LDP polygons and the number 12 revisited

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Abstract

We give a combinatorial proof of a lattice point identity involving a lattice polygon and its dual, generalizing the formula $\text{area}\,(\Delta) + \text{area}\,(\Delta^*) = 6$ for reflexive Δ. The identity is equivalent to the stringy Libgober–Wood identity for toric log del Pezzo surfaces.

Mathematics Subject Classifications: 52B20, 14M25, 11F20, 32J15

1 Introduction

The goal of this article is to give a combinatorial proof of the following combinatorial identity: Consider the convex hull $\Delta \subseteq \mathbb{R}^2$ of $n_1, \ldots, n_k \in \mathbb{Z}^2$. Assume that each n_i has coprime coordinates and that the origin is an interior point of Δ. This data defines a piecewise linear function $\kappa_\Delta \colon \mathbb{R}^2 \to \mathbb{R}$ via

$$\kappa_\Delta(x) = -\min\left\{\lambda \in \mathbb{R}_{\geqslant 0} \,|\, x \in \lambda\Delta\right\} .$$

The dual polygon is denoted by Δ^*. Then

$$6 \sum_{n \in \Delta \cap \mathbb{Z}^2} \left(\kappa_\Delta(n) + 1\right)^2 = \text{area}\,(\Delta) + \text{area}\,(\Delta^*) .$$

This identity has been proven by Batyrev and the third author [6, Corollary 4.5] using a string-theoretic (allowing mild singularities) variant of the Libgober–Wood identity for compact complex manifolds [18, Proposition 2.3]. Conversely, we can obtain the stringy Libgober–Wood identity for toric surfaces as a corollary of our combinatorial proof. In our proof, we reduce this (global) statement to a local and cone-wise statement, whose algebraic geometry analogue could be of independent interest.

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The formula as well as its variants in higher dimensions have a rich and colorful history. In the reflexive case where both Δ and Δ^* have only integral vertices the formula reduces to area (Δ)+area $(\Delta^*) = 6$. Rodriguez-Villegas & Poonen [20] as well as Hille & Skarke [13] prove non-convex and group theoretic generalizations of that latter formula, Kasprzyk & Nill [16] relax the reflexivity hypothesis. Still in dimension two, Haase & Schicho [14] and also Kołodziejczyk & Olszewska [17] prove refined inequalities, taking additional invariants into account. In the open problem collection [4], an equation for 3-dimensional reflexive polytopes is stated, a combinatorial proof was sought. Godinho, von Heymann & Sabatini [11] and Hofmann [12] provide combinatorial proofs and higher-dimensional analogues, also for non-convex generalizations, but under a smoothness assumption for the underlying toric varieties. Finally, Batyrev and the third author [6] remove this smoothness assumption, but need convexity.

2 Notation and preliminaries

This chapter fixes our notation while recalling basic notions and provides a short summary of fundamental concepts used throughout the paper. Almost all of its content is well-known and the main references are [7, 10].

2.1 Polygons and cones

Let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$ be a real vector space obtained by an extension of a 2-dimensional lattice $N \cong \mathbb{Z}^2$ (i.e., a free abelian group of rank 2). Furthermore, $M := \operatorname{Hom}(N, \mathbb{Z})$ denotes the dual lattice to N and $\langle \cdot, \cdot \rangle \colon M \times N \to \mathbb{Z}$ the natural pairing which extends to a pairing $\langle \cdot, \cdot \rangle \colon M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$, where $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$ is the corresponding real vector space to M.

Let $\Delta \subseteq N_{\mathbb{R}}$ be a 2-dimensional polytope (i.e., the convex hull $\operatorname{conv}(S)$ of a finite set $S \subseteq N_{\mathbb{R}}$), which will be called *polygon* in the following. A *face* $\theta \preceq \Delta$ *of* Δ is an intersection of Δ with an affine hyperspace, i.e., there exists $m \in M_{\mathbb{R}} \setminus \{0\}$ and $b \in \mathbb{R}$ such that $\theta = \Delta \cap H_{mb}$ with $H_{mb} := \{x \in N_{\mathbb{R}} \,|\, \langle m, x \rangle = b\}$. In particular, a *vertex* is a 0-dimensional face and an *edge* a 1-dimensional face.

If $\Delta \subseteq N_{\mathbb{R}}$ contains the origin $0 \in N$ in its interior, then one defines its *dual polygon* Δ^* as

$$\Delta^* := \{y \in M_{\mathbb{R}} \,|\, \langle y, x \rangle \geqslant -1 \;\forall x \in \Delta\} \subseteq M_{\mathbb{R}} \,.$$

If $\theta \preceq \Delta$ is a face of Δ, then

$$\theta^* := \{y \in \Delta^* |\, \langle y, x \rangle = -1 \;\forall x \in \theta\} \preceq \Delta^*$$

is a face of Δ^*, the so-called *dual face of* θ. Moreover, the duality between Δ and Δ^* implies a one-to-one order-reversing duality between k-dimensional faces $\theta \preceq \Delta$ of Δ and $(2-k-1)$-dimensional dual faces $\theta^* \preceq \Delta^*$ of Δ^* such that $\dim(\theta) + \dim(\theta^*) = 1$.

A polygon $\Delta \subseteq N_{\mathbb{R}}$ is called *lattice polygon* if

$$\Delta = \mathrm{conv}(\Delta \cap N)\,,$$

i.e., if all vertices of Δ belong to the lattice N.

If a lattice polygon $\Delta \subseteq N_{\mathbb{R}}$ contains the origin $0 \in N$ in its interior and its dual polygon Δ^* is also a lattice polygon, then it is called *reflexive*.

Definition 1. A lattice polygon $\Delta \subseteq N_{\mathbb{R}}$ containing the origin $0 \in N$ in its interior such that all vertices of Δ are primitive lattice points in N is called *LDP polygon* [15], where LDP is an abbreviation for 'log del Pezzo'.

In general, the vertices of the dual polygon $\Delta^* \subseteq M_{\mathbb{R}}$ to an LDP polygon Δ are not lattice points in M, i.e., Δ^* is in general a rational polygon. If $\Delta \subseteq N_{\mathbb{R}}$ is a reflexive polygon, then the origin $0 \in N$ is the only interior lattice point of Δ. Hence, all vertices of Δ are primitive lattice points in N, i.e., LDP polygons build a superclass of reflexive polygons.

Let $\Delta \subseteq N_{\mathbb{R}}$ be a lattice polygon. Then we define $\mathrm{v}(\Delta)$ to be the *normalized volume of* Δ, i.e., the positive integer

$$\mathrm{v}(\Delta) := 2! \cdot \mathrm{vol}_2(\Delta)\,,$$

where $\mathrm{vol}_2(\Delta)$ denotes the 2-dimensional volume of Δ with respect to the lattice N. Note that $\mathrm{vol}_2(\Delta) = \mathrm{area}(\Delta)$ if $N = \mathbb{Z}^2$. Similarly, we define the positive integer $\mathrm{v}(\theta) := k! \cdot \mathrm{vol}_k(\theta)$ for a k-dimensional face $\theta \preceq \Delta$ of Δ, where $\mathrm{vol}_k(\theta)$ denotes the k-dimensional volume of θ with respect to the sublattice $\langle\theta\rangle_{\mathbb{R}} \cap N$.

If Δ has vertices in $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e., Δ is a *rational polygon*, then we can similarly define the positive rational number $\mathrm{v}(\theta)$ for any face $\theta \preceq \Delta$. For this purpose, we consider an integer l such that $l\Delta$ is a lattice polygon and define for a k-dimensional face $\theta \preceq \Delta$ its normalized volume as $\mathrm{v}(\theta) := \frac{1}{l^k}\,\mathrm{v}(l\theta)$.

Let $U \subseteq N_{\mathbb{R}}$ be a finite set. Then a (*convex polyhedral*) *cone* generated by U is defined as the set

$$\sigma := \mathrm{cone}(U) = \{\textstyle\sum_{u\in U}\lambda_u u \,|\, \lambda_u \geqslant 0\}\,.$$

If U consists of $\mathbb{R}$-linear independent lattice vectors, then the corresponding (half-open) *fundamental parallelogram* of U is

$$\Pi := \Pi(U) = \{\textstyle\sum_{u\in U}\lambda_u u \,|\, 0 \leqslant \lambda_u < 1\}\,.$$

Note that the normalized volume of the fundamental parallelogram equals the number of lattice points contained in it. Moreover, a 2-dimensional cone σ is called *unimodular* if its ray generators u_1, u_2 form a part of a $\mathbb{Z}$-basis of N. Note that in this case the fundamental parallelogram $\Pi(u_1, u_2)$ contains only one lattice point.

Definition 2. Let $\sigma \subseteq N_{\mathbb{R}}$ be a 2-dimensional cone. We define ∇_σ to be the convex hull of the origin and the primitive ray generators u_1 and u_2 of the given cone σ, i.e.,

$$\nabla_\sigma := \operatorname{conv}(0, u_1, u_2)\,.$$

We denote the *relative interior of* ∇_σ by ∇_σ°.

