Shotgun Assembly of the Linial–Meshulam model

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Submitted: Nov 28, 2023; Accepted: Aug 8, 2025; Published: Sep 5, 2025

Abstract

In a recent paper, J. Gaudio and E. Mossel studied the shotgun assembly of the Erdős-Rényi graph $\mathcal{G}(n,p)$ with $p = n^{-\alpha}$, and showed that the graph is reconstructable from its 1-neighbourhoods with high probability if $0 < \alpha < 1/3$ and not reconstructable from its 1-neighbourhoods with high probability if $1/2 < \alpha < 1$. In this article, we generalise the notion of reconstruction of graphs to the reconstruction of simplicial complexes. We show that the Linial–Meshulam model $Y_d(n,p)$ on n vertices with $p = n^{-\alpha}$ is reconstructable from its 1-neighbourhoods with high probability when $0 < \alpha < 1/3$ and is not reconstructable from its 1-neighbourhoods with high probability when $1/2 < \alpha < 1$.

Mathematics Subject Classifications: 05C80, 60C05

1 Introduction

In [21], Mossel and Ross introduced the shotgun assembly of graphs. The shotgun assembly of a graph means reconstructing the graph from a collection of vertex neighbourhoods. The motivation comes from DNA shotgun assembly (determining a DNA sequence from multiple short nucleobase chains), reconstruction of neural networks (reconstructing a big neural network from subnetworks), and the random jigsaw puzzle problem. See [20] and references therein.

The recent development of random jigsaw puzzles can be found in [1] and [18]. The graph shotgun assembly for various models was studied extensively. For example, random regular graphs and labelled graphs were considered in [21] and [20], respectively. The reconstruction of the Erdős–Rényi graph is well studied.

The Erdős–Rényi graph [3] [4], denoted by $\mathcal{G}(n,p)$, is a random graph on n vertices, where each edge is added independently with probability $p \in [0,1]$. In [6], Gaudio and Mossel showed that $\mathcal{G}(n,p)$ with $p = n^{-\alpha}$ is reconstructable with high probability if

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$0 < \alpha < 1/3$ and not reconstructable from its 1-neighbourhoods with high probability if $1/2 < \alpha < 1$. Later, Huang and Tikhomirov showed that $\mathcal{G}(n,p)$ with $p = n^{-\alpha}$ is reconstructable with high probability if $0 < \alpha < 1/2$ and not reconstructable from its 1-neighbourhoods with high probability if $1/2 < \alpha < 1$ [11]. The reconstructions of $\mathcal{G}(n,p)$ from its 3 and 2-neighbourhoods are considered in [20, Theorem 4.5] and [6, Theorem 4] respectively. In [12], the authors have recently determined the exact thresholds for r-reconstructibility for $r \geqslant 3$, which also improves and generalises the known results for $r = 3$.

In this article, by generalising the notion of graph shotgun assembly, we introduce the notion of shotgun assembly of simplicial complexes. See Section 2 for a precise definition. The problem of shotgun assembly essentially tells us whether the local structure contains all the information about its global structure. Here we only consider the shotgun assembly problem for the Linial–Meshulam model. However, our notion of shotgun assembly of simplicial complexes can potentially be used for other simplicial complexes, for example, the multi-parameter random simplicial complexes [2, 5].

The Linial–Meshulam model [14], denoted by $Y_d(n,p)$, is a random d-dimensional simplicial complex on n vertices with a complete $(d-1)$-skeleton, in which each d-dimensional simplex is added independently with probability $p \in [0,1]$. See Section 2 for details. In [14], Linial and Meshulam introduced this model for $d = 2$. Later, it was extended for $d \geqslant 3$ by Meshulam and Wallach [19]. After that this model has been studied extensively, for example see [10, 7, 15, 13, 9, 22, 16, 8, 17]. Observe that $Y_1(n,p) = \mathcal{G}(n,p)$. In other words, $d = 1$ gives the Erdős–Rényi graph.

We show that $Y_d(n,p)$ for any $d \in \mathbb{N}$ with $p = n^{-\alpha}$ is reconstructable with high probability if $0 < \alpha < 1/3$ and not reconstructable with high probability if $1/2 < \alpha < 1$ from its 1-neighbourhoods. See Theorems 1 and 3. The meaning of reconstruction of a simplicial complex from its 1-neighbourhoods is given in Section 2. We believe that the range $0 < \alpha < 1/3$ of reconstruction is not optimal, the optimal range should be $0 < \alpha < 1/2$. See Conjecture 2.

The outline of the proof in the graph case and the one presented here for the Linial–Meshulam case are largely similar, but there are several important differences, particularly in terms of computational complexities and new insights. For instance, in the proof of Theorem 1, Lemma 6 plays a crucial role, and the idea of conditioning on the support of the simplicial complex represents a novel approach. Additionally, the proof of Lemma 6 is both new and non-trivial. Similarly, the proof of Lemma 9, which is used in the proof of Theorem 3, differs slightly in its computational approach from the version employed in the graph case by Gaudio and Mossel.

The rest of the article is organized as follows. In Section 2 we introduce the definition of the reconstruction of simplicial complexes, and relevant notation. The main two results are stated in Section 3. The proofs of Theorems 1 and 3 are given in Sections 4 and 5 respectively.

2 Preliminaries

Let X_0 be a finite set. A *finite abstract simplicial complex* X on X_0 is a collection of nonempty subsets of X_0 such that

1. $\{x\} \in X$ for all $x \in X_0$, and
2. if $\tau \subset \sigma$ and $\sigma \in X$ then $\tau \in X$.

For example, $X = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$ is an abstract simplicial complex on $\{1,2,3\}$. Note that, throughout this paper, the empty set is always excluded from the definition of a simplicial complex. We call a set $\sigma \in X$ with $|\sigma| = k+1$ a k-dimensional simplex (or simply k-simplex). For any finite set A, the notation $|A|$ will denote the number of elements in A. For ease of writing, we write *complex* instead of abstract simplicial complex in the rest of the article. The maximum of the dimensions of all simplices in X is called the dimension of complex X, denoted by $\dim(X)$. That is,

$$\dim(X) := \max\{\dim(\sigma) \ : \ \sigma \in X\},$$

where $\dim(\sigma) = |\sigma| - 1$. Note if $\dim(X) = 1$ then X can be viewed as a graph.

For $0 \leqslant j \leqslant \dim(X)$, the set of all j-simplices of X is denoted by

$$X_j := \{\sigma \in X \ : \ \dim(\sigma) = j\}.$$

Note that, with abuse of notation, X_0 denotes the collection of 0-simplices as well as the set of vertices. We say $\sigma, \sigma' \in X_j$ are neighbour if $\sigma \cup \sigma' \in X_{j+1}$. Then we write $\sigma \sim \sigma'$. A similar notion was introduced in [22]. We say the distance of $\sigma, \sigma' \in X_j$ is $k \in \mathbb{N} \cup \{0\}$ if k is the least possible number such that there exist $\sigma_0, \sigma_1 \dots, \sigma_k \in X_j$ with $\sigma = \sigma_0$ and $\sigma_k = \sigma'$ such that $\sigma_i \sim \sigma_{i+1}$ for $0 \leqslant i \leqslant k-1$. Then we write $\operatorname{dist}(\sigma, \sigma') = k$. For $\sigma \in X_j$, we define

$$X_{\sigma,k} := \{\sigma' \in X_j \ : \ \operatorname{dist}(\sigma, \sigma') \leqslant k\},$$

the set of all j-simplices which are within distance at most k from σ. For example see Figure 1. Clearly $\sigma \in X_{\sigma,k}$ for all $k \geqslant 0$. Note that if $k = 0$ or $\dim(\sigma) = \dim(X)$ then $X_{\sigma,k} = \{\sigma\}$, as there is no $\sigma'(\neq \sigma) \in X$ such that $\sigma' \sim \sigma$. Thus $k = 0$ and $\dim(\sigma) = \dim(X)$ are two trivial cases.

