Enumerating 1324-avoiders with few inversions

Svante Linusson Emil Verkama

Submitted: Sep 9, 2024; Accepted: Jul 3, 2025; Published: Sep 5, 2025

Abstract

We enumerate the numbers $\mathrm{av}_n^k(1324)$ of 1324-avoiding n-permutations with exactly k inversions for all k and $n \geqslant (k+7)/2$. The result depends on a structural characterization of such permutations in terms of a new notion of almost-decomposability. In particular, our enumeration verifies half of a conjecture of Claesson, Jelínek and Steingrímsson, according to which $\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324)$ for all n and k. Proving also the other half would improve the best known upper bound for the exponential growth rate of the number of 1324-avoiders from 13.5 to approximately 13.002.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

A permutation $\pi \in \mathfrak{S}_n$ *contains* a pattern $\tau \in \mathfrak{S}_m$ if there exist indices $i_1 < \ldots < i_m$ such that $\pi(i_a) < \pi(i_b)$ if and only if $\tau(a) < \tau(b)$ for all $a, b \in [m]$. Otherwise, π *avoids* τ. An *inversion* in π is a pair of indices (i, j) such that $i < j$ and $\pi_i > \pi_j$. We denote by $\mathrm{Av}_n(\tau)$ the set of all permutations of length n avoiding τ, and by $\mathrm{Av}_n^k(\tau) \subseteq \mathrm{Av}_n(\tau)$ those with exactly k inversions. Furthermore, we set $\mathrm{av}_n(\tau) = |\mathrm{Av}_n(\tau)|$ and $\mathrm{av}_n^k(\tau) = |\mathrm{Av}_n^k(\tau)|$. Two patterns σ and τ are called *Wilf equivalent* if $\mathrm{av}_n(\sigma) = \mathrm{av}_n(\tau)$ for all n.

1.1 Avoiding 1324

It is a well-known that

$$\mathrm{av}_n(\tau) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

for all patterns τ of length three, but determining $\mathrm{av}_n(\tau)$ for patterns of length four is much more difficult. The patterns of length four have three distinct Wilf equivalence classes (see [3, 22]), usually represented by 1234, 1342 and 1324. Exact formulas for $\mathrm{av}_n(1234)$ and $\mathrm{av}_n(1342)$ were found by Gessel in 1990 [17] and Bóna in 1997 [11], respectively,

Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden (linusson@kth.se, verkama@kth.se).

whereas 1324 has resisted all attempts at enumeration. The numbers $\mathrm{av}_n(1324)$ have been determined computationally up to $n = 50$ (sequence A061552 in the OEIS [21]), but in general not even the asymptotics are well-understood. For a thorough exposition of these topics, see Bóna [10], Kitaev [18] or Vatter [23]. The *Stanley–Wilf limit*

$$L(\tau) = \lim_{n\to\infty} \mathrm{av}_n(\tau)^{1/n},$$

exists for all patterns τ due to the Marcus–Tardos theorem [2, 20], but when $\tau = 1324$, only loose bounds are known. Table 1 shows the timeline of the evolution of these bounds; currently they are $10.27 < L(1324) < 13.5$ [5]. Since $L(1234) = 9$ and $L(1342) = 8$, 1324 is significantly easier to avoid than the other patterns of length four. Conway, Guttmann and Zinn-Justin have convincingly estimated that $L(1324) \approx 11.600 \pm 0.003$ [15, 16].

	Lower	Upper
2004. Bóna [9]		288
2005. Bóna [13]	9	
2006. Albert et al. [1]	9.47	
2012. Claesson, Jelínek and Steingrímsson [14]		16
2014. Bóna [8]		13.93
2015. Bóna [7]		13.74
2015. Bevan [4]	9.81	
2020. Bevan et al. [5]	10.27	13.5

Table 1: Best known upper and lower bounds for $L(1324)$ throughout history.

One possible avenue towards improvement is suggested by a conjecture of Claesson, Jelínek and Steingrímsson.

Conjecture 1 (Conjecture 13 in [14])**.** For all nonnegative integers n and k,

$$\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324).$$

As was demonstrated in [14], the conjecture implies a new upper bound $L(1324) \leqslant \exp\left(\pi\sqrt{2/3}\right) < 13.002$, using the fact that $\mathrm{av}_n^k(1324)$ is constant when the number k of inversions is fixed and $n \geqslant k+2$. Our main result proves half of the conjecture.

Theorem 2. *For all nonnegative integers k and $n \geqslant \frac{k+7}{2}$,*

$$\mathrm{av}_n^k(1324) = a(k) - 4a(k-n+1) - 6\sum_{i=0}^{k-n} a(i),$$

where $a(k) = \sum_{i=0}^{k} p(i)p(k-i)$ and $p(k)$ is the number of integer partitions of k. In particular,

$$\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = 4a(k-n+1) + 2a(k-n) \geqslant 0,$$

and this difference has a combinatorial interpretation.

Remark 3. In the language of generating functions, Theorem 2 states that

$$\mathrm{av}_n^k(1324) = [x^k]\left(P(x)^2 - \frac{R_n(x)}{1-x}\right)$$

whenever $n \geqslant \frac{k+7}{2}$, where $P(x) = \sum_{i\geqslant 0} p(i)x^i$ is the generating function for the partition numbers and $R_n(x) = 2(2+x)x^{n-1}P(x)^2$. In particular,

$$\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = [x^k]R_n(x).$$

This wording is more convenient for the proof, especially in Section 4.

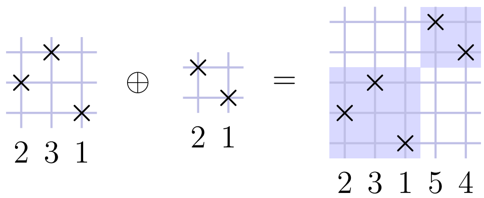
The proof relies on a new notion of *almost decomposable* permutations. The following subsection motivates this idea by defining *decomposable* permutations, and explains the constants $\mathrm{av}_{k+2}^k(1324)$.

1.2 Direct sums and decomposability

For two permutations $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$, we define the *direct sum* $\sigma \oplus \tau \in \mathfrak{S}_{n+m}$ by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } i \leqslant n, \\ n + \tau(i-n) & \text{if } i > n. \end{cases}$$

For example, $231 \oplus 21$ is obtained in the following way.



If a permutation π is the direct sum of two nonempty permutations, we call π *decomposable*, and otherwise *indecomposable*. Notice that π can be written uniquely as a direct sum

$$\pi = \pi^{(1)} \oplus \pi^{(2)} \oplus \ldots \oplus \pi^{(c)},$$

where each *component* $\pi^{(i)}$ is indecomposable. The formula (see [14, Lemma 8])

$$\mathrm{comp}(\pi) + \mathrm{inv}(\pi) \geqslant |\pi|,$$

where $\mathrm{comp}(\pi)$, $\mathrm{inv}(\pi)$ and $|\pi|$ denote the number of components, the number of inversions and the length of π, respectively, indicates that a permutation with few inversions should have many components. In particular, if $\mathrm{inv}(\pi) \leqslant |\pi| - 2$, then $\mathrm{comp}(\pi) \geqslant |\pi| - \mathrm{inv}(\pi) \geqslant 2$. It is easy to see that a decomposable permutation π avoids 1324 if and only if it is of the form

$$\pi = \pi^{(1)} \oplus 1 \oplus 1 \oplus \ldots \oplus 1 \oplus \pi^{(2)}, \tag{1}$$

where $\pi^{(1)}$ avoids 132 and $\pi^{(2)}$ avoids 213. The *inversion table* $b_1 b_2 \dots b_n$ of a 132-avoider of length n, defined by $b_i = |\{j > i : \pi_j < \pi_i\}|$, is weakly decreasing and therefore – with the exclusion of trailing 0's – a partition of $\mathrm{inv}(\pi)$. It follows that

$$\mathrm{av}_n^k(132) = \mathrm{av}_n^k(213) = p(k)$$

for all $n \geqslant k+1$. Hence, (1) gives

$$\mathrm{av}_n^k(1324) = a(k) = \sum_{i=0}^{k} p(i)p(k-i) = [x^k]P(x)^2$$

whenever $n \geqslant k+2$, where $P(x) = \sum_{i\geqslant 0} p(i)x^i$ [14, Proposition 15].

1.3 Interpreting the main result

The preceding discussion shows that Conjecture 1 holds trivially (with equality) for all $n \geqslant k+2$. Our main result, Theorem 2, improves this to $n \geqslant \frac{k+7}{2}$, and therefore proves half of the conjecture along with enumerating the corresponding values of $\mathrm{av}_n^k(1324)$. The strategy is to find an injection $\mathrm{Av}_n^k(1324) \to \mathrm{Av}_{n+1}^k(1324)$ and analyze it in order to enumerate the permutations not contained in its image. The injection relies on *almost decomposability*, which is related to normal decomposability, so it is not surprising that the partition numbers show up.

It is useful to keep in mind Table 2, in which the entry on row n and column k equals $\mathrm{av}_n^k(1324)$. Conjecture 1 is equivalent to the statement that each column of the diagram is weakly increasing as n increases. The blue cells indicate the constant parts of each column; the sequence $1, 2, 5, 10, 20, \dots$ comes from the generating function $P(x)^2$. The red cells contain the new numbers enumerated by Theorem 2. Specifically, the entry in a blue or red cell on row n and column k equals

$$a(k) - 4a(k-n+1) - 6\sum_{i=0}^{k-n} a(i).$$

The differences $\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324)$ are displayed in Table 3. The blue 0's come from the constant part of each column, and the numbers in the red cells are given by

$$4a(k-n+1) + 2a(k-n). \tag{2}$$

The diagram also shows that $n \geqslant \frac{k+7}{2}$ is the best possible bound for our method: if $n < \frac{k+7}{2}$ (and k is not too small), then $\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324)$ no longer equals (2). The data of Table 2 was provided to us by Anders Claesson; an expanded version is available at https://akc.is/inv-mono/ along with background on Conjecture 1.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	
1	1													
2	1	1												
3	1	2	2	1										
4	1	2	5	6	5	3	1							
5	1	2	5	10	16	20	20	15	9	4	1			
6	1	2	5	10	20	32	51	67	79	80	68	49	29	...
7	1	2	5	10	20	36	61	96	148	208	268	321	351	...
8	1	2	5	10	20	36	65	106	171	262	397	568	784	...
9	1	2	5	10	20	36	65	110	181	286	443	664	985	...
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	...
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	...
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	...

Table 2: The numbers $\mathrm{av}_n^k(1324)$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
1	0	1													
2	0	1	2	1											
3	0	0	3	5	5	3	1								
4	0	0	0	4	11	17	19	15	9	4	1				
5	0	0	0	0	4	12	31	52	70	76	67	49	29	14	...
6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	...
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	...
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	...
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	...
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	...
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	...
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	...

