n_t, m_2[$, then $[(x_{t_1-1}, n_t), (p, q), P_2]$ satisfies Lemma 19(2), a contradiction. Now assume that $m_2 \leqslant_T q$, then $[P_2, (p, q), (x_{t_1-1}, n_t)]$ satisfies Lemma 19(3), a contradiction. This means that q and m_2 are not ancestors. Let $\beta = \text{l.c.a}\{q, m_2\}$. Observe that $\beta \in T[y_{l_1-1}, x_{t_1-1}[$, since otherwise either $\beta \in T[n_1, m_2[$ and so $[P_2, (y_{l_1-1}, m_2), (p, q)]$ satisfies Lemma 19(1.b), or $\beta \in T[x_{t_1-1}, n_1[$ and so $[(x_{t_1-1}, n_t), (y_{l_1-1}, m_2), (p, q)]$ satisfies Lemma 19(1.b). Notice that if $\beta \neq y_{l_1-1}$, then $l(T[\beta, q]) < k$, since otherwise $[(p, q), (y_{l_1-1}, m_2)]$ satisfies Lemma 19(4). Hence, the structure of C and the above discussion imply that there exists (p_1, q_1) in $A(C)$ such that $\lambda \in T]y_{l_1-1}, x_{t_1-1}[$, $p_1 \in T]y_{l_1-1}, \lambda[$, q_1 and m_2 are not ancestors, with $\lambda = \text{l.c.a}\{q_1, m_2\}$. Thus, $[(y_{l_1-1}, m_2), (p_1, q_1)]$ satisfies Lemma 19(4) as $l(T[\lambda, m_2]) \geqslant k$, a contradiction. Assume now that (2) holds. If $q \in T]n_1, m_2[\setminus(P_1 \cup P_2)$, then $[(p, q), (x_{t_1-1}, n_t), P_2]$ satisfies Lemma 19(2), a contradiction. Else if $q \in T_{m_2}$, then $[P_2, (p, q), Q_1]$ satisfies Lemma 19(1.a), a contradiction. To end the proof, assume that (3) holds and consider the possible positions of q in $T]x_{t_1-1}, m_2[\cup T_{m_2}$. If $q \in T]x_{t_1-1}, n_t[$, then $[(x_{t_1-1}, n_t), (y_{l_1-1}, m_2), (p, q)]$ satisfies Lemma 19(3). Else if $q \in T]n_t, m_2[$, then $[(p, q), (x_{t_1-1}, n_t), P_2]$ satisfies Lemma 19(2). Else if $m_2 \leqslant_T q$, then $[P_2, (p, q), (x_{t_1-1}, n_t)]$ satisfies Lemma 19(3), a contradiction. This confirms our assertion. ♢

Notice that Assertion 28(1) together with the structure of C imply that $l(Q_2) = 1$.

Assertion 29. If all the vertices of C are ancestors, then $C \in C_6^1 \cup C_6^2$.

Proof of Assertion 29. We will prove a series of claims and conclude.

Claim 30. $z_2 \in T]x_1, x_2[$ *and* $z_2 \leqslant_T n_1$.

Proof of Claim 30. Notice that $z_2 \notin T]n_1, m_2[$ since else $[Q_1, (y_1, z_2), P_2]$ satisfies Lemma 19(2), a contradiction. Moreover, observe that $z_2 \notin T_{m_2}\setminus\{m_2\}$ since else $[P_2, (y_1, z_2), (x_{t_1-1}, n_t)]$ satisfies Lemma 19(3), a contradiction. This proves that $z_2 \leqslant_T n_1$. Now are going to show that $z_2 \in T]x_1, x_2[$. Assume first that $l(Q_1) = 1$. Then Assertion 28(1 and 3) together with the structure of C imply our claim. Assume now that $l(Q_1) > 1$. If $z_2 \leqslant_T x_1$, then Assertion 28(1 and 3) implies that there exists $(p, q) \in A(C)$ such that $p \in T]y_1, x_1[$ and $q \in T]x_1, x_{t_1-1}[$, and so $[(x_j, x_{j+1}), Q_2, (p, q)]$ satisfies Lemma 19(3), where j is maximal such that $x_j \leqslant_T q$, a contradiction. Else if $z_2 \in T]x_2, n_1[$, then Assertion 28 implies that $z_2 \notin T]x_{t_1-1}, n_1[$ and so $z_2 \in T]x_2, x_{t_1-1}[$. If there exists $(p, q) \in A(C)$ such that $p \in T]y_1, x_1[$ and $q \in T]x_1, x_{t_1-1}[$, then $[(x_j, x_{j+1}), Q_2, (p, q)]$ satisfies Lemma 19(3), where j is maximal such that $x_j \leqslant_T q$, a contradiction. Combining what precedes together with Assertion 28(1 and 3) and the structure of C, we guarantee the existence of an arc (p, q) of C such that $p \in T[x_1, x_2[$ and $q \in T]x_2, x_{t_1-1}[$. Hence, $[(x_j, x_{j+1}), Q_2, (p, q)]$ satisfies Lemma 19(3), where j is maximal such that $x_j \leqslant_T q$, a contradiction. This proves that $z_2 \in T]x_1, x_2[$ and thus confirms our claim. ♦

Claim 31. *For all* $p \in T[x_1, z_2[$*, there exists no vertex* $q \in T]z_2, n_1[$ *such that* $(p, q) \in A(C)$.

Proof of Claim 31. Assume otherwise. Then $[Q_1, (p, q), (y_1, z_2)]$ satisfies Lemma 19(1.a), a contradiction. ♦

In view of Assertion 28, Claim 31 and Lemma 19(3), one may easily see that $l(Q_3) = 1$. Consequently, Assertion 28 and Lemma 19(3) imply that $w_1 \in T[x_1, z_2[$. In what follows, assume that C is not a 6-blocks cycle, that is, $w_1 \neq x_1$.

Claim 32. *For all $p \in T[x_1, w_1[$, there exists no vertex $q \in T]w_1, z_2[$ such that $(p, q) \in A(C)$.*

Proof of Claim 32. Assume else and let j be maximal such that $w_j \leqslant_T q$. Then $[(w_j, w_{j+1}), Q_2, (p, q)]$ satisfies Lemma 19(3), a contradiction. ♦

In view of Assertion 28, Claims 31 and 32 and Lemma 19(3), one may easily see that the structure of C induces the existence of the arc (x_1, q) in $A(C)$ for some $q \in T]w_1, z_2[$, which contradicts Claim 32. As a result, C is a 6-blocks cycle and so $x_1 = w_1$. Hence, $C \in C_6^1 \cup C_6^2$. This completes the proof of Assertion 29. ♢

In what follows, we denote by $C = h_1, h_2, \ldots, h_\delta, h_1$.

Assertion 33. If there exist vertices h_{j_1} and h_{j_2} of C for some $j_1, j_2 \in \{1, \ldots, \delta\}$ such that h_{j_1} and h_{j_2} are not ancestors, then $C \in C_6^3 \cup C_6^4 \cup C_8$.

Proof of Assertion 33. We will prove a series of claims.

Claim 34. *Let $q_1 \in V(D_i^1)$ such that m_2 and q_1 are not ancestors and $v^* = l.c.a\{m_2, q_1\} \in T_{y_1} - y_1$. Let $p_j \in T]y_1, v^*]$ for $j = 1, 2$ with $p_1 \neq p_2$, and let $q_2 \in \bigcup_{z \in T]v^*, q_1]} T_z$. If $(p_1, q_1) \in A(C)$, then $(p_2, q_2) \notin A(C)$.*

Proof of Claim 34. Suppose otherwise and assume without loss of generality that $p_1 \leqslant_T p_2$. Notice that $v^* \in T]n_t, m_2[$, since else $[Q_2, (p_1, q_1)]$ satisfies Lemma 19(4) as $l(T[v^*, m_2]) \geqslant k$. If q_1 and q_2 are ancestors (possibly $q_1 = q_2$), then $[(p_1, q_1), (p_2, q_2), Q_2]$ satisfies Lemma 19($1.b$), a contradiction. This means that if such arcs exist in C then q_1 and q_2 are not ancestors and so the structure of C implies that $l(T[v^*, q_j]) \geqslant k$ for $j = 1, 2$. Now we are going to show that $p_j \in T]n_1, n_t[$ for $j = 1, 2$. Notice first that $n_1 \leqslant_T p_1$, since otherwise $[(p_1, q_1), P_2]$ satisfies Lemma 19(4), a contradiction. Now note that $p_2 \neq v^*$, since else $[P_2, (p_1, q_1)]$ satisfies Lemma 19(4), a contradiction. Moreover, observe that $p_2 \leqslant_T n_t$, since otherwise the union of $T[v^*, q_2]$, $T[v^*, m_2]$, $T[x_{t_1-1}, n_1] \cup P_2$ and $(x_{t_1-1}, n_t) \cup T[n_t, p_2] \cup (p_2, q_2)$ is a S-$C(k, 1, k, 1)$ in D, a contradiction. Thus, $p_j \in T]n_1, n_t[$ for $j = 1, 2$. Consequently, the structure of C induces the existence of an arc $(f_1, f_2) \in A(C)$, such that either $f_1 \in T]n_1, n_t[$ and $f_2 \in T]n_t, m_2[\cup T_{m_2} \backslash \{m_2\}$, or $f_1 \leqslant_T n_1$ and $f_2 \in T]n_1, n_t[$. If the former holds, then $[P_2, (f_1, f_2), (x_{t_1-1}, n_t)]$ satisfies Lemma 19($1.a$), a contradiction. Else if the latter holds, then $[(f_1, f_2), (x_{t_1-1}, n_t), P_2]$ satisfies Lemma 19(2), a contradiction. This completes the proof of our claim. ♦

Let β be minimal such that h_β and m_2 are not ancestors. According to Claim 34, it follows that $h_\beta = z_2$. Let $v^* = \text{l.c.a}\{m_2, z_2\}$. Indeed, Assertion 28(1) together with the structure of C imply that $v^* \neq y_1$ and so induce the existence of an arc $(p,q) \in A(C)$ such that $p \in T]y_1, v^*]$ and $q \in \bigcup_{z \in T]v^*, z_2]} T_z$.

Claim 35. *If $v^* \in T]n_t, m_2[$, then $C \in C_6^3$.*

Proof of Claim 35. Notice first that $l(T[v^*, z_2]) < k$, since else $[(y_1, z_2), P_2]$ satisfies Lemma 19(4), a contradiction. This implies that $q \in T_{z_2}$ and $p \neq v^*$. In fact, $q = z_2$, since else the union of $(y_1, z_2) \cup T[z_2, q]$, Q_2, $T[p, m_2]$ and (p,q) is a S-$C(k,1,k,1)$ in D, a contradiction. This gives that $p = h_{\beta+1} = w_{r-1}$. Now we will study the position of p. If $p \in T]x_{t_1-1}, n_1[$, then the union of $T[p, n_1] \cup P_2$, (p, z_2), $T[y_1, x_{t_1-1}] \cup (x_{t_1-1}, n_t) \cup T[n_t, z_2]$ and Q_2 is a S-$C(k,1,k,1)$ in D, a contradiction. Else if $p \in T]n_1, v^*[$, then the maximality of n_1 implies that $w_1 \leqslant_T n_1$, and so $[(x_{t_1-1}, n_t), Q_4[w_1, w_{r-1}], P_2]$ satisfies Lemma 19(2), a contradiction. Else if $p \in T]y_1, x_1[$, then Assertion 28(1), Assertion 28(3) and Claim 34 imply that there exists an arc $(h, h') \in A(C)$ such that $h \in T]y_1, x_1[$ and $h' \in T]x_1, x_{t_1-1}[$, and so $[(x_j, x_{j+1}), Q_2, (h, h')]$ satisfies Lemma 19(3), where j is maximal satisfying $x_j \leqslant_T h'$, a contradiction. Else if $p \in T[x_1, x_{t_1-1}[$ and $t_1 - 1 \neq 1$, then the union of $T[y_1, p] \cup (p, z_2)$, Q_2, $T[x_{t_1-1}, n_1] \cup P_2$ and $(x_{t_1-1}, n_t) \cup T[n_t, z_2]$ is a S-$C(k,1,k,1)$ in D, a contradiction. Then $p = x_{t_1-1} = x_1$, and so $C \in C_6^3$. This completes the proof. ♦

Claim 36. *If $v^* \notin T]n_t, m_2[$, then $v^* = p = x_1$.*

Proof of Claim 36. Since $v^* \notin T]n_t, m_2[$, then $v^* \in T]y_1, n_t]$ and so clearly $l(T[v^*, m_2]) \geqslant k$. Observe that $p = v^*$, since else $[Q_2, (p,q)]$ satisfies Lemma 19(4), a contradiction. This gives that $l(T[v^*, q]) \geqslant k$. If $v^* \in T]x_1, n_t[$, then $[(y_1, z_2), Q_1]$ satisfies Lemma 19(4), a contradiction. Else if $v^* \in T]y_1, x_1[$, then Assertion 28(1), Assertion 28(3) and Claim 34 imply that there exists an arc $(h, h') \in A(C)$ such that $h \in T]y_1, x_1[$ and $h' \in T]x_1, x_{t_1-1}[$, and so $[(x_j, x_{j+1}), Q_2, (h, h')]$ satisfies Lemma 19(3), where j is maximal satisfying $x_j \leqslant_T h'$, a contradiction. Hence, $v^* = p = x_1$. This confirms our claim. ♦

From now on, we will assume that $v^* = p = x_1$, since else $C \in C_6^3$, due to Claim 36 and Claim 35.