Let $k \geqslant 1$ and $j < \dim(X)$. The *k-neighbourhood* of $\sigma \in X_j$ is the $(j+1)$-subcomplex induced by $X_{\sigma,k}$, denoted by $N_{k,X}(\sigma)$. That is,

$$N_{k,X}(\sigma) := \{\tau \in X \ : \ \tau \subseteq \sigma' \cup \sigma'' \text{ for some } \sigma', \sigma'' \in X_{\sigma,k}\}.$$

For example see Figure 1. In particular, if $\dim(X) = 1$ and $v \in X_0$ then $N_{1,X}(v)$ refers to the sub-graph induced by v and its neighbours $\{w \in X_0 \ : \ v \sim w\}$. Note that $N_{k,X}(\sigma)$ may contain j-simplices that are not contained in any $(j+1)$-simplices and may be disconnected under the relation $\sim$.

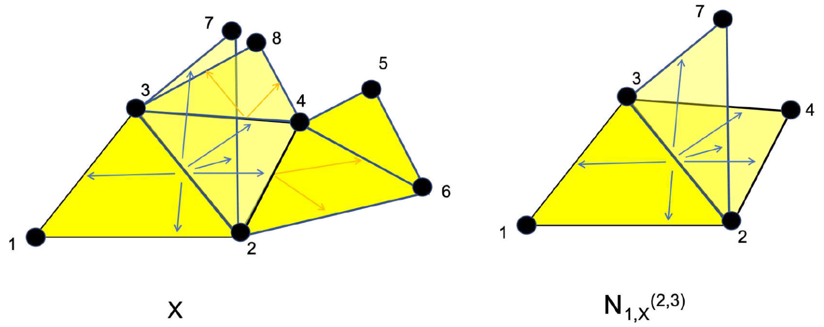

Figure 1: Here $d = 2$. For $\sigma = (2,3)$, for $k = 1, 2$,
$X_{\sigma,1} = \{(2,3),(1,2),(2,4),(2,7),(1,3),(3,4),(3,7),(3,8)\}$,
$X_{\sigma,2} = \{(2,3),(1,2),(2,4),(2,6),(2,7),(1,3),(3,4),(3,7),(3,8),(4,6),(4,8)\}$,
$N_{1,X}(\sigma) = \{$all subsets of simplices $(1,2,3),(2,3,4),(2,3,7)\}$ and
$N_{2,X}(\sigma) = \{$all subsets of simplices $(1,2,3),(2,3,4),(2,3,7),(2,4,6),(3,4,8)\}$

We say two complexes X and Y (on X_0 and Y_0 respectively) are *isomorphic* (denoted by $X \simeq Y$) if there exists a bijective function $f : X_0 \to Y_0$ such that

$$\{x_0, x_1, \ldots, x_k\} \in X \iff \{f(x_0), f(x_1), \ldots, f(x_k)\} \in Y,$$

for $0 \leqslant k \leqslant \dim(X)$ and for all $x_0, x_1, \ldots, x_k \in X_0$. It is clear that if $X \simeq Y$ then $|X_0| = |Y_0|$ and $\dim(X) = \dim(Y)$. If $\dim(X) = 1$ then $X \simeq Y$ means the two graphs X and Y are isomorphic.

We say two complexes X and $\widetilde{X}$ on X_0, with $\sigma \in X$ if and only if $\sigma \in \tilde{X}$, have same k-neighbourhoods if

$$N_{k,X}(\sigma) \simeq N_{k,\widetilde{X}}(\sigma) \text{ for all } \sigma \in X, \tag{1}$$

that is, the k-neighbourhoods of all simplices in both complexes are isomorphic. In this case we write $X \simeq_k \widetilde{X}$. Observe that if $\dim(\sigma) = \dim(X)$ then (1) holds trivially. In particular, if $\dim(X) = 1$ then $X \simeq_k \widetilde{X}$ implies that the k-neighbourhoods of $v \in X_0$ in X and $\widetilde{X}$ are isomorphic as graphs.

A complex X on X_0 is said to be *k-reconstructable* (in short, reconstructable) up to isomorphism from its k-neighbourhoods if $X \simeq_k \widetilde{X}$ implies $X \simeq \widetilde{X}$, for all complexes $\widetilde{X}$ on X_0. Further, we say X is *exactly reconstructable* if $X \simeq \widetilde{X}$ implies $X = \widetilde{X}$. We study whether the Linial–Meshulam model is reconstructable from its 1-neighbourhoods. The Linial–Meshulam model is a random complex of the form (2) (see below).

In the rest of the article, for $d \in \mathbb{N}$, the complex will be of the form

$$X := \left(\bigcup_{k=0}^{d-1} X_k \right) \cup X^d, \tag{2}$$

where $X_0 := \{1, 2, \ldots, n\}$, $X_k := \{\{i_0, \ldots, i_k\} : 1 \leqslant i_0 < \cdots < i_k \leqslant n\}$, for $1 \leqslant k \leqslant d$, and $X^d \subseteq X_d$. Note that X_k denotes the set of all k-simplices on X_0. In this model, X is a complex with complete $d-1$ dimensional skeleton. Observe that X is a graph when $d = 1$.

Note that if two complexes X and $\widetilde{X}$ on X_0 are of the form (2) then

$$N_{k,X}(\sigma) = N_{k,\widetilde{X}}(\sigma), \quad \text{whenever } \dim(\sigma) \leqslant d-2.$$

The neighbourhoods can differ only if $\dim(\sigma) = d-1$. Thus, in this case, the collection of k-neighbourhoods of X will be denoted by

$$\mathcal{N}_k(X) := \{N_{k,X}(\sigma) \ : \ \sigma \in X_{d-1}\}. \tag{3}$$

We say a complex X of the form (2) is reconstructable from its k-neighbourhoods if, for all $\widetilde{X}$ of the form (2),

$$X \simeq \widetilde{X} \text{ whenever } N_{k,X}(\sigma) \simeq N_{k,\widetilde{X}}(\sigma) \text{ for all } \sigma \in X_{d-1}.$$

Similarly, we say X is exactly reconstructable from its k-neighbourhoods if $X = \widetilde{X}$ whenever $N_{k,X}(\sigma) \simeq N_{k,\widetilde{X}}(\sigma)$ for all $\sigma \in X_{d-1}$. Figure 2 gives an example of a complex which is reconstructable but not exactly reconstructable.

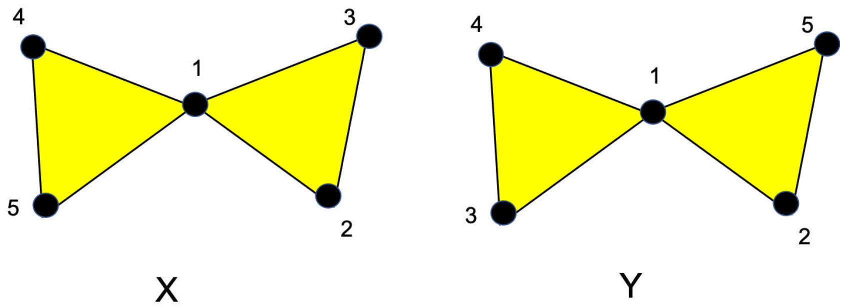

Figure 2: Here X is reconstructable from its 1-neighbourhoods but not exactly reconstructable because $X \simeq Y$ but $X \neq Y$.

The *degree* of a simplex $\sigma \in X_{d-1}$ is denoted by

$$\deg(\sigma) = \deg_X(\sigma) := \sum_{\tau \in X^d} \mathbf{1}_{\{\sigma \subset \tau\}},$$

the number of d-simplices containing σ. Observe that a $\tau \in X^d$ will contribute non-zero value in the last equation if $\tau = \sigma \cup \{v\}$ for some $v \in X_0 \backslash \sigma$. The set of neighbours of $\sigma \in X_{d-1}$ is denoted by S_σ, that is,

$$S_\sigma = \{\sigma' \in X_{d-1} \ : \ \sigma' \sim \sigma\}.$$

Note that the number of elements in S_σ is d times $\deg(\sigma)$, that is,

$$|S_\sigma| = d\deg(\sigma).$$

For an example see Figure 3.

