Forbidden pairs for traceability of 2-connected graphs

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Abstract

Let $\mathcal{H}$ be a set of connected graphs. A graph is said to be *$\mathcal{H}$-free* if it does not contain an induced subgraph isomorphic a member of $\mathcal{H}$. A graph is called *traceable* if it has a path containing all its vertices. In 1997, Faudree and Gould characterized all pairs R, S such that every connected $\{R, S\}$-free graph is traceable. In this paper, we extend this result by considering 2-connected graphs, and characterize all pairs R, S such that every 2-connected $\{R, S\}$-free graph is traceable. Furthermore, we characterize all 2-connected $\{K_{1,3}, N_{1,3,4}\}$-free non-traceable graphs.

Mathematics Subject Classifications: 05C38, 05C45

1 Introduction

We basically follow the most common graph-theoretical terminology and notation and for concepts not defined here we refer the reader to [4]. All graphs in this paper are simple, finite and undirected.

Let G be a graph, and $u, v \in V(G)$, $X \subseteq V(G)$, and let H be a subgraph of G. Then $N_G(v)$ denotes the set, and $d_G(v)$ the number, of neighbors of v in G, $d_H(v)$ the number of neighbors of v in H, $N_G(X)$ the set of vertices of $V(G) \setminus X$ having a neighbor in X, and $N_H(X)$ the set of vertices of $V(H) \setminus X$ having a neighbor in X. For $X \subset V(G)$, we use $\langle X\rangle_H$ to denote the subgraph of H induced by the set of vertices X in H. The *distance* between u and v in G is denoted $dist_G(u, v)$, and when $u, v \in V(H)$, $dist_H(u, v)$ denotes their distance in the subgraph H of G, i.e., the length of a shortest path between u and v in H. The *girth* (the *circumference*) of G, denoted by $g(G)$ ($c(G)$), is the length of a shortest (longest) cycle of G. A *pendant vertex* is a vertex of degree 1, and a *pendant edge* is an edge having a pendant vertex as an end vertex. As usual, we use $P_i (i \geqslant 1)$ to denote the path on i vertices. We use $N_{i,j,k}$ to denote the graph obtained by attaching

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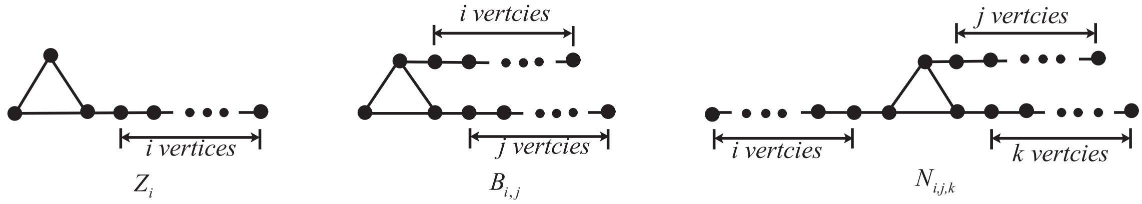

Figure 1: $Z_i, B_{i,j}$ and $N_{i,j,k}$

three vertex-disjoint paths of lengths $i, j, k \geqslant 0$ to a triangle. In the special case when $i, j \geqslant 1$ and $k = 0$ (or $i \geqslant 1$ and $j = k = 0$), $N_{i,j,k}$ is also denoted $B_{i,j}$ (or Z_i), respectively (see Figure 1).

A graph G is called *hamiltonian*, if it contains a Hamilton cycle, i.e., a cycle containing all vertices of G. A graph G is called *traceable*, if it contains a Hamilton path, i.e., a path containing all vertices of G. A graph G is called *Hamilton-connected* if it contains a Hamilton (x, y)-path for each pair x, y of vertices of G. A graph G is called *supereulerian* if it contains a spanning connected even subgraph of G. A graph G is called *pancyclic* if it contains a cycle C_l for all $3 \leqslant l \leqslant |G|$. A cycle C in a graph G is called *dominating* if every edge of G is incident with a vertex of C. A spanning subgraph H of graph G is called its *k-factor* if every vertex in H has edgree k, 1-factor is also called *perfect matching*. A graph G is called *homogeneously traceable* if it has a Hamilton path starting from any vertex of G.