Claim 37. *For all $z \in T_{z_2} - z_2$, there exists no vertex $w \in T[x_1, z_2[$ such that $(w, z) \in A(C)$.*

Proof of Claim 37. Assume the contrary is true. Then the union of $(y_1, z_2) \cup T[z_2, z]$, Q_2, $T[x_1, m_2]$ and $T[x_1, w] \cup (w, z)$ is a S-$C(k,1,k,1)$ in D, a contradiction. ♦

Now Claims 37 and 34, Assertion 28 and Lemma 19(3) imply that $l(Q_3) = 1$ and for all $j \geqslant \beta + 1$, $h_j \in \{x_1\} \cup \bigcup_{z \in T]x_1, z_2[} T_z$. Now it remains to prove two claims and conclude.

Claim 38. *If h_j and z_2 are ancestors for all $j \geqslant \beta + 1$, then $C \in C_6^4 \cup C_8$.*

Proof of Claim 38. Assume first that C is a 6-blocks cycle. Then $w_1 = x_1$ and so $C \in C_6^4$ as $l(Q_3) = 1$. Now assume that C is a cycle with at least eight blocks. If C is an 8-blocks cycle, then Claim 37 implies that $c_{\alpha_1} \leqslant_T z_2$ and $d_1 = x_1$. As $d_1 \leqslant_T w_1$, then Assertion 24 implies that $l(Q_4) = 1$ and $d_{\alpha_2-1} \leqslant_T w_1$. Hence, $C \in C_8$. Let us assume now that C is a cycle with at least ten blocks. Then $d_1 \neq x_1$. If $d_1 \leqslant_T w_1$, then Assertion 24 implies that $l(Q_4) = 1$ and $d_{\alpha_2-1} \leqslant_T w_1$, $c_{\alpha_1} \leqslant_T z_2$ and $v^* \leqslant_T d_1$. Observe that for all $f \in T[x_1, d_{\alpha_2-1}[$, there exists no $w \in T]d_{\alpha_2-1}, z_2[$ such that (f, w) is an arc of C. Assume else, then either $w \in T]d_{\alpha_2-1}, c_{\alpha_1}[$ and so $[(d_{\alpha_2-1}, c_{\alpha_1}), Q_3, (f, w)]$ satisfies Lemma 19(3), or $w \in T]c_{\alpha_1}, z_2[$ and so $[(f, w), Q_6, Q_4]$ satisfies Lemma 19(2), a contradiction. Moreover, observe that for all $f \in T[x_1, d_1[$, there exists no $w \in T]d_1, d_{\alpha_2-1}[$ such that (f, w) is an arc of C, since else $[(d_{j_1}, d_{j_1+1}), (y_1, z_2), (f, w)]$ satisfies Lemma 19(2), where j_1 is maximal satisfying $d_{j_1} \leqslant_T w$, a contradiction. In view of these two observations, it follows that $h_\delta \in T]d_1, z_2[$ and so $(x_1, h_\delta) \notin A(C)$, a contradiction. Thus, $w_1 \leqslant_T d_1$. Again we will notice two observations. For all $f \in T]c_1, c_{\alpha_1}[$, there exists no vertex $w \in T]c_{\alpha_1}, z_2[$ such that (f, w) is an arc of C, since else $[(f, w), Q_3, (c_{j_1-1}, c_{j_1})]$ satisfies Lemma 19(3), where j_1 is minimal satisfying $f \leqslant_T c_{j_1}$. Also observe that for all $f \in T[x_1, c_1[$, there exists no vertex $w \in T]c_1, c_{\alpha_1}[$ such that (f, w) is an arc of C, since else $[(c_{j_2}, c_{j_2+1}), Q_3, (f, w)]$ satisfies Lemma 19(3), where j_2 is maximal satisfying $c_{j_2} \leqslant_T w$. Hence, $h_\delta \in T]w_1, c_{\alpha_1}[$ and so $(x_1, h_\delta) \notin A(C)$, a contradiction. This completes the proof. ♦

Claim 39. *For all $j \geqslant \beta + 1$, h_j and z_2 are ancestors.*

Proof of Claim 39. Assume the contrary is true. Let $i > \beta + 1$ be minimal such that h_i and z_2 are not ancestors. Then $h_{i-1} \in T]x_1, z_2[\cap C$ and $(h_{i-1}, h_i) \in A(C)$. Set $x = \text{l.c.a}\{h_i, z_2\}$. The structure of C implies that there exists an arc (h, h^*) of C such that $h \in T[x_1, x]\backslash\{h_{i-1}\}$ and $h^* \in \bigcup_{z \in T]x, h_i]} T_z$. Assume that (h, h^*) is chosen to be the first arc of C with this property. Clearly, $x \notin V(D_i^1)$, since otherwise $[Q_3, (h, h^*)]$ or $[Q_3, (h_{i-1}, h_i)]$ satisfies Lemma 19(4), a contradiction. Observe that for all $z \in T[x_1, x[\backslash\{h_{i-1}\}$, there exists no vertex $w \in T]x, h_i] \cup T_{h_i}$ such that $(z, w) \in A(C)$, since otherwise $[(z, w), (h_{i-1}, h_i), Q_3]$ satisfies Lemma 19(1.b), a contradiction. This implies that h^* and h_i are not ancestors. Let $\rho > i$ be minimal such that h_ρ and h_i are not ancestors and let $\gamma = \text{l.c.a}\{h_\rho, h_i\}$. Clearly, $h_{\rho-1} \in T]x, \gamma]$ as $x \notin V(D_i^1)$. Moreover, the definition of D_i^1 and the structure of C imply that there exists i_1, with $i \leqslant i_1 < \rho - 1$, $h_{i_1} \in T]\gamma, h_i] \cup T_{h_i}$ and $h_{i_1+1} \in T]x, \gamma]$ such that $(h_{i_1+1}, h_{i_1}) \in A(C)$ (possibly $h_{i_1} = h_i$ and $h_{i_1+1} = h_{\rho-1}$). Indeed, for all $z \in T]\gamma, h_\rho] \cup T_{h_\rho}$, there exists no vertex $w \in T]h_{i-1}, \gamma]\backslash\{h_{\rho-1}\}$ such that $(w, z) \in A(C)$, since else $[(w, z), (h_{\rho-1}, h_\rho), (h_{i-1}, h_i)]$ satisfies Lemma 19(1.b), a contradiction. Furthermore, for all $z \in V(D_i^1)$ such that z and h_{i_1} are not ancestors and $\text{l.c.a}\{z, h_{i_1}\} \in T[h_{i_1+1}, h_i[$, there exists no vertex $w \in T[x_1, h_{i-1}[$ such that $(w, z) \in A(C)$, since else $[(h_{i-1}, h_i), (h_{i_1+1}, h_{i_1}), (w, z)]$ satisfies Lemma 19(1.b). In view of these observations together with the structure of C, we guarantee the existence of an arc $(w, z) \in A(C)$ such that z and h_ρ are not ancestors, $w_1 = \text{l.c.a}\{z, h_\rho\} \in T]\gamma, h_\rho[$ and $w \in T]h_{i-1}, \gamma]$. Let j be minimal such that (h_{j+1}, h_j) satisfies the properties of (w, z). Notice that $l(T[\gamma, h_i]) < k$, since otherwise $[(h_{i-1}, h_i), (h_{\rho-1}, h_\rho)]$ or $[(h_{i-1}, h_i), (h_{j+1}, h_j)]$

satisfies Lemma 19(4), a contradiction. Thus $\gamma \notin V(D_i^1)$, $h_{i_1} \in T_{h_i}$ and $h_{i_1+1} \in T]x, \gamma[$. One may easily check that the position of h_j, the structure of C and the definition of D_i^1 imply that $l(T[\gamma, h_j]) \geqslant k$ and $l(T[\gamma, h_\rho]) \geqslant k$. This gives that $h_{j+1} \in T]h_{i_1+1}, \gamma[$ and $h_\rho \in T[h_{i_1+1}, \gamma[$, since otherwise $[(h_{j+1}, h_j), (h_{i_1+1}, h_{i_1})]$ or $[(h_{\rho-1}, h_\rho), (h_{i_1+1}, h_{i_1})]$ satisfies Lemma 19(4), a contradiction. Now all the above explanation implies that if $\gamma_1 =$ l.c.a$\{h^*, h_i\} \in T[h_{i_1+1}, h_i[$, then $h \in T]h_{i_1+1}, \gamma_1[$, a contradiction. Hence, $\gamma_1 \in T]x, h_{i_1+1}[$. Notice that if $h_{i-1} \leqslant_T h$, then $[(h_{i-1}, h_i), (h, h^*)]$ satisfies Lemma 19(4), a contradiction. Hence, $h \leqslant_T h_{i-1}$ and so $l(T[\gamma_1, h^*]) < k$, since otherwise $[(h, h^*), (h_{i-1}, h_i)]$ satisfies Lemma 19(4), a contradiction. Now the minimality of (h, h^*), the fact that $\gamma_1 \notin V(D_i^1)$, the structure of C and the definition of D_i^1 imply that there exists an arc (p_1, q_1) of C such that $p_1 \in T]x, \gamma_1[$ annd $q_1 \in \bigcup_{z \in T]\gamma_1, h^*]} T_z$. Then $[(h_{i-1}, h_i), (p_1, q_1)]$ satisfies Lemma 19(4), a contradiction. This completes the proof of Claim 39. ♦

Therefore, Claims 34, 35, 36, 37, 38, and 39 complete the proof of Assertion 33. ♢

In the light of all the above Assertions, Lemma 20 is proved. □

3.1.3 The existence of 5-wheels in D_i^1

In this subsection, we provide an upper bound for the chromatic number of D_i^1 and complete the proof by proving that D_i^1 is a 5-wheel-free digraph.

Proposition 40. *$\chi(D_i^1) \leqslant 6$ for all $i \in \{1, \ldots, 2k\}$.*

Proof. Assume to the contrary that $\chi(D_i^1) > 6$. Then Corollary 18 implies that D_i^1 contains a 5-wheel with cycle C and center ω, denoted by $W = (C, \omega)$. Let $\{a_1, \ldots, a_5\} \subseteq N_C(\omega)$. By Lemma 20, $C \in \mathcal{C}$. Clearly $C \notin C_4^2$, since W is a 5-wheel and the cycles in C_4^2 are of 4 vertices. We will prove series of claims and conclude.

Claim 41. *$C \notin C_2$.*

Proof of Claim 41. Assume to the contrary that $C \in C_2$, and assume without loss of generality that $a_1 \leqslant_T a_2 \leqslant_T a_3 \leqslant_T a_4 \leqslant_T a_5$. Let $P_1 = n_1, \ldots, n_t$, $t \geqslant 2$; $P_2 = m_1, \ldots, m_l$, $l \geqslant 2$ with $n_1 = m_1$ and $n_t = m_l$ be the blocks of C. Notice that if $\omega \leqslant_T a_1$, or $a_5 \leqslant_T \omega$, then there exist at least two vertices in $\{a_2, a_3, a_4\}$ that belongs to the same block of C. Assume without loss of generality that a_2 and a_4 are vertices of P_1. Let i_1 be maximal satisfying $m_{i_1} \leqslant_T a_2$, and let i_2 be minimal satisfying $a_4 \leqslant_T m_{i_2}$. Assume first that $\omega \leqslant_T a_1$. Then either $\omega \leqslant_T m_{i_1}$ and so $[(\omega, a_2), (\omega, a_4), P_2[m_{i_1}, m_l]]$ satisfies Lemma 19(2), or $m_{i_1} \leqslant_T \omega$ and so $P_2 \cap T[\omega, a_2] = \phi$ and the union of $(\omega, a_4) \cup T[a_4, m_{i_2}]$, (ω, a_2), $P_1[m_1, a_1] \cup T[a_1, a_2]$ and $P_2[m_1, m_{i_2}]$ is a S-$C(k, 1, k, 1)$ in D, a contradiction. Now assume that $a_5 \leqslant_T \omega$. If $m_{i_2} \leqslant_T \omega$, or m_{i_2} and ω are not ancestors, then the union of $T[m_{i_1}, a_2] \cup (a_2, \omega)$, $P_2[m_{i_1}, m_{i_2}]$, $T[a_4, m_{i_2}]$ and (a_4, ω) is a S-$C(k, 1, k, 1)$ in D, a contradiction. And if $\omega \leqslant_T m_{i_2}$, then $P_2 \cap T[a_4, \omega] = \phi$ and so the union of $T[m_{i_1}, a_2] \cup (a_2, \omega)$, $P_2[m_{i_1}, m_l]$,