The *sail of* σ is the non-convex half-open lattice polygon defined as

$$\operatorname{sail}_\sigma := \nabla_\sigma \setminus \operatorname{conv}(\nabla_\sigma \cap N \setminus \{0\})$$

(cf. [2]) and its closure denoted by $\overline{\operatorname{sail}}_\sigma$.

The *normalized volume* $\mathrm{v}(\sigma)$ *of* a 2-dimensional cone σ is defined to be the normalized volume of the lattice polygon ∇_σ given by the convex hull of the origin and both primitive ray generators of the given cone σ, i.e.,

$$\mathrm{v}(\sigma) := \mathrm{v}(\nabla_\sigma)\,.$$

2.2 Toric surfaces

A *toric surface* X is a normal variety of dimension 2 over the field of complex numbers $\mathbb{C}$ containing a torus $\mathbb{T} \cong (\mathbb{C}^*)^2$ as a Zariski open set such that the action of $(\mathbb{C}^*)^2$ on itself extends to an action on X.

Definition 3. Let $\Delta \subseteq N_{\mathbb{R}}$ be a lattice polygon with $0 \in \Delta^\circ \cap N$. We define Σ_Δ to be the *spanning fan* of Δ in $N_{\mathbb{R}}$, i.e., $\Sigma_\Delta := \{\sigma_\theta \,|\, \theta \preceq \Delta\}$, where σ_θ is the cone $\mathbb{R}_{\geqslant 0}\theta$ spanned by the face $\theta \preceq \Delta$ of Δ with $\dim(\sigma_\theta) = \dim(\theta) + 1$. In particular, the spanning fan is a fan associated with an in general non-smooth normal projective toric surface X_{Σ_Δ}. Moreover, one obtains a resolution of singularities of X_{Σ_Δ} through the toric morphism $X_{\Sigma'_\Delta} \to X_{\Sigma_\Delta}$, where Σ'_Δ is a suitable refinement of Σ_Δ. In our 2-dimensional case, the rays of Σ'_Δ are spanned by all lattice points lying on the boundary of $\cup_{\sigma \in \Sigma_\Delta[2]} \operatorname{sail}_\sigma$, where $\Sigma_\Delta[i]$ denotes the set of i-dimensional cones in the fan Σ_Δ.

Remark 4. The structure we refer to as the *spanning fan* is also known as the *face fan* [22] or the *central fan* [8].

We briefly recap that a normal projective surface is a *log del Pezzo surface* if it has at worst log-terminal singularities and if its anticanonical divisor is an ample $\mathbb{Q}$-Cartier divisor. Moreover, toric log del Pezzo surfaces one-to-one correspond to LDP polygons [15]. The fan Σ defining a toric log del Pezzo surface X is the spanning fan Σ_Δ of the corresponding LDP polygon Δ. In particular, any LDP polygon Δ is the convex hull of all primitive ray generators of elements in $\Sigma_\Delta[1]$.

Let $\Delta \subseteq N_{\mathbb{R}}$ be a lattice polygon with $0 \in \Delta^\circ \cap N$. Then there exists a Σ_Δ-piecewise linear function $\kappa_\Delta : N_{\mathbb{R}} \to \mathbb{R}$ corresponding to the anticanonical divisor on X_{Σ_Δ} that is linear on each cone σ of Σ_Δ and has value -1 on every primitive ray generator of 1-dimensional cones of Σ_Δ.

In the rest of this subsection, we aim for introducing the stringy version of the Libgober–Wood identity from a geometric point of view, but restricted to log del Pezzo surfaces: If V is an arbitrary smooth projective surface, the *E-polynomial* of V is defined as

$$E\left(V;u,v\right) := \sum_{0 \leqslant p,q \leqslant 2} (-1)^{p+q} h^{p,q}\left(V\right) u^p v^q \, ,$$

where $h^{p,q}\left(V\right)$ denote the Hodge numbers of V. The *stringy E-function* of a normal projective $\mathbb{Q}$-Gorenstein variety X with at worst log-terminal singularities is a rational algebraic function in two variables u, v defined by the formula

$$E_{\mathrm{str}}(X;u,v) := \sum_{\emptyset \subseteq J \subseteq I} E(D_J;u,v) \prod_{j \in J} \left(\frac{uv-1}{(uv)^{a_j+1}-1} - 1 \right) ,$$

where $\rho : Y \to X$ is some desingularization of X, whose exceptional locus is a union of smooth irreducible divisors $D_1, \ldots, D_s$ with only simple normal crossings and $K_Y = \rho^* K_X + \sum_{i=1}^{s} a_i D_i$ for some rational numbers $a_i > -1$. For any non-empty subset $J \subseteq I := \{1, \ldots, s\}$, we define D_J to be the smooth subvariety $\cap_{j \in J} D_j$. As a special case, this formula implies $E_{\mathrm{str}}(X;u,v) = E(X;u,v)$ if X is smooth.

Let X be a toric log del Pezzo surface associated with a fan Σ. Then the *stringy version of the Libgober–Wood identity* is given as

$$\frac{d^2}{du^2} E_{\mathrm{str}}\left(X;u,1\right)\Big|_{u=1} = \frac{1}{6} c_2^{\mathrm{str}}(X) + \frac{1}{6} c_1(X).c_1^{\mathrm{str}}(X) = \frac{1}{6} c_2^{\mathrm{str}}(X) + \frac{1}{6} c_1(X)^2 \, ,$$

where $c_k^{\mathrm{str}}(X)$ denotes the *k-th stringy Chern class* introduced in [1, 9]. In particular, the k-th stringy Chern class of X can be computed purely combinatorial via

$$c_k^{\mathrm{str}}(X) = \sum_{\sigma \in \Sigma(k)} \mathrm{v}(\sigma) \cdot [X_\sigma]$$

[6], where $[X_\sigma]$ denotes the class of the closed torus orbit X_σ corresponding to a given cone $\sigma \in \Sigma$. The general stringy version of the Libgober–Wood identity holding for any projective variety with at worst log-terminal singularities can be found in [3, Theorem 3.8].

3 Main theorem and its reduction to a local version

We present a purely combinatorial proof of the following combinatorial identity that is equivalent to the stringy Libgober–Wood identity for log del Pezzo surfaces and relates LDP polygons to the number 12:

Theorem 5 [6, Corollary 4.5]**.** *Let $\Delta \subseteq N_{\mathbb{R}}$ be an LDP polygon. Then*

$$12 \sum_{n \in \Delta \cap N} \left(\kappa_\Delta(n) + 1\right)^2 = \mathrm{v}\left(\Delta\right) + \mathrm{v}\left(\Delta^*\right) \, ,$$

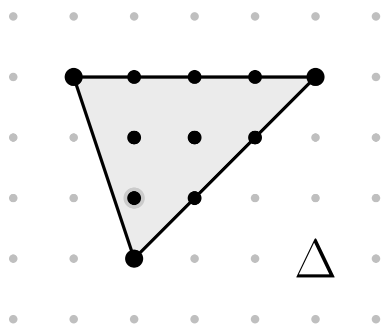

Figure 1: **LDP polygon Δ.** The origin is highlighted with a gray background. (A) Lattice polygon Δ with $\mathrm{v}(\Delta) = 12$. (B) Dual rational polygon Δ^* with rational vertex (blue) and $\mathrm{v}(\Delta^*) = 6$.

where $\kappa_\Delta : N_\mathbb{R} \to \mathbb{R},\, x \mapsto -\min\{\lambda \in \mathbb{R}_{\geqslant 0} \,|\, x \in \lambda\Delta\}$*. In particular, one always has* $\mathrm{v}(\Delta) + \mathrm{v}(\Delta^*) \geqslant 12$ *and equality holds if and only if* Δ *is reflexive.*

Example 6. Let $\Delta \subseteq N_\mathbb{R}$ be the LDP polygon given as the convex hull of $(0,-1)$, $(3,2)$, and $(-1,2)$ (cf. Figure 1A). Then Theorem 5 yields

$$12 \sum_{n \in \Delta \cap N} (\kappa_\Delta(n)+1)^2 = 12 \cdot (1^2 + 0.5^2 + 0.5^2) = 18 = 12 + 6 = \mathrm{v}(\Delta) + \mathrm{v}(\Delta^*)\,,$$

where the dual rational polygon $\Delta^* \subseteq M_\mathbb{R}$ is the convex hull of $(0,-0.5)$, $(3,1)$, and $(-1,1)$ (cf. Figure 1B).

Our strategy relies on a decomposition of the identity in Theorem 5 using the spanning fan Σ_Δ of the given LDP polygon Δ and considering its 2-dimensional cones separately.