Table 3: The numbers $\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324)$.

1.4 Structure of the paper

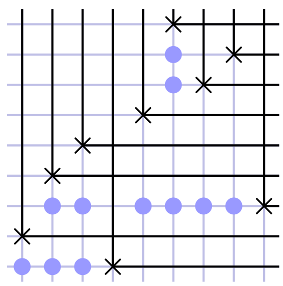
This paper is organized as follows. In Section 2, we introduce almost-decomposability and prove that all permutations in $\mathrm{Av}_n^k(1324)$ are either decomposable or almost decomposable whenever $n \geqslant \frac{k+7}{2}$. In Section 3, we construct an injection $\mathrm{Av}_n^k(1324) \to \mathrm{Av}_{n+1}^k(1324)$. The enumeration of $\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324)$ based on the injection is performed in Section 4. Finally, Section 5 contains a discussion of ideas to extend our method to prove more of Conjecture 1, reasons we have failed to do so, as well as possible improvements to the upper bound for $L(1324)$ given by the conjecture.

2 1324-avoiders with few inversions are almost decomposable

We will often utilize the *plots* $\{(i, \pi_i) : i \in [n]\}$ (in cartesian coordinates) of permutations $\pi \in \mathfrak{S}_n$. Inverting π corresponds with reflecting its plot across the line $y = x$, and the *reverse-complement* $\mathrm{rc}(\pi)_i = n + 1 - \pi_{n+1-i}$ rotates the plot by 180 degrees. Both π^{-1} and $\mathrm{rc}(\pi)$ preserve 1324-avoidance and the number of inversions of π, so these are useful operations for us.

We also use the *Rothe diagram* of π, which is obtained from the plot of π by drawing lines to north and east from each point (i, π_i), and marking the empty coordinate points – these points are the inversions of π. The following figure shows an example.

245169783 =



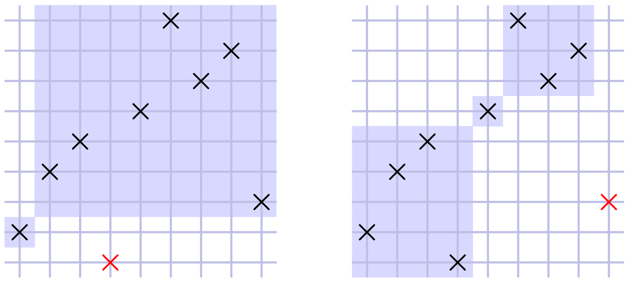
Definition 4. For $\pi \in \mathfrak{S}_n$ and $i \in [n]$, we denote by $\pi \setminus \pi_i$ the unique permutation in $\mathfrak{S}_{n-1}$ that is order-isomorphic to $\pi_1 \ldots \pi_{i-1}\pi_{i+1} \ldots \pi_n$. We say that $\pi \setminus \pi_i$ is obtained by *deleting* entry π_i from π. More generally, if $S \subseteq [n]$, then $\pi \setminus S$ is the unique permutation that is order-isomorphic to the sequence obtained by removing all entries contained in S from π.

Definition 5. A permutation $\pi \in \mathfrak{S}_n$ is called *almost decomposable* if it is indecomposable, but

$$\max\{\mathrm{comp}(\pi \setminus i) : i = 1, n, \pi_1, \pi_n\} \geqslant 2.$$

Example 6. Consider the indecomposable permutation $\pi = 245169783$. Since $\pi \setminus 1 = 13458672 = 1 \oplus 2348671$ is decomposable, π is almost decomposable. However, note that $\pi \setminus 3 = 23415867 = 2341 \oplus 1 \oplus 312$ is also decomposable.

Almost-decomposability means that deleting one of the points from the 'boundary' of the plot of the permutation makes it decomposable. Here are the plots of $\pi \setminus 1$ and $\pi \setminus 3$.



$\pi \setminus 1$ $\qquad$ $\pi \setminus 3$

An important detail in Section 3, where the injection $\text{Av}_n^k(1324) \to \text{Av}_{n+1}^k(1324)$ is constructed, is that e.g. *both* $\pi \setminus 1$ and $\pi \setminus n$ can be decomposable when π is almost decomposable. However, not all combinations are possible, and this is critical.

Proposition 7. *Let $\pi \in \mathfrak{S}_n$ be indecomposable. If $\pi \setminus 1$ is decomposable then $\pi \setminus \pi_1$ is indecomposable, and similarly if $\pi \setminus n$ is decomposable then $\pi \setminus \pi_n$ is indecomposable.*

Proof. The two parts of the statement are symmetrical, so it suffices to prove the first one. Suppose that $\pi \setminus 1 = \pi^{(1)} \oplus \pi'$, with $\pi^{(1)}$ an indecomposable permutation. We must have $\pi_1^{-1} > |\pi^{(1)}| + 1$ and $\pi_1 \leqslant |\pi^{(1)}| + 1$, so that in particular $\pi_1 < \pi_1^{-1}$. It is therefore not possible that also $\pi \setminus \pi_1$ is decomposable. □

All other combinations are, however, possible. Example 6 shows an almost decomposable permutation π for which $\pi \setminus 1$ are $\pi \setminus \pi_n$ decomposable, which is one of the four such combinations of two entries.

The goal of this section is to show that, up to the upper bound of $2n - 7$ inversions, every 1324-avoider of length n is either decomposable or almost decomposable. (We will rewrite the bound $n \geqslant \frac{k+7}{2}$ as $k \leqslant 2n - 7$ from here on.) This is the structural characterization that our proof of Theorem 2 relies on.

Theorem 8. *Each indecomposable permutation $\pi \in \text{Av}_n^k(1324)$ with $k \leqslant 2n-7$ is almost decomposable.*

Proof. If $\pi_1 < \pi_n$ and $\pi_1^{-1} < \pi_n^{-1}$, then Lemmas 9 and 13 below show that π is almost decomposable. Otherwise Lemma 14 applies. □

All of the facts needed for the above proof are obtained by counting inversions in a specific way, thus showing that all permutations with certain properties violate the bound $k \leqslant 2n - 7$. We will need to refer to certain 'regions' in the plots of our permutations. To start with, if a permutation π satisfies $\pi_1 < \pi_n$ and $\pi_1^{-1} < \pi_n^{-1}$, then any entry π_i such that

$$i < \pi_1^{-1} \qquad \text{and} \qquad \pi_i > \pi_n$$

is said to *lie in the northwestern region of* π, and if instead

$$i > \pi_n^{-1} \qquad \text{and} \qquad \pi_i < \pi_1$$

then π_i *lies in the southeastern region of* π. See Figure 1 for a visualization.

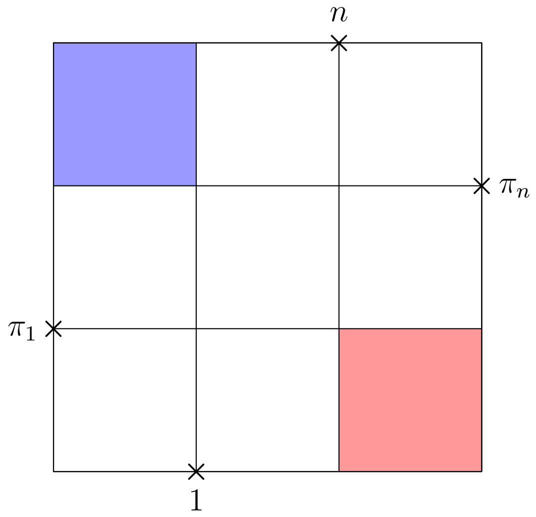


Figure 1: The northwestern and southeastern regions of a permutation π, colored blue and red, respectively.

Lemma 9. *Suppose $\pi \in \mathrm{Av}_n(1324)$ satisfies $\pi_1 < \pi_n$ and $\pi_1^{-1} < \pi_n^{-1}$. If π is indecomposable, it must have a point in its northwestern or southeastern region.*

Proof. Suppose this is not the case, and let m be the largest index such that $\pi_m < \pi_1$. We will show that $\pi_1 \dots \pi_m$ is a permutation, and therefore that π is decomposable. Indeed, we must have $\pi_i < \pi_m$ for all $\pi_1^{-1} < i < m$, as otherwise $1\pi_i\pi_m n$ forms a 1324-pattern. In particular $\pi_i < \pi_n$ for all $i \leqslant m$, and therefore $\pi_i < \pi_j$ for all $i \leqslant m$ and $j > m$; otherwise $\pi_1\pi_i\pi_j\pi_n$ is an occurrence of 1324. □

A point in the northwestern or southeastern region intuitively causes many inversions. Indeed, we will show that a 1324-avoider with at most $2n - 6$ inversions can have points in only one of the two regions.

For the remaining results, it will be convenient to make a distinction between inversions of the form (j, i) and (i, j) for a given index i. The former will be called *left-inversions* of index i, and the latter *right-inversions*. In the Rothe diagram of π, left-inversions are located to the left of the point (i, π_i), and right-inversions are below it.

Lemma 10. *If $\pi \in \mathrm{Av}_n^k(1324)$ with $k \leqslant 2n - 6$, $\pi_1 < \pi_n$ and $\pi_1^{-1} < \pi_n^{-1}$, then either the northwestern or the southeastern region of π contains no points.*

Proof. Suppose π has points π_i and π_j in the northwestern and southeastern regions, respectively. The index i has $\pi_i - i$ right-inversions, since if an index $k < i$ has $\pi_k > \pi_i$ then $\pi_1\pi_k\pi_i n$ forms a 1324-pattern. Furthermore, indices π_1^{-1} and n have $\pi_1^{-1} - 1$ and $n - \pi_n$ left-inversions, respectively. Of these inversions, (i, π_1^{-1}) and (i, n) were counted twice, so adding them up we get at least

$$\pi_i - i + \pi_1^{-1} - 1 + n - \pi_n - 2 = n + \underbrace{\pi_1^{-1} - i}_{>0} + \underbrace{\pi_i - \pi_n}_{>0} - 3 \geqslant n - 1.$$

Similarly, counting left-inversions of index 1, right-inversions of index j, and left-inversions of index π_n^{-1} gives another $n - 1$, out of which $(1, \pi_1^{-1})$, (i, j) and (π_n^{-1}, n) were counted twice. In total,

$$\mathrm{inv}(\pi) \geqslant 2(n-1) - 3 = 2n - 5.$$

The plot of π is illustrated in Figure 2. Points whose right-inversions are counted are colored blue, and points whose left-inversions are counted are colored red. The vertical blue rays indicate the possible positions of right-inversions of the corresponding point in the Rothe diagram of π; horizontal red rays contain the left-inversions. Intersections of red and blue rays correspond with double-counted inversions. The northwestern and southeastern regions are colored gray. □

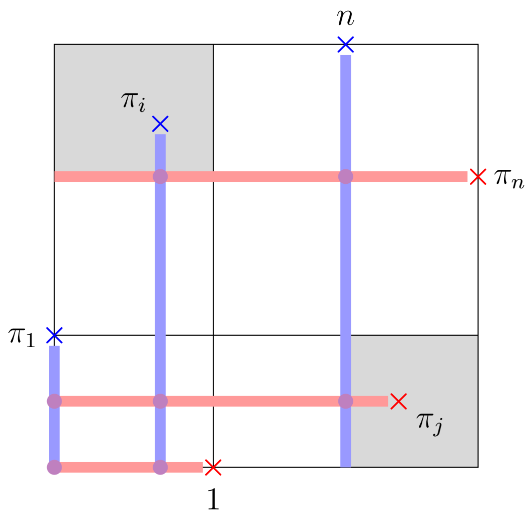


Figure 2: Counting inversions in a permutation with points in both the nortwestern and the southeastern region. Proof of Lemma 10.