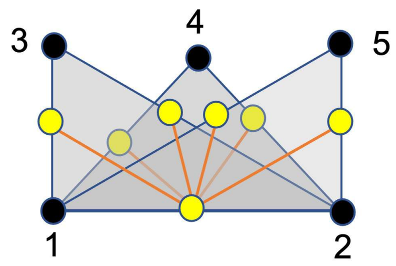

Figure 3: Here $d = 2$, $\deg_X(1,2) = 3$ and $|S_\sigma| = 6$

Next we recall the Linial–Meshulam model, which is a random simplicial complex $Y_d(n,p)$ on vertex set $\{1,2,\ldots,n\}$ with complete $(d-1)$-dimensional skeleton, and where each d-face appears independently with probability p.

It is easy to see that the degree of $\sigma \in X_{d-1}$ in $Y_d(n,p)$ is a Binomial random variable with parameters $(n-d,p)$, that is, $\deg_{Y_d(n,p)}(\sigma) \sim Bin(n-d,p)$. The use of the notation "$\sim$" will always be clear from the context as the same is also used for two neighbouring simplices.

3 Main Results

In this section we state our main results, and give the key idea of the proofs. Let us define the high probability events. We say a sequence of events A_n occurs *with high probability* if

$$\mathbf{P}(A_n^c) = o\left(\frac{1}{n^s}\right),$$

for some $s > 0$. We write $a_n = o(b_n)$ for two sequence of numbers $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ if $|a_n/b_n| \to 0$ as $n \to \infty$.

In [6], it was shown that the Erdős-Rényi graph $Y_1(n,p)$ with $p = n^{-\alpha}$ can be exactly reconstructed from its 1-neighbourhoods with high probability when $0 < \alpha < 1/3$. We extend this result to $d \in \mathbb{N}$.

Theorem 1. *The Linial–Meshulam model $Y_d(n,p)$ where $p = n^{-\alpha}$ for $0 < \alpha < 1/3$, is exactly reconstructable from its 1-neighbourhoods with high probability.*

A rough idea of the proof of Theorem 1 is the following. The Fingerprint Lemma (Lemma 4) establishes a sufficient condition for reconstructability. A union bound is taken over all events that cause the sufficient condition to fail. A failing event in our case is one subcomplex being embeddable in another. In particular, for the graph case (for

$d = 1$), a failing event is one subgraph being embeddable in another graph. We don't think the range $0 < \alpha < 1/3$ is optimal for the reconstruction of $Y_d(n, p)$. We have the following conjecture.

Conjecture 2. The Linial–Meshulam model $Y_d(n, p)$ with $p = n^{-\alpha}$ is exactly reconstructable from its 1-neighbourhoods with high probability if $0 < \alpha < 1/2$.

One can try to prove Conjecture 2 using the method used in [11]. Another direction of work would be considering the reconstruction problem from its 2-neighbourhoods using the method used in [6]. These remain for future works.

The next result is about non-reconstructibility of $Y_d(n, p)$. For $d = 1$, the graph $\mathcal{G}(n, p) \equiv Y_1(n, p)$ is non-reconstructible from its 1-neighbourhoods with high probability when $1/2 < \alpha < 1$. We show that the same result holds for all $d \geqslant 1$.

Theorem 3. *The Linial–Meshulam model $Y_d(n, p)$ where $p = n^{-\alpha}$ for $1/2 < \alpha < 1$, cannot be reconstructed from its 1-neighbourhoods with high probability.*

The failure of reconstructability in the above mentioned regime of p is established by a counting argument. The counting argument treats the collection of neighborhoods as the input to a given algorithm. Intuitively, the argument shows that the number of inputs is far smaller than the number of graphs generating those inputs. Some care is taken to account for the fact that the inputs are not uniformly distributed, although conditioning on the number of random simplices yields a uniform distribution. The fact that there are many more inputs than possible outputs implies that the probability that the algorithm outputs the correct complex vanishes asymptotically.

4 Proof of Theorem 1

In this section we prove Theorem 1. The following lemmas will be used in the proof. Throughout we use $p = n^{-\alpha}$, where $0 < \alpha < 1$.

We first state a generalization of the fingerprint lemma [6, Lemma 2]. Let $\sigma_1, \sigma_2 \in X_{d-1}$. We say there is an edge between σ_1 and σ_2 in X, denoted by (σ_1, σ_2), if $\sigma_1 \sim \sigma_2$ in X. For $\sigma_1 \sim \sigma_2$, $H_{\sigma_1,\sigma_2}(X)$ denotes the subcomplex induced by the simplices of $S_{\sigma_1} \cap S_{\sigma_2}$, that is,

$$H_{\sigma_1,\sigma_2} = H_{\sigma_1,\sigma_2}(X) := \{\tau \in X \ : \ \tau \subseteq \sigma \cup \sigma' \text{ for some } \sigma, \sigma' \in S_{\sigma_1} \cap S_{\sigma_2}\}.$$

See Figure 4 for an example. If $\dim(X) = 1$ and $v_1, v_2 \in X_0$ such that $v_1 \sim v_2$ then H_{v_1,v_2} is the subgraph induced by the common neighbours of v_1 and v_2. Two unordered edges (σ_1, σ_2) and (σ_3, σ_4) are said to be equal, denoted by $(\sigma_1, \sigma_2) = (\sigma_3, \sigma_4)$, if either $\sigma_1 = \sigma_3, \sigma_2 = \sigma_4$ or $\sigma_1 = \sigma_4, \sigma_2 = \sigma_3$. It is clear that if $(\sigma_1, \sigma_2) = (\sigma_3, \sigma_4)$ then $H_{\sigma_1,\sigma_2} \simeq H_{\sigma_3,\sigma_4}$.

Lemma 4 (Fingerprint Lemma). *Let X be a complex with complete $d-1$-dimensional skeleton. If two edges (σ_1, σ_2) and (σ_3, σ_4) are equal whenever H_{σ_1,σ_2} and H_{σ_3,σ_4} are isomorphic then X can be exactly reconstructed from the collection of its 1-neighbourhoods.*

In the next lemma, we give an upper bound (with high probability) on the number of simplices that are connected with both $\sigma_1, \sigma_2 \in X_{d-1}$.

Lemma 5. *Let $\sigma_1, \sigma_2 \in X_{d-1}$ such that $\sigma_1 \cup \sigma_2 \in X_{d,p}$, that is, $\sigma_1 \sim \sigma_2$ in $Y_d(n,p)$. The number of common neighbours of σ_1 and σ_2 is denoted by*

$$W_{\sigma_1,\sigma_2} := |\{\sigma \in X_{d-1} \,:\, \sigma \sim \sigma_1, \sigma \sim \sigma_2 \text{ in } Y_d(n,p)\}|.$$

Then there exists a positive constant C such that

$$\mathbf{P}(W_{\sigma_1,\sigma_2} \geqslant d - 1 + n^c(n-d-1)p^2) \leqslant \exp(-Cn^{1+c-2\alpha}). \tag{4}$$

In particular, if $c > 2\alpha - 1$, we obtain $W_{\sigma_1,\sigma_2} - d + 1 \leqslant n^{1+c-2\alpha}$ with high probability.

