

Clique Covers of Complete Graphs and Piercing Multitrack Intervals

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Abstract

Assume that $R_1, R_2, \dots, R_t$ are disjoint parallel lines in the plane. A t -interval (or t -track interval) is a set that can be written as the union of t closed intervals, each on a different line. It is known that pairwise intersecting 2-intervals can be pierced by two points, one from each line. However, it is not true that every set of pairwise intersecting 3-intervals can be pierced by three points, one from each line. For $k \geq 3$, Kaiser and Rabinovich asked whether k -wise intersecting t -intervals can be pierced by t points, one from each line. Our main result provides an asymptotic answer: in any set $S_1, \dots, S_n$ of k -wise intersecting t -intervals, at least $\frac{k-1}{k+1}n$ can be pierced by t points, one from each line. We prove this in a more general form, replacing intervals by subtrees of a tree. This leads to questions and results on covering vertices of edge-colored complete graphs by vertices of monochromatic cliques having distinct colors, where the colorings are chordal, or more generally induced C_4 -free graphs. For instance, we show that if the edges of a complete graph K_n are colored with red or blue so that both color classes are induced C_4 -free, then at least $\frac{4n}{5}$ vertices can be covered by a red and a blue clique, and this is best possible. We conclude by pointing to new Ramsey-type problems emerging from these restricted colorings.

Mathematics Subject Classifications: 05C15, 05C62

1 Introduction, results

1.1 t -intervals

Assume that $R_1, R_2, \dots, R_t$ are disjoint parallel lines in the plane. A t -interval (or t -track interval) is a set that can be written as the union of t closed intervals, $I_1, I_2, \dots, I_t$ with $I_j \subset R_j$. Gallai (in connection with a problem of piercing directed cycles in a digraph) asked how many points are needed to pierce pairwise intersecting 2-track intervals. Here

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we use *piercing* as an equivalent notion of *cover*: a family H of subsets of a set V is pierced (or covered) by $T \subset V$ if every set in H has a non-empty intersection with T .

Gyárfás and Lehel proved in [9] that the piercing number of pairwise intersecting t -track intervals is bounded by a function of t and for $t = 2, 3$ the smallest value of the piercing number is 2 and 4, respectively. Initiated by a breakthrough result of Tardos [14], topological methods and their simplifications resulted in sharp and approximate results about the smallest possible piercing number of t -intervals where the condition “pairwise intersecting” was replaced by “at most k pairwise disjoint”. For a survey of these results, see Chapter 30 in [4], Chapter 3 in [13].

The motivation of this paper is an interesting special piercing of t -intervals introduced by Kaiser and Rabinovich [12]: the piercing set is required to contain *at most one point* from every R_i . They call a family of t -track intervals *weakly intersecting* if there exists such a piercing set. A set system is *k-wise intersecting* or *k-intersecting*, if any k sets have a non-empty intersection ($k \geq 2$). Replacing pairwise intersection in the previous problem by k -wise intersection, the following was asked in [12].

Question 1 (Kaiser, Rabinovich). Assume that $k \geq 3$ and H is a family of k -wise intersecting t -intervals. Is H weakly intersecting?

For $k \geq \lfloor \log_2(t+1) \rfloor + 1$, Kaiser and Rabinovich proved (Theorem 4.5 in [12]) that k -wise intersecting t -intervals are weakly intersecting. In particular, the 3-wise intersecting t -intervals are weakly intersecting if $t \leq 6$.

Here we focus on a natural refinement towards Question 1: given a set H of k -wise intersecting t -intervals, how large subset F of H is weakly intersecting? Our main result shows that (independently of t), $|F|$ approaches $|H|$ as k grows.

Theorem 2. *Let t and $k \geq 2$ be positive integers. If H is a set of k -wise intersecting t -intervals, then at least $\frac{(k-1)|H|}{k+1}$ are weakly intersecting.*

We prove Theorem 2 in a more general form, instead of intervals, for subtrees of a tree (Theorem 5). We also show (see the discussion at the end of the proof of Theorem 5) that a similar statement holds if we replace k -wise intersecting with a different strengthening of pairwise intersecting. Namely, with *pairwise k -fold intersection*, that is, every pair of t -intervals intersect on at least k of the lines. For $k \geq \lfloor \frac{t+1}{2} \rfloor$, Kaiser and Rabinovich proved (Theorem 4.6 in [12]) that pairwise k -fold intersecting t -intervals are weakly intersecting. We focus on k -wise intersection in this paper. However, the corresponding problems are interesting for pairwise k -fold intersection as well.