Lemmas 9 and 10 together prove that if $\pi \in \mathrm{Av}_n^k(1324)$ is indecomposable, $k \leqslant 2n-6$, $\pi_1 < \pi_n$ and $\pi_1^{-1} < \pi_n^{-1}$, then π has a point either in the northwestern or the southeastern region, but not in both. Observe that π_i lies in the northwestern region of π if and only if i lies in the southeastern region of π^{-1}, so it always suffices to examine only one of the two cases.

Our following result will explain why $k \leqslant 2n-7$ is the best possible upper bound for our method. We introduce some more terminology based on Figure 3. Suppose π has a point π_i in the northwestern region. A point π_j satisfying

$$i < j < \pi_n^{-1} \qquad \text{and} \qquad 1 < \pi_j < \pi_1,$$

is said to *lie in the southern region in relation to* π_i. If

$$\pi_n^{-1} < j < n \qquad \text{and} \qquad \pi_1 < \pi_j < \pi_n,$$

then we say that π_j *lies in the eastern region in relation to* π_i. The points 1 and π_n are excluded from these regions. Lastly, notice that if π_j satisfies

$$i < j < \pi_n^{-1} \qquad \text{and} \qquad \pi_1 < \pi_j < \pi_n,$$

then $\pi_1\pi_i\pi_j n$ forms a 1324 pattern. This is why the central yellow region in Figure 3 must be empty if π avoids 1324.

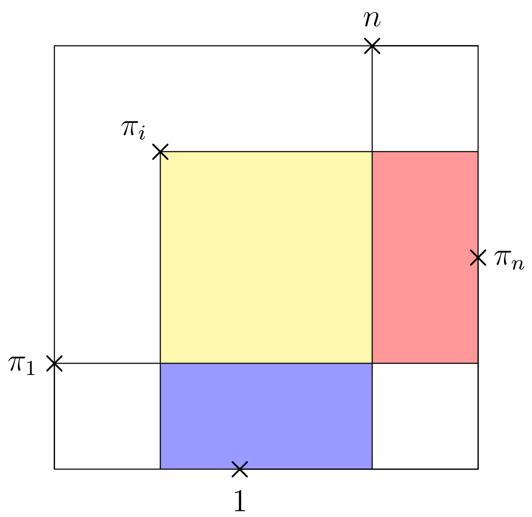


Figure 3: The southern region (in blue) and eastern region (in red) in relation to the northwestern point π_i of a permutation π. The central yellow region is empty if π avoids 1324.

Lemma 11. *If $\pi \in \mathrm{Av}_n^k(1324)$, $k \leqslant 2n-7$, with $\pi_1 < \pi_n$ and $\pi_1^{-1} < \pi_n^{-1}$ has a point π_i in the northwestern region, then there cannot exist two points π_{j_1} and π_{j_2} that lie in the southern and eastern regions in relation to π_i, respectively.*

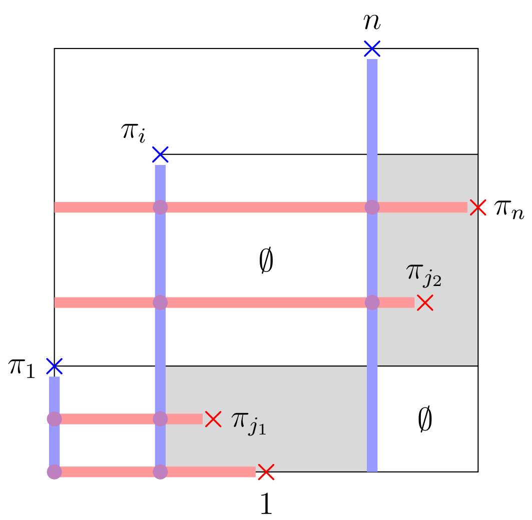


Figure 4: Counting inversions of a permutation with points in both the southern and eastern regions in relation to a point π_i in the northwestern region. Proof of Lemma 11.

Proof. Suppose that such indices j_1 and j_2 exist. See Figure 4 for the plot of π. There can be no point in the southeastern region of π by Lemma 10, and, as discussed above, the 'central' region in relation to π_i must be empty. It follows that each right-inversion of index i – except for (i, π_1^{-1}) and (i, n) – is caused by a point in the southern or eastern regions in relation to π_i. By counting the maximal number of points that can lie in these regions (and including 1 and π_n), we find that the number $\pi_i - i$ of right-inversions of index i is at most $\pi_1 - 1 + n - \pi_n^{-1}$.

Furthermore, index j_1 has at least i left-inversions, since if $j < i$ and $\pi_j < \pi_{j_1}$, then $\pi_j \pi_i \pi_{j_1} n$ forms a 1324-pattern. With a symmetrical argument, j_2 has at least $n - \pi_i + 1$

left-inversions. Adding together the right-inversions of indices i, 1 and π_n^{-1} with the left-inversions of indices π_1^{-1}, j_1, j_2 and n, accounting for the double counting of $(1, \pi_1^{-1})$, $(1, j_1)$, (i, π_1^{-1}), (i, j_1), (j_1, j_2), (i, n), (π_n^{-1}, j_1) and (π_n^{-1}, n), we finally get

$$\begin{aligned}\mathrm{inv}(\pi) \geqslant \pi_i - i \quad &+ \quad \underbrace{\pi_1 - 1 \quad + \quad n - \pi_n^{-1}}_{\geqslant \pi_i - i} \\ &+ \underbrace{\pi_1^{-1} - 1}_{\geqslant i} \quad + \quad i \quad + \quad n - \pi_i + 1 \quad + \quad \underbrace{n - \pi_n}_{\geqslant n - \pi_i + 1} \quad - \quad 8 \\ &\geqslant 2(n + i - \pi_i + 1) + 2(\pi_i - i) - 8 \\ &= 2n - 6. \end{aligned}$$

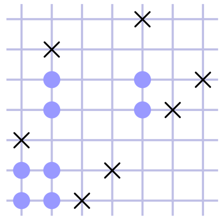
□

The bound $k \leqslant 2n - 7$ is tight, since e.g.

$$\pi = 3612 \quad 7 \dots n \quad 45$$

is 1324-avoiding with $2n - 6$ inversions, and neither decomposable nor almost decomposable. Here is the plot of the first such permutation, $\pi = 3612745$.

3612745 =



Interestingly, the same permutation π has been used before to exemplify that two 1324-avoiding permutations can have the same *profile*: π and π^{-1} have the same left-to-right minima and right-to-left maxima in the same positions [12].

Lemma 11 is sufficient for permutations with only one point in the northwestern region, but not strong enough to give almost-decomposability for permutations with several points in the northwestern region. However, as the following result shows, such a condition is even more limiting.

Lemma 12. *Let $\pi \in \mathrm{Av}_n^k(1324)$ with $k \leqslant 2n - 7$, $\pi_1 < \pi_n$, $\pi_1^{-1} < \pi_n^{-1}$, and two points $\pi_{i_1} < \pi_{i_2}$ in the northwestern region. There cannot exist two points π_{j_1} and π_{j_2} that lie in the southern region in relation to π_{i_1} and the eastern region in relation to π_{i_2}, respectively.*

Proof. Lemma 11 shows that there are no points in the eastern region in relation to π_{i_1} or the southern region in relation to π_{i_2}. It follows that the points causing right-inversions for indices i_1 and i_2 (excluding 1 and π_n) are contained in the southern and eastern gray regions of Figure 5, respectively, and none of the points creates a right-inversion for *both* i_1 and i_2. On the other hand, the points 1 and π_n create right-inversions for both i_1 and i_2, so the total number $\pi_{i_1} - i_1 + \pi_{i_2} - i_2$ of right-inversions of i_1 and i_2 satisfies

$$\pi_{i_1} - i_1 + \pi_{i_2} - i_2 \leqslant \pi_1 - 1 + n - \pi_n^{-1} + 2.$$

Counting inversions of π as in the previous proof, we get

$$\begin{aligned}\operatorname{inv}(\pi) \geqslant \pi_{i_1} - i_1 \;+\; \pi_{i_2} - i_2 \;+\; \underbrace{\pi_1 - 1 \;+\; n - \pi_n^{-1}}_{\geqslant \pi_{i_1} - i_1 + \pi_{i_2} - i_2 - 2} \\ +\; \underbrace{\pi_1^{-1} - 1}_{\geqslant i_2} \;+\; i_1 \;+\; n - \pi_{i_2} + 1 \;+\; \underbrace{n - \pi_n}_{\geqslant n - \pi_{i_1} + 1} \;-\; 10 \\ \geqslant 2(\pi_{i_1} - i_1 + \pi_{i_2} - i_2) + i_2 + i_1 + n - \pi_{i_2} + n - \pi_{i_1} - 10 \\ = 2n + \underbrace{\pi_{i_1} - i_1}_{\geqslant 3} + \underbrace{\pi_{i_2} - i_2}_{\geqslant 3} - 10 \\ \geqslant 2n - 4.\end{aligned}$$

Figure 5 shows the inversions we counted. □

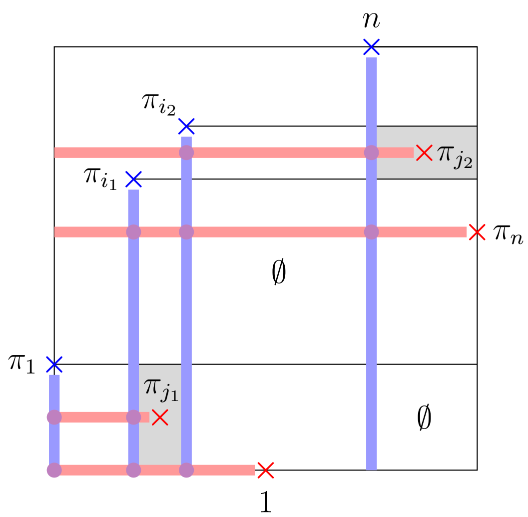


Figure 5: Counting inversions of a permutation with points in the southern and eastern regions in relation to the points π_{i_1} and π_{i_2} in the northwestern region. Proof of Lemma 12.