Definition 7. Let $\sigma \in \Sigma_\Delta[2]$ be a 2-dimensional cone with primitive ray generators u_1 and $u_2 \in N$. Moreover, let $m_\sigma \in M_\mathbb{R}$ be the vector dual to the edge $u_1 - u_2$, i.e., $\langle m_\sigma, u_1\rangle = -1 = \langle m_\sigma, u_2\rangle$. We consider all 2-dimensional cones $\sigma_1, \ldots, \sigma_{k_\sigma}$ in the refined fan Σ'_Δ of the given spanning fan Σ_Δ that are contained in σ. Therefore, we enumerate the corresponding k_σ edges of the sail_σ from u_2 to u_1 consecutively, denote the corresponding dual vectors by $m_{\sigma,1}, \ldots, m_{\sigma,k_\sigma}$, and define

$$\mathrm{conv}[m_\sigma, i] := \mathrm{conv}(m_\sigma, m_{\sigma,i}, m_{\sigma,i+1}) \ \text{ for } 1 \leqslant i \leqslant k_\sigma - 1\,.$$

For an illustration of these definitions, see Figure 2A and Figure 2B. Note that the rays of all cones in Σ'_Δ lying in σ are spanned by the non-zero lattice points of sail_σ. Moreover, $m_{\sigma,1}, \ldots, m_{\sigma,k_\sigma}$ have integer coordinates as the resolved cones are unimodular while m_σ may have rational coordinates.

Theorem 8. *Let* $\Delta \subseteq N_\mathbb{R}$ *be an LDP polygon and* $\sigma \in \Sigma_\Delta[2]$ *a 2-dimensional cone of the spanning fan* Σ_Δ*. Then*

$$12 \sum_{n \in \nabla_\sigma^\circ \cap N} (\kappa(n)+1)^2 = \mathrm{v}(\nabla_\sigma \setminus \mathrm{sail}_\sigma) + \sum_{i=1}^{k_\sigma - 1} \mathrm{v}(\mathrm{conv}[m_\sigma, i])\,, \tag{1}$$

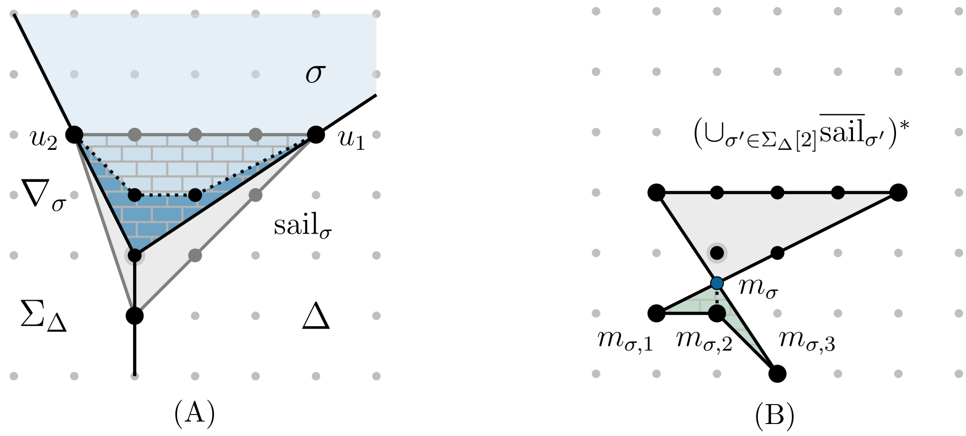

Figure 2: **Spanning fan Σ_Δ with sail_σ and dual polygon of $\cup_{\sigma'\in\Sigma_\Delta[2]}\overline{\mathrm{sail}}_{\sigma'}$.** (A) 2-dimensional cone σ (light and dark blue) of the spanning fan Σ_Δ of the LDP polygon Δ (gray), ∇_σ (brick) and sail_σ (dark blue). (B) Dual polygon of (non-convex) polygon $\cup_{\sigma'\in\Sigma_\Delta[2]}\overline{\mathrm{sail}}_{\sigma'}$ (gray) and $\cup_{i=1}^{k_\sigma-1}\mathrm{conv}[m_\sigma,i]$ (green).

where $\kappa := \kappa_\Delta|_\sigma$ is a linear function given as the restriction of the piecewise linear function κ_Δ to the cone σ.

Example 9. Let $\Delta \subseteq N_\mathbb{R}$ be the LDP polygon given in Example 6 and $\sigma \in \Sigma_\Delta[2]$ the 2-dimensional cone of the spanning fan Σ_Δ with primitive ray generators $u_1 = (3,2)$ and $u_2 = (-1,2) \in N$ (cf. Figure 2A). Then Theorem 8 yields

$$\begin{aligned}12\sum_{n\in\nabla_\sigma^\circ\cap N}(\kappa(n)+1)^2 = 12\sum_{n\in\{(0,1),(1,1)\}}(\kappa(n)+1)^2 &= 6 = 5+0.5+0.5\\ &= \mathrm{v}\left(\nabla_\sigma\setminus\mathrm{sail}_\sigma\right)+\mathrm{v}(\mathrm{conv}[m_\sigma,1])+\mathrm{v}(\mathrm{conv}[m_\sigma,2])\\ &= \mathrm{v}\left(\nabla_\sigma\setminus\mathrm{sail}_\sigma\right)+\sum_{i=1}^{k_\sigma-1}\mathrm{v}(\mathrm{conv}[m_\sigma,i])\,,\end{aligned}$$

where the $\nabla_\sigma = \mathrm{conv}((0,0),u_1,u_2)$, $\nabla_\sigma\setminus\mathrm{sail}_\sigma = \mathrm{conv}(u_1,u_2,(0,1),(1,1))$, $k_\sigma = 3$ (cf. Figure 2A, dotted edges), and $m_\sigma = (0,-0.5)$, $m_{\sigma,1} = (-1,-1)$, $m_{\sigma,2} = (0,-1)$, $m_{\sigma,3} = (1,-2)$ (cf. Figure 2B).

Theorem 8, which we prove combinatorially in Section 4, is our main ingredient for our combinatorial proof of the identity in Theorem 5. Summing up Equation (1) over all 2-dimensional cones of the spanning fan Σ_Δ of our given LDP polygon Δ, we obtain

$$12\sum_{n\in(\Delta\cap N)\setminus\{0\}}(\kappa(n)+1)^2 = \mathrm{v}(\Delta)-\mathrm{v}(\textstyle\bigcup_{\sigma\in\Sigma_\Delta[2]}\overline{\mathrm{sail}}_\sigma)+\displaystyle\sum_{\sigma\in\Sigma_\Delta[2]}\sum_{i=1}^{k_\sigma-1}\mathrm{v}(\mathrm{conv}[m_\sigma,i])\,.$$

Comparing this identity with the one in Theorem 5, it suffices to show

$$12 = \mathrm{v}\left(\Delta^*\right)+\mathrm{v}(\textstyle\bigcup_{\sigma\in\Sigma_\Delta[2]}\overline{\mathrm{sail}}_\sigma)-\displaystyle\sum_{\sigma\in\Sigma_\Delta[2]}\sum_{i=1}^{k_\sigma-1}\mathrm{v}(\mathrm{conv}[m_\sigma,i])\,. \qquad (2)$$

We will consider the union $\cup_{\sigma \in \Sigma_\Delta[2]} \overline{\text{sail}}_\sigma$ of all closed sails as a non-convex polygon and denote it by Δ_{sails}. Furthermore, we associate with it a fan $\Sigma_{\Delta_{\text{sails}}}$ having rays that are spanned by the boundary lattice points of Δ_{sails}. Note that this fan is unimodular, as all cones are unimodular cones by construction. In addition, this fan is *complete*, meaning the union of its cones is the whole space $\mathbb{R}^2$. For such a fan, every 1-dimensional cone τ with primitive ray generator v is contained in precisely two 2-dimensional cones $\sigma_l = \text{conv}(v, v_l)$ and $\sigma_r = \text{conv}(v, v_r)$ of this fan, where v_l and v_r are also primitive ray generators. Moreover, there exists a unique integer a_τ such that $v_l + v_r = a_\tau v$. Now we apply the following

Theorem 10 [20, Subsection 8.1]**.** *Let Σ be a complete unimodular fan in $\mathbb{R}^2$. Then*

$$12 = \sum_{\tau \in \Sigma[1]} (3 - a_\tau)\,.$$

Combining this theorem with the 2-dimensional property

$$\text{v}(\textstyle\bigcup_{\sigma \in \Sigma_\Delta[2]} \overline{\text{sail}}_\sigma) = \sum_{\tau \in \Sigma_{\Delta_{\text{sails}}}[1]} 1$$

and Equation (2), we arrive at

$$\sum_{\tau \in \Sigma_{\Delta_{\text{sails}}}[1]} (2 - a_\tau) = \text{v}\,(\Delta^*) - \sum_{\sigma \in \Sigma_\Delta[2]} \sum_{i=1}^{k_\sigma - 1} \text{v}(\text{conv}[m_\sigma, i])\,.$$