Theorem 2 with $k = 2$ shows that there is $F \subset H$, $|F| \geq \frac{|H|}{3}$ such that F is weakly intersecting. Our next result complements this with an upper bound on $|F|$.

Theorem 3. *For any $t \geq 2, n \geq 4t - 5$, there exists a set H of pairwise intersecting t -intervals such that $|H| = n$ and for any weakly intersecting subset $F \subset H$, we have*

$$|F| \leq 3(t-1) \left\lceil \frac{n}{4t-5} \right\rceil.$$

Pairwise intersecting 3-intervals are not necessarily weakly intersecting, constructions from [9] and [12] are shown in Figure 1. Note that the first example can be pierced by two points on R_1 (but not with three points from distinct lines).

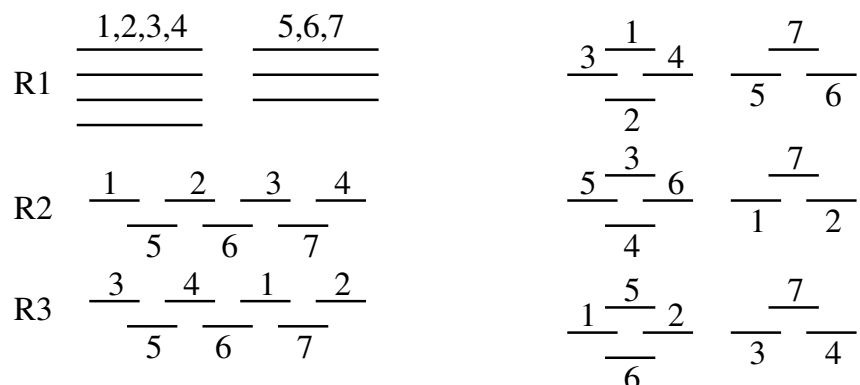


Figure 1: The constructions of [9] and [12].

1.2 Strong monochromatic clique covers

Here we formulate the geometric problems mentioned so far in a more general framework. This technique is used already in [9], [10], here we adopt it for the special piercing required in [12]. Consider a complete graph K_n , whose edge set is the union of t graphs, $G_1, G_2, \dots, G_t$. We consider this as an edge-coloring of K_n , where edges can have several colors, non-empty subsets of $[t]$. For $k \geq 2$, a (t, k) -coloring of K_n is a (multi)coloring with t colors, where each set of k vertices spans a monochromatic complete subgraph in some color. We often use the shorter term *clique* for complete graphs.

A covering of the vertices of K_n with vertices of monochromatic cliques is called a *strong cover* if the colors of the covering cliques are all different. If a t -coloring of K_n has a strong cover, then we define $\theta(K_n)$ as the minimum number of cliques needed for a strong cover. (Note that a strong cover can be reduced to a strong partition.)

One can easily see that any set of n k -wise intersecting t -intervals defines a (t, k) -coloring on K_n by representing the t -intervals by n vertices and assigning to edges the (non-empty) subset $S \subseteq [t]$ where for $i \in S$ the represented t -intervals have a non-empty intersection on R_i . In this coloring each G_i is an *interval graph*, i.e. can be represented as the intersection graph of a set of intervals on a line. We call such a coloring an *interval* (t, k) -coloring. This can be reversed, any interval (t, k) -coloring of K_n defines a system of n k -wise intersecting t -intervals by a (not unique) representation of G_i by intervals on the line R_i . Also, “weakly intersecting” corresponds to “strong covering”, since by the one-dimensional Helly theorem, a clique monochromatic in color i corresponds to a piercing point on R_i .

A well-known extension of interval graphs is the family of *chordal* (or *triangulated*) graphs, that do not contain cycles of length at least four as an induced subgraph. A (t, k) -coloring is *chordal* if each G_i is a chordal graph. Chordal graphs are interesting from a

geometric point of view as well, they can be characterized as intersection graphs of subtrees of a tree [6]. Using this characterization, chordal (t, k) -colorings can be represented with k -wise intersecting t -subtrees, i.e. sets that can be written as the union of t subtrees selected from t pairwise disjoint trees. Since pairwise intersecting subtrees of a tree also have the Helly property (see for example [7]), a strong cover by monochromatic cliques in a chordal (t, k) -coloring corresponds to weakly intersecting t -subtrees.

