The Multi-Generating Function for Intervals in Young's Lattice: some Comments on a Paper by Azam and Richmond

Jan Snellman[a]

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Abstract

Azam and Richmond studied the generating function $P_\lambda(y)$, which enumerates (by length) partitions in the lower ideal $[0, \lambda]$ in the Young lattice. They found a rational recursion for

$$Q_k(\mathbf{x}, y) = \sum_{\lambda \in \Lambda(k)} P_\lambda(y)\mathbf{x}^\lambda.$$

We show that their results can be extended to a multi-graded version.

By interpreting the original problem as one of enumerating plane partitions with two rows, we can describe the multi-graded version of Q_k using the integer transform of a certain rational pointed polyhedral cone. We furthermore relate Azam's and Richmond's result to those obtained by Andrews and Paule using MacMahon's Ω-operator.

Mathematics Subject Classifications: 05A17,05A15

1 Introduction

For a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0)$, let $|\lambda| = \sum_i \lambda_i$ denote its rank and k denote its length. We use $\Lambda(k)$ to denote the set of partitions with length k.

In [5] Azam and Richmond studied the rank-generating function

$$P_\lambda(y) = \sum_{\mu \in [0,\lambda]} y^{|\mu|}$$

of the lower order ideal $[0, \lambda]$ in the Young lattice. They obtained a rational recursion for

$$Q_k(\mathbf{x}, y) = \sum_{\lambda \in \Lambda(k)} P_\lambda(y)\mathbf{x}^\lambda,$$

[a]Department of Mathematics, Linköping University, Sweden (`jan.snellman@liu.se`)

and concluded that Q_k is a rational function, with denominator

$$D_k(x_1, \ldots, x_k, y) = \prod_{m=1}^{k} \prod_{j=0}^{m} (1 - y^j \prod_{\ell=1}^{m} x_\ell).$$

These results were used to establish asymptotics for the average cardinality of lower order ideals $[0, \lambda]$ of partitions λ of rank n.

2 Multigradings, pairs of partitions, and plane partitions with two rows

2.1 The generating functions Q_k and $\tilde{Q}_k$

Let us define

$$Q_k(\mathbf{x}, \mathbf{y}) = \sum_{\emptyset \leqslant \mu \leqslant \lambda \in \Lambda(k)} \mathbf{y}^\mu \mathbf{x}^\lambda.$$

Then specializing $y_1 = y_2 = \cdots = y_k = y$ we get back the previous $Q_k(\mathbf{x}, y)$. However, the multigraded version can be interpreted as the generating function of plane partitions with at most two rows, where the top row, representing λ, has $\lambda_k > 0$. Introducing

$$\tilde{Q}_k(\mathbf{x}, \mathbf{y}) = \sum_{\emptyset \leqslant \mu \leqslant \lambda \in \Lambda(\leqslant k)} \mathbf{y}^\mu \mathbf{x}^\lambda$$

where $\Lambda(\leqslant k)$ denotes partitions with length at most k, we have that

$$\tilde{Q}_k = \sum_{j=0}^{k} Q_j$$

and that

$$Q_k = \tilde{Q}_k - \tilde{Q}_{k-1}.$$

2.2 Cones, hyperplanes, and polytopes

The generating function $\tilde{Q}_k$ enumerates plane partitions contained in a 2-by-k box. Explicitly, the inequalities that the integer vectors $(\lambda, \mu) \in \mathbb{Z}^k \times \mathbb{Z}^k$ has to satisfy are as follows:

$$\lambda_i - \lambda_j \geqslant 0 \qquad \forall i < j \tag{1}$$
$$\mu_i - \mu_j \geqslant 0 \qquad \forall i < j \tag{2}$$
$$\lambda_i - \mu_i \geqslant 0 \qquad \forall i \tag{3}$$
$$\lambda_i \geqslant 0 \qquad \forall i \tag{4}$$
$$\mu_i \geqslant 0 \qquad \forall i \tag{5}$$

We let $C = C_k \subset \mathbb{R}^k \times \mathbb{R}^k$ denote the rational pointed polyhedral cone cut out in affine space by the above inequalities, and let A_k be its "integer transform", that is to say, the affine monoid $C_k \cap (\mathbb{Z}^k \times \mathbb{Z}^k)$.