$T[a_4, a_5] \cup P_1[a_5, n_t]$ and (a_4, ω) is a S-$C(k, 1, k, 1)$ in D, a contradiction. So either $3 \leqslant |N_C^+(\omega)| \leqslant 4$, or $3 \leqslant |N_C^-(\omega)| \leqslant 4$. Assume that the former holds and let $a_{j_1} \leqslant_T a_{j_2} \leqslant_T a_{j_3}$ be distinct out-neighbors of ω in C, and let a_{j_4} be an in-neighbors of ω in C. Assume without loss of generality that $a_{j_1} \in P_1$. We are going to prove that $m_2 \in T_{a_5}$, $\omega \leqslant_T n_2$, and so $|N_C^+(\omega)| = 4$. Notice that $P_2 \cap T]\omega, a_5[= \phi$, since else $[P_1[n_1, a_{j_1}], P_2[n_1, m_i], (\omega, a_5)]$ satisfies Lemma 19(2), where i is minimal satisfying $\omega \leqslant_T m_i$. Observe that $P_2 \cap T]n_1, \omega[= \phi$, since else $[P_2[m_2, m_l], (\omega, a_{j_3}), P_1[n_1, a_{j_1}]]$ satisfies Lemma 19(1.a). So $m_2 \in T_{a_5}$. Clearly, $\omega \leqslant_T n_2$ since else, $[(\omega, a_{j_2}), (m_1, m_2), (n_i, n_{i+1})]$ satisfies Lemma 19(3), where i is maximal satisfying $n_i \leqslant_T \omega$. Then $a_{j_4} = n_1 = m_1$. Since $m_2 \in T_{a_5}$, assume without loss of generality that $a_4 = a_{j_3} \leqslant_T a_{j_5} = a_5$. Then the union of $(\omega, a_{j_3}) \cup T[a_{j_3}, m_2]$, (ω, a_{j_2}), $(m_1, n_2) \cup T[n_2, a_{j_2}]$ and (m_1, m_2) is a S-$C(k, 1, k, 1)$ in D, a contradiction. So the latter holds. Let $a_{j_1} \leqslant_T a_{j_2} \leqslant_T a_{j_3}$ be distinct in-neighbors of ω in C, and let a_{j_4} be an out-neighbor of ω in C. Assume without loss of generality that $a_{j_3} \in P_1$, and let i is maximal satisfying $m_i \leqslant_T \omega$. We are going to prove that $m_{l-1} \in T[r, a_1]$ and $|N_C^-(\omega)| = 4$. Notice that $P_2 \cap T]a_1, \omega[= \phi$, since else $[P_1[a_{j_3}, n_t], P_2[m_i, m_l], (a_1, \omega)]$ satisfies Lemma 19(1.a). Also notice that $m_{l-1} \leqslant_T \omega$, since else $[P_2[m_1, m_{i+1}], (a_{j_1}, \omega), P_1[a_{j_3}, n_t]]$ satisfies Lemma 19(2). Now observe that $P_1 \cap T]\omega, m_l[= \phi$, since else $[(a_{j_2}, \omega), (n_{i_1}, n_{i_1+1}), (m_{l-1}, m_l)]$ satisfies Lemma 19(3), where i_1 is maximal satisfying $n_{i_1} \leqslant_T \omega$. Then $a_{j_4} = n_t = m_l$ and $n_{t-1} \leqslant_T \omega$. So $|N_C^-(\omega)| = 4$. Now since $m_{l-1} \in T[r, a_1]$, then assume without loss of generality that $a_{j_5} = a_1 \leqslant_T a_{j_1} = a_2$. So the union of $T[m_{l-1}, a_{j_1}] \cup (a_{j_1}, \omega)$, (m_{l-1}, m_l), $T[a_{j_2}, a_{j_3}] \cup P_1[a_{j_3}, m_l]$ and (a_{j_2}, ω) is a S-$C(k, 1, k, 1)$ in D, a contradiction. ♢

Claim 42. $C \notin C_4^1$.

Proof of Claim 42. Assume to the contrary that $C \in C_4^1$. First observe that $\omega \notin T[r, n_1[\backslash\{x_1\}$, since else there exist $a_j \in (P_1 \cup P_2)\backslash\{n_1\}$, such that $(\omega, a_j) \in A(W)$. Due to symmetry, we will assume that $a_j \in P_2$. Then $[Q_1, (\omega, a_j)]$ or $[(\omega, a_j), Q_1]$ satisfies Lemma 19(4), a contradiction. As $\omega \notin T[r, n_1[\backslash\{x_1\}$, then by symmetry we may assume that $N_C(\omega) \subseteq P_1 \cup \{x_1\}$. Assume now that either $\omega \in T_{n_t}\backslash\{n_t\}$, or ω and n_t are not ancestors (clearly if the latter holds, then l.c.a$\{\omega, n_t\} \in T[n_4, n_t[$ as W is a 5-wheel). Then in both cases there exist at least three in-neighbors of ω in $P_1\backslash\{n_t\}$, say $a_{j_1} \leqslant_T a_{j_2} \leqslant_T a_{j_3}$, and so $[(a_{j_1}, \omega), (a_{j_2}, \omega), Q_1]$ satisfies Lemma 19(1.a) or Lemma 19(1.b). So $\omega \in T]n_1, n_t[\backslash P_1$. Let i be minimal satisfying $\omega \leqslant_T n_i$. If $|N_{P_1}^+(\omega)| \geqslant 3$ or $|N_C^-(\omega)| \geqslant 3$ with $n_i \neq n_t$ in case $|N_C^-(\omega)| \geqslant 3$, then there exist $j \in [5]$ such that $a_j \in V(P_1)\backslash\{n_t, x_1, n_i, n_{i-1}\}$, and $[(\omega, a_j), Q_1, (n_{i-1}, n_i)]$, or $[(n_{i-1}, n_i), Q_1, (a_j, \omega)]$ satisfies Lemma 19(3), a contradiction. Then $|N_C^-(\omega)| \geqslant 4$. Let $a_{i_1} \leqslant_T a_{i_2} \leqslant_T a_{i_3} \leqslant_T a_{i_4}$ be distinct in-neighbors of ω in C. So the union of $T[x_1, a_{i_2}] \cup (a_{i_2}, \omega)$, Q_1, $T[a_{i_3}, a_{i_4}] \cup P_1[a_{i_4}, n_t]$ and (a_{i_3}, ω) is a S-$C(k, 1, k, 1)$, a contradiction. ♢

Claim 43. $C \notin C_4^3 \cup C_4^4$.

Proof of Claim 43. Assume to the contrary that $C \in C_4^3 \cup C_4^4$. Let $v =$ l.c.a$\{n_t, m_l\}$. Notice that $\omega \notin T[v, n_t[\cup T[v, m_l[$, since else $[Q_1, (m_{l-1}, m_l)]$ or $[Q_2, (n_{t-1}, n_t)]$ satisfies Lemma 19(4), a contradiction. Also notice that for all $p \in T[r, v[\backslash\{n_{t-1}\}$, there exist no vertex

$q \in T_{n_t}$ such that $(p,q) \in A(W)\backslash\{(x_1,n_t)\}$. Since else $[(y_1,q),(m_{l-1},m_l)]$ satisfies Lemma 19(4), or $[(n_{t-1},n_t),(p,q),Q_2]$ satisfies Lemma 19(1.b), or $[(m_{l-1},m_l),Q_2,(p,q)]$ satisfies Lemma 19(1.b), a contradiction. Similarly we prove that for all $p \in T[r,v[\backslash\{m_{l-1}\}$, there exist no vertex $q \in T_{m_l}$ such that $(p,q) \in A(W)\backslash\{(x_1,m_l)\}$. So $\omega \notin T[v,n_t] \cup T[v,m_l] \cup T_{n_t} \cup T_{m_l}$, and if $\omega \in T[r,v[\backslash V(C)$, then $\{n_t,m_l\} \cap N_C(\omega) = \phi$. If ω and n_t are not ancestors, then there exist two distinct in-neighbors a_{j_1},a_{j_2} of ω in $T[n_1,v[\cap V(C)$, and so $[(a_{j_1},\omega),(a_{j_2},\omega),Q_1]$ satisfies Lemma 19(1.b), a contradiction. Then $\omega \in T[r,v[$. Assume that $\omega \leqslant_T m_{l-1}$. Clearly, if there exist two out-neighbors of ω in $P_1[n_2,n_{t-1}]$, say a_{j_1},a_{j_2}, then $[(\omega,a_{j_1}),(\omega,a_{j_2}),(m_{l-1},m_l)]$ satisfies Lemma 19(2). So $|N_{P_2[n_1,m_{l-1}]}(\omega)| \geqslant 3$, and hence $C \in C_4^4$. Let $m_{i_1} \leqslant_T m_{i_2} \leqslant_T m_{i_3}$ be distinct neighbors of ω in $P_2[n_1,m_{l-1}]$. If $\omega \leqslant_T n_1$, then $[(\omega,m_{i_2}),(\omega,m_{i_3}),(n_1,n_2)]$ satisfies Lemma 19(2), a contradiction. Then $\omega \in T]n_1,m_{l-1}[\backslash V(C)$. Let i_4 be maximal satisfying $m_{i_4} \leqslant_T \omega$. As $|N_{P_2[n_1,m_{l-1}]}(\omega)| \geqslant 3$, then there exist $j \in \{i_1,i_2,i_3\}$ such that $[(\omega,m_j),Q_2,(m_{i_4},m_{i_4+1})]$ or $[(m_{i_4},m_{i_4+1}),Q_2,(m_j,\omega)]$ satisfies Lemma 19(3), a contradiction. So $m_{l-1} \leqslant_T \omega$. Clearly, if $C \in C_4^4$, then there exist no $p \in T[y_1,m_{l-1}[\cap V(C)$ such that $p \in N_C(\omega)$, since else $[(n_1,n_2),(p,\omega),(m_{l-1},m_l)]$ satisfies Lemma 19(2), a contradiction. So $|N_{P_1[n_2,n_{t-1}]}(\omega)| \geqslant 4$. Now similarly as in the above case we prove that if $\omega \leqslant_T n_{t-1}$, then Lemma 19(3) is satisfied, a contradiction. Hence $n_{t-1} \leqslant_T \omega$. Let $n_{i_1} \leqslant_T n_{i_2} \leqslant_T n_{i_3} \leqslant_T n_{i_4}$ be distinct in-neighbors of ω in $P_2[n_2,n_{t-1}]$, then the union of $T[y_1,n_{i_1}] \cup (n_{i_1},\omega)$, Q_1, $T[n_{i_2},n_{t-1}] \cup (n_{t-1},n_t)$ and (n_{i_2},ω) is a S-$C(k,1,k,1)$, a contradiction. This confirms our claim. ◊

Claim 44. *If $C \in (\bigcup_{i=1}^{3} C_6^i) \cup (\bigcup_{i=5}^{8} C_4^i)$, then $\omega \in T]y_1,m_l[$. If $C \in C_6^4$, then $\omega \in T]x_1,m_2[$. And if $C \in C_8$, then $\omega \in T]x_1,m_2[\cup T]x_1,z_2[$.*

Proof of Claim 44. First we will show that if $C \in C_6^4 \cup C_8$, then $\omega \notin T[r,x_1[\backslash\{y_1\}$, and if $C \in C_6^4$, then $\omega \notin T]x_1,z_2[$. Assume that $C \in C_6^4 \cup C_8$ and $\omega \in T[r,x_1[\backslash\{y_1\}$, then there exists $a_\alpha \in V(C)\backslash\{y_1,x_1\}$, such that $(\omega,a_\alpha) \in A(W)$ for some $\alpha \in [5]$, and so $[(\omega,a_\alpha),Q_j]$ or $[Q_j,(\omega,a_\alpha)]$ satisfies Lemma 19(4) with $j \in \{2,3\}$, a contradiction. Assume now that $C \in C_6^4$ and $\omega \in T]x_1,z_2[$, and let j_1 be minimal satisfying $\omega \leqslant_T w_{j_1}$. If either $|N^-_{Q_4 \cup \{y_1\}}(\omega)| \geqslant 3$ with $w_{j_1} \neq z_2$, or $|N^+_{Q_4}(\omega)| \geqslant 3$, then there exists $\alpha \in [5]$ such that $[(\omega,a_\alpha),Q_3,(w_{j_1-1},w_{j_1})]$ or $[(w_{j_1-1},w_{j_1}),Q_3,(a_\alpha,\omega)]$ satisfies Lemma 19(3), with $a_\alpha \in N_{Q_4\backslash\{z_2,w_{j_1-1},w_{j_1}\}}(\omega)$, a contradiction. Then $|N^-_{Q_4 \cup \{y_1\}}(\omega)| \geqslant 4$, and so the union of $T[y_1,a_{i_2}] \cup (a_{i_2},\omega)$, Q_3, $T[a_{i_3},a_{i_4}] \cup Q_4[a_{i_4},z_2]$ and (a_{i_3},ω) is a S-$C(k,1,k,1)$, where $a_{i_1} \leqslant_T a_{i_2} \leqslant_T a_{i_3} \leqslant_T a_{i_4}$ are distinct in-neighbors of ω in $Q_4 \cup \{y_1\}$, a contradiction. So if $C \in C_6^4$, then $\omega \notin T]x_1,z_2[$. Let's assume now that $C \in (\bigcup_{i=1}^{4} C_6^i) \cup (\bigcup_{i=5}^{8} C_4^i) \cup C_8$. Moreover assume that $\omega \notin T]y_1,m_l[$ in case $C \in (\bigcup_{i=1}^{3} C_6^i) \cup (\bigcup_{i=5}^{8} C_4^i)$, $\omega \notin T]x_1,m_2[$ in case $C \in C_6^4$, and $\omega \notin T]x_1,m_2[\cup T]x_1,z_2[$ in case $C \in C_8$. Then the above observations with our assumption implies that $|T]y_1,m_l[\cap N_C(\omega)| \geqslant 3$ or $|T[x_1,z_2[\cap N_C(\omega)| \geqslant 3$. Let $a_{i_j} \in V(C)\backslash\{y_1,m_l,z_2\}$ for $j = 1,2$, such that $a_{i_1} \leqslant_T a_{i_2}$ be two distinct neighbors of ω. If $l_T(\omega) > l_T(y_1)$, then $[(a_{i_1},\omega),(a_{i_2},\omega),Q_j]$ satisfies Lemma 19(1.a) or Lemma 19(1.b) with $j \in \{2,3\}$, a contradiction. Then $\omega \leqslant_T y_1$, and so $[(\omega,a_{i_1}),(\omega,a_{i_2}),Q_j]$ satisfies Lemma 19(2) with $j \in \{2,3\}$, a contradiction. ◊

Claim 45. $C \notin C_6^1 \cup C_6^2$.