A further extension is to consider C_4 -free graphs, where only induced cycles of length four are forbidden. In what follows, we use C_4 -free, C_5 -free in the induced sense. A simple but useful property of interval/chordal/ C_4 -free colorings is that they are closed for clique substitution. A *clique substitution* into a (t, k) -colored K_n is defined as follows. Replace a vertex v of K_n by a clique K where each edge of K is colored with all the t colors. All edges between K and $u \in V(K_n)$ are colored with the colors on the edge $(u, v) \in E(K_n)$. In case of $V(K) = \{v, w\}$ we say that w is a *replica* of v .

Proposition 4. *Assume that $\theta(K_n) = s$ in an interval/chordal/ C_4 -free (t, k) -coloring of K_n and K_m is obtained by a clique substitution into K_n . The obtained coloring is an interval/chordal/ C_4 -free (t, k) -coloring and $\theta(K_m) = s$.*

We prove Theorem 2 in the following stronger form.

Theorem 5. *Let $t, k \geq 2$ and consider a chordal (t, k) -coloring of the complete graph K_n . There is a strong cover on at least $\frac{k-1}{k+1}n$ vertices of K_n .*

For chordal $(2, 2)$ -colorings of K_n there is a strong cover for all vertices [9]. This holds for chordal $(3, 3)$ -colorings of K_n as well. (For interval $(3, 3)$ -colorings Theorem 6 is proved in [12].)

Theorem 6. *Assume that we have a chordal $(3, 3)$ -coloring of K_n . Then $\theta(K_n) \leq 3$ and Figure 2 shows that equality can occur for interval graphs.*

Theorem 7. *Assume that we have a chordal (t, t) -coloring of K_n . For even $t \geq 2$, $\theta(K_n) = 2$, and for odd $t \geq 3$, $\theta(K_n) \leq 3$.*

For C_4 -free $(2, 2)$ -colorings, we can determine the best strong cover of K_n .

Theorem 8. *In every C_4 -free $(2, 2)$ -coloring of K_n there is a strong cover on some $K_m \subset K_n$ with $m \geq \lceil \frac{4n}{5} \rceil$. This is best possible for every $n \geq 5$.*

2 Proofs

Proof of Theorem 3. We present the construction (a generalization of Figure 1 left part) in the framework of interval $(t, 2)$ -colorings. Set $n = 4t - 5$, and define the following t -coloring of K_n . Edges of color 1 form two vertex-disjoint cliques, $A = V(K_{2t-2})$ and $B = V(K_{2t-3})$, this is clearly an interval graph. The other $t - 1$ colors are defined by decomposing the complete bipartite graph $[A, B]$ into $t - 1$ Hamilton paths (which are

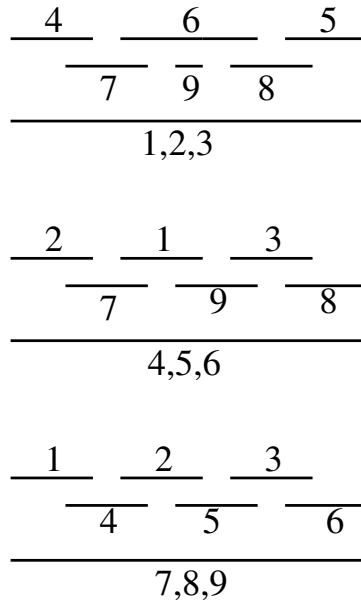


Figure 2: Nine 3-wise intersecting 3-intervals without piercing by two points.

also interval graphs). The easiest way to get this partition is to extend B with one vertex and use the well-known fact that $K_{2t-2,2t-2}$ can be partitioned into Hamilton cycles (an early reference is [2]). Removing one vertex from B , we get the required partition into Hamilton paths.

Observe that a partition of $V(K_n)$ into cliques of different colors can use only one of the two cliques in the first color. However, the vertices of the unused clique are independent in all other colors. Therefore, we can cover at most $2t - 2 + t - 1 = 3(t - 1)$ vertices by t cliques of distinct colors. At least $t - 2$ vertices are uncovered, giving the ratio $\frac{3(t-1)}{4t-5}$ for the covered vertices.