2.2.1 The case $k = 2$

For instance, when $k = 2$, the plane partitions in a 2×2-box are

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix} \qquad \lambda_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant 0,\ \lambda_1 \geqslant \mu_1 \geqslant \mu_2 \geqslant 0.$$

The corresponding integer transform is $\tilde{Q}_2(x_1, x_2, y_1, y_2)$; to get the plane partitions enumerated by $Q_2((x_1, x_2, y_1, y_2))$ we add the extra inequality $\lambda_2 > 0$. The resulting polyhedron has C_2 as its recession cone.

The cone $C_2 \subset \mathbb{R}^2 \times \mathbb{R}^2$ has 5 extremal rays, listed in Table 1. They can be calculated using Fourier-Motzkin elimination (see for instance [12] for a description of this algorithm). The software Normaliz [6] (which is conveniently available from within SageMath [10]) is able to do these calculations.

Table 1: Generating rays of plane partitions inside a 2 by 2 box

0	(1, 0, 0, 0)
1	(1, 0, 1, 0)
2	(1, 1, 0, 0)
3	(1, 1, 1, 0)
4	(1, 1, 1, 1)

2.2.2 General k

For a general k, we note that all extremal rays of $C = C_k$ intersect the affine hyperplane

$$H = \{(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k) \,:\, \lambda_1 = 1\}$$

in lattice points. Call the set of these points S_k. Let $P = P_k$ be the intersection $C \cap H$. Let

$$T_k = P \cap (\mathbb{Z}^k \times \mathbb{Z}^k).$$

Recall that we introduced the affine monoid

$$A = A_k = C_k \cap (\mathbb{Z}^k \times \mathbb{Z}^k)$$

whose generating function is $\tilde{Q}_k$.

Lemma 1. *Let C_k, A_k, P_k, S_k, T_k be as above. Then*

1. *The cone C is the disjoint union*

$$C = \cup_{t \geqslant 0} tP$$

   of dilations of P.

2. $S_k = T_k$.

3. *Denote the vector of length r consisting of all ones by $\mathbf{1}^r$, and the vector of length r consisting of all zeroes by $\mathbf{0}^r$. Put*

$$U_k = \{(\mathbf{1}^a, \mathbf{0}^b, \mathbf{1}^c, \mathbf{0}^d) \,:\, a+b = c+d = k, a \geqslant c, a \geqslant 1\}. \tag{6}$$

   Then $S_k = T_k = U_k$.

4. *The polytope P is the convex hull of S_k.*

5. *$\tilde{Q}_k$ is the multigraded Ehrhart series of P.*

6. *Let $D_k = \prod_{\mathbf{r} \in S_k} (1 - (\mathbf{xy})^{\mathbf{r}})$. Then $\tilde{Q}_k \times D_k$ is a polynomial.*

7. *S_k form a Hilbert basis for the affine monoid A_k.*

Proof. Let (λ, μ) be a plane partition in A_k. If (λ, μ) is non-zero, then $\lambda_1 \geqslant 1$. Let $(s(\lambda), s(\mu))$ be the support of the pair; here $s(\lambda)(i) = 1$ if $\lambda_i > 0$, and zero otherwise. Then it is easy to see that $(s(\lambda), s(\mu)) \in U_k$. Furthermore,

$$(\lambda, \mu) - (s(\lambda), s(\mu)) \in A_k.$$

Thus, every element in A_k is expressible as a sum of elements in U_k.