Proof of Claim 45. Assume the contrary is true, then Claim 44 implies that $\omega \in T]y_1, m_2[$. As W is a 5-wheel, then either $|N_C^+(\omega)| \geqslant 3$ or $|N_C^-(\omega)| \geqslant 3$. Assume first that $|N_C^+(\omega)| \geqslant 3$. Let $a_{i_1} \leqslant_T a_{i_2} \leqslant_T a_{i_3}$ be three out-neighbors of ω in C, let $p \in V(C)$ such that $T]\omega, p[\cap C = \phi$, and let $q \in N_C^-(p)$ (if exist). Clearly, $\omega \leqslant_T n_t$. Assume now that $\omega \leqslant_T z_2$. If $a_{i_3} \in T]z_2, m_2]$, then $[(\omega, a_{i_3}), Q_1, Q_3]$ satisfies Lemma 19(1.a), a contradiction. Then $a_{i_2} \in T]\omega, z_2[$, and so $[(x_1, x_2), Q_2, (\omega, a_{i_2})]$ or $\theta = [(\omega, a_{i_2}), Q_2, (q, p)]$ satisfies Lemma 19(3), a contradiction. If $\omega \in T]n_1, n_t[$ or $\omega \in T]z_2, x_{t_1-1}[$ (note that in case $C \in C_6^2$, we may have: $\omega \in T]z_2, x_{t_1-1}[$), then θ satisfies Lemma 19(3), a contradiction. So if $C \in C_6^1$ (resp. $C \in C_6^2$), then $\omega \in T]z_2, n_1[$ (resp. $\omega \in T]x_{t_1-1}, n_1[$). Then the union of $T[y_1, x_1] \cup Q_1$, Q_2, $T[\omega, n_1] \cup P_2$ and $(\omega, a_{i_2}) \cup T[a_{i_2}, n_t]$ is a S-$C(k, 1, k, 1)$ in D, a contradiction. Hence, $|N_C^-(\omega)| \geqslant 3$. Clearly $\omega \notin T]y_1, x_1[$. Let $a_{i_1} \leqslant_T a_{i_2} \leqslant_T a_{i_3}$ be three in-neighbors of ω in C, let $p \in V(C)$ such that $T]p, \omega[\cap C = \phi$, and let $q \in N_C^+(p)$ (if exist). Assume first that $n_1 \leqslant_T \omega$. If $a_{i_1} \in T[y_1, n_1[$, then $[(a_{i_1}, \omega), Q_1, P_2]$ satisfies Lemma 19(2), a contradiction. Then $a_{i_2} \in T]n_1, n_t[$, and so $[(a_{i_2}, \omega), Q_2, (x_{t_1-1}, n_t)]$ or $\theta_1 = [(p, q), Q_2, (a_{i_2}, \omega)]$ satisfies Lemma 19(3), a contradiction. Now if $\omega \in T]x_1, z_2[$ or $\omega \in T]x_2, n_1[$ (in case $C \in C_6^2$, we may have: $\omega \in T]x_2, n_1[$), then θ_1 satisfies Lemma 19(3), a contradiction. So if $C \in C_6^1$ (resp. $C \in C_6^2$), then $\omega \in T]z_2, n_1[$ (resp. $\omega \in T]z_2, x_2[$), and hence the union of $Q_1 \cup T[n_t, m_2]$, $T[x_1, a_{i_2}] \cup (a_{i_2}, \omega)$, $Q_3 \cup T[z_2, \omega]$ and Q_2 is a S-$C(k, 1, k, 1)$, a contradiction. This completes the proof. $\lozenge$

Claim 46. $C \notin C_6^3 \cup C_6^4 \cup C_8 \cup (\bigcup_{j=5}^{8} C_4^j)$.

Proof of Claim 46. Assume the contrary is true. Then Claim 44 implies that if $C \notin C_6^4 \cup C_8$, then $\omega \in T]y_1, m_l[$, if $C \in C_6^4$, then $\omega \in T]x_1, m_2[$, and if $C \in C_8$ then $\omega \in T]x_1, m_2[\cup T]x_1, z_2[$. By symmetry, if $C \in C_8$, then we will assume that $\omega \in T]x_1, m_2[$. Notice that if $C \in C_6^3$, then $z_2 \notin N_C(\omega)$ since otherwise $[(\omega, z_2), (x_1, z_2), Q_2]$ satisfies Lemma 19(1.b). We will prove a useful observation before taking all the possible positions of ω: For all $p \in T]y_1, n_1[\setminus V(C)$, there exist no $q \in T]n_1, m_l]\setminus\{n_t\}$ such that $(p, q) \in A(W)$. Assume else and notice that in case $C \in C_6^4 \cup C_8$, then clearly Claim 44 implies that $p \notin T]y_1, x_1[$. If $C \in C_4^5 \cup C_4^6 \cup C_6^3 \cup C_6^4 \cup C_8$, then $[(p, q), Q_1, P_2]$ satisfies Lemma 19(2) or $[(p, q), (n_1, m_2), Q_1]$ satisfies Lemma 19(1.a) or $[(p, m_2), P_2, Q_3]$ satisfies Lemma 19(1.b), a contradiction. And if $C \in C_4^7 \cup C_4^8$, then $[P_1, Q_2, (p, q)]$ satisfies Lemma 19(3) or $[(p, q), (m_{l-1}, m_l), Q_1]$ satisfies Lemma 19(1.a), a contradiction. This confirms our observation. Now we will discuss according to the position of ω. Assume first that $\omega \leqslant_T n_1$. Then our observations with the fact that W is a 5-wheel implies that $C \in C_4^5 \cup C_6^4 \cup C_8$, and $|N_C^+(\omega)| \geqslant 3$ or $|N_C^-(\omega)| \geqslant 3$. Assume that $|N_C^+(\omega)| \geqslant 3$, and let $a_{i_1} \leqslant_T a_{i_2} \leqslant_T a_{i_3}$ be three out-neighbors of ω in C. Clearly our observation implies that $\omega \leqslant_T x_{t_1-1}$, $a_{i_1} \in Q_1$, and $a_{i_3} \neq m_2$. So the union of $T[y_1, x_1] \cup Q_1[x_1, a_{i_1}] \cup T[a_{i_1}, a_{i_2}]$, Q_2, $(\omega, a_{i_3}) \cup T[a_{i_3}, m_2]$ and (ω, a_{i_2}) is a S-$C(k, 1, k, 1)$ in D, a contradiction. Then $|N_C^-(\omega)| \geqslant 3$, and so $[(x_i, x_{i+1}), Q_2, (a_{i_2}, \omega)]$ satisfies Lemma 19(3), where $a_{i_1} \leqslant_T a_{i_2} \leqslant_T a_{i_3}$ are three in-neighbors of ω in C, and i is maximal satisfying $x_i \leqslant_T \omega$. Hence $n_1 \leqslant_T \omega \leqslant_T m_l$. Clearly if there exist $p \in N_C(\omega) \cap T[y_1, n_1[$, then

$[Q_1, (p, \omega), P_2]$ satisfies Lemma 19(2), a contradiction. Then $N_C(\omega) \cap T[y_1, n_1[= \phi$. We will notice one more observation: For all $p \in T]n_1, n_t[$, there exist no $q \in T]n_t, m_l]$ such that $(p, q) \in A(W)$, since otherwise $[(p, q), P_2, Q_1]$ satisfies Lemma 19(1.a) or $[(p, q), P_1, Q_2]$ satisfies Lemma 19(3), a contradiction. Assume that $\omega \in T]n_t, m_l[$, then our observation with the fact that W is a 5-wheel implies that $C \in C_4^6$ and $|N_{P_2[m_2,m_l]}(\omega)| \geqslant 3$. If $\omega \in T]n_t, m_2[$, then our observations implies that $[(\omega, a_{j_2}), Q_2, (n_1, m_2)]$ satisfies Lemma 19(3), where $a_{j_1} \leqslant_T a_{j_2} \leqslant_T a_{j_3}$ are three out-neighbors of ω in $P_2[m_2, m_l]$, a contradiction. So $\omega \in T]m_2, m_l[$. Then $N_C^-(\omega) \cap \{n_1, n_t\} = \phi$, since else $[(n_t, \omega), Q_2, (n_1, m_2)]$ satisfies Lemma 19(3), or the union of $Q_1 \cup T[n_t, m_2]$, Q_2, $(n_1, \omega) \cup T[\omega, m_l]$, (n_1, m_2) is a S-$C(k, 1, k, 1)$ in D, a contradiction. Hence $N_C(\omega) \subseteq V(P_2) \backslash \{n_1\}$, and so there exists $\alpha \in [5]$ such that $[(\omega, a_\alpha), Q_2, (m_i, m_{i+1})]$ satisfies Lemma 19(3), where i is maximal satisfying $m_i \leqslant_T \omega$, or the union of $T[x_1, a_{j_1}] \cup (a_{j_1}, \omega)$, Q_2, $T[a_{j_2}, a_{j_3}] \cup P_2[a_{j_3}, m_l]$ and (a_{j_2}, ω) is a S-$C(k, 1, k, 1)$ in D, where $a_{j_1} \leqslant_T a_{j_2} \leqslant_T a_{j_3}$ are three in-neighbors of ω in $P_2[m_2, m_{l-1}]$, a contradiction. So $\omega \in T]n_1, n_t[$, and hence the above observations implies that $N_C(\omega) \subseteq T[n_1, n_t]$. Assume first that $|N_C^+(\omega)| \geqslant 3$, and let $a_{i_1} \leqslant_T a_{i_2} \leqslant_T a_{i_3}$ be three out-neighbors of ω in C. Let $p \in V(C)$ such that $T]\omega, p[\cap C = \phi$, and let $q \in N_C^-(p)$. Then either $C \notin C_4^8$ and so $[(\omega, a_{i_2}), Q_2, (q, p)]$ satisfies Lemma 19(3), or $C \in C_4^8$ and so $[(\omega, a_{i_3}), P_2, Q_1[x_1, p]]$ satisfies Lemma 19(1.a), a contradiction. Then $|N_C^-(\omega)| \geqslant 3$. Let $a_{i_1} \leqslant_T a_{i_2} \leqslant_T a_{i_3}$ be three in-neighbors of ω in C. Let $p \in V(C)$ such that $T]p, \omega[\cap C = \phi$, and let $q \in N_C^+(p)$. Then either $C \notin C_4^7$ and so $[(p, q), Q_2, (a_{i_2}, \omega)]$ satisfies Lemma 19(3), or $C \in C_4^7$ and so $[(a_{i_1}, \omega), P_1, P_2[p, m_l]]$ satisfies Lemma 19(2), a contradiction. This completes the proof. ♢

All the above discussion implies that $C \notin \mathcal{C}$, a contradiction. This completes the proof. □

3.2 Coloring D_i^2

In this section, we study the chromatic number of D_i^2. In fact, the coloring of D_i^2 heavily depends on the following observation:

Lemma 47. *Let D be an acyclic digraph. Then $G(D)$ is $\Delta^+(D)$-degenerate and thus $\chi(D) \leqslant \Delta^+(D) + 1$.*

Proof. Let G be a subgraph of $G(D)$ and let H be the subdigraph of D whose underlying graph is G. Let P be a longest directed path of H. One may easily see that the initial end of P, say u, has no in-neighbors in H, since otherwise we get either a directed path longer than P or a directed cycle in H. These are contradictions to the facts that P is a longest directed path of H and that D is acyclic. Hence, the only neighbors of u in G are its out-neighbors in H. This implies the desired result. □

Proposition 48. *$\chi(D_i^2) \leqslant 6$ for all $i \in [2k]$.*

Proof. Let B_1 and B_2 be a partition of the vertex-set of D_i^2, with $B_1 := \{v \in V_i | d^+_{D_i^2}(v) \leqslant 1\}$ and $B_2 := V_i \setminus B_1$. Obviously, $\Delta^+(D_i^2[B_1]) \leqslant 1$. Now we are going to prove that

$\Delta^+(D_i^2[B_2]) \leqslant 3$. Assume the contrary is true and let u be a vertex of B_2 whose out-degree in $D_i^2[B_2]$ is at least 4. By the definition of A_2, it is easy to see that all the out-neighbors of u belong to $T[r,u]$. This induces an ordering of the out-neighbors of u in $D_i^2[B_2]$ with respect to $\leqslant_T$, say $v_1, v_2, \ldots, v_t$ with $v_{i-1} \leqslant_T v_i$ for all $2 \leqslant i \leqslant t$. According to our assumption, note that t must be greater than 3. Moreover, the definition of B_2 forces the existence of an out-neighbor w_i of v_i other than v_1, for each $2 \leqslant i \leqslant t-1$. Due to the definition of A_2, w_i and v_1 must be ancestors. More precisely, $w_i \leqslant_T v_1$ for all $2 \leqslant i \leqslant t-1$, since otherwise if there exists $i_0 \in \{2, \ldots, t-1\}$ such that $v_1 \leqslant_T w_{i_0}$, then the union of $T[v_{i_0}, v_t], (v_{i_0}, w_{i_0}), (u, v_1) \cup T[v_1, w_{i_0}]$ and (u, v_t) is a S-$C(k,1,k,1)$ in D, a contradiction. To reach the final contradiction, we consider two out-neighbors v_i, v_j of u with $2 \leqslant i < j \leqslant t-1$ and their respective out-neighbors w_i, w_j. Note that the existence of v_i and v_j is guaranteed by the assumption that $t \geqslant 4$. Moreover, note that possibly $w_i = w_j$. In view of the above observation, $w_i \leqslant_T v_1$ and $w_j \leqslant_T v_1$. If $w_i \leqslant_T w_j$, then the union of $T[v_j, v_t], (v_j, w_j), (u, v_1) \cup T[v_1, v_i] \cup (v_i, w_i) \cup T[w_i, w_j]$ and (u, v_t) forms a S-$C(k,1,k,1)$ in D, a contradiction. Otherwise, the union of $T[v_j, v_t], (v_j, w_j) \cup T[w_j, w_i], (u, v_1) \cup T[v_1, v_i] \cup (v_i, w_i)$ and (u, v_t) is a S-$C(k,1,k,1)$ in D, a contradiction. This proves that $\Delta^+(D_i^2[B_2]) \leqslant 3$. Consequently, due to the fact that D_i^2 is acyclic together with Lemma 47, it follows that $D_i^2[B_1]$ is 2-colorable and $D_i^2[B_2]$ is 4-colorable. Therefore, by assigning the vertices of B_1 2 colors and those of B_2 4 new colors, we get a proper 6-coloring of D_i^2. This completes the proof. □

3.3 Coloring D_i^3

This section is devoted to color D_i^3 properly.