Applying Proposition 4, we can extend this example to an interval $(t, 2)$ -coloring of K_n , for any $n \geq 4t - 5$, because substitutions into interval graphs can be interpreted as repeating intervals. Thus partitioning n into $4t - 5$ parts as evenly as possible and repeating each t -interval accordingly, we get a construction for any $n \geq 4t - 5$. The upper bound for this construction is sharp if $4t - 5$ divides n . $\square$

Proof of Proposition 4. To see that substituting a clique K to $v \in V(K_n)$ preserves (t, k) -coloring, consider any $S \subset V(K_m)$ with $|S| = k$. Set $M = S \cap K$. If $|M| \leq 1$, then S spans a monochromatic clique by the assumption that K_n has a (t, k) -coloring. Otherwise, for any $w \in M$, $S' = M \setminus \{w\}$ extends the monochromatic clique $S \setminus S'$ by the definition of the substitution.

Substituting a clique K can create a chordless cycle C of length at least 4 only if $|V(C) \cap V(K)| = 2$, and the intersection is an edge $e = (v_i, v_{i+1})$ of C . Indeed, if $|V(C) \cap V(K)| \geq 3$, then C cannot be chordless. If $|V(C) \cap V(K)| \leq 1$, then C was present before the substitution. However, this implies that the next vertex v_{i+2} on C is

not in K , thus adjacent to v_i by the rule of substitution and creates a chord in C . Thus clique substitution preserves the property of being interval/chordal/ C_4 -free.

It is obvious that θ cannot decrease by clique substitution. But it cannot increase either, since the clique C covering the vertex v in an optimal strong cover of K_n is extended by the substituted K into a clique of the same color. $\square$

Proof of Theorem 5. The following proof idea is partly inspired by Theorem 3 of [8]. Consider a chordal (t, k) -coloring of the complete graph K_n . Let G_i denote the chordal graph on n vertices induced by color i . Chordal graphs are perfect ([7] which means that all their induced subgraphs have the same chromatic number and clique number.) Let us consider color 1, and let ω_1 be the clique number of G_1 . Since G_1 is perfect, the n vertices of K_n can be partitioned into ω_1 independent sets $X_1, \dots, X_{\omega_1}$. Since G_1 is chordal, the edges of color 1 between X_s and X_z form an acyclic subgraph of K_n for any two distinct indices s, z . Indeed, otherwise a cycle of minimum length in the bipartite graph $[X_s, X_z]$ would define a chordless cycle in G_1 . Therefore, we get an upper bound on the number of edges of color 1 induced by any subset of vertices. For the entire vertex set, we get:

$$\begin{aligned} |E(G_1)| &\leq \sum_{s < z} (|X_s| + |X_z| - 1) = (\omega_1 - 1)n - \binom{\omega_1}{2} \\ &\leq \omega_1 n - \omega_1 = \omega_1(n - 1). \end{aligned}$$

Similarly, if we consider an arbitrary subset R of the vertices, then (using the notation $G|_R$ for the subgraph induced by R),

$$|E(G_1|_R)| \leq \omega_1(|R| - 1).$$

Indeed, if ω'_1 denotes the number of those X_i -s that have a non-empty intersection with R , we get

$$|E(G_1|_R)| \leq (\omega'_1 - 1)|R| - \binom{\omega'_1}{2} \leq \omega'_1|R| - \omega'_1 = \omega'_1(|R| - 1) \leq \omega_1(|R| - 1).$$

In step 1, we select a clique W_1 of color 1 and size ω_1 and remove the vertices of W_1 from K_n to get $G^{(2)}$, a t -edge-colored complete graph on $n - \omega_1$ vertices. We repeat this step for the remaining colors $2, \dots, t$.

In step 2, we select a clique W_2 in $G^{(2)}$ of color 2 and size ω_2 and remove the vertices of W_2 from $G^{(2)}$ to get $G^{(3)}$. We also deduce that the number of edges of color 2 on any remaining vertex set R satisfies:

$$|E(G_2|_R)| \leq \omega_2(|R| - 1).$$

In step i , we select a clique W_i in $G^{(i)}$ of color i and size ω_i and remove the vertices of W_i from $G^{(i)}$ to get $G^{(i+1)}$. We also deduce that the number of edges of color i on any remaining vertex set R satisfies:

$$|E(G_i|_R)| \leq \omega_i(|R| - 1).$$

Suppose we covered cn vertices of K_n after t steps with cliques $W_1, \dots, W_t$ for some constant c . Denote the set of uncovered vertices by T . We get an upper bound on M , the sum of multiplicities of edges induced in T by some G_i , adding the upper bounds for each color.

$$M = \sum_{i=1}^t |E(G_i|_T)| \leq \sum_{i=1}^t \omega_i(|T| - 1) = \sum_{i=1}^t |W_i|(|T| - 1) = cn(|T| - 1).$$

On the other hand, we can get a lower bound on M using the following claim.