We are now prepared to prove one of our central lemmas.

Lemma 13. *If $\pi \in \operatorname{Av}_n^k(1324)$ with $k \leqslant 2n - 7$ and $\pi_1 < \pi_n$, $\pi_1^{-1} < \pi_n^{-1}$ has a point in its northwestern region, then $\pi \setminus 1$ or $\pi \setminus \pi_n$ is decomposable.*

Proof. Let $i_1 < i_2 < \ldots < i_m$ be the set of all indices whose points lie in the northwestern region of π. First, assume that no point lies in the southern region in relation to π_{i_1} and denote $\sigma = \pi \setminus 1$. We claim that $\sigma_1 \ldots \sigma_{i_1 - 1}$ is a permutation, which in turn would imply that σ is decomposable. Indeed, otherwise there exists indices j_1 and j_2 such that

$$j_1 < i_1 < j_2 \qquad \text{and} \qquad \pi_{j_1} > \pi_{j_2}.$$

However, we assumed that the southern region in relation to i_1 is empty, so $\pi_{j_2} > \pi_1$ and therefore $\pi_1 \pi_{j_1} \pi_{j_2} \pi_n$ forms a 1324 pattern. Figure 6 visualizes the argument: the red points π_{j_1} and π_{j_2} are in impossible positions, implying that the red region in the bottom right must be empty and therefore that the blue region to its left is a permutation in σ.

If there is a point in the southern region in relation to π_{i_1}, then Lemmas 11 and 12 prove that the eastern region in relation to π_{i_m} is empty instead. This case is symmetrical to the first one after reverse-complementation and inversion; more precisely, $\mathrm{rc}(\pi)^{-1}$ gets us back to the first case and therefore

$$\pi \setminus \pi_n = \mathrm{rc}\left(\mathrm{rc}(\pi)^{-1} \setminus 1\right)^{-1}$$

is decomposable. □

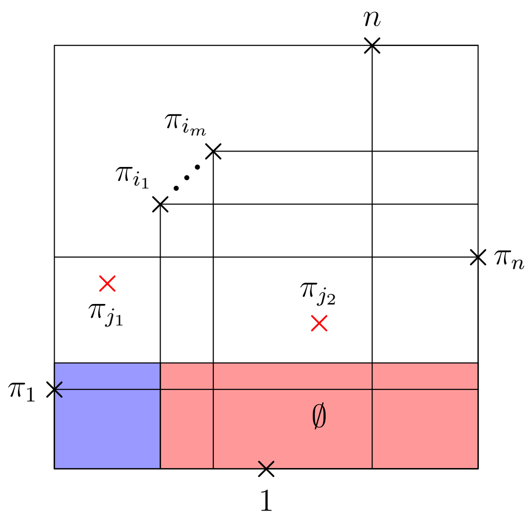


Figure 6: A permutation π satisfying the assumptions of Lemma 13 is almost decomposable, since $\pi \setminus 1$ or $\pi \setminus \pi_n$ is decomposable. The points π_{j_1} and π_{j_2} cannot be placed as they are in the picture, since they create a 1324 pattern.

So far, we have always assumed that $\pi_1 < \pi_n$ and $\pi_1^{-1} < \pi_n^{-1}$. We will now assume the opposite, leading to the other half of Theorem 8. This turns out to be much easier, since if e.g. $\pi_1 > \pi_n$, then deleting π_1 and π_n gets rid of a lot of inversions.

Lemma 14. *If $\pi \in \mathrm{Av}_n^k(1324)$, $k \leqslant 2n-5$ and $\pi_1 > \pi_n$, then $\pi \setminus \pi_1$ or $\pi \setminus \pi_n$ is decomposable.*

Proof. Let $\delta = \pi_1 - \pi_n$ and $\sigma = \pi \setminus \{\pi_1, \pi_n\}$. The right-inversions of index 1 and left-inversions of index n in π sum up to

$$\pi_1 - 1 + n - \pi_n - 1 = n - 2 + \delta,$$

so

$$\mathrm{comp}(\sigma) \geqslant |\sigma| - \mathrm{inv}(\sigma) \geqslant n - 2 - (2n - 5 - n + 2 - \delta) = \delta + 1 \geqslant 2.$$

Write $\sigma = \sigma^{(1)} \oplus 1 \oplus \ldots \oplus 1 \oplus \sigma^{(2)}$. We must have $\pi_1 < n - |\sigma^{(2)}| + 1$ or $\pi_n > |\sigma^{(1)}|$, since otherwise

$$\pi_1 - \pi_n \geqslant n - |\sigma^{(2)}| + 1 - |\sigma^{(1)}| = \delta + 2.$$

If $\pi_1 < n - |\sigma^{(2)}| + 1$ then $\pi \setminus \pi_n$ is decomposable, and if $\pi_n > |\sigma^{(1)}|$ then $\pi \setminus \pi_1$ is decomposable. □

3 The injection

Denote by $\mathcal{D}_n^k$ and $\mathcal{A}_n^k$ the sets of decomposable and almost decomposable permutations in $\mathrm{Av}_n^k(1324)$, respectively. In this section we will construct injections

$$g : \mathcal{D}_n^k \longrightarrow \mathrm{Av}_{n+1}^k(1324) \qquad \text{and} \qquad f : \mathcal{A}_n^k \longrightarrow \mathrm{Av}_{n+1}^k(1324),$$

with disjoint images, for all n and k. If $k \leqslant 2n-7$ then all permutations in $\mathrm{Av}_n^k(1324)$ are decomposable or almost decomposable by Theorem 8, so our mappings combine to an injection

$$\mathrm{Av}_n^k(1324) \longrightarrow \mathrm{Av}_{n+1}^k(1324).$$

In particular, this verifies Conjecture 1 for all $k \leqslant 2n-7$.

First of all, any $\pi \in \mathcal{D}_n^k$ can be written in the form

$$\pi = \pi^{(1)} \oplus \underbrace{1 \oplus \ldots \oplus 1}_{m \text{ times}} \oplus \pi^{(2)}$$

for some $m \geqslant 0$ by (1). This allows us to set

$$g(\pi) = \pi^{(1)} \oplus \underbrace{1 \oplus \ldots \oplus 1}_{m+1 \text{ times}} \oplus \pi^{(2)} \in \mathcal{D}_{n+1}^k.$$

The image $g(\mathcal{D}_n^k)$ is exactly the set of all permutations in $\mathrm{Av}_{n+1}^k(1324)$ with at least three components, and g is clearly injective. Note that when $n \geqslant k+2$, g is a bijection.

We will define f in a similar way. Let $\pi \in \mathcal{A}_n^k$.

1. If $\pi \setminus \pi_1$ is decomposable, let $f(\pi)$ be the permutation with $f(\pi)_1 = \pi_1$ and $f(\pi) \setminus \pi_1 = g(\pi \setminus \pi_1)$.

2. If $\pi \setminus 1$ is decomposable, let $f(\pi) = f(\pi^{-1})^{-1}$.

3. Otherwise, let $f(\pi) = (\mathrm{rc} \circ f \circ \mathrm{rc})(\pi)$, where $\mathrm{rc}(\pi)$ is the reverse-complement.

Remark 15. Let $\pi \in \mathcal{A}_n^k$.

(a) It is impossible for both $\pi \setminus 1$ and $\pi \setminus \pi_1$ to be decomposable by Proposition 7, so parts 1 and 2 of the definition are exclusive.

(b) If $\pi \setminus 1$ and $\pi \setminus \pi_1$ are indecomposable, then $\pi \setminus n$ or $\pi \setminus \pi_n$ must be decomposable, and it follows that $\mathrm{rc}(\pi) \setminus 1$ or $\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1$ is decomposable. This is why $(f \circ \mathrm{rc})(\pi)$ exists when it is used in part 3.

(c) If $\pi \setminus \pi_1$ is decomposable then $\pi_1 > \pi_2$, and therefore $f(\pi)$ avoids 1324. The 1324-avoiders are closed under inversion and taking reverse-complements, so $f(\pi)$ avoids 1324 also in parts 2 and 3.

(d) The number of inversions is preserved: in part 1,

$$\mathrm{inv}(f(\pi)) = \mathrm{inv}(g(\pi \setminus \pi_1)) + \pi_1 - 1 = \mathrm{inv}(\pi \setminus \pi_1) + \pi_1 - 1 = \mathrm{inv}(\pi).$$

Taking the inverse or the reverse-complement preserves the number of inversions, so this is true also for parts 2 and 3.

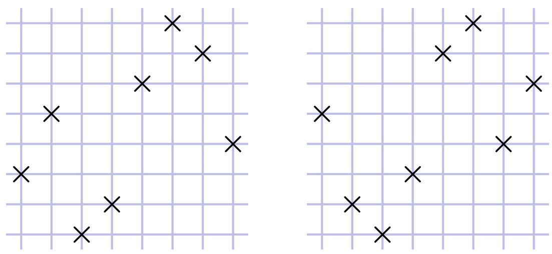
(e) $f(\pi)$ has at most two components: in part 1, if $f(\pi) = \pi^{(1)} \oplus 1 \oplus \pi'$ then $\pi = \pi^{(1)} \oplus \pi'$, a contradiction. Taking the inverse or the reverse-complement preserves the number of components, so this is true also for parts 2 and 3.

(f) In part 3, if $\pi \setminus \pi_n$ is decomposable then

$$\begin{aligned} f(\pi)_{n+1} = (\mathrm{rc} \circ f \circ \mathrm{rc})(\pi)_{n+1} &= n + 2 - (f \circ \mathrm{rc})(\pi)_{n+2-n-1} \\ &= n + 2 - \mathrm{rc}(\pi)_1 \\ &= n + 2 - (n + 1 - \pi_{n+1-1}) = \pi_n + 1. \end{aligned}$$

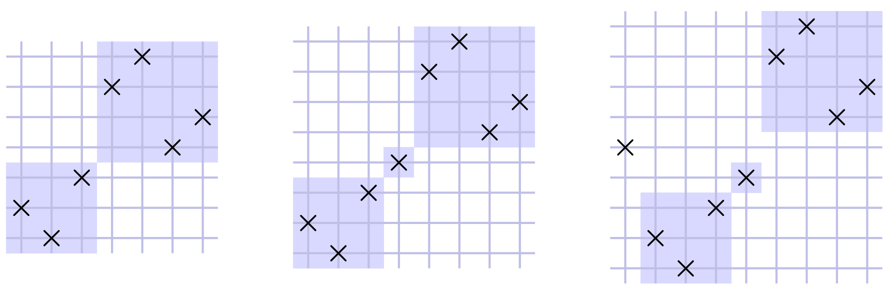
In part 2 it is clear that $f(\pi)^{-1}_1 = \pi^{-1}_1$, so if instead $\pi \setminus n$ is decomposable in part 3 then similarly $f(\pi)^{-1}_{n+1} = \pi^{-1}_n + 1$.