Let $\mathcal{H}$ be a set of connected graphs. A graph G is said to be $\mathcal{H}$-free if G does not contain an induced subgraph isomorphic to a member of $\mathcal{H}$. We call $\mathcal{H}$ a *forbidden pair* of G if $|\mathcal{H}| = 2$. If $\mathcal{H} = \{H\}$, then we simply say that G is *H-free* and G is *claw-free* if $H = K_{1,3}$. For a property $\mathcal{P}$, it is a popular research topic to give forbidden induced subgraphs condition forcing a graph to have the property $\mathcal{P}$. Many researchers characterized the forbidden pairs for the property $\mathcal{P}$ of k-connected graphs, we now summarize some known results in Table 1. When considering forbidden pairs for property $\mathcal{P}$ of k-connected graphs, the connectivity k must meet the necessity condition of $\mathcal{P}$ for otherwise the forbidden pairs are 'none'. The literal 'trivial' in Table 1 means that the forbidden pairs are arbitrary because every 4-edge-graph has been supereulerian. In 1984, Matthews and Sumner [36] conjectured that every 4-connected claw-free graph is hamiltonian, which has been proven to be equivalent to many other conjectures and still open. So the literal 'open' in Table 1 means that the forbidden pairs are related to Matthews-Sumner conjecture.

In this paper, we characterize all forbidden pairs for traceability of 2-connected graphs by proving the following.

Theorem 1. *Let R, S be a pair of connected graphs such that neither R nor S is an induced subgraph of P_3. Then every 2-connected $\{R, S\}$-free graph is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $B_{2,4}, N_{1,1,5}$ or $N_{1,3,3}$; $R = K_{1,4}$ and $S = P_4$.*

We actually obtain more general result than Theorem 1, and characterize all 2-connected $\{K_{1,3}, N_{1,3,4}\}$-free non-traceable graphs. Before state our next result, we need

Table 1: Characterizing forbidden pairs for properties $\mathcal{P}$ of k-connected graphs

$\mathcal{P}$	$k=1$	$k=2$	$k=3$	$k \geqslant 4$
hamiltonian	none	full [2][13]	partical [7][18][26][30][33][40]	open
traceable	full [13]	this paper	unknown	open
Hamilton-connected	none	none	partical [3][13][15][25][31][32][38]	open
perfect mathching	full [20]	full [19]	full[19]	full [19]
2-factor	none	full [14]	unknown	open
homogenously traceable	none	full [27]	unknown	open
supereulerian	none	full [34][35]	unknown	trivial
pancyclic	none	none	full [22]	partical [16][17]
dominating cycle	none	partical [8][9][12]	unknown	open

the following definitions.

Let G be a claw-free graph. A vertex $x \in V(G)$ is *locally connected* if the neighborhood of x induces a connected subgraph in G. For $x \in V(G)$, the graph G'_x obtained from G by adding the edges $\{yz : y, z \in N(x) \text{ and } yz \notin E(G)\}$ is called the *local completion* of G at x. The *closure* of G, denoted by $cl(G)$, is obtained from G by recursive performing local completions at any locally connected vertex with non-complete neighborhood, as long as it is possible. Ryjáček [37] proved the closure $cl(G)$ is the line graph of a triangle-free graph. To *split* a vertex v is to replace v by two adjacent vertices, v' and v'', and to replace each edge incident to v by an edge incident to either v' and v'' (but not both). We now define the following six graphs depicted in Figure 2. Let

- F_1 be obtained from a complete bipartite graph $K_{2,t}(t \geqslant 2)$ by splitting one vertex of degree t into two new vertices, and adding some pendant edges (possible zero) to the two new vertices and the other vertex of degree t (denoted as the special vertex v)};

- F_2 be obtained from $K_{2,2t+1}(t \geqslant 1)$ by adding some pendant edges (possible zero) to the exactly one vertex of degree two (denoted as the special vertex v) and the two vertices of degree $2t+1$;

- F_3 be obtained from $K_{2,2t}(t \geqslant 1)$ by adding some pendant edges (possible zero) to the exactly one vertex of degree two (denoted as the special vertex v) and the two vertices of degree $2t$;

- F_4 be obtained from $K_{2,2t}(t \geqslant 2)$ by adding some pendant edges (possible zero) to all vertices (one of the two vertices of degree $2t$ is denoted as the special vertex v);

- F_5 be obtained from $K_{2,2t+1}(t \geqslant 2)$ by adding some pendant edges (possible zero) to all vertices except one vertex of degree two (one of the two vertices of degree $2t+1$ is denoted as the special vertex v);

- F_6 be obtained from $K_{2,2t+1}(t \geqslant 2)$ by adding at least one pendant edge to each vertex of degree two and by adding some pendant edges (possible zero) to the two vertices of degree $2t+1$ (one of the two vertices of degree $2t+1$ is denoted as the special vertex v).