In order to verify this identity, we again use the fact that $\Sigma_{\Delta_{\text{sails}}}$ is a complete unimodular fan in $\mathbb{R}^2$. The reasoning at the end of Section 8.1 in [20] can be applied to our case and states in particular that the sum

$$\sum_{\tau \in \Sigma_{\Delta_{\text{sails}}}[1]} (2 - a_\tau)$$

equals the sum of signed lengths of dual edges τ^* corresponding to τ. Furthermore, the proof also shows that the sum of signed lengths of dual edges can be expressed as a sum of signed volumes. In particular,

$$\sum_{\tau \in \Sigma_{\Delta_{\text{sails}}}[1]} (2 - a_\tau) = \sum_{\tau \in \Sigma_{\Delta_{\text{sails}}}[1]} \det(\tau^*)\,,$$

where $\det(\tau^*)$ is the determinant of the 2×2 matrix with the two vertices of τ^* as columns (respecting the direction of the edges in the chain) so that $|\det(\tau^*)| = \text{v}(\text{conv}(0, \tau^*))$. It remains to deduce

$$\sum_{\tau \in \Sigma_{\Delta_{\text{sails}}}[1]} \det(\tau^*) = \text{v}\,(\Delta^*) - \sum_{\sigma \in \Sigma_\Delta[2]} \sum_{i=1}^{k_\sigma - 1} \text{v}(\text{conv}[m_\sigma, i])\,. \tag{3}$$

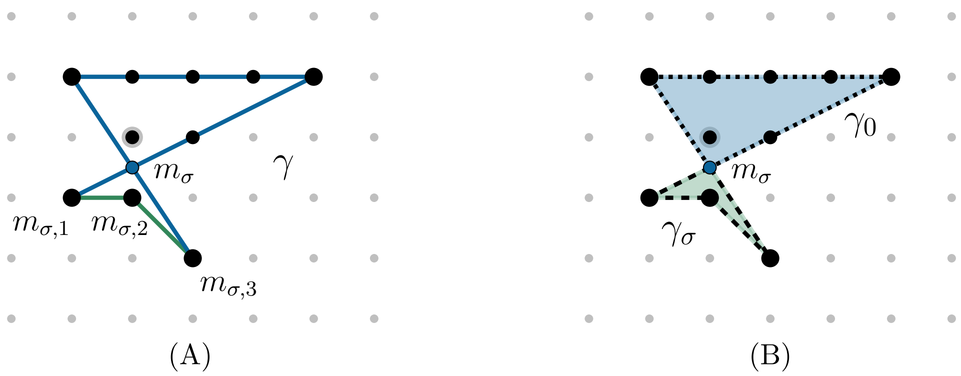

Figure 3: **Dual polygon $(\cup_{\sigma' \in \Sigma_\Delta[2]} \overline{\mathrm{sail}}_{\sigma'})^*$ with closed curve γ.** (A) Edges of $(\cup_{\sigma' \in \Sigma_\Delta[2]} \overline{\mathrm{sail}}_{\sigma'})^*$ given as a closed chain γ marked according to sign: blue for positive sign, green for negative sign. The interior intersection point of the two positive edges is m_σ. (B) Signed volumes for the curve γ split into γ_0 (blue area counted positive) and γ_σ (green area counted negative).

Let γ be the closed curve corresponding to the chain of dual edges τ^* whose orientation is induced by the signs. Observe that $\sum_{\tau \in \Sigma_{\Delta_{\mathrm{sails}}}[1]} \det(\tau^*) = \int_\gamma \alpha$, where α is a 1-form such that $\frac{1}{2} d\alpha$ is the standard volume form on $\mathbb{R}^2$, see literature on differential forms, e.g., [19, Section 37.3]. We split the curve γ into simple closed curves γ_0 and γ_σ for $\sigma \in \Sigma_\Delta[2]$: γ_0 runs through the boundary of Δ^* and γ_σ through $m_\sigma, m_{\sigma,1}, \dots, m_{\sigma,k_\sigma}, m_\sigma$. The integral splits into a sum of integrals over these simple closed curves, where $\int_{\gamma_0} \alpha = \mathrm{v}(\Delta^*)$ and $\int_{\gamma_\sigma} \alpha$ is the negative normalized volume of the area bounded by γ_σ (the winding number is -1). This area is subdivided into the triangles $\mathrm{conv}[m_\sigma, i]$ (cf. Definition 7).

This shows Equation (3) and thus finishes our combinatorial proof of the identity in Theorem 5.

Example 11. We continue with the LDP polygon Δ studied in Example 6 and 9 and consider the dual polygon to $\cup_{\sigma' \in \Sigma_\Delta[2]} \overline{\mathrm{sail}}_{\sigma'}$ (cf. Figure 3). Equation (3) holds because

$$\sum_{\tau \in \Sigma_{\Delta_{\mathrm{sails}}}[1]} \det(\tau^*) = 4 + 2 - 1 - 1 + 1 = 6 - 0.5 - 0.5$$

$$= \mathrm{v}\left(\Delta^*\right) - \sum_{\sigma \in \Sigma_\Delta[2]} \sum_{i=1}^{k_\sigma - 1} \mathrm{v}(\mathrm{conv}[m_\sigma, i]) \, .$$

4 Proving the cone-wise identity

We distinguish two cases in our proof of Theorem 8. For unimodular cones, we easily see that the left hand side and the right hand side of Equation (1) both vanish. For non-unimodular cones, a combinatorial proof by induction will be given in the rest of this section.

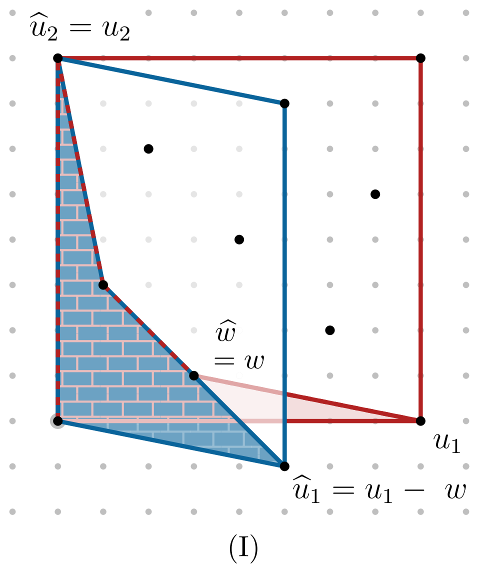

Figure 4: **Illustration of both cone generation operations.** Both cases show the fundamental parallelograms $\Pi(u_1, u_2)$ of the given cone σ and $\Pi(\widehat{u}_1, \widehat{u}_2)$ of the cone $\widehat{\sigma}$ from which σ is constructed together with the generating vectors w and $\widehat{w}$ for the respective base changes. (I) Case I: $V = 8, a = 3$ and $\widehat{V} = 5, \widehat{a} = 3$. (II) Case II: $V = 5, a = 3$ and $\widehat{V} = 3, \widehat{a} = 1$.

Our reasoning is based on the fact that for any non-unimodular cone σ with primitive ray generators u_1 and u_2 all interior lattice points of the fundamental parallelogram $\Pi(u_1, u_2)$ can be generated by some vector $w = \frac{1}{V}(au_1 + u_2)$, where $a \in \mathbb{N}$ and $V := \mathrm{v}(\sigma) = \mathrm{v}(\Pi(u_1, u_2))/2$. More precisely, these lattice points are represented in the (u_1, u_2)-basis as

$$\lfloor iw \rfloor = \frac{1}{V}((ia \bmod V)u_1 + iu_2) \tag{4}$$

for $i = 0, 1, \ldots, V$, see Figure 4.