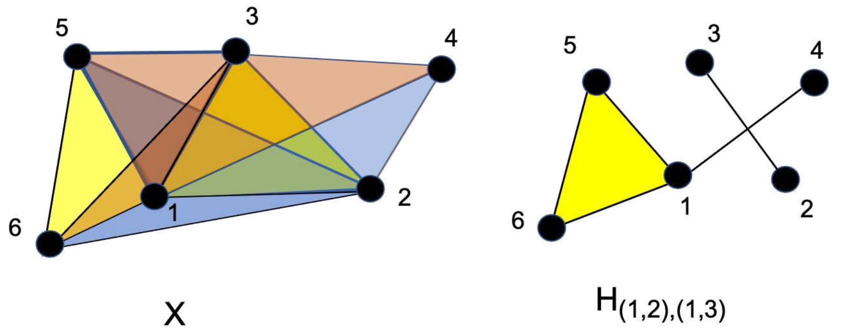

Figure 4: $W_{(1,2),(1,3)} = |\{(1,4),(1,5),(1,6),(2,3)\}| = 4$.

In the next lemma, for $\sigma_1 \sim \sigma_2$ and $\sigma_3 \sim \sigma_4$, we derive a lower bound (with high probability) on the number of simplices that are connected only with $\sigma_1, \sigma_2 \in X_{d-1}$, not with σ_3, σ_4. We write $a_n = \Theta(b_n)$ if there exist $C_1, C_2 > 0$ such that $C_1 b_n \leqslant a_n \leqslant C_2 b_n$ for all large n.

Lemma 6. *Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in X_{d-1}$ such that $\sigma_1 \sim \sigma_2$ and $\sigma_3 \sim \sigma_4$. Define*

$$S = S_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} := \{\sigma \in X_{d-1} \,:\, \sigma \sim \sigma_i \text{ for } i = 1,2,3,4\},$$
$$Z = Z_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} := \mathbf{1}\{\sigma_1 \sim \sigma_3, \sigma_1 \sim \sigma_4\} + \mathbf{1}\{\sigma_2 \sim \sigma_3, \sigma_2 \sim \sigma_4\}.$$

If $(1-2\alpha) > 0$ then there exists $c > 0$ such that, for large n,

$$P\left(W_{\sigma_1,\sigma_2} - |S| - Z \leqslant \frac{1}{2}np^2\right) \leqslant \exp(-cn^{1-2\alpha}). \tag{5}$$

The proofs of Lemmas 5 and 6 are given at the end of this section. We note down [6, Lemma 3, Lemma 4] which will be used in the proofs.

Lemma 7. *[Chernoff's bound] Let $T_1, T_2, \dots, T_n$ be independent indicator random variables and call $T = \sum_{i=1}^{n} T_i$. Then for any $\delta > 0$,*

$$\mathbf{P}(T \leqslant (1-\delta)\mathbf{E}(T)) \leqslant \exp\left(-\frac{\delta^2}{2}\mathbf{E}(T)\right) \text{ and}$$

$$\mathbf{P}(T \geqslant (1+\delta)\mathbf{E}(T)) \leqslant \exp\left(-\frac{\delta^2}{2+\delta}\mathbf{E}(T)\right).$$

Lemma 8 (Gaudio and Mossel)**.** *[6, Lemma 1.11] Let X and Y be random variables such that conditioned on Y, $X \sim Bin(Y, p)$. Let $Z(m) \sim Bin(m, p)$. Then*

$$\mathbf{P}\left(X \leqslant t_1 \,\middle|\, Y \geqslant t_2\right) \leqslant \mathbf{P}(Z(t_2) \leqslant t_1\Big) \text{ and } \mathbf{P}\left(X \geqslant t_1 \,\middle|\, Y \leqslant t_2\right) \leqslant \mathbf{P}(Z(t_2) \geqslant t_1\Big).$$

Now we proceed to prove Theorem 1.

Proof of Theorem 1. Let $\sigma_1, \sigma_2 \in X_{d-1}$ such that $\sigma_1 \sim \sigma_2$. Recall that H_{σ_1,σ_2} denotes the subcomplex induced by the vertices of $S_{\sigma_1} \cap S_{\sigma_2}$ in $Y_d(n,p)$. Note that the subcomplex H_{σ_1,σ_2} is random. For $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in X_{d-1}$ such that $\sigma_1 \sim \sigma_2$, $\sigma_3 \sim \sigma_4$, we show that, for some $s > 0$,

$$\mathbf{P}(H_{\sigma_1,\sigma_2} \simeq H_{\sigma_3,\sigma_4}) = o(e^{-n^s}) \text{ whenever } (\sigma_1,\sigma_2) \neq (\sigma_3,\sigma_4). \tag{6}$$

Then the result follows by combining Lemma 4, (6) followed by an union bound argument. It remains to prove (6).

Let S be as defined in Lemma 6 and Y_1 be the subcomplex induced by the simplices of $S_{\sigma_1} \cap S_{\sigma_2} \backslash (S \cup \{\sigma_3, \sigma_4\})$, the shared neighbours of σ_1 and σ_2 (excluding σ_3 and σ_4) that are not neighbours of both σ_3 and σ_4. Let Y_2 be the subcomplex induced by the simplices of $S_{\sigma_3} \cap S_{\sigma_4}$. Note that Y_1 and Y_2 are disjoint by construction.

Observe that if $H_{\sigma_1,\sigma_2} \simeq H_{\sigma_3,\sigma_4}$ then $W_{\sigma_1,\sigma_2} = W_{\sigma_3,\sigma_4}$ and Y_1 can be embedded into Y_2 as a subcomplex of Y_2 (we write $Y_1 \subset Y_2$ with the abuse of notation). Thus

$$\mathbf{P}(H_{\sigma_1,\sigma_2} \simeq H_{\sigma_3,\sigma_4}) \leqslant \mathbf{P}(Y_1 \subset Y_2).$$

We claim that there exists $\max\{0, 2\alpha - 1\} < c < 1 - 3\alpha$ and $a, b, C > 0$ such that

$$P\left(Y_1 \subset Y_2\right) \leqslant n^4 (n^{an^{1+c-2\alpha} - bn^{2-5\alpha}} + \exp(-Cn^{1+c-2\alpha})). \tag{7}$$

The right hand side of the above equation will go to zero if $2 - 5\alpha > 1 + c - 2\alpha$, which is equivalent to say that $c < 1 - 3\alpha$. This is a consistent condition if $\alpha < \frac{1}{3}$. Applying a union bound,

$$\begin{aligned}
&\mathbf{P}\left(\exists \sigma_1 \sim \sigma_2, \sigma_3 \sim \sigma_4 \ : \ H_{\sigma_1,\sigma_2} \simeq H_{\sigma_3,\sigma_4}\right) \\
&\leqslant n^{4d} \mathbf{P}\left(H_{\sigma_1,\sigma_2} \simeq H_{\sigma_3,\sigma_4}\right) \\
&\leqslant n^{4(d+1)} (n^{an^{1+c-2\alpha} - bn^{2-5\alpha}} + \exp(-Cn^{1+c-2\alpha})) \\
&= o(n^{-s}),
\end{aligned}$$

for any $s > 0$ as $\alpha < 1/3$. Thus, for any $(\sigma_1, \sigma_2) \neq (\sigma_3, \sigma_4)$, we have $H_{\sigma_1,\sigma_2} \not\simeq H_{\sigma_3,\sigma_4}$ with high probability if $\alpha < 1/3$. This proves result.