Claim 9. *In $K_n|_T$ every edge has at least $k - 1$ colors.*

Indeed, for $k = 2$, the claim is trivial, since K_n is a complete graph. For $k \geq 4$, we can prove a slightly stronger claim, namely that *every* edge in K_n (not just in $K_n|_T$) has at least $k-1$ colors. Otherwise K_n has a strong cover by at most 3 monochromatic cliques. Indeed, suppose to the contrary that an edge $e = (u, v)$ in K_n has only colors $1, \dots, k-2$. Consider the subgraph $G' = G_1 \cup \dots \cup G_{k-2}$ on $V(K_n)$. We claim that G' is $(k-2)$ -intersecting (in *these* $k-2$ colors). Indeed, consider a set $S \subset V(K_n)$ of $k-2$ vertices. We add $\{u, v\}$ and potentially 1 or 2 more vertices (if $S \cap \{u, v\} \neq \emptyset$) to get a set S' of k vertices. We had a chordal (t, k) -coloring by assumption, so S' is contained in a monochromatic clique. Since this clique contains the edge e , this could only be one of the colors $[k-2]$. Thus indeed S is contained in a monochromatic clique in these colors. Furthermore, this implies that every edge of K_n is covered by G' . However, G' is $(k-2)$ -colored and $(k-2)$ -intersecting, therefore by Theorem 7 $V(G') = V(K_n)$ can be covered by at most 3 monochromatic cliques, a much stronger result than what we are looking for; we cover *all* vertices of $V(K_n)$ and by *at most 3 cliques*.

For $k = 3$, suppose to the contrary that edge $e = (u, v)$ has only color i in $K_n|_T$, say $u \in X_s$ and $v \in X_z$ in the decomposition above. Since u is in T (after removing W_i), originally $|X_s| \geq 2$ was true. Add a vertex $u' \in X_s$, $u' \neq u$ to get $S = \{u, u', v\}$. The graph K_n was 3-intersecting, so S is contained in a monochromatic clique. However, this could only be color i , since e only has color i . On the other hand, the edge (u, u') cannot have color i since X_s is an independent set in color i , a contradiction, finishing the proof of the claim.

Finally, comparing the bounds on M , we get

$$(k-1) \binom{(1-c)n}{2} \leq M \leq cn(|T| - 1) = cn((1-c)n - 1). \quad (1)$$

Hence

$$(k-1)(1-c) \leq 2c,$$

or

$$\frac{k-1}{k+1} \leq c. \quad \square$$

As we mentioned in the introduction, we can prove a statement similar to Theorem 2 and Theorem 5 in a slightly different setting, where k -wise intersection is replaced with pairwise k -fold intersection. More precisely, we have the following.

Theorem 10. *Let $t, k \geq 2$, and consider a chordal t -coloring of the complete graph K_n , where every edge has at least $k-1$ colors. Then there is a strong cover of at least $\frac{k-1}{k+1}n$ vertices of K_n .*

The proof is virtually identical to that of Theorem 5 above. For the upper bound on $|E(G|_T)|$ in (1) we just used the fact that we have a chordal t -coloring. For the lower bound we used Claim 9, which now holds by assumption. The rest of the proof is the same.

Proof of Theorem 6. Consider a chordal $(3, 3)$ -coloring of K_n . Let E_j denote the set of edges colored with c_j for $j = 1, 2, 3$. We can either find an edge $e = (u, v) \in E_1$ colored only with c_1 , or $c_2 \cup c_3$ is a chordal $(2, 2)$ -coloring of K_n . In the latter case, $V(K_n)$ can be covered by two monochromatic cliques by Theorem 3 of [9]. In the former case, we notice that $T = \{u, v, w\}$ with any $w \notin \{u, v\}$ spans a monochromatic triangle, which must be in color c_1 . Therefore, E_1 induces a connected graph in c_1 . If it is a clique, then one monochromatic clique covers $V(K_n)$. Otherwise, we can apply a basic property of chordal graphs (see e.g. [7] Theorem 4.1): there is a clique cut-set Q , i.e. $V(K_n)$ has a non-trivial partition into A, Q, B such that Q spans a clique in c_1 and no edge of the complete bipartite graph $[A, B]$ has color c_1 . Note that neither A nor B induces an edge e having color c_1 only. Otherwise we could extend e to a monochromatic triple intersecting both A and B . This must be of color c_1 , a contradiction since no edge between A, B can be of color c_1 . Hence $c_2 \cup c_3$ is a chordal 2-coloring of the complete graph on $A \cup B$. Again, by Theorem 3 of [9], $A \cup B$ can be covered by 2 monochromatic cliques, one in color c_2 and one in color c_3 . Therefore, together with Q (in color c_1), $V(K_n)$ has a strong cover with three colors.