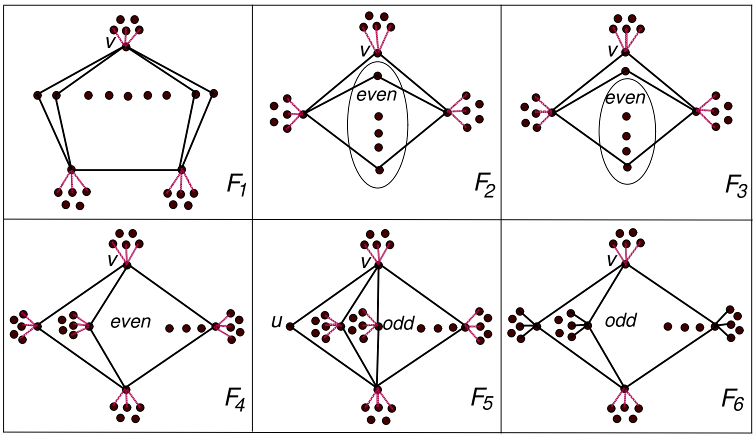

Figure 2: Graph F_i for $1 \leqslant i \leqslant 6$

Let $F_\star$ be obtained three copies of F_6 by identifying their special vertex v, and let $\mathcal{F} = \{F : F$ is obtained from $F_\star$ and a sequence of $F_i(1 \leqslant i \leqslant 6)$ by identifying their special vertex $v\}$. We may state our next main result.

Theorem 2. *Let G be a 2-connected $\{K_{1,3}, N_{1,3,4}\}$-free graph. Then either G is traceable or $cl(G)$ is the line graph of a member in $\mathcal{F}$.*

In the next section, we will introduce the properties of Ryjáček closure and some useful results. In Section 3, we will prove Theorem 1. In Section 4, we will prove Theorems 2 and 11 which are used to prove the sufficiency of Theorem 1. In the last section, we give some concluding remarks.

2 Preliminaries and basic results

2.1 The stable properties under Ryjáček closure

Ryjáček prove that the closure of claw-free graphs preserves the hamiltonicity.

Theorem 3. *([37]) Let G be a claw-free graph. Then*

(i) $cl(G)$ is uniquely determined;

(ii) $cl(G)$ is the line graph of a triangle-free graph;

(iii) G is hamiltonian if and only if $cl(G)$ is hamiltonian.

Brandt, Favaron and Ryjáček proved that the closure of claw-free graphs preserves the traceability.

Theorem 4. *([5]) Let G ba a claw-free graph. Then G is traceable if and only if $cl(G)$ is traceable.*

We say that a class $\mathcal{H}$ of claw-free graphs is *stable* if for every graph in $\mathcal{H}$, its closure is also in $\mathcal{H}$. Brousek, Ryjáček and Favaron proved that the following classes of claw-free graphs are stable.

Theorem 5. *([6]) If $S \in \{P_i : i > 0\} \cup \{Z_i : i > 0\} \cup \{N_{i,j,k} : i, j, k > 0\}$, then the class of $\{K_{1,3}, S\}$-free graphs is stable.*

In [6], Brousek, Ryjáček and Favaron also pointed out that the class of $\{K_{1,3}, B_{i,j}\}$-free graphs ($i, j \geqslant 1$) is not stable. Later, Du and the second author dealt with this case by considering $\{K_{1,3}, B_{i,j}\}(i, j \geqslant 1)$-free graphs with three pendant vertices.