Proposition 49. $\chi(D_i^3) \leqslant 4k+2$ *for all* $i \in [2k]$.

Proof. Assume to the contrary that $\chi(D_i^3) \geqslant 4k+3$. Due to Theorem 4, D_i^3 contains a copy Q of $P(2k+1, 2k+1)$, which is the union of two directed paths Q_1 and Q_2 which are disjoint except in their initial vertex, say $Q_1 = y_0, y_1, \ldots, y_{2k}$ and $Q_2 = z_0, z_1, \ldots, z_{2k}$ with $y_0 = z_0$. We need to prove a series of assertions as follows:

Assertion 50. For all $i \in [2k-1]$ and $j \in [2k]$, y_i is not an ancestor of z_j and z_i is not an ancestor of y_j.

Proof of Assertion 50. Due to symmetry, we are going to show that y_i is not an ancestor of z_j for all $1 \leqslant i \leqslant 2k-1$ and $1 \leqslant j \leqslant 2k$. Assume the contrary is true. Then there exists $i \in [2k-1]$ such that $y_i \leqslant_T z_j$ for some $j \in [2k]$. Suppose that y_i and z_j are chosen so that $T[y_i, z_j] \cap Q_2[y_0, z_j] = \{z_j\}$. By the definition of A_3, note that $y_{i+1} \notin T[y_i, z_j]$, as $(y_i, y_{i+1}) \in A_3$. Observe now that $T[r, y_{i+1}] \cap (Q_1[y_0, y_i] \cup Q_2[y_0, z_j]) \neq \emptyset$, since otherwise the union of $T[\beta, y_{i+1}], T[\beta, y_0] \cup Q_2[y_0, z_j], T[y_i, z_j]$ and (y_i, y_{i+1}) forms a S-$C(k,1,k,1)$ in D, with $\beta = \text{l.c.a}\{y_0, y_{i+1}\}$. This is a contradiction to the fact that D is $C(k,1,k,1)$-subdivision-free. Let $\alpha \in T[r, y_{i+1}]$ such that $T[\alpha, y_{i+1}] \cap (Q_1[y_0, y_i] \cup Q_2[y_0, z_j]) = \{\alpha\}$. Clearly, $\alpha \notin \{z_j, y_i\}$. If $\alpha \in Q_1$, then the union of $Q_1[y_0, \alpha] \cup T[\alpha, y_{i+1}], Q_2[y_0, z_j], T[y_i, z_j]$

and (y_i, y_{i+1}) is a S-$C(k,1,k,1)$ in D, a contradiction. This implies that α must belong to $Q_2 - y_0$. But the union of $T[\alpha, y_{i+1}], Q_2[\alpha, z_j], T[y_i, z_j]$ and (y_i, y_{i+1}) is a S-$C(k,1,k,1)$ in D, a contradiction. This confirms our assertion. $\lozenge$

In what follows, we denote by $x_1 = \text{l.c.a}\{y_0, y_1\}$, $x_2 = \text{l.c.a}\{y_0, z_1\}$ and $x_3 = \text{l.c.a}\{y_1, z_1\}$.

Assertion 51. $x_3 \notin \{x_1, x_2\}$.

Proof of Assertion 51. Suppose the contrary is true, that is, $x_3 = x_1$ or $x_3 = x_2$. Without loss of generality, assume that $x_3 = x_2$. This means that $x_2 \leqslant_T x_1$. By the definition of D_i^3, note that $T[x_1, y_1]$ and $T[x_2, z_1]$ are of length at least $2k$. Throughout the proof of this assertion, we denote by $T_1 = T[x_1, y_1] \cup T[x_2, z_1] \cup T[x_2, y_0]$.

Claim 52. *y_0 is not an ancestor neither of y_i nor of z_i for all $i \in [2k]$.*

Proof of Claim 52. Due to symmetry, we are going to prove that y_0 is not an ancestor of y_i for all $i \in [2k]$. Assume the contrary is true. Then there exists $i \in [2k]$ such that $y_0 \leqslant_T y_i$. Assume that y_i is chosen so that y_0 is not an ancestor of y_j for all $j < i$. Clearly, $i > 1$. Then the union of $T[x_2, z_1], T[x_2, y_1] \cup Q_1[y_1, y_i], T[y_0, y_i]$ and (y_0, z_1) is a S-$C(k,1,k,1)$, a contradiction. This affirms our claim. $\blacklozenge$

Claim 53. *For all $0 \leqslant i \leqslant k$ and $j \in [2k]$, z_i is not an ancestor of z_j.*

Proof of Claim 53. We proceed by induction on i. The base case $i = 0$ follows by Claim 52. Suppose now that z_t is not an ancestor of z_j for all $0 \leqslant t < i$ and $j \in [2k]$. Our aim is to prove that z_i is not an ancestor of z_j for all $j \in [2k]$. Assume the contrary is true, that is, there exists $j \in [2k]$ such that $z_i \leqslant_T z_j$. Assume that z_j is chosen so that $l_T(z_j)$ is maximal and $T[z_i, z_j] \cap Q_2 = \{z_i, z_j\}$. Clearly, $z_{i+1} \notin T[z_i, z_j]$. Let α_1 be the vertex of $T[r, z_{i+1}]$ such that $T[\alpha_1, z_{i+1}] \cap T_1 = \{\alpha_1\}$ if $x_2 \in T[r, z_{i+1}]$ and $\alpha_1 = \text{l.c.a}\{x_2, z_{i+1}\}$ otherwise. Note that $T[\alpha_1, z_{i+1}] \cap Q_2[y_0, z_i] = \emptyset$, due to the induction hypothesis. Moreover, $T[\alpha_1, z_{i+1}] \cap Q_1[y_1, y_{2k-1}] = \emptyset$, according to Assertion 50. Now we are going to show that $\alpha_1 \in T]x_1, y_1]$. In fact, if $\alpha_1 \notin T]x_1, y_1]$, we consider two possibilities: If $\alpha_1 \in T[x_2, z_1]$, then the union of $T[x_2, y_1], T[x_2, z_{i+1}] \cup Q_2[z_{i+1}, z_j], Q_2[y_0, z_i] \cup T[z_i, z_j]$ and (y_0, y_1) is a S-$C(k,1,k,1)$, a contradiction. Else if $\alpha_1 \notin T[x_2, z_1]$, then α_1 and x_1 are ancestors and so the union of $T[\beta, y_1], T[\beta, z_{i+1}] \cup Q_2[z_{i+1}, z_j], Q_2[y_0, z_i] \cup T[z_i, z_j]$ and (y_0, y_1) is a S-$C(k,1,k,1)$ in D with $\beta = \min_T\{x_1, \alpha_1\}$, a contradiction. Now we consider the vertex α_2 of $T[r, y_2]$ such that $T[\alpha_2, y_2] \cap (T_1 \cup T[\alpha_1, z_{i+1}] \cup T[z_i, z_j]) = \{\alpha_2\}$ if $x_2 \in T[r, y_2]$ and $\alpha_2 = \text{l.c.a}\{x_2, y_2\}$ otherwise. Note that $T[\alpha_2, y_2] \cap Q_2[y_0, z_{2k-1}] = \emptyset$, according to Assertion 50 and Claim 52. Moreover, $T[\alpha_2, y_2] \cap T[z_i, z_j] = \emptyset$, according to Assertion 50. Actually, $\alpha_2 \in T]\alpha_1, z_{i+1}]$. If not, then the union of $T[\beta, y_2]$, $T[\beta, z_{i+1}] \cup Q_2[z_{i+1}, z_j], Q_2[y_0, z_i] \cup T[z_i, z_j]$ and $Q_1[y_0, y_2]$ is a S-$C(k,1,k,1)$ in D, where $\beta = \min_T\{\alpha_1, \alpha_2\}$ if α_1 and α_2 are ancestors and $\beta = \text{l.c.a}\{\alpha_1, \alpha_2\}$ otherwise. This is a contradiction to the fact that D is $C(k,1,k,1)$-subdivision-free digraph. $*\{$Note here that $l(T[\alpha_2, y_2]) < k$ and $l(T[\alpha_2, z_{i+1}]) < k$, since otherwise the union of $T[\alpha_2, y_2], T[\alpha_2, z_{i+1}] \cup Q_2[z_{i+1}, z_j], T[x_2, z_1] \cup Q_2[z_1, z_i] \cup T[z_i, z_j]$ and $T[x_2, y_1] \cup (y_1, y_2)$ is a S-$C(k,1,k,1)$, a contradiction. This implies that $l(T[\alpha_1, \alpha_2]) \geqslant k$, as $l(T[\alpha_1, y_2]) \geqslant 2k$. Moreover, note that $l(Q_2[z_{i+1}, z_j]) \leqslant k-2$, since otherwise the union of $T[\alpha_2, z_{i+1}] \cup Q_2[z_{i+1}, z_j], T[\alpha_2, y_2], T[x_2, y_1] \cup (y_1, y_2)$ and $T[x_2, z_1] \cup Q_2[z_1, z_i] \cup T[z_i, z_j]$ is a S-$C(k, 1, k, 1)$, a contradiction. This guarantees the existence of z_{j+1}. Let α_3 be the vertex of $T[r, z_{j+1}]$ such that $T[\alpha_3, z_{j+1}] \cap (T_1 \cup T[\alpha_1, z_{i+1}] \cup T[\alpha_2, y_2] \cup T[z_i, z_j]) = \{\alpha_3\}$ if $x_2 \in T[r, z_{j+1}]$ and α_3 is the l.c.a of x_2 and z_{j+1} otherwise. Observe that $T[\alpha_3, z_{j+1}] \cap Q_2[y_0, z_j] = \emptyset$. In fact, $T[\alpha_3, z_{j+1}] \cap Q_2[y_0, z_{i-1}] = \emptyset$, due to the induction hypothesis. Moreover, $T[\alpha_3, z_{j+1}] \cap Q_2[z_{i+1}, z_{j-1}] = \emptyset$, since otherwise if there exists $i+1 \leqslant t \leqslant j-1$ such that $z_t \leqslant_T z_{j+1}$, then the union of $T[\alpha_2, z_{i+1}] \cup Q_2[z_{i+1}, z_t] \cup T[z_t, z_{j+1}], T[\alpha_2, y_2], T[x_2, y_1] \cup (y_1, y_2)$ and $T[x_2, z_1] \cup Q_2[z_1, z_i] \cup T[z_i, z_j] \cup (z_j, z_{j+1})$ is a S-$C(k, 1, k, 1)$, a contradiction. This together with the maximality of z_j imply that z_i is also not an ancestor of z_{j+1}. To reach the final contradiction, we study the possible positions of α_3. If $\alpha_3 \in T]\alpha_1, z_{i+1}]$, then the union of $T[\beta, z_{j+1}], T[\beta, y_2], T[x_2, y_1] \cup (y_1, y_2)$ and $T[x_2, z_1] \cup Q_2[z_1, z_i] \cup T[z_i, z_j] \cup (z_j, z_{j+1})$ with $\beta = \min_T\{\alpha_2, \alpha_3\}$ is a S-$C(k, 1, k, 1)$, a contradiction. If $\alpha_3 \in T]\alpha_2, y_2]$, then $l(T[\alpha_2, z_{j+1}]) \geqslant 2k$. But $l(T[\alpha_2, z_{j+1}]) = l(T[\alpha_2, \alpha_3]) + l(T[\alpha_3, z_{j+1}])$ and $l(T[\alpha_2, \alpha_3]) < l(T[\alpha_2, y_2]) < k$, then $l(T[\alpha_3, z_{j+1}]) \geqslant k$. Consequently, the union of $T[\alpha_3, z_{j+1}], T[\alpha_3, y_2], T[x_2, y_1] \cup (y_1, y_2)$ and $T[x_2, z_1] \cup Q_2[z_1, z_i] \cup T[z_i, z_j] \cup (z_j, z_{j+1})$ is a S-$C(k, 1, k, 1)$, a contradiction. Otherwise, let $\beta = \min_T\{\alpha_1, \alpha_3\}$ if α_1 and α_3 are ancestors and $\beta = \text{l.c.a}\{\alpha_1, \alpha_3\}$ otherwise. Then the union of $T[\beta, z_{i+1}], T[\beta, z_{j+1}], T[z_i, z_j] \cup (z_j, z_{j+1})$ and (z_i, z_{i+1}) is a S-$C(k, 1, k, 1)$ in D, a contradiction$\}*$ affirming that z_i is not an ancestor of z_j for all $j \in [2k]$. This ends the proof. ♦