The rest of the proof is dedicated to prove the claim stated in (7). We have

$$\begin{aligned}&\mathbf{P}(Y_1 \subset Y_2)\\ &\leqslant \sum_{\lambda,\mu,k} \mathbf{P}(\{Y_1 \subset Y_2 \ : \ W_{\sigma_1,\sigma_2} = W_{\sigma_3,\sigma_4} = \lambda + Z, |S| = \mu, |\operatorname{Supp}_d(Y_1^{d-1})| = k\}),\end{aligned}$$

where λ is the number of elements in the set $S_{\sigma_1} \cap S_{\sigma_2} \setminus \{\sigma_3, \sigma_4\}$, Z as in Lemma 6 and $\operatorname{Supp}_d(A) = \{\sigma_1 \cup \sigma_2 \in X_{d,p} | \sigma_1, \sigma_2 \in A\}$ for $A \subseteq X_{d-1}$. Note that, given $|S| = \mu$, at most $2\mu + 1$ d-simplices are revealed in $X_{d,p}$ because σ_1 and σ_2 are connected and each $\sigma \in S$ is connected with both σ_1 and σ_2. Therefore

$$\begin{aligned}&P\left(Y_1 \subset Y_2 \ : \ W_{\sigma_1,\sigma_2} = W_{\sigma_3,\sigma_4} = \lambda + Z, |S| = \mu, |\operatorname{Supp}_d(Y_1^{d-1}|) = k\right)\\ \leqslant&\binom{\lambda+2}{\lambda-\mu}(\lambda-\mu)! p^{k-2\mu-1}\\ \leqslant&(\lambda+2)^{\lambda-\mu}\left(n^{-\alpha}\right)^{k-2\lambda-1},\end{aligned}$$

as $\mu \leqslant \lambda$ and $p = n^{-\alpha}$. The first inequality follows from the fact that $\lambda + Z \leqslant \lambda + 2$, so we can choose $(\lambda - \mu)$ simplices in at most $\binom{\lambda+2}{\lambda-\mu}$ ways and their numbering is possible in at most $(\lambda - \mu)!$ ways. The factor $p^{k-2\mu-1}$ appears because we need to add at least $(k - 2\mu - 1)$ many d-simplices in the complex. Since $\lambda - \mu \leqslant \lambda$ we have

$$\begin{aligned}&P\left(Y_1 \subset Y_2 \ : \ W_{\sigma_1,\sigma_2} = W_{\sigma_3,\sigma_4} = \lambda + Z, |S| = \mu, |\operatorname{Supp}_d(Y_1^{d-1}|) = k\right)\\ &\leqslant \exp\left(\lambda \log(n) - \alpha(k - 2\lambda - 1)\log(n)\right)\\ &\leqslant \exp\{((2\alpha + 1)\lambda + \alpha - \alpha k)\log n\}\\ &= n^{(2\alpha+1)\lambda+\alpha-\alpha k}.\end{aligned} \tag{8}$$

Next we complete the proof of (7) using the following two claims, which estimate λ and k respectively,

$$\mathbf{P}(|S_{\sigma_1} \cap S_{\sigma_2} \setminus \{\sigma_3, \sigma_4\}| \leqslant n^{1+c-2\alpha}) \geqslant 1 - \exp(-Cn^{1+c-2\alpha}). \tag{9}$$

$$\mathbf{P}(|\operatorname{Supp}_d(Y_1^{d-1})| \geqslant C_1 n^{2-5\alpha}) \geqslant 1 - \exp(-C_2 n^{2-5\alpha}), \tag{10}$$

where C, C_1 and C_2 are positive constants. Using (9) and (10) from (8) we get

$$\mathbf{P}\left(Y_1 \subset Y_2\right) \leqslant n^4 \cdot n^{an^{1+c-2\alpha} - bn^{2-5\alpha}} + n^4 \exp(-Cn^{1+c-2\alpha}),$$

as $\lambda, \mu \leqslant n$ and $k \leqslant n^2$. This completes the proof of (7). It remains to prove (9) and (10).

Proof of (9)*:* Observe that (9) follows from Lemma 5.

Proof of (10)*:* Clearly, given $W_{\sigma,\sigma'} - |S| - Z$, $|\operatorname{Supp}_d(Y_1^{d-1})| \sim Bin(W_{\sigma,\sigma'} - |S| - Z, p)$. From Lemma 6, for some $c > 0$, we have

$$\mathbf{P}(W_{\sigma,\sigma'} - |S| - Z \geqslant \frac{1}{2}np^2) \geqslant 1 - e^{-cn^{1-2\alpha}}.$$

The right hand side goes to 1 if $1-2\alpha > 0$. Lemma 8 and Lemma 7 imply that

$$\begin{aligned}
&\mathbf{P}\left(|\operatorname{Supp}_d(Y_1^{d-1})| \leqslant (1-\epsilon)p\binom{\frac{1}{2}np^2}{2} \,\middle|\, (W_{\sigma,\sigma'} - |S| - Z) \geqslant \frac{1}{2}np^2\right) \\
&\leqslant \mathbf{P}\left(Bin(\frac{1}{2}np^2, p) \leqslant (1-\epsilon)p\binom{\frac{1}{2}np^2}{2}\right) \\
&\leqslant \exp\left(-\frac{\epsilon^2}{2}p\binom{\frac{1}{2}np^2}{2}\right) \\
&= \exp(-C_2 n^{2-5\alpha}),
\end{aligned}$$

for some positive constant C_2. Thus we have

$$\mathbf{P}(|\operatorname{Supp}_d(Y_1^{d-1})| \geqslant C_1 n^{2-5\alpha}) \geqslant 1 - \exp(-C_2 n^{2-5\alpha}),$$

for some positive constant C_1. This complete the proof. ■

Next we give the proofs of Lemmas 4, 5 and 6. The proof of Lemma 4 can be derived from [6, Lemma 2], for sake of completeness we give a proof.

Proof of Lemma 4. Since X has complete $(d-1)$-dimensional skeleton, in order to reconstruct X it is enough to check whether any two simplices $\sigma_1, \sigma_2 \in X_{d-1}$ $(\sigma_1 \neq \sigma_2)$ are connected in X. To determine that, we examine the neighbourhoods of σ_1, σ_2 by observing the subcomplexes H_{σ_1,σ_3} and H_{σ_2,σ_4} for neighbours $\sigma_1 \sim \sigma_3$ and $\sigma_2 \sim \sigma_4$. The reconstruction algorithm is as follows: We conclude that $\sigma_1 \sim \sigma_2$ in X if there exist $\sigma_3, \sigma_4 \in X_{d-1}$ such that $\sigma_1 \sim \sigma_3$, $\sigma_2 \sim \sigma_4$ and H_{σ_1,σ_3} is isomorphic with H_{σ_2,σ_4}.

Suppose $\sigma_1 \sim \sigma_2$ in X. We choose $\sigma_4 = \sigma_1$ and $\sigma_3 = \sigma_2$. Then, $H_{\sigma_1,\sigma_3} = H_{\sigma_2,\sigma_4} = H_{\sigma_1,\sigma_2}$. Conversely, suppose there are some $\sigma_3, \sigma_4 \in X_{d-1}$ such that $\sigma_1 \sim \sigma_3$, $\sigma_2 \sim \sigma_4$ and H_{σ_1,σ_3} is isomorphic with H_{σ_2,σ_4}. Then, the hypothesis of the lemma says that $(\sigma_1, \sigma_3) = (\sigma_2, \sigma_4)$. Therefore, $\sigma_1 = \sigma_4$ and $\sigma_3 = \sigma_2$ because $\sigma_1 \neq \sigma_2$. Hence, (σ_1, σ_2) is an edge in X or in other words $\sigma_1 \sim \sigma_2$. So, continuing the process described in the algorithm, we recover the complex. ■

Proof of Lemma 5. For $\sigma_1 \sim \sigma_2$, define $S_{\sigma_1,\sigma_2} = S'_{\sigma_1,\sigma_2} \cup S''_{\sigma_1,\sigma_2}$ where

$$\begin{aligned}
S'_{\sigma_1,\sigma_2} &= \{\sigma \in X_{d-1} \ : \ \sigma \sim \sigma_1, \sigma \sim \sigma_2, \sigma \subset \sigma_1 \cup \sigma_2\} \\
S''_{\sigma_1,\sigma_2} &= \{\sigma \in X_{d-1} \ : \ \sigma \sim \sigma_1, \sigma \sim \sigma_2, \sigma \nsubseteq \sigma_1 \cup \sigma_2\}
\end{aligned}$$

Clearly, $W_{\sigma_1,\sigma_2} = |S'_{\sigma_1,\sigma_2}| + |S''_{\sigma_1,\sigma_2}|$. Observe that if $\sigma_1, \sigma_2 \neq \sigma \in X_{d-1}$ such that $\sigma \subset \sigma_1 \cup \sigma_2$ then $\sigma \sim \sigma_1$ and $\sigma \sim \sigma_2$. Therefore

$$|S'_{\sigma_1,\sigma_2}| = d-1. \tag{11}$$

Again $\sigma_1 \sim \sigma_2$ implies that $\sigma_1 \cap \sigma_2 \in X_{d-2}$. Therefore if $\sigma \sim \sigma_1, \sigma_2$ but $\sigma \nsubseteq \sigma_1 \cup \sigma_2$ then σ will be of the form $(\sigma_1 \cap \sigma_2) \cup \{v\}$ for some $v \in X_0 \backslash (\sigma_1 \cup \sigma_2)$. See Figure 5.