Lemma 6. *([11]) Let G be a connected claw-free graph. Suppose G contains a connected induced subgraph H with three pendant vertices v_1, v_2, v_3. Then for any pair of $v_i, v_j \in \{v_1, v_2, v_3\}$, H has an induced subgraph $B_{l,k}(l \geqslant k \geqslant 1)$ containing v_i, v_j.*

Lemma 7. *Let G be a connected claw-free graph. Suppose G has a connected induced subgraph H containing two pendant edges u_1v_1, u_2v_2 with $u_1 \neq u_2$ and $d_H(v_1) = d_H(v_2) = 1$. If $|V(H)\backslash\{v_1, v_2\}| \geqslant 3$, then H has an induced $B_{1,1}$ containing v_1 and v_2, or $dist_H(u_1, u_2) \geqslant 2$.*

Proof. Suppose that $dist_H(u_1, u_2) \leqslant 1$. Then $u_1u_2 \in E(G)$. Since $|V(H)\backslash\{v_1, v_2\}| \geqslant 3$, there exists a vertex $w \in V(H)\backslash\{v_1, v_2, u_1, u_2\}$ such that w is adjacent to one of u_1, u_2, say $u_1w \in E(G)$. Since G is claw-free, we have $u_2w \in E(G)$, then $\langle\{v_1, u_1, u_2, v_2, w\}_G$ is an induced $B_{1,1}$. This proves Lemma 7. □

2.2 Useful results

A subgraph H of a graph G is *dominating* if every edge of G has at least one end in H. A subgraph H of a graph G is *even* if every vertex of H has even degree. A *trail* in a graph G is a sequence $W := v_0e_1v_1 \cdots v_{l-1}e_lv_l$, whose terms are alternately vertices (not necessarily distinct) and distinct edges of G, such that v_{i-1} and v_i are ends of $e_i, 1 \leqslant i \leqslant l$. For convenience, we sometimes abbreviate the term of $v_0e_1v_1 \cdots v_{l-1}e_lv_l$ to $v_0v_1 \cdots v_{l-1}v_l$. Harary and Nash-Williams [23] showed that for a graph H with $|E(H)| \geqslant 3$, $L(H)$ is hamiltonian if and only if H has a dominating connected even subgraph. Li, Lai and Zhan obtained similar result for traceability.

Theorem 8. *([29]) Let G be a graph with $|E(G)| \geqslant 3$. Then the line graph $L(G)$ is traceable if and only if G has a dominating trail.*

We now give the following definitions introduced in [39]. Let G be a 2-connected graph and let C be a cycle of G, and let D be a component of $G - V(C)$. Clearly D has at least two distinct neighbors on C. For any path P in D, if the two ends (probably only one if P is itself a vertex) of P have two distinct neighbors x_1, x_2 on C, then P is called a *2-attaching path* of C in D, and $\{x_1, x_2\}$ is called a *2-attaching pair* of P on C. Note that if D is a K_1 or K_2, then D is itself the 2-attaching path of C. Furthermore, if a longest 2-attaching path of D has order k, then D is called a *k-component* of $G - V(C)$.

Lemma 9. *([39]) Let G be a 2-connected graph with circumference $c(G)$ and let C be a longest cycle of G. Then*

(i) if D is a k-component of $G - V(C)$, then $k \leqslant \lfloor \frac{c(G)}{2} \rfloor - 1$;

(ii) every 2-component of $G - V(C)$ is a star;

(iii) if $c(G) \leqslant 5$, then G has a spanning trail starting from any vertex and every vertex lies on a circumference cycle;

(iv) if $c(G) \leqslant 7$, then G has a spanning trail.

3 Proof of Theorem 1

3.1 The necessity part of Theorem 1

We may construct eight non-traceable 2-connected graphs G_i with $1 \leqslant i \leqslant 8$, as shown in Figure 3. Then each $G_i(1 \leqslant i \leqslant 8)$ contains at least one of R, S as an induced subgraph.

Claim 10. *Either R or S is a $K_{1,3}$ or a $K_{1,4}$.*

Proof. Suppose, by contradiction, that neither R nor S is a $K_{1,3}$ or a $K_{1,4}$. Consider the following fact: If a connected graph is not a graph in $\{P_1, P_2, P_3, K_{1,3}, K_{1,4}\}$, then it contains one of the graphs in $\{K_3, C_4, P_4, K_{1,5}\}$ as an induced subgraph. Consider the graph G_1, we may assume that R is an induced subgraph of G_1 without loss of generality. Since G_1 is $\{K_3, P_4\}$-free, it follows that R contains one of graphs in $\{C_4, K_{1,5}\}$ as an induced subgraph. Suppose first that R contains an induced subgraph C_4. Note that G_2 and G_7 are C_4-free, implying that both are R-free. Then S is a common induced subgraph of G_2 and G_7. Note that the maximal common induced subgraph of G_2 and G_7 is $K_{1,3}$, implying that S is an induced subgraph of $K_{1,3}$, a contradiction.