Equality is possible for $n = 9$, because the interval $(3, 3)$ -coloring represented in Figure 2 cannot be pierced by two points from different lines (in fact two points from the same line cannot pierce them either). The example can be extended by Proposition 4 for any $n > 9$. $\square$

Proof of Theorem 7. Assume that t is even. For any pair of colors, there is an edge e of K_n missing both colors, otherwise the chordal coloring of K_n can be considered as a $(2, 2)$ -coloring and $\theta(K_n) \leq 2$ follows from Theorem 3 in [9] and the proof is finished. Thus there are edges $e_1, \dots, e_{\frac{t}{2}}$, each missing a different pair of colors. However, their union has at most t vertices, which cannot span a monochromatic clique, contradicting the assumption that we have a (t, t) -coloring.

For odd t , the argument is similar, but here we partition the colors into $\frac{t-3}{2}$ distinct pairs and one triple. Again, for any pair of colors, there is an edge e_i of K_n missing both colors, otherwise the chordal coloring of K_n can be considered as a $(2, 2)$ -coloring and $\theta(K_n) \leq 2$ follows from Theorem 3 in [9] and the proof is finished. Also, there exists a

triangle T in K_n , which is not a monochromatic triangle in any of the three colors of the color-triple, otherwise $\theta(K_n) \leq 3$ follows from Theorem 6, finishing the proof. However,

$$T \cup e_1 \cup \dots \cup e_{\frac{t-3}{2}}$$

covers at most t vertices in K_n , which cannot span a monochromatic clique, contradicting the assumption that we have a (t, t) -coloring. $\square$

Proof of Theorem 8. We apply the following theorem of Gyárfás and Lehel (which strengthens a result of Aharoni, Berger, Chudnovsky and Ziani [3]). Let K_5^* denote the 2-colored K_5 , where each edge has exactly one color and both color classes form a C_5 . In fact, substitution of nearly equal complete graphs into the vertices of a K_5^* shows that Theorem 8 is indeed best possible.

Theorem 11. ([10]) *Assume that K_n has a $(2, 2)$ -coloring containing no (induced) C_4 and K_5^* subgraphs. Then $\theta(K_n) \leq 2$.*

By Theorem 11, we may assume that K_n contains a copy of K_5^* (otherwise we have a strong cover on K_n). Let K be a maximal subgraph of K_n , which can be obtained from K_5^* by substituting cliques with vertex sets $X_1, \dots, X_5$ to its vertices, i.e. no further vertex substitution into K is a subgraph of K_n . Assume w.l.o.g. that the complete bipartite graphs $[X_i, X_{i+1}]$ ($[X_i, X_{i+2}]$) have only red (only blue) edges in (mod 5) index arithmetic (and the edges within each X_i have both colors).

Lemma 12. *Assume that $w \in V(K_n) \setminus V(K)$. Then all edges from w to $V(K)$ have a common color.*

Proof of Lemma 12. Suppose to the contrary that there exists $w \in V(K_n) \setminus V(K)$ and two vertices, $p, q \in V(K)$ such that (w, p) is red only and (w, q) is blue only. We may assume $p \in X_1$. If $q \in X_1$, then we can find $q' \in V(K) \setminus X_1$ such that (w, q') is also blue only. Indeed, otherwise for any $v_2 \in X_2, v_5 \in X_5$ the edges $(w, v_2), (w, v_5)$ have color red and (w, v_2, q, v_5) is a red induced C_4 , a contradiction. Thus we may assume $q \in X_2$ or $q \in X_3$.

Claim 13. (i) *If $q \in X_2$ then w is a replica of a vertex of X_5 .* (ii) *If $q \in X_3$ then w is a replica of a vertex of X_4 .*

Note that Claim 13 implies Lemma 12 because the existence of w contradicts the assumption that K is maximal.