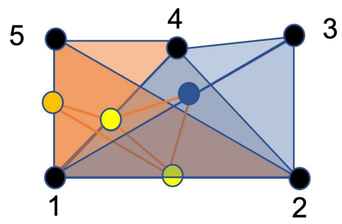

Figure 5: simplices $(1,5)$ and $(1,3)$ are connected with both $(1,4),(1,2)$, as $(125),(145),(134),(123)$ simplices are included in the complex.

Therefore S''_{σ_1,σ_2} can be written as

$$S''_{\sigma_1,\sigma_2} = \{(\sigma_1 \cap \sigma_2) \cup \{v\} \ : \ v \in X_0 \backslash \{\sigma_1 \cup \sigma_2\}, (\sigma_1 \cap \sigma_2) \cup \{v\} \in X_{d,p}\}.$$

Which implies that $|S''_{\sigma_1,\sigma_2}| \sim Bin(n-d-1, p^2)$, as we need to add two simplices to get an element in S''_{σ_1,σ_2}. See Figure 5. Let $c > 0$. Using Lemma 7, we have

$$\begin{aligned}\mathbf{P}(|S''_{\sigma_1,\sigma_2}| \geqslant n^c(n-d-1)p^2) &\leqslant \exp\left(-\frac{(n^c-1)^2}{1+n^c}(n-d-1)n^{-2\alpha}\right) \\ &\leqslant \exp(-Cn^{1+c-2\alpha}), \qquad (12)\end{aligned}$$

for some constant $C > 0$. We get the result by combining (11) and (12). Clearly, the right hand side goes to zero if $c > 2\alpha - 1$. ■

Proof of Lemma 6. Let $\sigma \in S$. Then $\sigma \sim \sigma_1, \sigma_2$ and $\sigma \sim \sigma_3, \sigma_4$. Therefore

$$\sigma = (\sigma_1 \cap \sigma_2) \cup \{v\} \text{ and } \sigma = (\sigma_3 \cap \sigma_4) \cup \{v'\}$$

for some $v, v' \in X_0$. Thus we get the following identity

$$(\sigma_1 \cap \sigma_2) \cup \{v\} = (\sigma_3 \cap \sigma_4) \cup \{v'\}. \qquad (13)$$

Case-I: Suppose $|\cap_{i=1}^4 \sigma_i| = d-1$, that is, $(\sigma_1 \cap \sigma_2) = (\sigma_3 \cap \sigma_4)$. Then any $v = v' \in X_0 \backslash (\cup_{i=1}^4 \sigma_i)$ satisfies (13). Thus

$$\mathbf{P}((\sigma_1 \cap \sigma_2) \cup \{v\} \in S''_{\sigma_1,\sigma_2} \backslash S) = p^2(1-p^2),$$

where S is as defined in Lemma 6. Therefore we get $|S''_{\sigma_1,\sigma_2}| - |S| - Z \sim Bin(n-d-3, p^2(1-p^2))$. Fix $0 < \epsilon < 1/2$. Lemma 7 implies that

$$\mathbf{P}(|S''_{\sigma_1,\sigma_2}| - |S| - Z \leqslant \epsilon(n-d-3)p^2(1-p^2)) \leqslant \exp(-\frac{\epsilon^2}{2}(n-d-3)p^2(1-p^2)).$$

If $1 - 2\alpha > 0$, then the last equation implies that

$$\mathbf{P}(W_{\sigma_1,\sigma_2} - |S| - Z \leqslant \frac{1}{2}np^2) \leqslant \exp(-\Theta(n^{1-2\alpha})), \qquad (14)$$

as $|S'_{\sigma_1,\sigma_2}| = d-1$.

Case-II: Suppose $|\cap_{i=1}^{4} \sigma_i| = d-2$. Then there is only one choice of v, v' which satisfies (13), namely, $v = (\sigma_1 \cap \sigma_2) \backslash (\sigma_3 \cap \sigma_4)$ and $v' = (\sigma_3 \cap \sigma_4) \backslash (\sigma_1 \cap \sigma_2)$. Thus $|S| \leqslant 1$. We have

$$W_{\sigma_1,\sigma_2} - |S| - Z \leqslant W_{\sigma_1,\sigma_2}.$$

Next we give bound on W_{σ_1,σ_2}. We have

$$W_{\sigma_1,\sigma_2} = |S'_{\sigma_1,\sigma_2}| + |S''_{\sigma_1,\sigma_2}|,$$

where S'_{σ_1,σ_2} and S''_{σ_1,σ_2} are as defined in the proof of Lemma 5. Since $|S''_{\sigma_1,\sigma_2}| \sim Bin(n-d-1, p^2)$, by Chernoff's bound (Lemma 7), for $0 < \epsilon < 1/2$,

$$\mathbf{P}(|S''_{\sigma_1,\sigma_2}| \leqslant \epsilon(n-d-1)p^2) \leqslant \exp(-\frac{\epsilon^2}{2}(n-d-1)p^2).$$

If $1-2\alpha > 0$ then the last equation implies that

$$\mathbf{P}(W_{\sigma_1,\sigma_2} \leqslant \frac{1}{2}np^2) \leqslant \exp(-\Theta(n^{1-2\alpha})),$$

as $|S'_{\sigma_1,\sigma_2}| = d-1, |S| \leqslant 1, |Z| \leqslant 2$. Thus we get, for large n,

$$P\left(W_{\sigma_1,\sigma_2} - |S| - Z \leqslant \frac{1}{2}np^2\right) \leqslant \exp(-\Theta(n^{1-2\alpha})). \tag{15}$$

Case-III: Suppose $|\cap_{i=1}^{4} \sigma_i| \leqslant d-3$. Then there is no $v, v' \in X_0$ which satisfies (13). Thus $|S| = 0$. Hence the result follows as in Case II.

Similar analysis can be done when the edges are of the form (σ_1, σ_2) and (σ_2, σ_3). It can be shown that $|S''_{\sigma_1,\sigma_2}| - |S| - Z \sim Bin(n-d-3, p^2(1-p))$ if $|\cap_{i=1}^{3} \sigma_i| = d-1$. Otherwise, $|S| \leqslant 1$. Thus, following the calculation as in Case-I,II, if $1-2\alpha > 0$ we get

$$\mathbf{P}\left(W_{\sigma_1,\sigma_2} - |S| - Z \leqslant \frac{1}{2}np^2\right) \leqslant \exp(-\Theta(n^{1-2\alpha})).$$

We skip the details here. Hence the result. ∎

5 Proof of Theorem 3

This section is dedicated for the proof of Theorem 3. Let X be a complex, where

$$X := \{X_0, \ldots, X_{d-1}, X^d\},$$

and $X^d \subseteq X_d$. Recall S_σ denotes the the set of neighbours of $\sigma \in X_{d-1}$. Define

$$D_\sigma = D_\sigma(S_\sigma) := \{\sigma \cup \sigma' \ : \ \sigma' \in S_\sigma\} \text{ and } \mathrm{Supp}_d(S_\sigma) := \{\sigma_1 \cup \sigma_2 \ : \ \sigma_1, \sigma_2 \in S_\sigma\}.$$

Observe that each simplex in $D_\sigma(S_\sigma)$ contains σ. However, not every simplex in $\mathrm{Supp}_d(S_\sigma)$ contains σ. The set of d-simplices which do not contain σ is denoted by

$$D_\sigma^* = D_\sigma^*(S_\sigma) := \mathrm{Supp}_d(S_\sigma) \backslash D_\sigma(S_\sigma).$$

Observe that $|D_\sigma| = \deg(\sigma)$. The 1-neighbourhood of σ in X can be written as

$$N_{1,X}(\sigma) = (S_\sigma, D_\sigma, D_\sigma^*).$$

An illustration of $N_{1,X}(\sigma), D_\sigma, D_\sigma^*$ is given in Figure 6.