Elements in T_k are irreducible; if the partition $(\mathbf{1}^a, \mathbf{0}^b)$ is to written as a sum of elements in $\mathbb{N}^k \times \mathbb{N}^k$, one of the summands would have to start with a zero — but this is impossible.

By Gordan's lemma (see for instance [7]) we have that the Hilbert basis of A_k consists of the irreducible elements in the monoid. Any element in $(\lambda, \mu) \in A_k$ with $\lambda_1 > 1$ can be written as

$$(\lambda, \mu) = (s(\lambda), s(\mu)) + ((\lambda, \mu) - (s(\lambda), s(\mu)))$$

and is thus reducible. Hence, the Hilbert basis consists precisely of T_k, and this set is equal to U_k and S_k. □

For a simplicial rational cone, the generating function has numerator 1, and denominator given by the extremal rays. Our cone C is not simplicial, though; it has more generators than the embedding dimension $2k$. Thus the numerator is some multivariate polynomial. However, from general theory [7, 9] it follows that

Corollary 2. *The denominator of $\tilde{Q}_k$, and hence of Q_k, is precisely D_k.*

Specializing $y_1 = \cdots = y_k = y$ we recover Proposition 15 of [5]. In the multigraded case there can be no cancellation between the numerator and the denominator of Q_k, so we can assert that this D_k is *the* denominator, not just divisible by the denominator.

2.3 Calculating $\tilde{Q}_k$ by triangulating C_k

2.3.1 $k = 2$

Let us consider C_2 again. It lives in $\mathbb{R}^2 \times \mathbb{R}^2$ but, as was shown in Table 1, it is spanned by 5 extremal rays, hence it is it not simplicial. We can, however, triangulate it into a union of simplicial cones. We used Normaliz [6] to find such a triangulation, shown in Table 2 (rows indicate subsets of rays).

Table 2: Triangulation of C_2

0	1	2	4
1	2	3	4

So $C = C_2 = K_1 \cup K_2$, where K_1, K_2 and $K_3 = K_1 \cap K_2$ are rational simplicial cones. K_3 is generated by the intersection of the generating rays of K_1 and of K_2, that is to say, by r_1, r_1, r_4.

A rational polyhedral simplicial cone generated by the rays $\mathbf{r}$ will have generating function

$$\frac{1}{\prod_{\mathbf{r}}(1-(\mathbf{xy})^{\mathbf{r}}))}.$$

Hence, by inclusion-exclusion,

$$\begin{aligned}\frac{N}{(1-(\mathbf{xy})^{r_0})(1-(\mathbf{xy})^{r_1})(1-(\mathbf{xy})^{r_2})(1-(\mathbf{xy})^{r_3})(1-(\mathbf{xy})^{r_4})}\\ = \frac{1}{(1-(\mathbf{xy})^{r_0})(1-(\mathbf{xy})^{r_1})(1-(\mathbf{xy})^{r_2})(1-(\mathbf{xy})^{r_4})}\\ + \frac{1}{(1-(\mathbf{xy})^{r_1})(1-(\mathbf{xy})^{r_2})(1-(\mathbf{xy})^{r_3})(1-(\mathbf{xy})^{r_4})}\\ - \frac{1}{(1-(\mathbf{xy})^{r_1})(1-(\mathbf{xy})^{r_2})(1-(\mathbf{xy})^{r_4})}\end{aligned}$$

hence

$$N = (\mathbf{xy})^{r_3} + (\mathbf{xy})^{r_0} - (\mathbf{xy})^{r_0}(\mathbf{xy})^{r_3},$$

which evaluates to

$$(-x_0y_0x_1 + 1) + (-x_0 + 1) - ({x_0}^2y_0x_1 - x_0y_0x_1 - x_0 + 1) = -{x_0}^2y_0x_1 + 1.$$

2.3.2 $k = 3$

For $k = 3$ the plane partitions are

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{pmatrix}$$

with inequalities ensuring that the entries are non-negative and non-increasing in rows and columns.