In a similar way, we can prove that y_i is not an ancestor of y_j, for all $0 \leqslant i \leqslant k$ and $j \in [2k]$. To complete the proof, we consider the vertices α_1 and α_2 of $T[r, z_k]$ and $T[r, y_k]$ respectively such that $T[\alpha_1, z_k] \cap T_1 = \{\alpha_1\}$ if $x_2 \in T[r, z_k]$ and $\alpha_1 = \text{l.c.a}\{x_2, z_k\}$ otherwise, and $T[\alpha_2, y_k] \cap T_1 = \{\alpha_2\}$ if $x_2 \in T[r, y_k]$ and $\alpha_2 = \text{l.c.a}\{x_2, y_k\}$ otherwise. Note that $(T[\alpha_1, z_k] \cup T[\alpha_2, y_k]) \cap (Q_1[y_0, y_{2k-1}] \cup Q_2[y_0, z_{2k-1}]) = \emptyset$. Let $\beta_1 = \min_T\{\alpha_1, \alpha_2\}$ if α_1 and α_2 are ancestors and $\beta_1 = \text{l.c.a}\{\alpha_1, \alpha_2\}$ otherwise. Given that $\beta_2 = \text{l.c.a}\{y_k, z_k\}$, we study two cases: If $\beta_1 = \beta_2$, then at least one of $T[\beta_1, y_k]$ and $T[\beta_1, z_k]$ has length greater than k. Consequently, the union of $T[\beta_1, z_k], T[\beta_1, y_k], Q_1[y_0, y_k]$ and $Q_2[y_0, z_k]$ is a S-$C(k, 1, k, 1)$ in D, a contradiction. Otherwise, we get $\alpha_1 = \alpha_2 = \beta_1$ and $\beta_2 \in T]\alpha_1, z_k]$. Clearly, $l(T[\beta_2, y_k]) < k$ and $l(T[\beta_2, z_k]) < k$, since otherwise the union of $T[\beta_2, z_k], T[\beta_2, y_k], Q_1[y_0, y_k]$ and $Q_2[y_0, z_k]$ would be a S-$C(k, 1, k, 1)$ in D, a contradiction. To reach the final contradiction, let α_3 be the vertex of $T[r, z_{k+1}]$ such that $T[\alpha_3, z_{k+1}] \cap (T_1 \cup T[\alpha_1, y_k] \cup T[\alpha_1, z_k]) = \{\alpha_3\}$ if $x_2 \in T[r, z_{k+1}]$ and $\alpha_3 = \text{l.c.a}\{x_2, z_{k+1}\}$ otherwise. If $\alpha_3 \in T[\beta_2, z_k]$, then the union of $T[\beta_2, z_{k+1}], T[\beta_2, y_k], Q_1[y_0, y_k]$ and $Q_2[y_0, z_{k+1}]$ forms a S-$C(k, 1, k, 1)$ in D, a contradiction. Else if $\alpha_3 \in T]\beta_2, y_k]$, then $l(T[\alpha_3, z_{k+1}]) \geqslant k$, since otherwise $l(T[\beta_2, y_k]) > T[\beta_2, \alpha_3] \geqslant k$, a contradiction. Thus the union of $T[\alpha_3, z_{k+1}], T[\alpha_3, y_k], Q_1[y_0, y_k]$ and $Q_2[y_0, z_{k+1}]$ forms a S-$C(k, 1, k, 1)$ in D, a contradiction. Else if $\alpha_3 \notin T[\beta_2, y_k] \cup T[\beta_2, z_k]$, then consider $\beta_3 = \{\alpha_3\}$ if $\alpha_3 \leqslant_T \beta_2$ and $\beta_3 = \text{l.c.a}\{\alpha_3, \beta_2\}$ otherwise. It is easy to check that β_3 is the least common ancestor of z_k and z_{k+1} as well as of y_k and z_{k+1}. Thus $l(T[\beta_3, z_{k+1}]) \geqslant 2k$ and so the union of $T[\beta_3, z_{k+1}], T[\beta_3, y_k], Q_1[y_0, y_k]$ and $Q_2[y_0, z_{k+1}]$ forms a S-$C(k, 1, k, 1)$ in D, a contradiction. This finishes the proof of Assertion 51. ♢

In view of Assertion 51, we get that $x_1 = x_2$. In what follows, we denote by $T_1 = T[x_1, y_1] \cup T[x_1, z_1] \cup T[x_1, y_0]$.

Assertion 54. For all $i \in [k]$ and $j \in [2k]$, z_i is not an ancestor of z_j.

Proof of Assertion 54. Assume the contrary is true, that is, there exists $i \in [k]$ and $j \in [2k]$ such that $z_i \leqslant_T z_j$. Assume that i is chosen to be minimal and j is chosen so that $l_T(z_j)$ is maximal and $T[z_i, z_j] \cap Q_2 = \{z_i, z_j\}$. Clearly, $z_{i+1} \notin T[z_i, z_j]$. Let α_1 be the vertex of $T[r, z_{i+1}]$ such that $T[\alpha_1, z_{i+1}] \cap T_1 = \{\alpha_1\}$ if $x_1 \in T[r, z_{i+1}]$ and $\alpha_1 = \text{l.c.a}\{x_1, z_{i+1}\}$ otherwise. Due to the choice of z_i together with Assertion 50, keep in mind that $T[\alpha_1, z_{i+1}] \cap (Q_1[y_1, y_{2k-1}] \cup Q_2[z_1, z_i]) = \emptyset$.

Claim 55. $\alpha_1 \in T]x_1, y_1]$.

Proof of Claim 55. Assume the contrary is true. First, assume that $\alpha_1 = y_0$. Let α_2 be the vertex of $T[r, y_2]$ such that $T[\alpha_2, y_2] \cap (T_1 \cup T[y_0, z_{i+1}] \cup T[z_i, z_j]) = \{\alpha_2\}$ if $x_1 \in T[r, y_2]$ and $\alpha_2 = \text{l.c.a}\{x_1, y_2\}$ otherwise. Note that $T[r, y_2] \cap T[z_i, z_j] = \emptyset$, since otherwise z_i would be an ancestor of y_2, a contradiction to Assertion 50. If $\alpha_2 \notin T[y_0, z_{i+1}]$, we consider two cases: If $\alpha_2 \in T]x_3, z_1]$, then the union of $T[\alpha_2, z_1] \cup Q_2[z_1, z_i] \cup T[z_i, z_j], T[\alpha_2, y_2], T[x_1, y_1] \cup (y_1, y_2)$ and $T[x_1, z_{i+1}] \cup Q_2[z_{i+1}, z_j]$ forms a S-$C(k, 1, k, 1)$, a contradiction. Otherwise, consider $\beta = \min_T\{\alpha_2, x_3\}$ if α_2 and x_3 are ancestors and $\beta = \text{l.c.a}\{\alpha_2, x_3\}$ otherwise. Then the union of $T[\beta, y_2], T[\beta, z_1] \cup Q_2[z_1, z_i] \cup T[z_i, z_j], T[y_0, z_{i+1}] \cup Q_2[z_{i+1}, z_j]$ and $Q_1[y_0, y_2]$ forms a S-$C(k, 1, k, 1)$, a contradiction. Else if $\alpha_2 \in T[y_0, z_{i+1}]$, then $\alpha_2 \neq y_0$, since otherwise the union of $T[y_0, y_2], T[y_0, z_{i+1}] \cup Q_2[z_{i+1}, z_j], T[x_3, z_1] \cup Q_2[z_1, z_i] \cup T[z_i, z_j]$ and $T[x_3, y_1] \cup (y_1, y_2)$ is a S-$C(k, 1, k, 1)$ in D, a contradiction. Moreover, $l(T[\alpha_2, y_2]) \leqslant k - 1$ and $l(T[\alpha_2, z_{i+1}]) \leqslant k - 1$, since otherwise the union of $T[\alpha_2, y_2], T[\alpha_2, z_{i+1}] \cup Q_2[z_{i+1}, z_j], T[x_1, y_1] \cup (y_1, y_2)$ and $T[x_1, y_0] \cup Q_2[y_0, z_i] \cup T[z_i, z_j]$ is a S-$C(k, 1, k, 1)$ in D, a contradiction. Furthermore, $Q_2[z_{i+1}, z_j]$ has length at most $k - 2$, since otherwise the union of $T[\alpha_2, z_{i+1}] \cup Q_2[z_{i+1}, z_j], T[\alpha_2, y_2], T[x_1, y_1] \cup (y_1, y_2)$ and $T[x_1, y_0] \cup Q_2[y_0, z_i] \cup T[z_i, z_j]$ is a S-$C(k, 1, k, 1)$ in D, a contradiction. This induces the existence of z_{j+1}. Let α_3 be the vertex of $T[r, z_{j+1}]$ such that $T[\alpha_3, z_{j+1}] \cap (T_1 \cup T[\alpha_2, y_2] \cup T[y_0, z_{i+1}] \cup T[z_i, z_j]) = \{\alpha_3\}$ if $x_1 \in T[r, z_{j+1}]$ and $\alpha_3 = \text{l.c.a}\{x_1, z_{j+1}\}$ otherwise. Observe that $T[\alpha_3, z_{j+1}] \cap Q_2]y_0, z_j] = \emptyset$. In fact, $T[\alpha_3, z_{j+1}] \cap Q_2]y_0, z_{i-1}] = \emptyset$, due to the choice of z_i. Moreover, $T[\alpha_3, z_{j+1}] \cap Q_2[y_{i+1}, z_{j-1}] = \emptyset$, since otherwise if there exists $i + 1 \leqslant t \leqslant j - 1$ such that $z_t \leqslant_T z_{j+1}$, then the union of $T[\alpha_2, z_{i+1}] \cup Q_2[z_{i+1}, z_t] \cup T[z_t, z_{j+1}], T[\alpha_2, y_2], T[x_1, y_1] \cup (y_1, y_2)$ and $T[x_1, y_0] \cup Q_2[y_0, z_i] \cup T[z_i, z_j] \cup (z_j, z_{j+1})$ is a S-$C(k, 1, k, 1)$, a contradiction. This together with the maximality of z_j imply that z_i is also not an ancestor of z_{j+1}. To reach the final contradiction, we study the possible positions of α_3: If $\alpha_3 \in T]\alpha_2, y_2]$, then $l(T[\alpha_3, z_{j+1}]) \geqslant k$, since otherwise $l(T[\alpha_3, y_2]) \geqslant k$, a contradiction. Consequently, the union of $T[\alpha_3, z_{j+1}], T[\alpha_3, y_2], T[x_1, y_1] \cup (y_1, y_2)$ and $T[x_1, y_0] \cup Q_2[y_0, z_i] \cup T[z_i, z_j] \cup (z_j, z_{j+1})$ is a S-$C(k, 1, k, 1)$, a contradiction. Else if $\alpha_3 \notin T]\alpha_2, y_2]$ and $l_T(\alpha_3) > l_T(y_0)$, then the union of $T[\beta, z_{j+1}], T[\beta, y_2], T[x_1, y_1] \cup (y_1, y_2)$ and $T[x_1, y_0] \cup Q_2[y_0, z_i] \cup T[z_i, z_j] \cup (z_j, z_{j+1})$ is a S-$C(k, 1, k, 1)$ in D, with $\beta = \min_T\{\alpha_2, \alpha_3\}$. This is a contradiction to the fact that D is $C(k, 1, k, 1)$-subdivision-free. Else, let $\beta = \min_T\{\alpha_3, y_0\}$ if α_3 and y_0 are ancestors and let $\beta = \text{l.c.a}\{\alpha_3, y_0\}$ otherwise. Note that possibly $\alpha_3 = y_0$. Hence, the union of $T[\beta, z_{i+1}], T[\beta, z_{j+1}], T[z_i, z_j] \cup (z_j, z_{j+1})$ and (z_i, z_{i+1}) is a S-$C(k,1,k,1)$ in D, a contradiction affirming that y_0 is not an ancestor of z_{i+1} and thus $\alpha_1 \neq y_0$. But $\alpha_1 \notin T]x_1, y_1]$, then x_1 and α_1 are ancestors. Consequently, the union of $T[\beta, y_1], T[\beta, z_{i+1}] \cup Q_2[z_{i+1}, z_j], Q_2[y_0, z_i] \cup T[z_i, z_j]$ and (y_0, y_1) is a S-$C(k,1,k,1)$ in D, with $\beta = \min_T\{x_1, \alpha_1\}$. This a contradiction to the fact that D is $C(k,1,k,1)$-subdivision-free and thus a confirmation to our claim. ♦

Now we consider the vertex α_2 of $T[r, y_2]$ such that $T[\alpha_2, y_2] \cap (T_1 \cup T[\alpha_1, z_{i+1}] \cup T[z_i, z_j]) = \{\alpha_2\}$ if $x_1 \in T[r, y_2]$ and $\alpha_2 = \text{l.c.a}\{x_1, y_2\}$ otherwise. Note that $T[\alpha_2, y_2] \cap (Q_2]y_0, z_j] \cup T[z_i, z_j]) = \emptyset$, according to Assertion 50.