Proof of Claim 13. To prove (i), we consider the set of red 5-cycles of the form $\{p, q, v_3, v_4, v_5, p\}$, where $v_3 \in X_3, v_4 \in X_4, v_5 \in X_5$ are arbitrary. Note that (w, v_3) is blue only, otherwise (w, p, q, v_3) is an induced red C_4 , a contradiction. Also, (w, v_4) is red only, otherwise (w, v_4, p, v_3) is an induced blue C_4 a contradiction. Moreover, $e = (w, v_5)$ must be colored with both red and blue. Indeed, if e is red only, then (w, q, v_5, v_3) spans a blue induced C_4 ; if e is blue only, then (w, p, v_5, v_4) spans a red induced C_4 . Since the choice of v_3, v_4, v_5 was arbitrary, all edges from w to X_3 are blue only, to X_4 red only, and to X_5 both red and blue. We show that all edges from w to X_1 are red only and all edges

from w to X_2 are blue only. This is true for $p \in X_1$ and $q \in X_2$. Suppose to the contrary $v_1 \in X_1, v_1 \neq p$, and the edge (w, v_1) is also blue. For any $v_4 \in X_4$, (w, v_1, v_4, q) is an induced blue C_4 , a contradiction. Similarly, if for $v_2 \in X_2, v_2 \neq q$ the edge (w, v_2) is also red, then for any $v_3 \in X_3, v_4 \in X_4$, (w, v_2, v_3, w_4) is an induced red C_4 , a contradiction, finishing the proof of (i).

Note that the proof of (ii) can be obtained from the proof of (i) by a color switch, considering the set of blue 5-cycles of the form $\{p, q, v_5, v_2, v_4, p\}$, where $v_5 \in X_5, v_2 \in X_2, v_4 \in X_4$ are arbitrary. After that we continue the proof accordingly. This concludes the proof of Claim 13 and the proof of Lemma 12. $\square$

By Lemma 12, $V(K_n) \setminus V(K)$ can be partitioned into R, B , where all edges of the complete bipartite graph $[V(K), R]$ are red and all edges of the complete bipartite graph $[V(K), B]$ are blue. Also, any edge (r_1, r_2) in R is red, otherwise for any $v_1 \in X_1, v_3 \in X_3$, (r_1, v_1, r_2, v_3) is an induced red C_4 . The same argument can be applied to show that all edges within B are blue. Let $|X_i| = \min\{|X_j| : j \in [5]\}$. Now $X_{i+2} \cup X_{i-2} \cup R$ is a red clique and $X_{i+1} \cup X_{i-1} \cup B$ is a blue clique, and their union covers all but at most $|X_i| \leq n/5$ vertices of K_n . This finishes the proof of Theorem 8. $\square$

3 Conclusion

Theorem 2 is a step towards deciding Question 1 motivating our research. Its more general form, Theorem 5 leads to further questions on strong coverings. Also, clique covers lead to Ramsey-type questions in interval/chordal/ C_4 -free $(t, 2)$ -colorings of K_n . For these restricted colorings the t -color classical Ramsey numbers are linear in n . Indeed, chordal and C_4 -free graphs with n vertices and $c\binom{n}{2}$ edges contain cliques of size $c'n$. The best value of c' for chordal graphs is $\alpha(c) = 1 - \sqrt{1-c}$, see [1], [8]. For C_4 -free graphs the existence of c' is established in [8], and its best value we know is $\alpha^2(c)$, proved by Holmsen [11].

3.1 Strong covers

- (i) We do not know what is the largest subset of weakly intersecting t -intervals among n pairwise intersecting t -intervals. Theorems 2, 3 place it between $\frac{n}{3}$ and $\frac{3n}{4}$.
- (ii) Perhaps $\theta(K_n) \leq 2$ for chordal (t, t) -colorings of K_n for large enough odd t , improving Theorem 7. Figure 2 shows that this is false for $t = 3$.
- (iii) For general t , C_4 -free $(t, 2)$ -colorings of K_n are essentially different from chordal $(t, 2)$ -colorings, where there are $n/3$ vertices with a strong cover by Theorem 5. In a C_4 -free $(t, 2)$ -coloring the size of the covered part must depend on t as well. Indeed, Chung and Graham [5] proved that the t -color Ramsey number of C_4 is asymptotic to t^2 . Thus there exist C_4 -free (consequently K_4 -free) t -colorings of K_n , where $n \sim t^2$. By clique substitutions, we get C_4 -free $(t, 2)$ -colorings with the largest monochromatic clique having $\sim \frac{3n}{t^2}$ vertices, thus t monochromatic cliques can cover

at most $m \sim \frac{3n}{t}$ vertices. For $t = 2$, Theorem 8 gives a sharp bound $(\frac{4n}{5})$ for a strong cover. However, for $t = 3$, we do not know the asymptotic, our best example is $\frac{3n}{4}$, based on the coloring of K_8 described in (vii) below.