Extremal rays There are now 9 extremal rays, generating the cone $C = C_3 \subset \mathbb{R}^3 \times \mathbb{R}^3$. We display these vectors in $\mathbb{R}^3 \times \mathbb{R}^3$ as 2 by 3 matrices.

Table 3: Extremal rays of plane partitions with 2 rows and 3 columns

0	$\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}$	3	$\begin{pmatrix}1&1&0\\1&0&0\end{pmatrix}$	6	$\begin{pmatrix}1&1&1\\1&0&0\end{pmatrix}$
1	$\begin{pmatrix}1&0&0\\1&0&0\end{pmatrix}$	4	$\begin{pmatrix}1&1&0\\1&1&0\end{pmatrix}$	7	$\begin{pmatrix}1&1&1\\1&1&0\end{pmatrix}$
2	$\begin{pmatrix}1&1&0\\0&0&0\end{pmatrix}$	5	$\begin{pmatrix}1&1&1\\0&0&0\end{pmatrix}$	8	$\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}$

Triangulation A (regular) triangulation of the cone, with rays numbered as in Table 3, is shown in Table 4.

Table 4: Triangulation of the cone of plane partitions with 2 rows and 3 columns, rows are subcones

0	1	2	4	5	8
0	1	4	5	7	8
1	2	3	4	5	8
1	3	4	5	6	8
1	4	5	6	7	8

2.3.3 General k

It is feasible to use inclusion-exclusion to find $\tilde{Q}_3$, the generating function of the cone C_3. However, this is not an efficient way of calculating $\tilde{Q}_k$ for general k. The number of extremal rays of C_k is, as we shown, equal to one less the number of plane partitions inside a $2 \times k \times 1$ box. From [8], this number is $\binom{2+k}{2} - 1$. The number of simplicial subcones in the triangulation grows swiftly; it is equal to the Catalan number. We tabulate the number of extremal rays and subcones in a triangulation of C_k in Table 5.

3 The rational recursion of Azam and Richmond

3.1 Original version

We state the main result of [5]. Recall that their Q_k is multi-graded in $\mathbf{x}$ but simply-graded in y, so depends on $k + 1$ variables.

Table 5: nr of cones in triangulation of C

k	dim(C)	nr rays	nr cones in tri
2	4	5	2
3	6	9	5
4	8	14	14
5	10	20	42
6	12	27	132
7	14	35	429
8	16	44	1430
9	18	54	4862

Theorem 3 (Azam and Richmond Thm 1). *Let $p_k = x_1 \cdots x_k$, and for a sequence of parameters $Z = (z_1, \ldots, z_{k+1})$, let*

$$Q_k(Z) = Q_k(z_1, \ldots, z_{k+1}).$$

- *If $Z = (x_1, \ldots, x_k, y)$, then denote $Q_k = Q_k(Z)$.*
- *For $0 < r \leqslant k$, we put $Z_r = (y^r p_{r+1}, x_{r+2}, x_{r+3}, \ldots, x_k, y)$.*

Then $Q_0 = 1$ and for $k \geqslant 1$ we have

$$(1 - p_k)Q_k = x_k Q_{k-1} + \sum_{0 \leqslant i < r \leqslant k} \left(\frac{y^r p_k}{1 - y^r p_r} \right) Q_{k-r}(Z_r) \cdot Q_i \tag{7}$$

In particular, Q_k is a rational function in the variables $x_1, \ldots, x_k, y$.

They go on to prove

Proposition 4 (Azam and Richmond Proposition 15). *Let $p_k = x_1 \cdots x_k$, and $D_k = D_k(x_1, \ldots, x_k, y) = \prod_{m=1}^{k} \prod_{j=0}^{m} (1 - y^j p_m)$. Then $Q_k \cdot D_k$ is a polynomial.*

As we have seen, this latter results is a straight-forward consequence of classification of the generating rays of C_k.