Claim 56. $\alpha_2 \in T]\alpha_1, z_{i+1}]$.

Proof of Claim 56. Assume the contrary is true. First, assume that $\alpha_2 = y_0$. If $\alpha_1 \in T]x_1, x_3]$, then the union of $T[\alpha_1, z_{i+1}], T[\alpha_1, y_1] \cup (y_1, y_2), T[y_0, y_2]$ and $Q_2[y_0, z_{i+1}]$ forms a S-$C(k,1,k,1)$, a contradiction. Else if $\alpha_1 \in T]x_3, y_1]$, then $l(T[\alpha_1, z_{i+1}]) \leqslant k-1$, since otherwise the union of $T[\alpha_1, z_{i+1}], T[\alpha_1, y_1] \cup (y_1, y_2), T[y_0, y_2]$ and $Q_2[y_0, z_{i+1}]$ is a S-$C(k,1,k,1)$ in D, a contradiction. This implies that $l(T[x_3, \alpha_1]) \geqslant k$. Moreover, $Q_2[z_{i+1}, z_j]$ has length at most $k-2$, since otherwise the union of $T[\alpha_1, z_{i+1}] \cup Q_2[z_{i+1}, z_j]$, $T[\alpha_1, y_1] \cup (y_1, y_2), T[y_0, y_2]$ and $Q_2[y_0, z_i] \cup T[z_i, z_j]$ is a S-$C(k,1,k,1)$ in D, a contradiction. This guarantees the existence of z_{j+1}. Let α_3 be the vertex of $T[r, z_{j+1}]$ such that $T[\alpha_3, z_{j+1}] \cap (T_1 \cup T[y_0, y_2] \cup T[\alpha_1, z_{i+1}] \cup T[z_i, z_j]) = \{\alpha_3\}$ if $x_1 \in T[r, z_{j+1}]$ and $\alpha_3 = \text{l.c.a}\{x_1, z_{j+1}\}$ otherwise. Observe that $T[\alpha_3, z_{j+1}] \cap Q_2]y_0, z_j] = \emptyset$. In fact, $T[\alpha_3, z_{j+1}] \cap Q_2]y_0, z_{i-1}] = \emptyset$, due to the choice of z_i. Moreover, $T[\alpha_3, z_{j+1}] \cap Q_2[z_{i+1}, z_{j-1}] = \emptyset$, since otherwise if there exists $i+1 \leqslant t \leqslant j-1$ such that $z_t \leqslant_T z_{j+1}$, then the union of $T[\alpha_1, z_{i+1}] \cup Q_2[z_{i+1}, z_t] \cup T[z_t, z_{j+1}], T[\alpha_1, y_1] \cup (y_1, y_2), T[y_0, y_2]$ and $Q_2[y_0, z_i] \cup T[z_i, z_j] \cup (z_j, z_{j+1})$ is a S-$C(k,1,k,1)$, a contradiction. This together with the maximality of z_j imply that z_i is also not an ancestor of z_{j+1}. To reach the final contradiction, we study the possible positions of α_3: If $\alpha_3 \in T[y_0, y_2]$, then the union of $T[y_0, z_{j+1}], (y_0, y_1), T[x_3, y_1], T[x_3, z_1] \cup Q_2[z_1, z_{j+1}]$ is a S-$C(k,1,k,1)$ in D, a contradiction. Else if $\alpha_3 \notin T]\alpha_1, y_1]$, let $\beta = \min_T\{\alpha_1, \alpha_3\}$ if α_1 and α_3 are ancestors and let $\beta = \text{l.c.a}\{\alpha_1, \alpha_3\}$ otherwise. Then the union of $T[\beta, z_{j+1}], T[\beta, y_1] \cup (y_1, y_2), T[y_0, y_2]$ and $Q_2[y_0, z_{j+1}]$ is a S-$C(k,1,k,1)$, a contradiction. Else, $\alpha_3 \in T]\alpha_1, y_1]$. Note that $l(T[\alpha_3, z_{j+1}]) \leqslant k-1$, since otherwise the union of $T[\alpha_3, z_{j+1}], T[\alpha_3, y_1] \cup (y_1, y_2), T[y_0, y_2]$ and $Q_2[y_0, z_{j+1}]$ is a S-$C(k,1,k,1)$, a contradiction. But $l(T[\alpha_1, z_{j+1}]) \geqslant 2k$, then $l(T[\alpha_1, \alpha_3]) \geqslant k$ and so the union of $T[\alpha_1, y_1], T[\alpha_1, z_{i+1}] \cup Q_2[z_{i+1}, z_j], (y_0, y_1)$ and $Q_2[y_0, z_i] \cup T[z_i, z_j]$ is a S-$C(k,1,k,1)$, a contradiction. This confirms that $\alpha_2 \neq y_0$. To complete the proof, we assume to the contrary that $\alpha_2 \notin T]\alpha_1, z_{i+1}]$ and we consider $\beta = \min_T\{\alpha_1, \alpha_2\}$ if α_1 and α_2 are ancestors and $\beta = \text{l.c.a}\{\alpha_1, \alpha_2\}$ otherwise. Then the union of $T[\beta, y_2], T[\beta, z_{i+1}] \cup Q_2[z_{i+1}, z_j], Q_1[y_0, y_2]$ and $Q_2[y_0, z_i] \cup T[z_i, z_j]$ is a S-$C(k,1,k,1)$, a contradiction. This implies Claim 56. ♦

The rest of the proof of Assertion 54 is exactly the same as $*\{\ldots\}*$ in Claim 53, with exactly two differences. The first difference is that each place we have used $T[x_2, z_1] \cup Q_2[z_1, z_i]$ in $*\{\ldots\}*$ must be replaced by $T[x_2, y_0] \cup Q_2[y_0, z_i]$ in the proof of Assertion

54. The second one is that in the proof of Claim 53 $T[\alpha_3, z_{j+1}] \cap Q_2[y_0, z_{i-1}] = \emptyset$ due to the induction hypothesis. However, in the proof of this assertion we have $T[\alpha_3, z_{j+1}] \cap Q_2]y_0, z_{i-1}] = \emptyset$ by the choice of z_i. Indeed, the case where $y_0 \in T[\alpha_3, z_{j+1}]$ in the proof of this assertion will be prevented by the last contradiction of Claim 53. This ends the proof. $\Diamond$

In a similar way, we can show that y_i is not an ancestor of y_j for all $i \in [k]$ and $j \in [2k]$. To complete the proof, let α_1 (resp. α_2) be the vertex of $T[r, y_k]$ (resp. $T[r, z_k]$) such that $T[\alpha_1, y_k] \cap T_1 = \{\alpha_1\}$ (resp. $T[\alpha_2, z_k] \cap T_1 = \{\alpha_2\}$) if $x_1 \in T[r, y_k]$ (resp. $x_1 \in T[r, z_k]$) and $\alpha_1 = \text{l.c.a}\{x_1, y_k\}$ (resp. $\alpha_2 = \text{l.c.a}\{x_1, z_k\}$) otherwise.

Assertion 57. $y_0 \in \{\alpha_1, \alpha_2\}$.

Proof of Assertion 57. Assume the contrary is true. This, together with Assertion 50 and Assertion 54, implies that $(T[\alpha_1, y_k] \cup T[\alpha_2, z_k]) \cap (Q_1[y_0, y_k] \cup Q_2[y_0, z_k]) = \emptyset$, since otherwise, without loss of generality, there would exist $0 \leqslant i \leqslant k$ such that $z_i \in (T[\alpha_1, y_k] \cup T[\alpha_2, z_k]) \cap (Q_1[y_0, y_k] \cup Q_2[y_0, z_k])$. In that case, if $i \in [k]$, z_i would be an ancestor of either y_k or z_k, which would contradict Assertion 50 and Assertion 54 respectively. Else if $i = 0$, z_0 would belong to $\{\alpha_1, \alpha_2\}$, a contradiction to our assumption. Set $\alpha_3 = \text{l.c.a}\{y_k, z_k\}$ and $\alpha_4 = \min_T\{\alpha_1, \alpha_2\}$ if α_1 and α_2 are ancestors and $\alpha_4 = \text{l.c.a}\{\alpha_1, \alpha_2\}$ otherwise. Assume that $\alpha_3 = \alpha_4$, then $l(T[\alpha_1, y_k]) \geqslant k$ and $l(T[\alpha_2, z_k]) \geqslant k$ unless $\alpha_1 \in T]x_3, z_1]$ and $\alpha_2 \in T]x_3, y_1]$. In the later case, $\alpha_3 = x_3$ and so $l(T[x_3, z_k]) \geqslant k$ and $l(T[x_3, y_k]) \geqslant k$. Then the union of $T[\alpha_3, z_k], T[\alpha_3, y_k], Q_1[y_0, y_k]$ and $Q_2[y_0, z_k]$ is a $S\text{-}C(k, 1, k, 1)$, a contradiction. This implies that $\alpha_3 \neq \alpha_4$ and thus $\alpha_1 = \alpha_2 = \alpha_4$ and $\alpha_3 \in T[\alpha_1, z_k]$. Clearly, $T[\alpha_3, y_k]$ and $T[\alpha_3, z_k]$ have length at most $k-1$, since otherwise the union of $T[\alpha_3, y_k], T[\alpha_3, z_k], Q_2[y_0, z_k]$ and $Q_1[y_0, y_k]$ is a $S\text{-}C(k, 1, k, 1)$, a contradiction. This gives that $l(T[\alpha_1, \alpha_3]) \geqslant k$, since at least one of $T[\alpha_1, y_k]$ and $T[\alpha_1, z_k]$ has length at least $2k$. Let α_5 be the vertex of $T[r, z_{k+1}]$ such that $T[\alpha_1, z_{k+1}] \cap (T_1 \cup T[\alpha_1, y_k] \cup T[\alpha_1, z_k]) = \{\alpha_5\}$ if $x_1 \in T[r, z_{k+1}]$ and $\alpha_5 = \text{l.c.a}\{x_1, z_{k+1}\}$ otherwise. Due to Assertion 50 and Assertion 54, it follows that $T[\alpha_5, z_{k+1}] \cap (Q_1[y_1, y_k] \cup Q_2[z_1, z_k]) = \emptyset$. If $\alpha_5 = y_0$, let $\beta = \alpha_1$ if $\alpha_1 \leqslant_T \beta_1$ and $\beta = \text{l.c.a}\{\alpha_1, z_1\}$ otherwise. Then the union of $Q_1[y_0, y_k], T[y_0, z_{k+1}], T[\beta, z_1] \cup Q_2[z_1, z_{k+1}]$ and $T[\beta, y_k]$ is a $S\text{-}C(k, 1, k, 1)$, a contradiction. Hence, $\alpha_5 \neq y_0$. Let $\beta = \alpha_5$ if $\alpha_5 \leqslant_T y_k$ and $\beta = \text{l.c.a}\{\alpha_5, y_k\}$ otherwise. Then $l(T[\beta, z_{k+1}]) \geqslant k$ if $\alpha_5 \in T_{\alpha_1}$ and $l(T[\beta, y_k]) \geqslant k$ otherwise. This implies that the union of $Q_1[y_0, y_k], Q_2[y_0, z_{k+1}], T[\beta, z_{k+1}]$ and $T[\beta, y_k]$ is a $S\text{-}C(k, 1, k, 1)$, a contradiction. This ends the proof. $\Diamond$

To reach the final contradiction, we consider two principle cases: If $\alpha_1 = \alpha_2 = y_0$, then the union of $T[y_0, z_k], T[y_0, y_k], T[x_3, y_1] \cup Q_1[y_1, y_k]$ and $T[x_3, z_1] \cup Q_2[z_1, z_k]$is a $S\text{-}C(k, 1, k, 1)$, a contradiction. Otherwise, due to symmetry, assume that $\alpha_1 = y_0$ and $\alpha_2 \neq y_0$. If $\alpha_2 \notin T[x_3, z_1] \cup T[x_3, y_1]$, let $\beta = \alpha_2$ if $\alpha_2 \leqslant_T y_0$ and $\beta = \text{l.c.a}\{\alpha_2, y_0\}$ otherwise. Then the union of $T[\beta, y_k], T[\beta, z_k], T[x_3, z_1] \cup Q_2[z_1, z_k]$ and $T[x_3, y_1] \cup Q_1[y_1, y_k]$ is a $S\text{-}C(k, 1, k, 1)$, a contradiction. Hence, $\alpha_2 \in T[x_3, z_1] \cup T[x_3, y_1]$. Let $\beta = \alpha_2$ if $\alpha_2 \leqslant_T y_1$ and $\beta = x_3$ otherwise. Then the union of $T[\beta, y_1] \cup Q_1[y_1, y_k], T[\beta, z_k], Q_2[y_0, z_k]$ and $T[y_0, y_k]$ is a $S\text{-}C(k, 1, k, 1)$, a contradiction. This finishes the proof. $\square$

3.4 Main Theorem

Now we are ready to prove Theorem 12 that we restate:

Theorem 58. *Let D be a strongly connected digraph having no subdivisions of $C(k_1, 1, k_3, 1)$ and let $k = max\{k_1, k_3\}$, then the chromatic number of D is at most $36 \cdot (2k) \cdot (4k + 2)$.*