Example 16. Consider the permutation $\pi = 35126874 \in \mathrm{Av}^8_8(1324)$. Here are the plots of π and $\mathrm{rc}(\pi)$.



π $\qquad$ $\mathrm{rc}(\pi)$

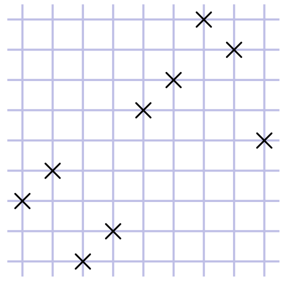
We can see that $\pi \setminus 1$ and $\pi \setminus \pi_1$ are both indecomposable, whereas $\pi \setminus \pi_n$ is decomposable. Therefore $\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1$ is decomposable. The following figure shows the permutations $\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1$, $g(\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi_1))$ and $(f \circ \mathrm{rc})(\pi)$.



$\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1$ $\qquad$ $g(\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1)$ $\qquad$ $(f \circ \mathrm{rc})(\pi)$

Finally, we get the following.

$$f(\pi) = (\mathrm{rc} \circ f \circ \mathrm{rc})(\pi) = 341267985 \quad =$$



In order to show that f is injective, we will construct its inverse. The natural way to obtain π from $f(\pi) =: \sigma$ is by reversing the steps used to construct σ: first delete a well-chosen entry in $\{1, n+1, \sigma_1, \sigma_{n+1}\}$ from σ, then apply g^{-1} on the resulting permutation (which has at least three components), and finally insert back the deleted entry. The choice of entry to delete should mirror the definition of f: deleting 1 or σ_1 is always prioritized over deleting n or σ_n.

However, it is not obvious that this works. Suppose, for example, that $\pi \setminus 1$ and $\pi \setminus \pi_1$ are indecomposable, and $\pi \setminus n$ is decomposable. This means that $\sigma \setminus (n+1)$ has at least three components, and deleting $n+1$ is indeed correct. But is it possible that $\sigma \setminus 1$ or $\sigma \setminus \sigma_1$ also have at least three components? If, say, $\sigma \setminus 1$ had at least three components, we would prioritize deleting 1 over deleting $n+1$, and (after applying g^{-1} and inserting back the entry 1) obtain some permutation $\tau \in \mathcal{A}_n^k$ such that $\tau \setminus 1$ is decomposable. In particular $\tau \neq \pi$, so our supposed inverse of f would not work.

To show that f is injective, it is therefore crucial to show that the situation we described never happens. In other words, if $\pi \setminus \pi_1$ is indecomposable, then $f(\pi) \setminus f(\pi)_1$ must have at most two components. We prove this in Lemma 18, after a simple intermediate result.

Lemma 17. *Let $\pi \in \mathcal{A}_n^k$ and $i \in [n]$.*

(a) If $\pi \setminus \pi_1$ is decomposable and $\pi_i > \pi_1$, then $f(\pi)_{i+1} = \pi_i + 1$.

(b) If $\pi \setminus \pi_1$ and $\pi \setminus 1$ are indecomposable, $\pi \setminus \pi_n$ is decomposable, and $\pi_i < \pi_n$, then $f(\pi)_i = \pi_i$.

Proof. For (a) we have $(\pi \setminus \pi_1)_{i-1} = \pi_i - 1$. The length of the first component of $\pi \setminus \pi_1$ is strictly less than π_1, so $(f(\pi) \setminus \pi_1)_i = \pi_i$ and $f(\pi)_{i+1} = \pi_i + 1$.

For part (b), $f(\pi) = (\mathrm{rc} \circ f \circ \mathrm{rc})(\pi)$ by definition of f, where $\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1$ is decomposable. Since $\mathrm{rc}(\pi)_1 = n + 1 - \pi_n$ and

$$\mathrm{rc}(\pi)_{n+1-i} = n + 1 - \pi_{n+1-(n+1-i)} = n + 1 - \pi_i,$$

our assumption implies that $\mathrm{rc}(\pi)_{n+1-i} > \mathrm{rc}(\pi)_1$. Part (a) gives

$$(f \circ \mathrm{rc})(\pi)_{n+2-i} = \mathrm{rc}(\pi)_{n+1-i} + 1 = n + 2 - \pi_i,$$

and therefore

$$f(\pi)_i = (\mathrm{rc} \circ f \circ \mathrm{rc})(\pi)_i = n + 2 - (f \circ \mathrm{rc})(\pi)_{n+2-i} = \pi_i. \qquad \square$$

Lemma 18. *Let $\pi \in \mathcal{A}_n^k$. If $\pi \setminus \pi_1$ is indecomposable, then $f(\pi) \setminus f(\pi)_1$ has at most two components.*

Proof. Denote $\sigma = f(\pi)$ and $\tau = \sigma \setminus \sigma_1$. First, if $\pi \setminus 1$ is decomposable then $\sigma \setminus 1$ has at least three components and the claims follows from Proposition 7.

Suppose instead that $\pi \setminus \pi_n$ is decomposable, and assume for the sake of a contradiction that τ has at least three components. Denote the length of the first component of τ by m. Since the last $n - \tau_n + 1$ entries of τ must be contained in one component we have $\tau_n > m + 1$, which implies that $\sigma_{n+1} > m + 1$ and therefore that $\pi_n > m$ by Remark 15 (f). Lemma 17 (a) shows that $\sigma_i = \pi_i$ for all i for which $\pi_i \leqslant m$, and we know that $\sigma_2 \dots \sigma_{m+1} = \tau_1 \dots \tau_m$ is a permutation, so we get $\pi_2 \dots \pi_{m+1} = \tau_1 \dots \tau_m$. Therefore $\pi \setminus \pi_1$ is decomposable, contradicting our original assumption.

Lastly, suppose that $\pi \setminus n$ is decomposable, and assume again that τ has at least three components, the first of which has length m. In this case we have $\tau_n^{-1} > m + 1$ which implies that $\sigma_{n+1}^{-1} > m + 2$ (since an entry is added to the beginning) and therefore that $\pi_n^{-1} > m + 1$ by Remark 15 (f). Applying Lemma 17 (b) to π^{-1} shows that $\sigma_i^{-1} = \pi_i^{-1}$ for all i such that $\pi_i^{-1} \leqslant m+1$. Since $\sigma_2 \dots \sigma_{m+1}$ is a permutation, the condition $\sigma_i^{-1} \leqslant m+1$ holds for all $i \in [m]$ and hence $\pi_1^{-1} \dots \pi_m^{-1} = \sigma_1^{-1} \dots \sigma_m^{-1}$. We conclude that $\pi_2 \dots \pi_{m+1} = \tau_1 \dots \tau_m$ is a permutation, and thus $\pi \setminus \pi_1$ is decomposable. □

Theorem 19. *The function $f : \mathcal{A}_n^k \to \mathrm{Av}_{n+1}^k(1324)$ is injective for all n and k. Furthermore, $\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324)$ whenever $k \leqslant 2n - 7$.*

Proof. We will define a function $h : f(\mathcal{A}_n^k) \to \mathcal{A}_n^k$ by reversing the steps used to define f, and afterwards prove that h is the inverse of f. Suppose that $\sigma \in f(\mathcal{A}_n^k)$.

- If $\sigma \setminus \sigma_1$ has at least three components, let $h(\sigma)$ be the permutation with $h(\sigma)_1 = \sigma_1$ and $h(\sigma) \setminus \sigma_1 = g^{-1}(\sigma \setminus \sigma_1)$.
- If $\sigma \setminus 1$ has at least three components, let $h(\sigma) = h(\sigma^{-1})^{-1}$.
- Otherwise, let $h(\sigma) = (\mathrm{rc} \circ h \circ \mathrm{rc})(\sigma)$.

Since at least one of the permutations $\sigma \setminus 1$, $\sigma \setminus \sigma_1$, $\sigma \setminus (n+1)$ or $\sigma \setminus \sigma_{n+1}$ has at least three components, we can always construct $h(\sigma)$ according to the rules above. We will now prove that h is the inverse of f. Let $\pi \in \mathcal{A}_n^k$.

- If $\pi \setminus \pi_1$ is decomposable then $f(\pi) \setminus f(\pi)_1$ has at least three components by the definition of f, and it is easy to see that $(h \circ f)(\pi) = \pi$.
- If $\pi \setminus 1$ is decomposable then $f(\pi) \setminus 1$ has at least three components, and $f(\pi) \setminus f(\pi)_1$ has at most two components by Lemma 18, so
$$(h \circ f)(\pi) = h\big(f(\pi)^{-1}\big)^{-1} = h\big(f\big(\pi^{-1}\big)\big)^{-1} = \pi.$$
- If $\pi \setminus 1$ and $\pi \setminus \pi_1$ are indecomposable, then both $f(\pi) \setminus 1$ and $f(\pi) \setminus f(\pi)_1$ have at most two components by Lemma 18. It follows that
$$(h \circ f)(\pi) = (\mathrm{rc} \circ h \circ \mathrm{rc} \circ \mathrm{rc} \circ f \circ \mathrm{rc})(\pi) = \pi.$$

We conclude that f has a left-inverse, and is therefore injective. All permutations in its image have at most two components by Remark 15 (e), whereas the permutations in the image of g all have at least three components. Hence the images of f and g are disjoint, and the mapping

$$\mathcal{D}_n^k \cup \mathcal{A}_n^k \longrightarrow \mathrm{Av}_{n+1}^k(1324)$$

given by combining f and g is injective. If $k \leqslant 2n-7$ then $\mathcal{D}_n^k \cup \mathcal{A}_n^k = \mathrm{Av}_n^k(1324)$ by Theorem 8, so $\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324)$. □

4 Enumerating the difference

The goal of this section is to describe the set of permutations

$$\mathcal{R}_{n+1}^k := \mathrm{Av}_{n+1}^k(1324) \setminus \big(g(\mathcal{D}_n^k) \cup f(\mathcal{A}_n^k)\big)$$

for all $k \leqslant 2n-7$. One obvious set of permutations in $\mathcal{R}_{n+1}^k$ are all $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ with $\sigma_1 = n+1$ or $\sigma_{n+1} = 1$. To understand the remaining, first recall that g is a bijection from $\mathcal{D}_n^k$ to permutations in $\mathrm{Av}_{n+1}^k(1324)$ with at least three components. Key to the discussion is a natural extension of f^{-1}. Let $\mathcal{B}_{n+1}^k$ denote the set of permutations $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ such that $\mathrm{comp}(\sigma) \leqslant 2$, $\sigma_1 \neq n+1$, $\sigma_{n+1} \neq 1$, and

$$\max\{\mathrm{comp}(\sigma \searrow i) : i = 1, n+1, \sigma_1, \sigma_{n+1}\} \geqslant 3. \tag{3}$$

Then, as for f^{-1}, we can define $h : \mathcal{B}_{n+1}^k \to \mathcal{A}_n^k$ as follows:

- If $\mathrm{comp}(\sigma \searrow \sigma_1) \geqslant 3$, let $h(\sigma)$ be the permutation with $h(\sigma)_1 = \sigma_1$ and $h(\sigma) \searrow \sigma_1 = g^{-1}(\sigma \searrow \sigma_1)$.
- If $\mathrm{comp}(\sigma \searrow 1) \geqslant 3$, let $h(\sigma) = h(\sigma^{-1})^{-1}$.
- Otherwise, let $h(\sigma) = (\mathrm{rc} \circ h \circ \mathrm{rc})(\sigma)$.