Our main idea is to argue by induction over the volume $V = \mathrm{v}(\sigma)$ of a cone σ with primitive ray generators u_1 and u_2, i.e., $\sigma = \mathrm{cone}(u_1, u_2)$. For $V = 1$ the cone is unimodular and thus Theorem 8 holds. We proceed by assuming that we are given a non-unimodular cone σ with volume $V > 1$. We construct another cone $\widehat{\sigma}$ with primitive ray generators $\widehat{u}_1$ and $\widehat{u}_2$, whose volume is strictly smaller, i.e., $\widehat{V} := \mathrm{v}(\widehat{\sigma}) < \mathrm{v}(\sigma) = V$. Furthermore, we establish the validity of our formula for σ by deducing it from the validity for $\widehat{\sigma}$. In order to determine this cone $\widehat{\sigma}$, we consider the three consecutive lattice points u_1, w, and v on the boundary of sail_σ. As the two cones $\mathrm{cone}(u_1, w)$ and $\mathrm{cone}(w, v)$ are unimodular, we deduce that $u_1 + v = \lambda w$ for some $\lambda \in \mathbb{N}$ with $\lambda \geqslant 2$. If $\lambda > 2$ (Case I), we define $\widehat{\sigma}$ to be the cone generated by $\widehat{u}_1 := u_1 - w$ and $\widehat{u}_2 := u_2$. Note that $\widehat{\sigma}$ has the same lattice points in its sail as σ except u_1 (which is replaced by $u_1 - w$) and has strictly smaller volume, see Figure 4I for an illustration. If $\lambda = 2$ (Case II), the three lattice points u_1, w, and v are collinear and we define $\widehat{\sigma}$ to be the cone generated by $\widehat{u}_1 := w$ and $\widehat{u}_2 := u_2$. Note that

$\overline{\text{sail}}_{\widehat{\sigma}} \cap N = \overline{\text{sail}}_{\sigma} \cap N$ and $\widehat{\sigma}$ has strictly smaller volume than σ, see Figure 4II. Therefore, for every non-unimodular cone σ, we can construct a cone $\widehat{\sigma} = \text{cone}(\widehat{u}_1, \widehat{u}_2)$ with strictly smaller volume by applying one of the following two operations:

$$\begin{aligned} &\text{I. } \sigma \to \widehat{\sigma} = \text{cone}(u_1 - w, u_2), \quad u_1 \mapsto \widehat{u}_1 = u_1 - w, \quad u_2 \mapsto \widehat{u}_2 = u_2\,, \\ &\text{II. } \sigma \to \widehat{\sigma} = \text{cone}(w, u_2), \quad u_1 \mapsto \widehat{u}_1 = w, \quad u_2 \mapsto \widehat{u}_2 = u_2\,. \end{aligned}$$

Both these operations are reversible. The inverse operations are the following:

$$\begin{aligned} &\text{I. } \widehat{\sigma} \to \sigma = \text{cone}(\widehat{u}_1 + \widehat{w}, \widehat{u}_2), \quad \widehat{u}_1 \mapsto u_1 = \widehat{u}_1 + \widehat{w}, \quad \widehat{u}_2 \mapsto u_2 = \widehat{u}_2\,, \\ &\text{II. } \widehat{\sigma} \to \sigma = \text{cone}(2\widehat{u}_1 - \widehat{w}, \widehat{u}_2), \quad \widehat{u}_1 \mapsto u_1 = 2\widehat{u}_1 - \widehat{w}, \quad \widehat{u}_2 \mapsto u_2 = \widehat{u}_2\,, \end{aligned}$$

where $\widehat{w}$ is defined analogously as w above, see Figure 4. By construction, we can easily deduce the properties

$$\text{I. } V = \widehat{V} + \widehat{a}, \quad a = \widehat{a}, \quad w = \widehat{w}\,, \tag{5}$$

$$\text{II. } V = 2\widehat{V} - \widehat{a}, \quad a = \widehat{V}, \quad w = \widehat{u}_1\,. \tag{6}$$

As induction hypothesis we assume that Equation (1) holds for all cones $\widehat{\sigma}$ with volume $\widehat{V}$ strictly smaller than V. Therefore, it suffices to show that Equation (1) still holds when the cone $\widehat{\sigma}$ is changed to σ. We consider the left hand side (LHS) and the right hand side (RHS) of Equation (1) separately and determine the differences of the new and old values associated to σ and $\widehat{\sigma}$, that is $\text{LHS} - \widehat{\text{LHS}}$ and $\text{RHS} - \widehat{\text{RHS}}$, respectively. Comparing these differences, we will see that they coincide. This finishes the induction step.

4.1 Left hand side of Equation (1)

First, we will express the left hand side of Equation (1) in terms of the volume and the sawtooth function.

Definition 12 [21, Chapter 1, Introduction]**.** Let x be a rational number. Then

$$(\!(x)\!) := \begin{cases} x - [x] - 1/2 & \text{if } x \in \mathbb{Q} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

defines the *sawtooth function* of period 1, where $[x]$ denotes the greatest integer not exceeding x. Given integers h, k with $\gcd(h, k) = 1$ and $k \geqslant 1$, the *Dedekind sum* is defined as

$$\text{s}(h, k) := \sum_{i=1}^{k} \left(\!\!\left(\frac{hi}{k}\right)\!\!\right) \left(\!\!\left(\frac{i}{k}\right)\!\!\right) .$$

Remark 13. Let h, k, m, and i be integers. By the periodicity of the sawtooth function we immediately see that $\left(\!\left(\frac{(h-mk)i}{k}\right)\!\right) = \left(\!\left(\frac{hi}{k}\right)\!\right)$. Thus

$$\text{s}(h, k) = \text{s}(h - mk, k) \quad \text{and} \quad \text{s}(-h, k) = -\,\text{s}(h, k)\,,$$

where the last equation holds since the sawtooth function is odd, that is $(\!(-x)\!) = -\,(\!(x)\!)$ [21, Chapter 3, Elementary Properties].

Lemma 14 [21, Chapter 2, Lemma 2, Theorem 1]**.** *Let h and k be two integers with* $\gcd(h,k)=1$*. Then*

$$\mathrm{s}(1,k)=-\frac{1}{4}+\frac{1}{6k}+\frac{k}{12}=\frac{1}{12k}(k-1)(k-2)$$

and

$$\mathrm{s}(h,k)+\mathrm{s}(k,h)=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{1}{hk}+\frac{k}{h}\right).$$

Lemma 15. *For every 2-dimensional cone σ with primitive ray generators u_1 and u_2, we have*

$$12\sum_{n\in\nabla_\sigma^\circ\cap N}(\kappa(n)+1)^2=\frac{(V-1)(V-2)}{V}+12\cdot\mathrm{s}(a,V)\,, \tag{7}$$

where $V=\mathrm{v}(\sigma)$ and $a\in\mathbb{N}$ is such that all interior lattice points of $\Pi(u_1,u_2)$ are generated by $w=\frac{1}{V}(au_1+u_2)$.

Proof. Without loss of generality, we restrict ourselves to non-unimodular cones $\sigma=\mathrm{cone}(u_1,u_2)$. Since u_1 and u_2 are primitive and $Vw=au_1+u_2$, we have $\gcd(V,a)=1$. Furthermore, we denote by (u_1^*,u_2^*) the dual basis to (u_1,u_2) with respect to the standard scalar product. Observe that $\kappa=-u_1^*-u_2^*$ by construction. Therefore, we get

$$2\sum_{n\in\nabla_\sigma^\circ\cap N}(\kappa(n)+1)^2=\sum_{i=1}^{V-1}(\kappa(\lfloor iw\rfloor)+1)^2$$

which follows from the symmetry of κ. Furthermore, as $\kappa=-u_1^*-u_2^*$ and $\gcd(V,a)=1$, we deduce from Equation (4) and Definition 12 that

$$\begin{aligned}2V\left(1+\kappa(\lfloor iw\rfloor)+\left(\left(\frac{ai}{V}\right)\right)\right)&=2V-2(ia \bmod V)-2i+2ai-2V\left\lceil\frac{ai}{V}\right\rceil-V\\&=V-2i+2\left(ai-(ia \bmod V)-V\left\lceil\frac{ai}{V}\right\rceil\right)\\&=2V-i\end{aligned}$$

holds for $i=1,\dots,V-1$. This yields

$$\kappa(\lfloor iw\rfloor)+1=-\left(\left(\frac{ai}{V}\right)\right)+\frac{1}{2}-\frac{i}{V}.$$

As $\gcd(V,a)=1$, the set of values of $\left(\left(\frac{ai}{V}\right)\right)^2$ for $i=1,\dots,V-1$ agrees with the set of values of $\left(\left(\frac{j}{V}\right)\right)^2$ for $j=1,\dots,V-1$. Therefore,

$$\sum_{i=1}^{V-1}\left(\left(\frac{ai}{V}\right)\right)^2=\sum_{j=1}^{V-1}\left(\left(\frac{j}{V}\right)\right)^2=\mathrm{s}(1,V)=\frac{1}{12V}(V-1)(V-2)\,,$$

where we used Definition 12, Lemma 14, and the fact $\left(\left(\frac{V}{V}\right)\right) = 0$ for the second equality. It is well known that $\sum_{i=1}^{V-1} \left(\left(\frac{ai}{V}\right)\right) = 0$ (as $\left(\left(\frac{ai}{V}\right)\right) + \left(\left(\frac{a(V-i)}{V}\right)\right) = 0$ for $i = 1, \dots, V-1$ if $\gcd(V,a) = 1$, analogously to Remark 13). Thus,

$$\sum_{i=1}^{V-1} \frac{i}{V} \left(\left(\frac{ai}{V}\right)\right) = \sum_{i=1}^{V-1} \left(\frac{i}{V} - \left\lfloor\frac{i}{V}\right\rfloor - \frac{1}{2}\right) \left(\left(\frac{ai}{V}\right)\right) = \sum_{i=1}^{V-1} \left(\left(\frac{i}{V}\right)\right) \left(\left(\frac{ai}{V}\right)\right) = \mathrm{s}(a,V)\,.$$