Proof. Let T be a spanning out-tree of D. Indeed, the existence of T is guaranteed due to the fact that D is strongly connected digraph. According to Proposition 16, we may assume that T is final. Define D^i_j as before for $i \in [2k]$ and $j \in [3]$. Due to Lemma 15 together with Proposition 40, Proposition 48 and Proposition 49, we get that $\chi(D_i) \leqslant 36 \cdot (4k + 2)$ for all $i \in [2k]$. As $V(D) = \bigcup_{i=1}^{2k} V(D_i)$, by assigning $36(4k - 2)$ distinct colors to each D_i, we obtain a proper coloring of D with $36 \cdot (2k) \cdot (4k + 2)$ colors. □

4 The existence of S-$C(k_1, 1, k_3, 1)$ in Hamiltonian digraphs

The previous bound can be strongly improved in case that the digraph contains a Hamiltonian directed cycle. In this section, we provide a tighter bound for the chromatic number of Hamiltonian digraphs containing no subdivisions of $C(k, 1, k, 1)$. Before proving Theorem 13, we need the following lemma:

Lemma 59. *Let k be a positive integer and let D be a $C(k, 1, k, 1)$-subdivision-free digraph with a Hamiltonian directed cycle C. Assume that u, v, w, x, x' are five vertices of D such that $uv \in E(G(D)) \setminus E(C), w \in C]u, v[$ and $x, x' \in C]v, u[$ in a way that $v(C[v, x]) = k$ and $v(C[x', u]) = k$. If $(v, u) \in E(D)$, then $|N_{G(D)}(w) \cap C]x, x'[| \leqslant 2$.*

Proof. We are going to prove first that w has at most one out-neighbor in $V(C]x, x'[)$. If w has two out-neighbors in $V(C]x, x'[)$, say v_i, v_j with $i < j$, then the union of $(w, v_j) \cup C[v_j, u]$, (w, v_i), $C[v, v_i]$ and (v, u) forms a S-$C(k, 1, k, 1)$ in D, a contradiction. Thus,

$$|N^+_D(w) \cap C]x, x'[| \leqslant 1. \tag{1}$$

Now we will prove that w has at most one in-neighbor in $V(C]x, x'[)$. If w has two in-neighbors in $V(C]x, x'[)$, say v_i, v_j with $i < j$, then the union of $C[v, v_i] \cup (v_i, w)$, (v, u), $C[v_j, u]$ and (v_j, w) is a S-$C(k, 1, k, 1)$, a contradiction. Thus,

$$|N^-_D(w) \cap C]x, x'[| \leqslant 1. \tag{2}$$

Hence, according to the inequalities 1 and 2, we get that $|N_{G(D)}(w) \cap C]x, x'[| \leqslant 2$. This ends the proof. □

Now we are ready to prove Theorem 13, that we restate:

Theorem 60. *Let D be a Hamiltonian digraph having no subdivisions of $C(k_1, 1, k_3, 1)$ and let $k = \max\{k_1, k_3\}$. Then D is $(6k-1)$-degenerate and thus $\chi(D) \leqslant 6k$.*

Proof. Let G be a subgraph of $G(D)$ and let H be the subdigraph of D whose underlying graph is G. If $\delta(G) \leqslant 6k-1$, then we are done. Otherwise, we will prove that D contains a S-$C(k,1,k,1)$, which means that the case where $\delta(G) > 6k-1$ does not hold. Thus it suffices now to prove that if $\delta(G) \geqslant 6k$ for a subgraph G of $G(D)$, then D contains a S-$C(k,1,k,1)$. Suppose the contrary is true and let $C = v_1v_2\cdots v_n$ be a Hamiltonian directed cycle of D, where $n = |V(D)| \geqslant |V(G)| \geqslant \delta(G)+1 \geqslant 6k+1$. Since $\delta(G) \geqslant 6k$, then there exist two vertices u, v of G such that $uv \in E(G) \setminus E(C)$ and $|V(C[u,v]) \cap V(G)| \geqslant 3k+1$. Assume that u, v are chosen such that $|V(C[u,v]) \cap V(G)|$ is minimal but at least $3k+1$. This implies that $|N_G(u) \cap V(C[u,v])| = 3k$. Hence, $|N_G(u) \cap V(C]v,u[)| \geqslant 3k$, since otherwise we get that $d_G(u) \leqslant 6k-1 < \delta(G)$, a contradiction. Thus, we guarantee the existence of two distinct vertices t and t' of $C]v,u[$ such that $l(C[v,t]) = k-1$ and $l(C[t',u]) = k-1$. Now we will consider the possible directions of the edge uv in H. If $(v,u) \in E(H)$, we define v' to be the vertex of G such that $C[v',v] \cap V(G) = \{v', v\}$. By the choice of the edge uv, note that v' has at most $3k-1$ neighbors in $C[u,v'] \cap V(G)$ and thus $|N_G(v') \cap C[u,v]| \leqslant 3k$. Moreover, Lemma 59 gives that $|N_G(v') \cap C]t,t'[| \leqslant 2$. Combining all these together, we get

$$\begin{aligned} d_G(v') &= |N_G(v') \cap C[u,v]| + |N_G(v') \cap C]v,t]| + |N_G(v') \cap C]t,t'[| + |N_G(v) \cap C[t',u[| \\ &\leqslant 3k + (k-1) + 2 + (k-1) \\ &= 5k, \end{aligned}$$

contradicting the fact that $\delta(G) \geqslant 6k$. Therefore, $(v,u) \notin E(H)$ and so $(u,v) \in E(H)$.

Now we consider the vertices w and w' of G with $|V(C[u,w]) \cap V(G)| = k+1$ and $|V(C[w',v]) \cap V(G)| = k+1$. Due to the fact that $|V(C[u,v]) \cap V(G)| \geqslant 3k+1$, it is clear that $w \neq w'$ and $|V(C[w,w']) \cap V(G)| \geqslant k+1$. To reach the final contradiction, we need to prove a series of claims as follows.

Claim 61. *If $N_G^+(w') \cap C]t,t'[\neq \emptyset$, then $N_G(w) \cap C]t,t'[= \emptyset$.*

Proof of Claim 61. Let v_i be the out-neighbor of w' in $G \cap C]t,t'[$ such that i is minimal. We are going to show now that $|N_G^+(w) \cap C]t,t'[| = 0$. Suppose not and let $v_j \in |N_G^+(w) \cap C]t,t'[$. If $i \leqslant j$, then the union of $C[w',v]$, $(w',v_i) \cup C[v_i,v_j]$, $C[u,w] \cup (w,v_j)$ and (u,v) is a S-$C(k,1,k,1)$ in D, a contradiction. Otherwise, the union of $C[w',v]$, (w',v_i), $C[u,w] \cup (w,v_j) \cup C[v_j,v_i]$ and (u,v) is a S-$C(k,1,k,1)$, a contradiction. This proves that $N_G^+(w) \cap C]t,t'[= \phi$. Now we shall show that $|N_G^-(w) \cap C]t,t'[| = 0$. Suppose not and let $v_j \in |N_G^-(w) \cap C]t,t'[$. If $i \leqslant j$, then the union of $C[w',v]$, $(w',v_i) \cup C[v_i,v_j] \cup (v_j,w)$, $C[u,w]$ and (u,v) is a S-$C(k,1,k,1)$, a contradiction. Otherwise, we are going to argue on the neighbors of w' in G. First, consider the directed cycle $C[w,v_j] \cup (v_j,w)$. Since $|V(C[v_j,w])| \geqslant |V(C[v_j,v_i])| + |V(C[t',u])| + |V(C[u,w])| \geqslant 1 + (k-1) + (k+1) = 2k+1$, then w' has at most 2 neighbors in $C]v_{j+k-1}, u[\cap G$, due to Lemma 59. Consequently, it follows that w' has at most $k+1$ neighbors in $C]v_j, u[\cap G$. Moreover, w' has no neighbors in $C]t, v_j] \cap G$. In fact, by the choice of the out-neighbor v_i of w', it is clear to see that w' has no out-neighbors in $C]t, v_j] \cap G$. Also, w' has no in-neighbors in $C]t, v_j] \cap G$, since otherwise the union of $(z, w') \cup C[w', v]$, $C[z, v_j] \cup (v_j, w)$, $C[u, w]$ and (u, v) is a S-$C(k, 1, k, 1)$ in D, with z is an in-neighbor of w' in $C]t, v_j] \cap G$. This is a contradiction to the fact that D is $C(k, 1, k, 1)$-subdivision-free. Furthermore, by the choice of the edge uv, note that w' has at most $3k-1$ neighbors in $C[u, w'] \cap G$. This together with the fact that $|V(C[w', v]) \cap V(G)| = k+1$ imply that w' has at most $4k-1$ neighbors in $C[u, v] \cap G$. Therefore, according of all what precedes, we get

$$\begin{aligned}
&d_G(w') = |N_G(w') \cap C[u, v]| + |N_G(w') \cap C]v, t]| + |N_G(w') \cap C]t, v_j]| + |N_G(w') \cap C]v_j + u[| \\
&\leqslant (4k-1) + (k-1) + 0 + (k+1) \\
&= 6k-1,
\end{aligned}$$

a contradiction to the fact that $\delta(G) \geqslant 6k$, affirming our claim. ⧫

Claim 62. $|N_G^+(w') \cap C]t, t'[| = 0$.

Proof of Claim 62. Suppose to the contrary that w' has an out-neighbor in $G \cap C]t, t'[$. Thus, according to Claim 61, we get that $|N_G^+(w) \cap C]t, t'[| = 0$. Hence,

$$\begin{aligned}
&d_G(w) = |N_G(w) \cap C[u, w[| + |N_G(w) \cap C]w, v]| + |N_G(w) \cap C]v, t]| + |N_G(w) \cap C]t, t'[| + \\
&|N_G(v) \cap C[t', u[| \\
&\leqslant k + (3k-1) + (k-1) + 0 + (k-1) \\
&= 6k-3,
\end{aligned}$$

a contradiction to the fact that $\delta(G) \geqslant 6k$. This proves our claim. ⧫

Claim 63. *If $N_H^-(w') \cap C]t, t'[\neq \emptyset$, then $N_H^-(w) \cap C]t, t'[= \emptyset$.*

Proof of Claim 63. Suppose the contrary is true and let $v_i \in |N_H^-(w) \cap C]t, t'[$ such that i is minimal. Let v_j be an in-neighbor of w' in $H \cap C]t, t'[$. If $i \leqslant j$, then the union of $C[v_i, v_j] \cup (v_j, w') \cup C[w', v]$, (v_i, w), $C[u, w]$ and (u, v) is a S-$C(k, 1, k, 1)$ in D, a contradiction. Otherwise, the union of $(v_j, w') \cup C[w', v]$, $C[v_j, v_i] \cup (v_i, w)$, $C[u, w]$ and (u, v) is a S-$C(k, 1, k, 1)$ in D, a contradiction. This confirms Claim 63. ⧫

Claim 64. *If $|N_H^-(w') \cap C]t, t'[| \geqslant 2$, then $N_H^+(w) \cap C]t, t'[= \emptyset$.*

Proof of Claim 64. Since $|N_H^-(w') \cap C]t, t'[| \geqslant 2$, then there exist two distinct vertices z and z' in $N_H^-(w') \cap C]t, t'[$. Assume that z, z' are chosen so that $z = v_i$ such that i is minimal and $z' = v_j$ for some $j > i$. Suppose now that $N_H^+(w) \cap C]t, t'[\neq \emptyset$ and let v_p be an out-neighbor of w in $C]t, t'[\cap H$. If $p \geqslant j$, then the union of $(v_i, w') \cup C[w', v]$, $C[v_i, v_p]$, $C[u, w] \cup (w, v_p)$ and (u, v) is a S-$C(k, 1, k, 1)$ in D, a contradiction. Otherwise, the union of $(v_j, w') \cup C[w', v]$, $C[v_j, u] \cup (u, v) \cup C[v, v_p]$, $C[w, w']$ and (w, v_p) forms a S-$C(k, 1, k, 1)$, a contradiction. This proves Claim 64. ⧫

To complete the proof, we are going to prove that w' has at most one in-neighbor in $C]t, t'[\cap H$. Suppose not, then Claim 63 and Claim 64 imply that w has no neighbors in

$C]t, t'[\cap G$. Hence,

$d_G(w) = |N_G(w) \cap C[u, w[| + |N_G(w) \cap C]w, v]| + |N_G(w) \cap C]v, t]| + |N_G(w) \cap C]t, t'[| + |N_G(v) \cap C[t', u[|$
$\leqslant k + (3k-1) + (k-1) + 0 + (k-1)$
$= 6k - 3,$

a contradiction to the fact that $\delta(G) \geqslant 6k$. Thus, $|N_H^-(w') \cap C]t, t'[| \leqslant 1$. Consequently, according to Claim 63, $|N_G(w') \cap C]t, t'[| \leqslant 1$. Therefore,

$$\begin{aligned} d_G(w') &= |N_G(w') \cap C[u, v]| + |N_G(w') \cap C]v, t]| + |N_G(w') \cap C]t, t'[| + |N_G(w') \cap C[t', u[| \\ &\leqslant (4k-1) + (k-1) + 1 + (k-1) \\ &= 6k - 2, \end{aligned}$$

a contradiction to the fact that $\delta(G) \geqslant 6k$. This completes the proof. □

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