The numerators N_k of $Q_k = N_k / D_k$ are given below, for $k = 1, 2, 3$. We show the multi-graded case, with denominator $D_k = \prod_{m=1}^{k} \prod_{j=0}^{m} (1 - p_m \prod_{\ell=1}^{j} y_\ell)$. The y-simplygraded case, (as studied by Azam and Richmond) can be recovered by setting the different y_i's to y.

$k = 1$: $-x_1^2 y_1 + x_1 y_1 + x_1$.

$k = 2$:

$$x_1^3 y_1^2 x_2^3 y_2 - x_1^2 y_1^2 x_2^2 y_2 - x_1^2 y_1 x_2^2 y_2 - x_1^2 y_1 x_2^2 - x_1^2 y_1 x_2 + x_1 y_1 x_2 y_2 + x_1 y_1 x_2 + x_1 x_2$$

$k = 3$:

$$\begin{aligned}
x_1^6y_1^4x_2^5y_2^2x_3^4y_3 - x_1^5y_1^4x_2^4y_2^2x_3^3y_3 - x_1^5y_1^3x_2^4y_2^2x_3^3y_3 - x_1^4y_1^3x_2^4y_2^2x_3^4y_3 - x_1^5y_1^3x_2^4y_2x_3^3y_3 \\
- x_1^5y_1^3x_2^4y_2x_3^3 - x_1^5y_1^3x_2^4y_2x_3^2 + x_1^4y_1^3x_2^3y_2^2x_3^2y_3 + x_1^3y_1^3x_2^3y_2^2x_3^3y_3 + x_1^4y_1^3x_2^3y_2x_3^2y_3 \\
+ x_1^3y_1^2x_2^3y_2^2x_3^3y_3 + x_1^4y_1^3x_2^3y_2x_3^2 + x_1^4y_1^2x_2^3y_2x_3^2y_3 + x_1^3y_1^2x_2^3y_2x_3^3y_3 + x_1^4y_1^2x_2^3y_2x_3^2 + x_1^3y_1^2x_2^3y_2x_3^3 \\
+ x_1^4y_1^2x_2^3x_3^2 + x_1^3y_1^2x_2^3y_2x_3^2 - x_1^2y_1^2x_2^2y_2^2x_3^2y_3 + x_1^3y_1^2x_2^3y_2x_3 - x_1^3y_1^2x_2^2y_2x_3y_3 - x_1^2y_1^2x_2^2y_2x_3^2y_3 \\
- x_1^2y_1^2x_2^2y_2x_3^2 - x_1^2y_1x_2^2y_2x_3^2y_3 - x_1^2y_1^2x_2^2y_2x_3 - x_1^2y_1x_2^2y_2x_3^2 - x_1^2y_1x_2^2y_2x_3 \\
- x_1^2y_1x_2^2x_3^2 - x_1^2y_1x_2^2x_3 + x_1y_1x_2y_2x_3y_3 - x_1^2y_1x_2x_3 \\
+ x_1y_1x_2y_2x_3 + x_1y_1x_2x_3 + x_1x_2x_3
\end{aligned}$$

We note, for future reference, that

$$\tilde{Q}_2 = 1 + Q_1 + Q_2 = \frac{1 - x_1^2x_2y_1}{(1-x_1)(1-x_1x_2)(1-x_1y_1)(1-x_1x_2y_1)(1-x_1x_2y_1y_2)} \tag{8}$$

3.2 Multigraded version

We come to the main purpose of this note: the rational recursion of Q_k works multigradedly!