Now suppose that R contains an induced subgraph $K_{1,5}$. Note that G_6 and G_7 are $K_{1,5}$-free, implying that both are R-free. Then S is a common induced subgraph of G_6 and G_7. Note that the maximal common induced subgraph of G_6 and G_7 is $K_{1,4}$, implying that S is an induced subgraph of $K_{1,4}$, a contradiction. This proves Claim 10. $\square$

By Claim 10, we may assume that $R = K_{1,3}$ or $K_{1,4}$ without loss of generality. If $R = K_{1,4}$, then considering graphs G_4, G_5, G_7, each one is $K_{1,4}$-free and then it contains S as an induced subgraph. Note that the common induced subgraph of G_5 and G_7 is a path, implying that S is a path. Since the largest induced path of G_4 is P_4, it follows that S is an induced subgraph of P_4.

Hence we assume that $R = K_{1,3}$. Considering graphs G_3, G_5, G_8, each one is $K_{1,3}$-free and then it contains S as an induced subgraph. If S is a tree, then since G_8 is claw-free, S is a path. Note that the largest induced path of G_8 is P_8, implying S is an induced path of P_8. Hence assume that S contains an induced cycle. Note that the length of any common induced cycle of G_3 and G_5 is three, implying that any induced cycle of S should be a triangle. We further claim that S contains only one triangle. Otherwise, S

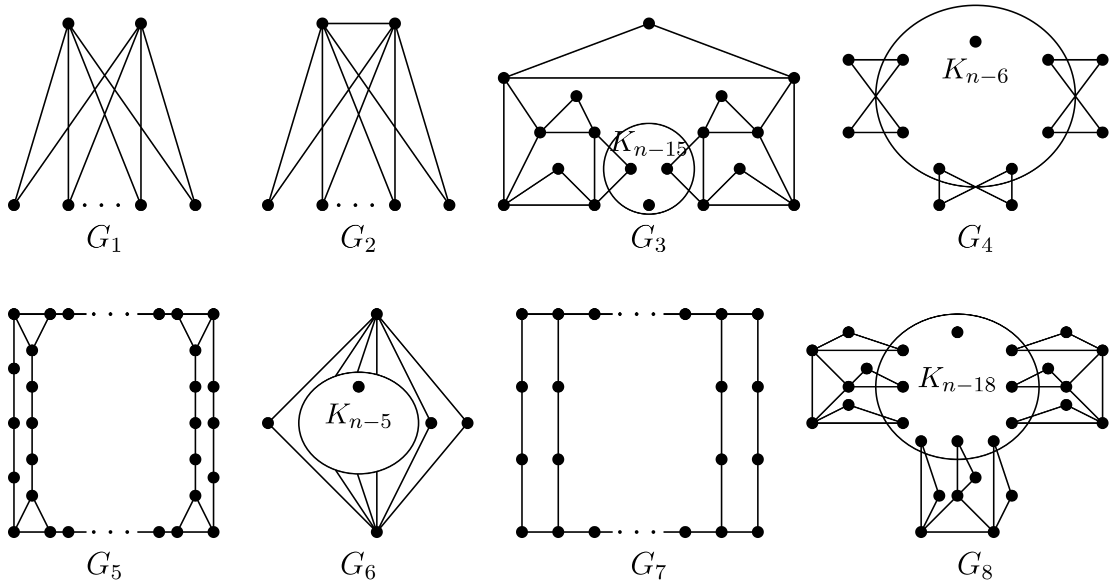

Figure 3: $G_i, i = 1, \cdots, 8$

contains at least two triangles. Note that the length of any induced path in G_5 joining any two triangles is at least four. But the length of any induced path in G_8 joining any two triangles is at most three, contradicting the fact that S is a common induced subgraph of G_8 and G_5.