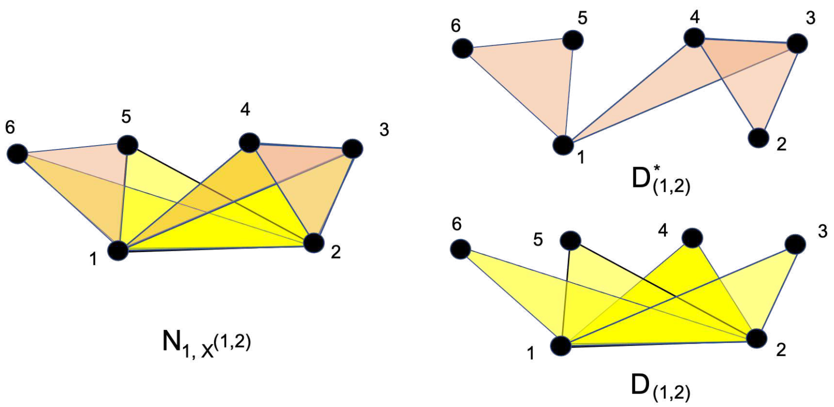

Figure 6: An illustration for $N_{1,X}(\sigma), D_\sigma, D_\sigma^*$.

Fix $0 < \epsilon < 1$ and $q_n = (1+\epsilon)p$. For $c > 0$, suppose $t_n = (1+n^c)p$. Define

$$\mathcal{S} = \left\{ \{N_{1,X}(\sigma), \sigma \in X_{d-1}\} \ : \ |D_\sigma| < nq_n, |D_\sigma^*| < \frac{1}{2}n^2q_n^2t_n, \forall \sigma \in X_{d-1} \right\}, \tag{16}$$

the set of all possible 1-neighbourhood collections where the degree of each central vertex is less than nq_n and each neighbourhood has fewer than $\frac{1}{2}n^2q_n^2t_n$ neighbouring d-faces those are not counted in $\deg(\sigma)$.

Lemma 9. *Let $\mathcal{N}_1(X)$ and $\mathcal{S}$ be as defined in* (3) *and* (16) *respectively. Let $X \in Y_d(n,p)$ with $p = n^{-\alpha}$ and $0 < \alpha < 1$. If $c > 3\alpha - 2$ then*

$$\mathbf{P}(\mathcal{N}_1(X) \in \mathcal{S}) \geqslant 1 - e^{-an^b}$$

for some positive constants a and b.

Lemma 10. *Let $I := \left\{m \in \mathbb{N} : \left|m - \binom{n}{d+1}p\right| < \epsilon\binom{n}{d+1}p\right\}$, where $p = n^{-\alpha}$. Then*

$$\mathbf{P}(|X_{d,p}| \in I) \geqslant 1 - e^{-n^d},$$

where $|X_{d,p}|$ denotes the number of d-simplices in $Y_d(n,p)$.

Lemma 11. *Let $\mathcal{S}$ be as defined in* (16)*. If $c < 2\alpha - 1$ then*

$$\max_{m\in I} \frac{n^d|\mathcal{S}|}{\binom{\binom{n}{d+1}}{m}} = o(e^{-n^s}),$$

for some $s > 0$.

Proof of Theorem 3. We consider a particular complex reconstruction algorithm which outputs a complex when a collection of 1-neighbourhoods is given. Let $\mathcal{S}$ be a collection of 1 neighbourhoods as defined in (16). The algorithm maps each element of $\mathcal{S}$ to an isomorphism class, which corresponds to at most $n^d!$ labelled complexes. The algorithm fails if $X \in Y_d(n,p)$ such that $\mathcal{N}_1(X) \in \mathcal{S}$ but the output of the algorithm of $\mathcal{N}_1(X)$ is not isomorphic to X.

We condition on the event $|X_{d,p}| = m$ for some $m \in \mathbb{N}$. Given this information, there are $\binom{\binom{n}{d+1}}{m}$ possible labelled d-simplices which may be chosen with equal probability. Therefore, conditioned on $|X_{d,p}| = m$, the algorithm fails when any complex X is not achieved by the algorithm output. Let p_m denote the probability of failure given $|X_{d,p}| = m$. Thus,

$$\begin{aligned} p_m &= \mathbf{P}(\text{ algorithm fails } \Big| \, |X_{d,p}| = m) \\ &\geqslant \mathbf{P}\left(\mathcal{N}_1(X) \in \mathcal{S} | |X_{d,p}| = m\right) - \frac{n^d!|\mathcal{S}|}{\binom{\binom{n}{d+1}}{m}} \\ &= \frac{\binom{\binom{n}{d+1}}{m} - n^d!|\mathcal{S}|}{\binom{\binom{n}{d+1}}{m}} - P\left(\mathcal{N}_1(X) \notin \mathcal{S} | |X_{d,p}| = m\right). \end{aligned}$$

Let $I \subseteq \left\{1, 2, \ldots, \binom{n}{d+1}\right\}$ be as defined in Lemma 10. Let p^* denote the overall failure probability of the algorithm. Then

$$\begin{aligned} p^* &\geqslant \sum_{m\in I} p_m \\ &\geqslant \sum_{m\in I} \frac{\binom{\binom{n}{d+1}}{m} - n^d!|\mathcal{S}|}{\binom{\binom{n}{d+1}}{m}} \mathbf{P}(|X_{d,p}| = m) - P\left(\{\mathcal{N}_1(X) \notin \mathcal{S}\} \cap \{|X_{d,p}| \in I\}\right) \\ &\geqslant \mathbf{P}(|X_{d,p}| \in I) \min_{m\in I} \frac{\binom{\binom{n}{d+1}}{m} - n^d!|\mathcal{S}|}{\binom{\binom{n}{d+1}}{m}} - \mathbf{P}(\mathcal{N}_1(X) \notin \mathcal{S}). \end{aligned}$$

Therefore Lemmas 9, 10 and 11 implies that

$$p^* \geqslant (1 - e^{-n^d})(1 - e^{-n^s}) - e^{-an^b} \geqslant 1 - e^{-a'n^{b'}},$$

for some $s, a, b, a', b' > 0$, if the constant c satisfies

$$\max\{0, 3\alpha - 2\} < c < \min\{\alpha, 2\alpha - 1\}.$$

The above is satisfied when $1/2 < \alpha < 1$ as required. The proof is then completed by choosing $c = 1/2(\max 0, 3\alpha - 2 + 2\alpha - 1)$ since $\min\{\alpha, 2\alpha - 1\} = 2\alpha - 1$. ■

The rest of the section is dedicated to prove Lemmas 9, 10 and 11.

Proof of Lemma 9. Recall S_σ denotes the set of neighbours of $\sigma \in X_{d-1}$ in $Y_d(n,p)$. Consequently we define D_σ and D^*_σ as defined above. Note that S_σ is a random set, hence D_σ and D^*_σ are random. We show that if $c > 3\alpha - 2$ then

$$\mathbf{P}\left(\bigcap_{\sigma \in X_{d-1}} \left(\{|D_\sigma| < nq_n\} \cap \{|D^*_\sigma| < \frac{1}{2}n^2q_n^2t_n\}\right)\right) \geqslant 1 - e^{-an^b}, \tag{17}$$

for some positive constants a and b. Clearly (17) gives the result.