Combining everything, we obtain

$$\begin{aligned}
\sum_{i=1}^{V-1} (\kappa(\lfloor iw \rfloor) + 1)^2 &= \sum_{i=1}^{V-1} \left(- \left(\left(\frac{ai}{V}\right)\right) + \frac{1}{2} - \frac{i}{V} \right)^2 \\
&= \sum_{i=1}^{V-1} \left(\left(\frac{ai}{V}\right)\right)^2 - \sum_{i=1}^{V-1} \left(\left(\frac{ai}{V}\right)\right) + 2 \sum_{i=1}^{V-1} \frac{i}{V} \left(\left(\frac{ai}{V}\right)\right) \\
&\qquad - \frac{1}{V} \sum_{i=1}^{V-1} i + \frac{1}{4}(V-1) + \frac{1}{V^2} \sum_{i=1}^{V-1} i^2 \\
&= \frac{1}{12V}(V-1)(V-2) + 2\,\mathrm{s}(a,V) + \frac{1}{12V}(V-1)(V-2) \\
&= \frac{1}{6V}(V-1)(V-2) + 2\,\mathrm{s}(a,V)\,.
\end{aligned}$$

□

Remark 16. Using Lemma 15 and solving for the Dedekind sum $s(a,V)$, one obtains an expression for $s(a,V)$ in terms of the area V of the cone and a sum involving the lattice points in ∇°_σ of the cone σ. Since such a cone can be constructed for any given $s(a,V)$, this provides an explicit geometric interpretation of Dedekind sums. See also [5] for related discussions.

In the following, we will consider the difference $\mathrm{LHS} - \widehat{\mathrm{LHS}}$ for Case I and II separately:

Case I Using Lemma 15 and Equation (5), we get

$$\begin{aligned}
\mathrm{LHS} - \widehat{\mathrm{LHS}} = &\frac{1}{\widehat{V} + \widehat{a}}(\widehat{V} + \widehat{a} - 1)(\widehat{V} + \widehat{a} - 2) + 12\,\mathrm{s}(\widehat{a}, \widehat{V} + \widehat{a}) \\
&- \frac{1}{\widehat{V}}(\widehat{V} - 1)(\widehat{V} - 2) - 12\,\mathrm{s}(\widehat{a}, \widehat{V})\,.
\end{aligned}$$

The reciprocity law (Lemma 14) and the periodicity for Dedekind sums (Remark 13) imply

$$12\,\mathrm{s}(\widehat{a}, \widehat{V} + \widehat{a}) = 12 \left(- \mathrm{s}(\widehat{V} + \widehat{a}, \widehat{a}) - \frac{1}{4} + \frac{1}{12} \left(\frac{\widehat{a}}{\widehat{V} + \widehat{a}} + \frac{1}{\widehat{a}(\widehat{V} + \widehat{a})} + \frac{\widehat{V} + \widehat{a}}{\widehat{a}} \right) \right)$$

$$= -12\,\mathrm{s}(\widehat{V},\widehat{a}) - 3 + \frac{\widehat{a}}{\widehat{V}+\widehat{a}} + \frac{1}{\widehat{a}(\widehat{V}+\widehat{a})} + \frac{\widehat{V}+\widehat{a}}{\widehat{a}}$$

$$= 12\left(\mathrm{s}(\widehat{a},\widehat{V}) + \frac{1}{4} - \frac{1}{12}\left(\frac{\widehat{a}}{\widehat{V}} + \frac{1}{\widehat{a}\widehat{V}} + \frac{\widehat{V}}{\widehat{a}}\right)\right)$$

$$- 3 + \frac{\widehat{a}}{\widehat{V}+\widehat{a}} + \frac{1}{\widehat{a}(\widehat{V}+\widehat{a})} + \frac{\widehat{V}+\widehat{a}}{\widehat{a}})$$

$$= 12\,\mathrm{s}(\widehat{a},\widehat{V}) + \frac{-\widehat{a}^2 + \widehat{a}\widehat{V} + \widehat{V}^2 - 1}{\widehat{V}(\widehat{V}+\widehat{a})}\,.$$

Simplifying

$$\frac{1}{\widehat{V}+\widehat{a}}(\widehat{V}+\widehat{a}-1)(\widehat{V}+\widehat{a}-2) = \frac{1}{\widehat{V}}(\widehat{V}-1)(\widehat{V}-2) + \frac{\widehat{a}(\widehat{a}\widehat{V}+\widehat{V}^2-2)}{\widehat{V}(\widehat{V}+\widehat{a})}\,,$$

we arrive at

$$\mathrm{LHS} - \widehat{\mathrm{LHS}} = \frac{(\widehat{a}+1)(\widehat{V}-1)(\widehat{V}+\widehat{a}+1)}{\widehat{V}(\widehat{V}+\widehat{a})} = (\widehat{a}+1)\left(1 - \frac{\widehat{a}+1}{\widehat{V}(\widehat{V}+\widehat{a})}\right)\,. \tag{8}$$

Case II Using Lemma 15 and Equation (6), we similarly obtain

$$\mathrm{LHS} - \widehat{\mathrm{LHS}} = \frac{1}{2\widehat{V}-\widehat{a}}(2\widehat{V}-\widehat{a}-1)(2\widehat{V}-\widehat{a}-2) + 12\,\mathrm{s}(\widehat{V},2\widehat{V}-\widehat{a})$$
$$- \frac{1}{\widehat{V}}(\widehat{V}-1)(\widehat{V}-2) - 12\,\mathrm{s}(\widehat{a},\widehat{V})\,.$$

Again, the reciprocity law (Lemma 14) and elementary properties of Dedekind sums (Remark 13) imply

$$12\,\mathrm{s}(\widehat{V},2\widehat{V}-\widehat{a}) = -12\,\mathrm{s}(2\widehat{V}-\widehat{a},\widehat{V}) - 3 + \frac{\widehat{a}}{2\widehat{V}-\widehat{a}} + \frac{1}{\widehat{a}(2\widehat{V}-\widehat{a})} + \frac{2\widehat{V}-\widehat{a}}{\widehat{a}}$$
$$= 12\ \mathrm{s}(\widehat{a},\widehat{V}) - 3 + \frac{\widehat{V}^2 + (2\widehat{V}-\widehat{a})^2 + 1}{\widehat{V}(2\widehat{V}-\widehat{a})}\,.$$

As

$$\frac{1}{2\widehat{V}-\widehat{a}}(2\widehat{V}-\widehat{a}-1)(2\widehat{V}-\widehat{a}-2) = \frac{1}{\widehat{V}}(\widehat{V}-1)(\widehat{V}-2) + \frac{(\widehat{V}-\widehat{a})(-\widehat{a}\widehat{V}+2\widehat{V}^2-2)}{\widehat{V}(2\widehat{V}-\widehat{a})}\,,$$

we finally obtain

$$\mathrm{LHS} - \widehat{\mathrm{LHS}} = \frac{(\widehat{V}+1)(\widehat{V}-\widehat{a}-1)(2\widehat{V}-\widehat{a}-1)}{\widehat{V}(2\widehat{V}-\widehat{a})}$$
$$= (\widehat{V}-\widehat{a}-1)\left(1 + \frac{\widehat{V}-\widehat{a}-1}{\widehat{V}(2\widehat{V}-\widehat{a})}\right)\,. \tag{9}$$

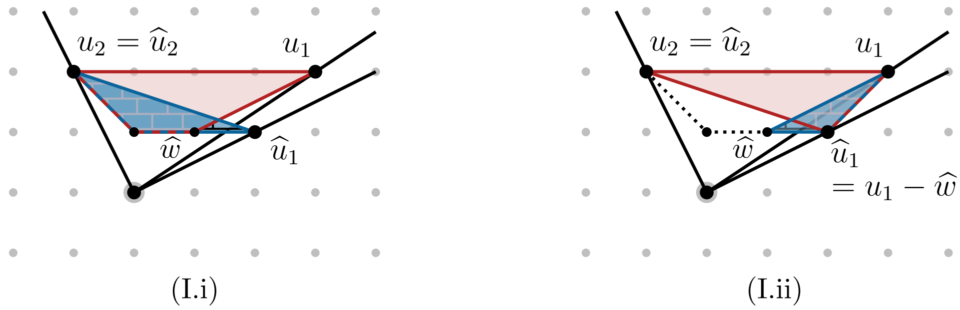

Figure 5: **Case I – Illustration of considered domains in decomposition of $\boldsymbol{\nabla_\sigma \setminus \mathrm{sail}_\sigma}$.** Here: $V = 8, a = 3$ and $\widehat{V} = 5, \widehat{a} = 3$. (I.i) $\nabla_{\widehat{\sigma}} \setminus \mathrm{sail}_{\widehat{\sigma}}$ (blue) and $\nabla_\sigma \setminus \mathrm{sail}_\sigma$ (red). (I.ii) $\mathrm{conv}(\widehat{u}_1, \widehat{u}_1 + \widehat{w}, \widehat{w})$ (blue) and $\mathrm{conv}(\widehat{u}_1, \widehat{u}_1 + \widehat{w}, \widehat{u}_2)$ (red).