Clearly $h|_{f(\mathcal{A}_n^k)} = f^{-1}$. Furthermore, Proposition 24 will show that whenever $k \leqslant 2n-7$, *all* permutations $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ with $\mathrm{comp}(\sigma) \leqslant 2$ satisfy condition (3). Hence

$$f(\mathcal{A}_n^k) \cup \{\sigma \in \mathcal{R}_{n+1}^k : \sigma_1 \neq n+1 \text{ and } \sigma_{n+1} \neq 1\} = \mathcal{B}_{n+1}^k,$$

and we get

$$\{\sigma \in \mathcal{R}_{n+1}^k : \sigma_1 \neq n+1 \text{ and } \sigma_{n+1} \neq 1\} = \{\sigma \in \mathcal{B}_{n+1}^k : f(h(\sigma)) \neq \sigma\}. \tag{4}$$

Determining the permutations in the right-hand side of (4) is a manageable task.

We will assume that $k \leqslant 2n-7$ throughout the remainder of this section. With the preceding discussion in mind, we reiterate that $\mathcal{R}_{n+1}^k$ consists of the following collections.

R1 Permutations $\sigma \in \mathrm{Av}^k_{n+1}(1324)$ with $\sigma_1 = n+1$ or $\sigma_{n+1} = 1$. In the first case

$$\mathrm{inv}(\sigma \searrow \sigma_1) = k - n \leqslant n - 7,$$

i.e. $\sigma \searrow \sigma_1$ is decomposable, and the other case is symmetrical, so these permutations are enumerated by

$$[x^k]\big(2x^n P(x)^2\big),$$

where $P(x) = \sum_{i \geqslant 0} p(i)x^i$ is the generating function for the partition numbers.

R2 Permutations $\sigma \in \mathrm{Av}^k_{n+1}(1324)$ with $\mathrm{comp}(\sigma) \leqslant 2$, $\sigma_1 \neq n+1$, $\sigma_{n+1} \neq 1$, and $f(h(\sigma)) \neq \sigma$. Again, $h(\sigma)$ is always well-defined by Proposition 24.

We will further split class R2 into two subcollections to make its treatment easier:

R2a Permutations $\sigma \in \mathrm{Av}^k_{n+1}(1324)$ with $\sigma_2 = n+1$ or $\sigma_{n+1} = 2$. In the first case $\mathrm{comp}(\sigma \backslash n+1) \geqslant 3$, so $h(\sigma)_1 = n$, implying that $h(\sigma) \searrow h(\sigma)_1$ is decomposable and thus $f(h(\sigma))_1 = n$. This means that $f(h(\sigma)) \neq \sigma$, because if $\sigma_1 = n$ then $\sigma \searrow (n+1)$ can not be decomposable. The other case is symmetrical. Similarly to R1, this class is enumerated by

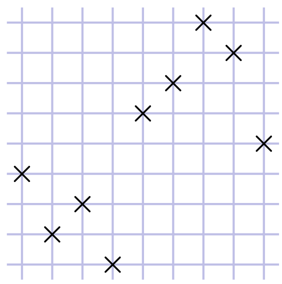
$$[x^k]\big(2x^{n-1} P(x)^2\big).$$

R2b Permutations $\sigma \in \mathrm{Av}^k_{n+1}(1324)$ with $\mathrm{comp}(\sigma) \leqslant 2$ and $\sigma_1, \sigma_2 \neq n+1$, $\sigma_{n+1} \neq 1, 2$, such that $f(h(\sigma)) \neq \sigma$. These permutations are described below, and their enumeration

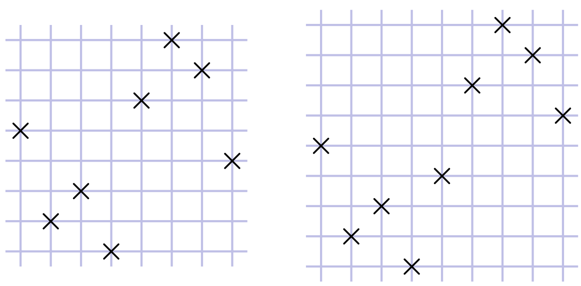
$$[x^k]\big(2x^{n-1} P(x)^2\big)$$

is obtained in Proposition 21.

We will start with an example of a class R2b permutation.

$\sigma = 423167985 \quad =$ 

Using the definitions of f, with $\mathrm{comp}(\sigma \backslash \sigma_9) \geqslant 3$ and h, with $\mathrm{comp}(h(\sigma) \backslash h(\sigma)_1) \geqslant 3$, we obtain the following permutations.



$h(\sigma)$ $\qquad$ $f(h(\sigma))$

Clearly $f(h(\sigma)) \neq \sigma$. Observe the structure of σ: σ_1 is placed precisely above the first component of $\sigma \setminus \{\sigma_1, \sigma_{n+1}\}$, and $\sigma_{n+1} = \sigma_1 + 1$. This is not a coincidence, as it turns out that up to taking inverses, all class R2b permutations have this structure.

Lemma 20. *If $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ is a class R2b permutation, then*

(a) $\mathrm{comp}(\sigma \setminus \sigma_{n+1}) \geqslant 3$*, $\sigma_1 = m+1$ and $\sigma_{n+1} = m+2$, where m is the length of the first component of $\sigma \setminus \sigma_{n+1}$, or*

(b) $\mathrm{comp}(\sigma \setminus (n+1)) \geqslant 3$*, $\sigma_1^{-1} = m+1$ and $\sigma_{n+1}^{-1} = m+2$, where m is the length of the first component of $\sigma \setminus (n+1)$.*

Proof. Let $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ be a class R2b permutation and denote $\pi = h(\sigma)$. First observe that if $\sigma \setminus \sigma_1$ (resp. $\sigma \setminus 1$) has at least three components, then $\pi \setminus \pi_1$ (resp. $\pi \setminus 1$) is decomposable and $f(\pi) = \sigma$, i.e. σ is not a class R2b permutation – this is easy to see from the definitions of f and h. We will therefore assume throughout the proof that $\sigma \setminus \sigma_1$ and $\sigma \setminus 1$ have at most two components.

Therefore $\sigma \setminus \sigma_{n+1}$ or $\sigma \setminus (n+1)$ has at least three components. Assume the former. Observe that if both $\pi \setminus \pi_1$ and $\pi \setminus 1$ are indecomposable then $f(\pi) = \sigma$, so one of the two must be decomposable.

- Suppose first that $\pi \setminus \pi_1$ is decomposable. Consider the permutation $\tau = \pi \setminus \{\pi_1, \pi_n\}$ and write $\tau = \tau^{(1)} \oplus 1 \oplus \ldots \oplus 1 \oplus \tau^{(2)}$ for its decomposition into components, $m = |\tau^{(1)}|$ and $\ell = |\tau^{(2)}|$. We must have $\pi_1 > \pi_n$ in order for both $\pi \setminus \pi_1$ and $\pi \setminus \pi_n$ to be decomposable. Furthermore, if $\pi_1 \leqslant m+1$ then $\pi_n \leqslant m$ and $\pi \setminus \pi_1$ cannot be decomposable. Therefore $\pi_1 \geqslant m+2$, and with a symmetrical argument $\pi_n \leqslant n - \ell - 1$. Figure 7 visualizes the structure of π. Using the definition of h, if $b \geqslant 1$ (resp. $b = 0$) then σ is of the form described in Figure 8 (resp. Figure 9), where $b = \pi_1 - \pi_n - 1$.

  If $b \geqslant 1$, the increasing sequence after $\tau^{(1)}$ in σ is of length $a + 1 \geqslant 1$, which means that $\sigma \setminus \sigma_1$ has at least three components, contradicting our assumption. Therefore $b = 0$. Similarly in this case, if $a \geqslant 1$ then $\sigma \setminus \sigma_1$ has at least three components. Hence $a = 0$, giving us condition (a).

- One can show that $\pi \setminus 1$ cannot be decomposable with a similar structural analysis.

If instead $\mathrm{comp}(\sigma \setminus (n+1)) \geqslant 3$, condition (b) follows from a symmetrical argument, concluding the proof. □

Proposition 21. *If $k \leqslant 2n - 7$, the number of class R2b permutations in $\mathrm{Av}_{n+1}^k(1324)$ equals*

$$[x^k]\big(2x^{n-1}P(x)^2\big).$$

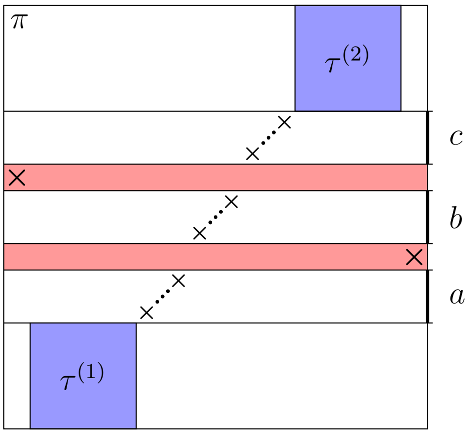


Figure 7: A permutation $\pi = h(\sigma)$ satisfying assumptions from the proof of Lemma 20. We know that π_n lies above $\tau^{(1)}$, π_1 lies below $\tau^{(2)}$ and $\pi_1 > \pi_n$. The lengths of the increasing subsequences located (in height) from $\tau^{(1)}$ to π_n, π_n to π_1 and π_1 to $\tau^{(2)}$ are denoted a, b and c, respectively.

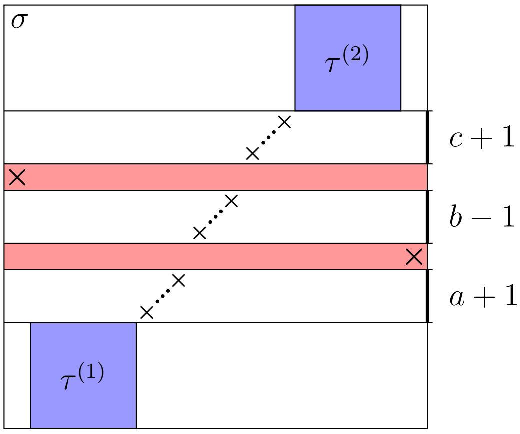


Figure 8: The permutation σ corresponding to π from Figure 7. If $b \geqslant 1$, the lengths of the increasing sequences change from a, b and c in π to $a+1$, $b-1$ and $c+1$ in σ, respectively.