Proof of (17)*:* Observe that $|D_\sigma| = \deg(\sigma) \sim Bin(n-d,p)$. Recall $q_n = (1+\epsilon)p$. Then Lemma 7 gives

$$\begin{aligned}\mathbf{P}\left(|D_\sigma| < nq_n\right) &\geqslant \mathbf{P}\left(|D_\sigma| < (n-d)q_n\right) \\ &= 1 - \mathbf{P}\left(|D_\sigma| \geqslant (1+\epsilon)(n-d)q_n\right) \\ &\geqslant 1 - \exp\left(-\frac{\epsilon^2 p(n-d)}{3}\right). \end{aligned} \tag{18}$$

Next we derive a bound of $|D^*_\sigma|$ with high probability. We have

$$\begin{aligned}&\mathbf{P}\left(|D^*_\sigma| \geqslant \frac{1}{2}n^2q_n^2t_n)\right) \\ &\leqslant \mathbf{P}\left(|D^*_\sigma| \geqslant \frac{1}{2}n^2q_n^2t_n \Big| |D_\sigma| < nq_n)\right) + \mathbf{P}(|D_\sigma| \geqslant nq_n).\end{aligned} \tag{19}$$

Note that, conditioned on $|D_\sigma|$, $|D^*_\sigma| \sim Bin(\binom{|D_\sigma|}{2}, p)$. Lemmas 7 and 8 imply

$$\begin{aligned}\mathbf{P}\left(|D^*_\sigma| \geqslant \frac{1}{2}n^2q_n^2t_n \;\Big|\; |D_\sigma| \leqslant nq_n)\right) &\leqslant \mathbf{P}\left(|D^*_\sigma| \geqslant \frac{1}{2}n^2q_n^2t_n \;\Big|\; |D_\sigma| = nq_n)\right) \\ &\leqslant \exp\left(-\frac{n^{2c}}{2+n^c}\binom{nq_n}{2}p\right). \end{aligned} \tag{20}$$

Combining the bounds from (18) and (20) and substituting in (19), we obtain

$$\mathbf{P}\left(|D^*_\sigma| \geqslant \frac{1}{2}n^2q_n^2t_n)\right) \leqslant \exp\left(-\frac{n^{2c}}{2+n^c}\binom{nq_n}{2}p\right) + \exp\left(-\frac{\epsilon^2 p(n-d)}{3}\right).$$

By the union bound, the required probability is given by

$$\begin{aligned}&\mathbf{P}\left(\bigcap_{\sigma \in X_{d-1}} \left(\{|D_\sigma| < nq_n\} \cap \{|D^*_\sigma| < \frac{1}{2}n^2q_n^2t_n\}\right)\right) \\ &\geqslant 1 - \binom{n}{d}\exp\left(-\frac{n^{2c}d}{2+n^c}\binom{nq_n}{2}p\right) + \binom{n}{d}\exp\left(-\frac{\epsilon^2 p(n-1)}{3}\right) \\ &\geqslant 1 - n^d \exp\left(-C_3 n^{2+c-3\alpha}\right) - n^d \exp\left(-C_4 n^{1-\alpha}\right),\end{aligned}$$

for some positive constants C_3, C_4. The above will give us a high probability bound when $2 + c - 3\alpha > 0$. Hence the result if $c > 3\alpha - 2$, as $\alpha < 1$. ■

Proof of Lemma 10. Note that $|X_{d,p}| \sim Bin\left(\binom{n}{d+1}, p\right)$. Then Lemma 7 gives

$$\mathbf{P}(|X_{d,p}| \notin I) = P\left(\left||X_{d,p}| - \binom{n}{d+1}p\right| \geqslant \epsilon\binom{n}{d+1}p\right) \leqslant 2\exp\left(-\frac{\epsilon^2}{3}\binom{n}{d+1}p\right).$$

Using $p = n^{-\alpha}$ and $\binom{n}{d+1} \leqslant n^{d+1}$, we get the result. ■

Proof of Lemma 11. By the definition of $\mathcal{S}$, we have $|D_\sigma| \in \{1, \dots, nq_n\}$ and $|D^*_\sigma| \in \{1, \dots, \frac{1}{2}n^2q_n^2t_n\}$ for all $\sigma \in X_{d-1}$. Again the choices for the number of neighbouring d-simplices is upper bounded by $\binom{\binom{nq_n}{2}}{\frac{1}{2}n^2q_n^2t_n}$ for each σ. Therefore

$$|\mathcal{S}| \leqslant \left(nq_n \cdot \frac{1}{2}n^2q_n^2t_n \cdot \binom{\binom{nq_n}{2}}{\frac{1}{2}n^2q_n^2t_n}\right)^{\binom{n}{d}} \leqslant \left(\frac{1}{2}n^3q_n^3t_n\left(\frac{e}{t_n}\right)^{\frac{1}{2}n^2q_n^2t_n}\right)^{\binom{n}{d}}, \tag{21}$$

in the last inequality we use the fact that $\left(\frac{n}{k}\right)^k \leqslant \binom{n}{k} \leqslant \left(\frac{en}{k}\right)^k$ for positive integers n and k with $1 \leqslant k \leqslant n$. Now,

$$\min_{m\in I}\binom{\binom{n}{d+1}}{m} = \binom{\binom{n}{d+1}}{(1-\epsilon)\binom{n}{d+1}p} \geqslant \left(\frac{1}{(1-\epsilon)p}\right)^{(1-\epsilon)\binom{n}{d+1}p}. \tag{22}$$

Therefore, using (21), (22) and the fact that $n^d! \leqslant \exp\left(dn^d\log(n)\right)$, we get

$$\begin{aligned}
\max_{m\in I}\frac{n^d!|\mathcal{S}|}{\binom{\binom{n}{d+1}}{m}} &\leqslant \frac{\exp\left(dn^d\log(n)\right)\left(\frac{1}{2}n^3q_n^3t_n\left(\frac{e}{t_n}\right)^{\frac{1}{2}n^2q_n^2t_n}\right)^{\binom{n}{d}}}{\left(\frac{1}{(1-\epsilon)p}\right)^{(1-\epsilon)\binom{n}{d+1}p}} \\
&= \exp\left\{dn^d\log(n) + \binom{n}{d}\log(\frac{1}{2}n^3q_n^3t_n) + \binom{n}{d}\frac{1}{2}n^2q_n^2t_n\log\left(\frac{e}{t_n}\right)\right) \\
&\qquad - (1-\epsilon)\binom{n}{d+1}p\log\left(\frac{1}{(1-\epsilon)p}\right)\Bigg\} \\
&\leqslant \exp\Big\{dn^d\log(n) + C_5n^d\log(n^{3+c-4\alpha}) + C_6n^{c-3\alpha+d+2}\log(n) \qquad (23) \\
&\qquad - C_7n^{1+d-\alpha}\log(n)\Big\},
\end{aligned}$$

for some positive constants C_5, C_6, C_7. The right hand side of (23) goes to zero exponentially when $d < d+1-\alpha$ (which is always true since $\alpha < 1$) and $c-3\alpha+d+2 < 1+d-\alpha$. In other words, the required condition is $c < 2\alpha - 1$. ■

Acknowledgements

The authors would like to express their sincere gratitude to the anonymous referees for their careful reading and insightful suggestions, which have greatly improved the clarity and quality of this paper. The research of KA was partially supported by the Inspire Faculty Fellowship: DST/INSPIRE/04/2020/000579. SC's research was supported by the NBHM postdoctoral fellowship (order no. 0204/19/2021/R&D-II/11871).

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