4.2 Right hand side of Equation (1)

The two summands

$$\mathrm{v}\left(\nabla_\sigma \setminus \mathrm{sail}_\sigma\right) \quad \text{and} \quad \sum_{i=1}^{k_\sigma - 1} \mathrm{v}(\mathrm{conv}[m_\sigma, i])$$

on the right hand side of Equation (1) will be considered separately for each case.

Case I By our construction, we have

$$\nabla_\sigma \setminus \mathrm{sail}_\sigma = \overline{((\nabla_{\widehat{\sigma}} \setminus \mathrm{sail}_{\widehat{\sigma}}) \cup \mathrm{conv}(\widehat{u}_1, \widehat{u}_1 + \widehat{w}, \widehat{u}_2)) \setminus \mathrm{conv}(\widehat{u}_1, \widehat{u}_1 + \widehat{w}, \widehat{w})}$$

as illustrated in Figure 5. Since

$$\mathrm{conv}(\widehat{u}_1, \widehat{u}_1 + \widehat{w}, \widehat{w}) \subseteq (\nabla_{\widehat{\sigma}} \setminus \mathrm{sail}_{\widehat{\sigma}}) \cup \mathrm{conv}(\widehat{u}_1, \widehat{u}_1 + \widehat{w}, \widehat{u}_2)$$

and

$$\mathrm{v}\left((\nabla_{\widehat{\sigma}} \setminus \mathrm{sail}_{\widehat{\sigma}}) \cap \mathrm{conv}(\widehat{u}_1, \widehat{u}_1 + \widehat{w}, \widehat{u}_2)\right) = 0\,,$$

we obtain

$$\begin{aligned}
&\mathrm{v}\left(\nabla_\sigma \setminus \mathrm{sail}_\sigma\right) - \mathrm{v}\left(\nabla_{\widehat{\sigma}} \setminus \mathrm{sail}_{\widehat{\sigma}}\right) \\
&\qquad = \mathrm{v}\left(\mathrm{conv}(\widehat{u}_1, \widehat{u}_1 + \widehat{w}, \widehat{u}_2)\right) - \mathrm{v}\left(\mathrm{conv}(\widehat{u}_1, \widehat{u}_1 + \widehat{w}, \widehat{w})\right) \\
&\qquad = \det(\widehat{w}, \widehat{u}_2 - \widehat{u}_1) - \det(\widehat{w}, \widehat{w} - \widehat{u}_1) = \det(\widehat{w}, \widehat{u}_2 - \widehat{w}) = \det(\widehat{w}, \widehat{u}_2) \\
&\qquad = \frac{\widehat{a}}{\widehat{V}} \cdot \widehat{V} = \widehat{a}
\end{aligned}$$

by utilizing $\det(\widehat{u}_1, \widehat{u}_2) = \widehat{V}$.

For the second summand of the right hand side of Equation (1), we need to determine how the involved functionals behave when we change from $\widehat{\sigma}$ to σ. Recall $m_{\widehat{\sigma}} = -\widehat{u}_1^* - \widehat{u}_2^*$. Since $u_1 = \widehat{u}_1 + \ \widehat{w}$ and $u_2 = \widehat{u}_2$, we have

$$m_\sigma = -\frac{\widehat{V} - 1}{\widehat{V} + \widehat{a}} \widehat{u}_1^* - \widehat{u}_2^* = m_{\widehat{\sigma}} + \frac{\widehat{a} + 1}{\widehat{V} + \widehat{a}} \widehat{u}_1^*\,.$$

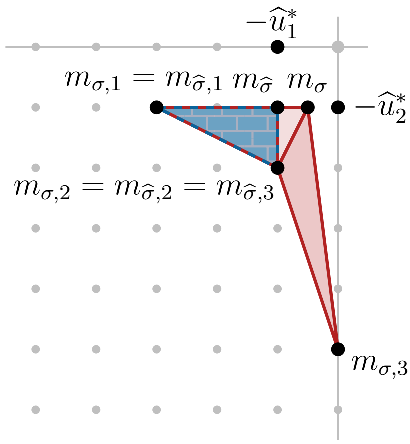

Figure 6: **Case I – dual perspective** for illustration in Figure 5 with $\cup_{i=1}^{k_{\widehat{\sigma}}-1}\operatorname{conv}[m_{\widehat{\sigma}}, i]$ (blue) and $\cup_{i=1}^{k_\sigma-1}\operatorname{conv}[m_\sigma, i]$ (red).

As explained in Section 3, the functionals $m_{\widehat{\sigma},1}, m_{\widehat{\sigma},2}, \ldots, m_{\widehat{\sigma},k_\sigma}$ may be associated with edges of $\overline{\mathrm{sail}}_{\widehat{\sigma}}$ not incident to 0 (and similarly for $\overline{\mathrm{sail}}_\sigma$). We enumerate these edges of the sail starting from the edge incident to $\widehat{u}_2$ (and finishing with the edge incident to $\widehat{u}_1$). By construction, $\overline{\mathrm{sail}}_\sigma$ has the same edges as $\overline{\mathrm{sail}}_{\widehat{\sigma}}$, except for the last edge connecting $\widehat{w}$ to $\widehat{u}_1$ and $\widehat{w}$ to $\widehat{u}_1 + \ \widehat{w}$, respectively (cf. Figure 5I.i). Accordingly, this is also true for the functionals, except that $m_{\widehat{\sigma},k_{\widehat{\sigma}}}$ is replaced by $m_{\sigma,k_\sigma} = -\widehat{V}\widehat{u}_2^*$ (cf. Figure 6). As

$$\langle m_{\widehat{\sigma},1}, \widehat{u}_2\rangle = -1 \quad \text{and} \quad \langle m_{\widehat{\sigma},1}, \frac{1}{\widehat{V}}\widehat{u}_1 + \frac{\widehat{b}}{\widehat{V}}\widehat{u}_2\rangle = -1\,,$$

we have

$$m_{\widehat{\sigma},1} = (\widehat{b} - \widehat{V})\widehat{u}_1^* - \widehat{u}_2^*\,, \tag{10}$$

where $\widehat{b} \in [1, V]$ is the multiplicative inverse of $\widehat{a}$ modulo $\widehat{V}$, i.e., $\widehat{a} \cdot \widehat{b} = 1 \mod \widehat{V}$. Furthermore,

$$m_{\widehat{\sigma},k_{\widehat{\sigma}}} = -\widehat{u}_1^* + (\widehat{a} - \widehat{V})\widehat{u}_2^* \tag{11}$$

because $\langle m_{\widehat{\sigma},k_{\widehat{\sigma}}}, \widehat{u}_1\rangle = -1$ and $\langle m_{\widehat{\sigma},k_{\widehat{\sigma}}}, \widehat{w}\rangle = \langle m_{\widehat{\sigma},k_{\widehat{\sigma}}}, \frac{\widehat{a}}{\widehat{V}}\widehat{u}_1 + \frac{1}{\widehat{V}}\widehat{u}_2\rangle = -1$. Additionally, $m_{\widehat{\sigma}}$, m_σ and $m_{\widehat{\sigma},1}$ are collinear (as they all take value -1 on $\widehat{u}_2$), see Figure 6. Therefore, with $k_{\widehat{\sigma}} = k_\sigma$, we have

$$\bigcup_{i=1}^{k_{\widehat{\sigma}}-1}\operatorname{conv}[m_{\widehat{\sigma}}, i] \subseteq \bigcup_{i=1}^{k_\sigma-1}\operatorname{conv}[m_\sigma, i]\,.$$