3.2 Ramsey type problems

- (iv) In every chordal $(2, 2)$ -coloring of K_n , there is a monochromatic clique K_m with $m \geq \lceil \frac{n}{2} \rceil$. This follows immediately from Theorem 11. Equality is possible for every $n \geq 4$ with interval colorings by substituting vertices into the $(2, 2)$ -coloring of K_4 , where both colors form a path with three edges.
- (v) In every interval $(3, 2)$ -coloring of K_n , there is a monochromatic clique K_m with $m \geq \frac{n}{4}$. This follows from the result that pairwise intersecting 3-intervals can be pierced by four points (Theorem 4 in [9]). On the other hand, Figure 3 shows an interval $(3, 2)$ -coloring of K_{10} with no monochromatic K_4 . One can blow-up each vertex by Proposition 4 to get an interval $(3, 2)$ -coloring of K_n where any monochromatic clique can cover at most $\lceil \frac{3n}{10} \rceil + 1$ ($3n/10$ if $n \equiv 0 \pmod{10}$) of the vertices.

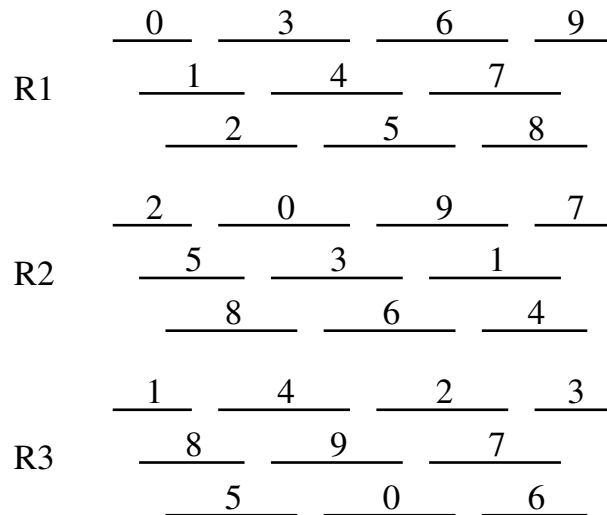


Figure 3: Covering K_{10} with 3 interval graphs.

- (vi) For $t \geq 3$, in every chordal $(t, 2)$ -coloring of K_n , there is a monochromatic clique K_m with $m \geq \alpha(t^{-1})n$. This follows from the result of [1],[8] cited above. There exists an interval t -coloring of K_n with no monochromatic clique larger than $\lceil \frac{n}{t} \rceil + 1$: partition $V(K_n)$ into t parts $S_1, \dots, S_t$ evenly. For every i the edges within S_i and for $j > i$ the edges in the complete bipartite graph $[S_i, S_j]$ are colored with color i .
- (vii) In every C_4 -free $(2, 2)$ -coloring of K_n , there is a monochromatic clique K_m with $m \geq \lceil \frac{2n}{5} \rceil$. Equality is possible for every $n \geq 5$. This follows immediately from Theorem 8.

For C_4 -free $(3, 2)$ -colorings, we have no asymptotic result. Our best lower bound is the general one from (viii). Our upper bound is $\lceil \frac{n}{4} \rceil + 1$ ($n/4$ if $n \equiv 0 \pmod{8}$) obtained by substitutions into the following 3-coloring of K_8 with vertex set $[8]$. Color 1 is the C_7 defined by the cyclic order (1234567) and the edges (4, 8), (7, 8); color 2 is the C_7 defined by the cyclic order (1835746) and the edges (2, 5), (2, 6); color 3 is the C_8 defined by the cyclic order (14273685) and the diagonals (1, 3), (2, 8).

- (viii) In every C_4 -free $(t, 2)$ -coloring of K_n , there is a monochromatic clique K_m with $m \geq \alpha^2(t^{-1})n$. This follows from the result of [11] cited above. On the other hand, as shown in (iii), there exists C_4 -free $(t, 2)$ -colorings with no larger monochromatic cliques than $\sim \frac{3n}{t^2}$.

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