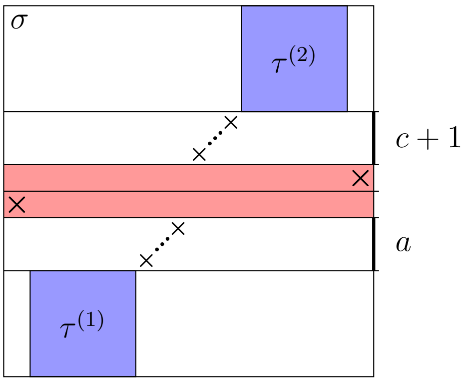


Figure 9: The permutation σ corresponding to π from Figure 7, in the case $b = 0$.

Proof. Let $\sigma \in \mathrm{Av}_{n+1}^{k}(1324)$ be a class R2b permutation satisfying Lemma 20 (a) and let $\tau = \sigma \setminus \{\sigma_1, \sigma_{n+1}\}$. Then $\mathrm{inv}(\tau) = k - n + 1 \leqslant n - 6$, so $\tau \in \mathrm{Av}_{n-1}^{k-n+1}(1324)$ and

$$\mathrm{av}_{n-1}^{k-n+1}(1324) = [x^{k-n+1}]P(x)^2 = [x^k]\big(x^{n-1}P(x)^2\big).$$

Conversely, for any permutation $\tau \in \mathrm{Av}_{n-1}^{k-n+1}(1324)$, we can construct a unique class R2b permutation $\sigma \in \mathrm{Av}_{n+1}^{k}(1324)$ satisfying Lemma 20 (a) by placing σ_1 and σ_{n+1} appropriately. The case of Lemma 20 (b) is symmetrical, and the result follows. □

We still need to show that condition (3) holds for all permutations in $\mathrm{Av}_{n+1}^{k}(1324)$

whenever $k \leqslant 2n-7$ in order to justify (4). This is the last part of the proof of Theorem 2. We first show two lemmas we need.

Lemma 22. *If $\pi \in \mathrm{Av}_n^k(132)$ is indecomposable and $k \leqslant 2n-5$, then $\pi_1 = n$ and $\pi \setminus n$ is decomposable; or $\pi_n = 1$ and $\pi \setminus 1$ is decomposable.*

Proof. Suppose that $\pi_1 \neq n$ and $\pi_n \neq 1$. Since π is indecomposable, $\pi_n \neq n$. It follows that $\pi_1 > \pi_n$, since $\pi_1 n \pi_n$ forms a 132-pattern otherwise. Similarly $\pi_1^{-1} > \pi_n^{-1}$ to avoid the 132-pattern $1n\pi_n$. Counting inversions like in Section 2 gives

$$\mathrm{inv}(\pi) \geqslant \pi_1 - 1 + n - \pi_n + \pi_1^{-1} - 1 + n - \pi_n^{-1} - 2 \geqslant 2n - 4.$$

If $\pi_1 = n$, then $\mathrm{inv}(\pi \setminus n) \leqslant n - 6$ and and $\pi \setminus n$ must be decomposable. The other case is symmetrical. □

Lemma 23. *If $\sigma \in \mathrm{Av}_m^k(1324)$ is indecomposable, $k \leqslant 2m-9$, and $\sigma \setminus \sigma_1$ has exactly two components, then $\sigma \setminus \sigma_m$ or $\sigma \setminus m$ has at least five components.*

Proof. Write $\sigma \setminus \sigma_1 = \sigma^{(1)} \oplus \sigma^{(2)}$ and let $\delta = \sigma_1 - |\sigma^{(1)}|$. Since $\sigma^{(1)}$ is indecomposable we have $\mathrm{inv}\left(\sigma^{(1)}\right) \geqslant |\sigma^{(1)}| - 1$, and therefore

$$\begin{aligned} \mathrm{inv}\left(\sigma^{(2)}\right) &= k - (\sigma_1 - 1) - \mathrm{inv}\left(\sigma^{(1)}\right) \\ &\leqslant 2m - 9 - |\sigma^{(1)}| - \delta + 1 - |\sigma^{(1)}| + 1 \\ &= 2\big(m - |\sigma^{(1)}|\big) - 7 - \delta \\ &= 2|\sigma^{(2)}| - 5 - \delta. \end{aligned}$$

Since $\sigma^{(2)}$ is 213-avoiding, $\sigma_1^{(2)} = |\sigma^{(2)}|$ or $\sigma_{|\sigma^{(2)}|}^{(2)} = 1$ by Lemma 22, using the fact $213 = \mathrm{rc}(132)$. In the first case

$$\begin{aligned} \mathrm{comp}\left(\sigma^{(2)} \setminus \sigma_1^{(2)}\right) &\geqslant |\sigma^{(2)} \setminus \sigma_1^{(2)}| - \mathrm{inv}\left(\sigma^{(2)} \setminus \sigma_1^{(2)}\right) \\ &\geqslant |\sigma^{(2)}| - 1 - \big(2|\sigma^{(2)}| - 5 - \delta - |\sigma^{(2)}| + 1\big) = \delta + 3. \end{aligned}$$

Denoting $\tau = \sigma \setminus \sigma_1$, this shows that $\tau \setminus (m-1)$ has at least $\delta + 4$ components, the first of which is $\sigma^{(1)}$. Since $\sigma_1 = |\sigma^{(1)}| + \delta$, 'placing back' σ_1 in front of $\tau \setminus (m-1)$ combines δ components into one, so we conclude that $\sigma \setminus m$ has at least five components. The case $\sigma_{|\sigma^{(2)}|}^{(2)} = 1$ is similar. □

Proposition 24. *If $\sigma \in \mathrm{Av}_m^k(1324)$ has at most two components and $k \leqslant 2m-9$, then*

$$\max\{\mathrm{comp}(\sigma \setminus i) : i = 1, m, \sigma_1, \sigma_m\} \geqslant 3.$$

Proof. If $\sigma = \sigma^{(1)} \oplus \sigma^{(2)}$, then $\mathrm{inv}\left(\sigma^{(i)}\right) \leqslant 2n-5$ for $i=1$ or $i=2$, and Lemma 22 proves the claim. If σ is indecomposable, then it is almost decomposable, i.e. one of the entries $i \in \{1, m, \sigma_1, \sigma_m\}$ satisfies $\mathrm{comp}(\sigma \setminus i) \geqslant 2$. If $\sigma \setminus i$ has exactly two components, then (taking the inverse and reverse-complement if necessary) Lemma 23 proves that another entry $j \in \{1, m, \sigma_1, \sigma_m\}$ satisfies the claim. □

Proof of Theorem 2. Summing together the enumerations for classes R1, R2a and R2b yields

$$|\mathcal{R}_{n+1}^k| = \mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = [x^k]\big(2(2+x)x^{n-1}P(x)^2\big) =: [x^k]R_n(x)$$

whenever $k \leqslant 2n-7$. It follows that

$$\begin{aligned}\mathrm{av}_n^k(1324) &= [x^k]\big(P(x)^2\big) - \sum_{i\leqslant k}[x^i]R_n(x)\\ &= [x^k]\left(P(x)^2 - \frac{R_n(x)}{1-x}\right),\end{aligned}$$

and extracting the coefficient yields the result. □

5 Further directions and conjectures

This section contains discussion of three separate topics: extending our method; repeated differences of the numbers $\mathrm{av}_n^k(1324)$; and the unimodality of the sequences

$$\mathrm{av}_n^0(1324),\quad \mathrm{av}_n^1(1324),\quad \ldots,\quad \mathrm{av}_n^{\binom{n}{2}}(1324).$$

Unimodality could improve the upper bound for $L(1324)$ that Conjecture 1 gives.

5.1 Extending almost-decomposability

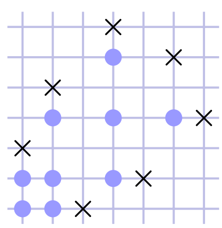
A natural idea is to delete more than one point from the boundary of the plot of a permutation to make it decomposable. For example, one could imagine handling our earlier counterexample $\pi = 3612745$ by deleting entries 1 and 2. However, the critical Proposition 7 does not appear to have a counterpart, since e.g. $\pi = 3412$ satisfies both

$$\mathrm{comp}(\pi \setminus \{1,2\}) = 2 \qquad \text{and} \qquad \mathrm{comp}(\pi \setminus \{\pi_1, \pi_2\}) = 2$$

Furthermore, some permutations with as few as $2n-5$ have no natural point combination to remove. Consider the permutation

$$\pi = 34\ldots(n-4)\ \ (n-2)1n2(n-1)(n-3).$$

Here is the first such permutation.

3517264 = 

It is difficult to see which entries should be deleted. The permutation $\pi \setminus \{1,2\}$ is certainly decomposable, but applying g to it and inserting back the entries 1 and 2 yields a permutation containing 1324. One could instead try to delete different pairs of points, such as n and π_n, but it is hard to imagine that the resulting mapping could be injective.

It should, still, be possible to improve the upper bound from $k \leqslant 2n-7$ to $k \leqslant 2n-6$ by analyzing the permutations in $\mathrm{Av}_n^{2n-6}(1324)$ that are neither decomposable nor almost decomposable. These permutations are essentially characterized in the proof of Lemma 11; all other intermediate results in Section 2 yield higher bounds for k. There are not many such permutations, and they can probably be counted.

5.2 Repeated differences

One way to prove Conjecture 1 would be understanding the numbers $\mathrm{av}_k^n(1324)$ also for $k > 2n-7$. Studying the numbers when $k \leqslant 3n-15$ we have found a tantalising pattern. We here extend the study of differences from Theorem 2 to repeated differences. In what follows we will write av_n^k for $\mathrm{av}_n^k(1324)$. For $n \geqslant 10$ it seems for example to always be the case that

$$(\mathrm{av}_{n+3}^{2n-3} - \mathrm{av}_{n+2}^{2n-3}) - (\mathrm{av}_{n+2}^{2n-4} - \mathrm{av}_{n+1}^{2n-4}) - \left((\mathrm{av}_{n+2}^{2n-5} - \mathrm{av}_{n+1}^{2n-5}) - (\mathrm{av}_{n+1}^{2n-6} - \mathrm{av}_{n}^{2n-6})\right) = 4.$$

In fact, for a fixed $r \geqslant 0$, the following always seems to be constant for $n \geqslant 10 + r$:

$$\begin{aligned} b_{r,n} := {} & (\mathrm{av}_{n+3}^{2n+r-3} - \mathrm{av}_{n+2}^{2n+r-3}) - (\mathrm{av}_{n+2}^{2n+r-4} - \mathrm{av}_{n+1}^{2n+r-4}) \\ & - \left((\mathrm{av}_{n+2}^{2n+r-5} - \mathrm{av}_{n+1}^{2n+r-5}) - (\mathrm{av}_{n+1}^{2n+r-6} - \mathrm{av}_{n}^{2n+r-6})\right). \end{aligned}$$

Anders Claesson has kindly provided us with data of av_n^k for $k, n \leqslant 45$ and with these we may compute numbers $b_{r,n}$ up to $r = 9$ as $4, 8, 14, 28, 52, 88, 150, 244, 390, 612$ and we offer the following conjecture.