Hence, we get (cf. Figure 6)

$$
\begin{aligned}
&\sum_{i=1}^{k_\sigma-1} \mathrm{v}(\mathrm{conv}[m_\sigma, i]) - \sum_{i=1}^{k_{\widehat{\sigma}}-1} \mathrm{v}(\mathrm{conv}[m_{\widehat{\sigma}}, i]) \\
&= \mathrm{v}(\mathrm{conv}(m_\sigma, m_{\widehat{\sigma}}, m_{\widehat{\sigma},k_{\widehat{\sigma}}})) + \mathrm{v}(\mathrm{conv}(m_\sigma, m_{\widehat{\sigma},k_{\widehat{\sigma}}}, m_{\sigma,k_\sigma})) \\
&= \det(m_{\widehat{\sigma},k_{\widehat{\sigma}}} - m_{\widehat{\sigma}}, m_\sigma - m_{\widehat{\sigma}}) + \det(m_{\sigma,k_\sigma} - m_{\widehat{\sigma},k_{\widehat{\sigma}}}, m_\sigma - m_{\widehat{\sigma},k_{\widehat{\sigma}}}) \\
&= \det\left((\widehat{a}+1-\widehat{V})\widehat{u}_2^*, \frac{\widehat{a}+1}{\widehat{V}+\widehat{a}}\widehat{u}_1^*\right) + \det\left(\widehat{u}_1^* - \widehat{a}\widehat{u}_2^*, (\widehat{V}-\widehat{a}-1)\widehat{u}_2^* + \frac{\widehat{a}+1}{\widehat{V}+\widehat{a}}\widehat{u}_1^*\right) \\
&= -\frac{(\widehat{V}-\widehat{a}-1)(\widehat{a}+1)}{\widehat{V}+\widehat{a}}\det(\widehat{u}_2^*, \widehat{u}_1^*) - \frac{\widehat{a}(\widehat{a}+1)}{\widehat{V}+\widehat{a}}\det(\widehat{u}_2^*, \widehat{u}_1^*) + (\widehat{V}-\widehat{a}-1)\det(\widehat{u}_1^*, \widehat{u}_2^*) \\
&= \frac{(\widehat{V}-\widehat{a}-1)(\widehat{a}+1)}{\widehat{V}(\widehat{V}+\widehat{a})} + \frac{\widehat{a}(\widehat{a}+1)}{\widehat{V}(\widehat{V}+\widehat{a})} + \frac{\widehat{V}-\widehat{a}-1}{\widehat{V}} = 1 - \frac{(\widehat{a}+1)^2}{\widehat{V}(\widehat{V}+\widehat{a})}\,.
\end{aligned}
$$

Combining everything yields

$$
\mathrm{RHS} - \widehat{\mathrm{RHS}} = \widehat{a} + 1 - \frac{(\widehat{a}+1)^2}{\widehat{V}(\widehat{V}+\widehat{a})},
$$

which coincides with the corresponding difference $\mathrm{LHS} - \widehat{\mathrm{LHS}}$ in Equation (8).

Case II By our assumption (as illustrated in Figure 7), we have

$$
\nabla_\sigma \setminus \mathrm{sail}_\sigma = (\nabla_{\widehat{\sigma}} \setminus \mathrm{sail}_{\widehat{\sigma}}) \cup \mathrm{conv}(\widehat{u}_1, \widehat{u}_2, 2\widehat{u}_1 - \widehat{w})\,.
$$

Using again $\det(\widehat{u}_1, \widehat{u}_2) = \widehat{V}$, we obtain

$$
\begin{aligned}
&v\left(\nabla_\sigma \setminus \mathrm{sail}_\sigma\right) - v\left(\nabla_{\widehat{\sigma}} \setminus \mathrm{sail}_{\widehat{\sigma}}\right) = v\left(\mathrm{conv}(\widehat{u}_1, \widehat{u}_2, 2\widehat{u}_1 - \widehat{w})\right) = \det(\widehat{u}_1 - \widehat{w}, \widehat{u}_2 - \widehat{u}_1) \\
&= \det\left(\left(1 - \frac{\widehat{a}}{\widehat{V}}\right)\widehat{u}_1 - \frac{1}{\widehat{V}}\widehat{u}_2, \widehat{u}_2 - \widehat{u}_1\right) = \left(1 - \frac{\widehat{a}}{\widehat{V}}\right)\cdot\widehat{V} - \frac{1}{\widehat{V}}\cdot\widehat{V} = \widehat{V} - \widehat{a} - 1\,.
\end{aligned}
$$

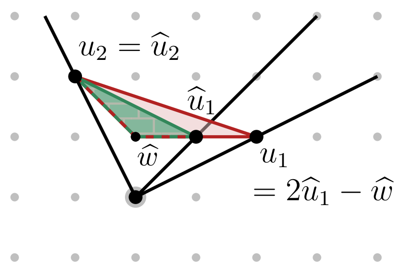

Figure 7: **Case II – Illustration of considered domains in decomposition of $\boldsymbol{\nabla_\sigma \setminus \mathrm{sail}_\sigma}$.** Here: $V = 5, a = 3$ and $\widehat{V} = 3, \widehat{a} = 1$. $\nabla_{\widehat{\sigma}} \setminus \mathrm{sail}_{\widehat{\sigma}}$ (green) and $\nabla_\sigma \setminus \mathrm{sail}_\sigma$ (red).

As in Case I, we need to determine how the functionals involved in the second summand of the RHS of Equation (1) behave when we change from $\widehat{\sigma}$ to σ. First recall that $m_{\widehat{\sigma}} = -\widehat{u}_1^* - \widehat{u}_2^*$. As $u_1 = 2\widehat{u}_1 - \widehat{w}$ and $u_2 = \widehat{u}_2$, we have

$$m_\sigma = -\frac{\widehat{V}+1}{2\widehat{V}-\widehat{a}}\widehat{u}_1^* - \widehat{u}_2^* = m_{\widehat{\sigma}} + \frac{\widehat{V}-\widehat{a}-1}{2\widehat{V}-\widehat{a}}\widehat{u}_1^* .$$

As above, we enumerate the edges of the sail that are not incident to 0 starting from the edge incident to $\widehat{u}_2$ (finishing with the edge incident to $\widehat{u}_1$). By construction, $\overline{\text{sail}}_\sigma$ has the same edges as $\overline{\text{sail}}_{\widehat{\sigma}}$ and an additional edge with functional $m_{\sigma,k_{\widehat{\sigma}}+1} = m_{\widehat{\sigma},k_{\widehat{\sigma}}}$. The other functionals are identical, see Figure 8 for an illustration. As in Case I, the functionals $m_{\widehat{\sigma},1}$ and $m_{\widehat{\sigma},k_{\widehat{\sigma}}}$ can be expressed by Equation (10) and (11). Hence,

$$\begin{aligned}
&\sum_{i=1}^{k_\sigma-1} \mathrm{v}(\mathrm{conv}[m_\sigma, i]) - \sum_{i=1}^{k_{\widehat{\sigma}}-1} \mathrm{v}(\mathrm{conv}[m_{\widehat{\sigma}}, i]) \\
&= \sum_{i=1}^{k_{\widehat{\sigma}}-1} \mathrm{v}(\mathrm{conv}(m_\sigma, m_{\widehat{\sigma},i}, m_{\widehat{\sigma},i+1})) - \sum_{i=1}^{k_{\widehat{\sigma}}-1} \mathrm{v}(\mathrm{conv}(m_{\widehat{\sigma}}, m_{\widehat{\sigma},i}, m_{\widehat{\sigma},i+1})) \\
&= \mathrm{v}(\mathrm{conv}(m_\sigma, m_{\widehat{\sigma}}, m_{\widehat{\sigma},k_{\widehat{\sigma}}})) = \det(m_\sigma - m_{\widehat{\sigma}}, m_{\widehat{\sigma},k_{\widehat{\sigma}}} - m_{\widehat{\sigma}}) \\
&= \det\left(\frac{\widehat{V}-\widehat{a}-1}{2\widehat{V}-\widehat{a}}\widehat{u}_1^*, (\widehat{V}-\widehat{a}-1)\widehat{u}_2^*\right) = \frac{(\widehat{V}-\widehat{a}-1)^2}{2\widehat{V}-\widehat{a}}\det(\widehat{u}_1^*, \widehat{u}_2^*) = \frac{(\widehat{V}-\widehat{a}-1)^2}{\widehat{V}(2\widehat{V}-\widehat{a})} .
\end{aligned}$$

Combining everything yields

$$\mathrm{RHS} - \widehat{\mathrm{RHS}} = \widehat{V} - \widehat{a} - 1 + \frac{(\widehat{V}-\widehat{a}-1)^2}{\widehat{V}(2\widehat{V}-\widehat{a})} ,$$

which coincides with the corresponding difference $\mathrm{LHS} - \widehat{\mathrm{LHS}}$ in Equation (9).

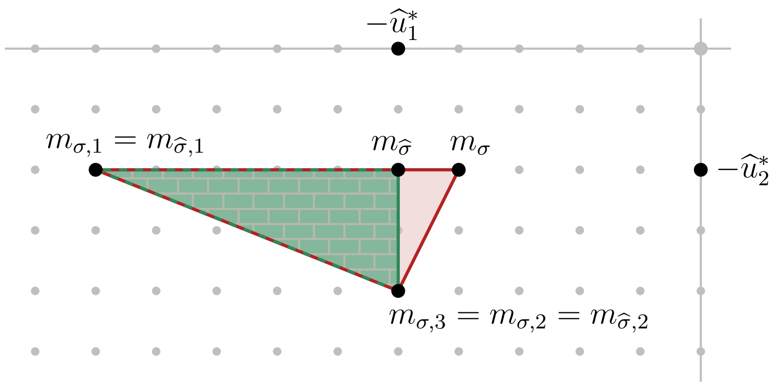

Figure 8: **Case II – dual perspective** for illustration in Figure 7 with $\cup_{i=1}^{k_{\widehat{\sigma}}-1} \mathrm{conv}[m_{\widehat{\sigma}}, i]$ (green) and $\cup_{i=1}^{k_\sigma-1} \mathrm{conv}[m_\sigma, i]$ (red).

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