Corollary 5. *For $i, r, k \geqslant 0$, define*

- $Q_k = Q_k(\mathbf{x}, \mathbf{y})$
- $p_r = x_1 \cdots x_r$
- $q_r = y_1 \cdots y_r$
- $\hat{Z}_{r,k} = (p_{r+1}q_r, x_{r+2}, \ldots, x_k, y_{r+1}, \ldots, y_k)$
- $R_{i,r,k} = \frac{p_kq_r}{(1-p_k)(1-p_rq_r)} Q_i Q_{k-r}(\hat{Z}_{r,k})$

Then $Q_0 = 1$ and for $k > 0$

$$Q_k = \frac{x_kQ_{k-1}}{1-p_k} + \sum_{0 \leqslant i < r \leqslant k} R_{i,r,k} \tag{9}$$

Proof (sketch). The difference to the original theorem is that $Q_k = Q_k(\mathbf{x}, \mathbf{y})$ is a function of $2k$ variables whereas $\mathbf{Q}_k(\mathbf{x}, y)$ is a function of $k+1$. Furthermore, the substitution in Q_{k-r} is refined to

$$Q_{k-r}(x_1 \cdots x_{r+1} \cdot y_1 \cdots y_r, x_{r+2}, \ldots, x_k, y_{r+1}, \ldots, y_k)$$

rather than

$$Q_{k-r}(x_1 \cdots x_{r+1} \cdot y^r, x_{r+2}, \ldots, x_k, y).$$

The various lemmas and propositions in Section 2 of [5] that prove the recursion are based on bijections, and can be modified so to work multigradedly. Specifically:

- Replace $P_\lambda(y) = \sum_{\mu\in[\emptyset,\lambda]} y^{|\mu|}$ with $P_\lambda(\mathbf{y}) = \sum_{\mu\in[\emptyset,\lambda]} \mathbf{y}^\mu$
- Replace $P_{\mu,\lambda}(y) = \sum_{\nu\in[\mu,\lambda]} y^{|\nu|}$ with $P_{\mu,\lambda}(\mathbf{y}) = \sum_{\nu\in[\mu,\lambda]} \mathbf{y}^\nu$
- Replace
$$Q_{k,m}(\mathbf{x}, y) = \sum_{\lambda\in\Lambda(k,m)} P_\lambda(y)\mathbf{x}^\lambda$$
with
$$Q_{k,m}(\mathbf{x}, \mathbf{y}) = \sum_{\lambda\in\Lambda(k,m)} P_\lambda(\mathbf{y})\mathbf{x}^\lambda$$
- Proposition 12: Replace y^k with $y_1 \cdots y_k$.
- Proposition 14: Also replace
$$Q_{k-r,m-1}(y^r p_{r+1}, x_{r+2}, \ldots, x_k, y) \text{ with } Q_{k-r,m-1}(q_r p_{r+1}, x_{r+2}, \ldots, x_k, y_{r+1}, \ldots, y_k)$$
- Lemma 13, Theorem 1: Do the above replacements. □

4 Relation to prior work by Andrews and Paule and MacMahon

4.1 Geometric interpretation of the rational recursion

The rational recursion above yields an efficient way of calculating Q_ℓ, and hence $\tilde{Q}_\ell$. Explicitly,

$$\begin{aligned}
\tilde{Q}_\ell &= \sum_{k=0}^{\ell} Q_k \\
&= \sum_{k=0}^{\ell} \left(\frac{x_k Q_{k-1}}{1-p_k} + \sum_{0\leqslant i<r\leqslant k} R_{i,r,k} \right) \\
&= \sum_{k=0}^{\ell} \left(\frac{x_k Q_{k-1}}{1-p_k} + \sum_{0\leqslant i<r\leqslant k} \frac{p_k q_r}{(1-p_k)(1-p_r q_r)} Q_i Q_{k-r}(\hat{Z}_{r,k}) \right)
\end{aligned}$$

This is a description how to slice up the affine monoid A_ℓ into disjoint pieces; Q_k enumerates lattice points in $C \cap H^+$, C being the polyhedral cone, and $H+$ the open half-space $\lambda_k > 0$. The term $Q_{k-1}\frac{x_k}{1-p_k}$ enumerates lattice points in the translation of the projection of C in a certain direction, et etcetera. It is not a triangulation of C into subcones, nor is it a "disjoint decomposition" as is computed by Normaliz; it is much more complicated.