Conjecture 25. The numbers $b_{r,n}$ are equal for a fixed r with $n \geqslant 10 + r$ (call them b_r) and they satisfy

$$\sum_{r \geqslant 0} b_r x^r = 2\left(1 - x^2\right)\left(2 - x^2\right) P(x)^2,$$

where $P(x)$ is again the generating function for the partition numbers.

Remark 26. To guess the formula in the conjecture one really only needs five numbers b_0, b_1, b_2, b_3, b_4 since the polynomial is of degree four, but it is true for all ten numbers we have.

Remark 27. If the conjecture is proven we could still not determine the numbers av_n^k for all $k \leqslant 3n - 15$ since we also would need starting values when $n = 10 + r$ of $(\mathrm{av}_{n+2}^{2n+r-5} - \mathrm{av}_{n+1}^{2n+r-5}) - (\mathrm{av}_{n+1}^{2n+r-6} - \mathrm{av}_{n}^{2n+r-6})$. That sequence starts

$$12, 24, 41, 120, 274, 553, 1098, 2055$$

but we have no conjecture for what they are in general.

To add an extra level of intrigue there seems to be a pattern also for fourth differences when trying to understand how the third differences $b_{r,n}$ deviate for $n < 10 + r$. More precisely, $b_{r,r+9}$ and $b_{r,r+8}$ seem to be $4(r-2)$ and $16(r-5)+32$ larger than in Conjecture 25 for $r \geqslant 3$ and 6, respectively.

5.3 Improved bounds and unimodality

Let $c \leqslant 1$ be a constant such that the maximal value of each sequence $\left(\mathrm{av}_n^k(1324)\right)_k$ occurs with $k \leqslant c \cdot \binom{n}{2}$, and assume that Conjecture 1 is true. Denote $m_n = \binom{n}{2}$, $c_n = \lfloor c \cdot m_n \rfloor$ and $\rho = \exp\left(\pi\sqrt{2/3}\right)$. Then, using the same technique as in [14, Theorem 17],

$$\begin{aligned}\mathrm{av}_n(1324) = \sum_k \mathrm{av}_n^k(1324) &\leqslant (m_n+1) \max_k \mathrm{av}_n^k(1324)\\ &\leqslant (m_n+1)[x^{c_n}]P(x)^2\\ &\leqslant (m_n+1)(c_n+1)\rho^{\sqrt{2c_n}}\\ &\leqslant (m_n+1)(c_n+1)\rho^{\sqrt{c}n\sqrt{1-1/n}}.\end{aligned}$$

Taking the nth root and the limit as $n \to \infty$,

$$L(1324) \leqslant \rho^{\sqrt{c}} = \exp\left(\pi\sqrt{2c/3}\right).$$

Incidentally, $c = 21/23 \approx 0.913$ gives $L(1324) \leqslant 11.6004$, which is in the range of the estimation $L(1324) = 11.600 \pm 0.003$ from [15, 16]. We do not believe that an improvement this drastic will be possible with this approach, since the sequences $\left(\mathrm{av}_n^k(1324)\right)_k$ should be very top-heavy. It is possible to produce large random 1324-avoiders using the Monte Carlo method in [19], and they seem to support this intuition.

The method of estimation should, furthermore, be too rough to get an upper bound very close to the actual value of $L(1324)$. Note that $c > 0.813$, since $c = 0.813$ gives $L(1324) \leqslant 10.263$, contradicting the known lower bound 10.27 – again, assuming Conjecture 1.

Question 28. Is there a constant $c < 1$ such that the maximal value of each sequence $\left(\mathrm{av}_n^k(1324)\right)_k$ occurs with $k \leqslant c \cdot \binom{n}{2}$?

This line of thinking leads to another natural question: are the sequences $\left(\mathrm{av}_n^k(1324)\right)_k$ unimodal? It is well-known (see [6] for a nice proof) that $\left(s_n^k\right)_k$ is log-concave, where s_n^k denotes the number of all permutations (not required to avoid any pattern) of length n with k inversions. As far as we know, there are no similar nontrivial results for the pattern avoiding case.

Log-concavity does not hold for $\left(\mathrm{av}_n^k(1324)\right)_k$, since e.g. $2^2 < 1 \cdot 5$. On the other hand, if we remove the first $n-1$ entries of each sequence, they do seem to be log-concave. Unimodality holds for the full sequences in the data we have, but we have not found a proof. Of special interest would be the position of the 'tops' of the unimodal sequences, due to the discussion above.

Conjecture 29. The sequence $\left(\mathrm{av}_n^k(1324)\right)_{k=0}^{\binom{n}{2}}$ is unimodal for each n.

Acknowledgements

We are grateful to Anders Claesson and Bjarki Ágúst Guðmundsson, who provided us with data for the numbers $\mathrm{av}_n^k(1324)$, for $k, n \leqslant 45$, and correctly guessed the generating function $R_n(x) = 2(2+x)x^{n-1}P(x)^2$ for the differences $\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324)$. The data is available at https://akc.is/inv-mono/. We are also grateful to the anonymous referee for their careful reading and helpful suggestions. Both authors were funded by the Swedish Research Council, VR, grant 2022-03875.

References

[1] M. H. Albert, M. Elder, A. Rechnitzer, P. Westcott, and M. Zabrocki. "On the Stanley-Wilf limit of 4231-avoiding permutations and a conjecture of Arratia". In: *Adv. Appl. Math.* 36.2 (2006), pp. 96–105. DOI: 10.1016/j.aam.2005.05.007.

[2] R. Arratia. "On the Stanley-Wilf conjecture for the number of permutations avoiding a given pattern". In: *Electron. J. Comb.* 6 (1999), #N1, 4. DOI: 10.37236/1477.

[3] J. Backelin, J. West, and G. Xin. "Wilf-equivalence for singleton classes". In: *Adv. Appl. Math.* 38.2 (2007), pp. 133–148. DOI: 10.1016/j.aam.2004.11.006.

[4] D. Bevan. "Permutations avoiding 1324 and patterns in Łukasiewicz paths". In: *J. Lond. Math. Soc. (2)* 92.1 (2015), pp. 105–122. DOI: 10.1112/jlms/jdv020.

[5] D. Bevan, R. Brignall, A. Elvey Price, and J. Pantone. "A structural characterisation of Av(1324) and new bounds on its growth rate". In: *Eur. J. Comb.* 88 (2020). 103115. DOI: 10.1016/j.ejc.2020.103115.

[6] M. Bóna. "A combinatorial proof of the log-concavity of a famous sequence counting permutations". In: *Electron. J. Comb.* 11.2 (June 2004), #N2, 4. DOI: 10.37236/1889.

[7] M. Bóna. "A new record for 1324-avoiding permutations". In: *Eur. J. Math.* 1.1 (2015), pp. 198–206. DOI: 10.1007/s40879-014-0020-6.

[8] M. Bóna. "A new upper bound for 1324-avoiding permutations". In: *Comb. Probab. Comput.* 23.5 (2014), pp. 717–724. DOI: 10.1017/S0963548314000091.

[9] M. Bóna. "A simple proof for the exponential upper bound for some tenacious patterns". In: *Adv. Appl. Math.* 33.1 (2004), pp. 192–198. DOI: 10.1016/j.aam.2003.07.003.

[10] M. Bóna. *Combinatorics of permutations.* Third edition. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2022. xix+507. DOI: 10.1201/9780429274107.

[11] M. Bóna. "Exact enumeration of 1342-avoiding permutations: a close link with labeled trees and planar maps". In: *J. Comb. Theory – A* 80.2 (1997), pp. 257–272. DOI: 10.1006/jcta.1997.2800.

[12] M. Bóna. “Permutations avoiding certain patterns: the case of length 4 and some generalizations”. In: *Discrete Math.* 175.1 (1997), pp. 55–67. DOI: 10.1016/S0012-365X(96)00140-9.

[13] M. Bóna. “The limit of a Stanley-Wilf sequence is not always rational, and layered patterns beat monotone patterns”. In: *J. Comb. Theory – A* 110.2 (2005), pp. 223–235. DOI: 10.1016/j.jcta.2004.07.014.

[14] A. Claesson, V. Jelínek, and E. Steingrímsson. “Upper bounds for the Stanley-Wilf limit of 1324 and other layered patterns”. In: *J. Comb. Theory – A* 119.8 (2012), pp. 1680–1691. DOI: 10.1016/j.jcta.2012.05.006.

[15] A. R. Conway and A. J. Guttmann. “On 1324-avoiding permutations”. In: *Adv. Appl. Math.* 64 (2015), pp. 50–69. DOI: 10.1016/j.aam.2014.12.004.

[16] A. R. Conway, A. J. Guttmann, and P. Zinn-Justin. “1324-avoiding permutations revisited”. In: *Adv. Appl. Math.* 96 (2018), pp. 312–333. DOI: 10.1016/j.aam.2018.01.002.

[17] I. M. Gessel. “Symmetric functions and P-recursiveness”. In: *J. Comb. Theory – A* 53.2 (1990), pp. 257–285. DOI: 10.1016/0097-3165(90)90060-A.

[18] S. Kitaev. *Patterns in permutations and words.* Monographs in Theoretical Computer Science. An EATCS Series. Springer, Heidelberg, 2011. xxii+494. DOI: 10.1007/978-3-642-17333-2.

[19] N. Madras and H. Liu. “Random pattern-avoiding permutations”. In: *Algorithmic probability and combinatorics.* Vol. 520. Contemp. Math. Amer. Math. Soc., Providence, RI, 2010, pp. 173–194. DOI: 10.1090/conm/520/10259.

[20] A. Marcus and G. Tardos. “Excluded permutation matrices and the Stanley-Wilf conjecture”. In: *J. Comb. Theory – A* 107.1 (2004), pp. 153–160. DOI: 10.1016/j.jcta.2004.04.002.

[21] OEIS Foundation Inc. *The On-Line Encyclopedia of Integer Sequences*. Published electronically at https://oeis.org. 2025.

[22] Z. E. Stankova. “Forbidden subsequences”. In: *Discrete Math.* 132.1 (1994), pp. 291–316. DOI: 10.1016/0012-365X(94)90242-9.

[23] V. Vatter. “Permutation classes”. In: *Handbook of enumerative combinatorics*. Discrete Math. Appl. (Boca Raton). CRC Press, Boca Raton, FL, 2015, pp. 753–833. DOI: 10.1201/b18255-18.