4.2 Generating functions for plane partitions in a box using the Omega operator

In a long series of papers, starting with [1], [2], Andrews and Paule revisits MacMahon's [8] method of partition analysis. Their twelfth entry [3] applies this method to plane partitions. They define

$$p_{m,n}(X) = \sum_{a_{i,j} \in P_{m,n}} x_{1,1}^{a_{1,1}} \cdots x_{m,n}^{a_{m,n}}$$

where $P_{m,n}$ consists of all $m \times n$ matrices $(a_{i,j})$ over non-negative integers $a_{i,j}$ such that $a_{i,j} \geqslant a_{i,j+1}$ and $a_{i,j} \geqslant a_{i+1,j}$. Putting $m = 2$, we get our objects of interest.

They then (pages 650-651) illustrate MacMahon's method using his Ω operator by calculating $p_{2,2}(X)$. This is of course the same as $\tilde{Q}_2$.

Andrews and Paule have developed a Mathematica package called Omega [4]. The Maple package LinDiophantus [11], by Doron Zeilberger, can perform similar calculations. We will briefly illustrate how to use the Omega package (written by Daniel Krenn) which is included in recent versions of SageMath [10]. For more information about this package, type `help(MacMahonOmega)` at the SageMath command prompt.

```
L.<mu11,mu12,l11,l21,x11,x12,x21,x22> = LaurentPolynomialRing(ZZ)
p22setup = [1-x11*l11*mu11, 1-x21*l21/mu11, 1-x12*mu12/l11, 1-x22/(l21*mu12)]
M = MacMahonOmega
p22 = M(l21, M(l11, M(mu12, M(mu11, 1, p22setup))))
[(t[0],t[1]) for t in p22]
```

yields the result

```
[(-x11^2*x12*x21 + 1, 1),
 (-x11 + 1, -1),
 (-x11*x12 + 1, -1),
 (-x11*x12*x21*x22 + 1, -1),
 (-x11*x21 + 1, -1),
 (-x11*x12*x21 + 1, -1)]
```

We recognize from (8) the numerator (first line) and denominator of $\tilde{Q}_2$, with renamed variables.

The most interesting part, for us, in [3], is their Lemma 2.3, which provides a recursion for plane partitions in an $m \times n$ box. Specialising to $m = 2$ we get

Corollary 6 (Andrews and Paule Lemma 2.3)**.**

$$p_{2,n+1}\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} & x_{1,n+1} \\ x_{2,1} & \cdots & x_{2,n} & x_{2,n+1} \end{pmatrix} = \left(1 - x_{1,n+1}x_{2,n+1} \prod_{1 \leqslant i \leqslant 2, 1 \leqslant j \leqslant n} x_{i,j}\right)^{-1}$$
$$\times \Omega_{\geqslant} \quad p_{2,n}\begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} & \lambda_0 x_{1,n} \\ x_{2,1} & \cdots & x_{2,n-1} & \lambda_1 x_{2,n} \end{pmatrix}$$
$$\times \frac{1}{(1 - \frac{x_{1,n+1}}{\lambda_0})(1 - \frac{x_{1,n+1}x_{2,n+1}}{\lambda_0 \lambda_1})} \quad (10)$$

Without going into details regarding the Ω operator, we will mention that it operates of formal Laurent polynomials and transforms the expression under its purvey so that the "spurious" λ variables (not related to partitions, we are using Andrews' and Paule's notations here) gets eliminated, and what is left is the desired generating function.

Question 7. Is there a relation between the "rational recursion" (9) and Andrews' and Paule's Lemma 2.3?

Remark 8. The SageMath code used by the author to implement the rational recursion is included in the ancillary section of the ArXiv preprint, arXiv:2401.04030.

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