Then S is a $Z_i, B_{i,j}$ or an $N_{i,j,k}$. If S is a Z_i, then since the maximal Z_i of G_8 is Z_5, S is an induced subgraph of Z_5. If S is a $B_{i,j}$, then since all maximal $B_{i,j}$ of G_5 are $B_{2,4}, B_{1,5}$, S is an induced subgraph of $B_{2,4}$ or $B_{1,5}$. If S is an $N_{i,j,k}$, then since all maximal $N_{i,j,k}$ of G_8 are $N_{1,1,5}, N_{3,3,3}$, and G_3 is $N_{2,2,2}$-free, S is an induced subgraph of $N_{1,1,5}$ or $N_{1,3,3}$. Note that $P_8, Z_5, B_{1,5}$ are three induced subgraphs of $N_{1,1,5}$, we summarize that S is an induced subgraph of $B_{2,4}, N_{1,1,5}$, or $N_{1,3,3}$. The proof is complete. □

3.2 The sufficiency part proof of Theorem 1

We now state the following results to prove the sufficiency of Theorem 1.

Theorem 11. *Every 2-connected $\{K_{1,3}, N_{1,1,5}\}$-free graph is traceable.*

A graph is called a *block-chain* if its connectivity is at least 2, or its connectivity is 1 and it has exactly two end-blocks. Li, Broersma and Zhang proved the following.

Theorem 12. *([28] Every $\{K_{1,4}, P_4\}$-free block-chain is traceable.*

Proof the sufficiency of Theorem 1. By Theorms 11 and 12, we only prove that every 2-connected $\{K_{1,3}, S\}$-free graph is traceable for the cases $S = B_{2,4}$ or $N_{1,3,3}$. Suppose not, and let G be a counter-example to this. Then G is a 2-connected non-traceable

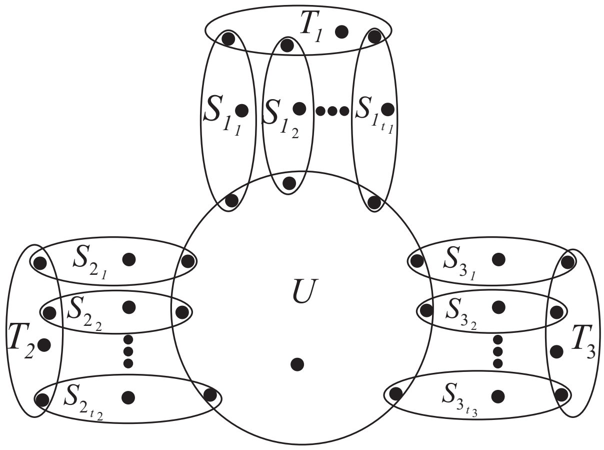

Figure 4: $L(F_\star)$

$\{K_{1,3}, N_{1,3,4}\}$-free graph. By Theorem 2, we assume that $cl(G) = L(H)$ where $H \in \mathcal{F}$. By the definition of $\mathcal{F}$, H contains a subgraph $F_\star$ which is obtained by three copies F_6 be identifying the special vertices, where F_6 is depicted in Fighre 2. Then $cl(G)$ contains an induced subgraph $L(F_\star)$ where $L(F_\star)$ is depicted in Figure 4. Let W be the vertex set of $L(F_\star)$. By the definition of $L(F_\star)$, W has a partition $\{U, S_{1_1}, \cdots, S_{1_{t_1}}, S_{2_1}, \cdots, S_{2_{t_2}}, S_{3_1}, \cdots, S_{3_{t_3}}, T_1, T_2, T_3\}$ such that each one induces a clique of size at least three.

Note that for any two distinct elements $X_1, X_2 \in \{U, S_{1_1}, \cdots, S_{1_{t_1}}, S_{2_1}, \cdots, S_{2_{t_2}}, S_{3_1}, \cdots, S_{3_{t_2}}, T_1, T_2, T_3\}$ with $X_1 \cap X_2 \neq \emptyset$, it holds that $|X_1 \cap X_2| = 1$ and let $X_1 \cap X_2 = \{v_{X_1 X_2}\}$. For $i = 1, 2, 3$, since $|S_{i_{t_i}}| \geqslant 3$, we have $S_{i_{t_i}} \setminus (U \cup T_i) \neq \emptyset$. Let w_1, w_2, w_3 be three vertices such that $w_j \in S_{j_2} \setminus (U \cup T_j)$ for $j = 1, 2, 3$. The graph $\langle \{v_{US_{1_1}}, v_{US_{2_1}}, v_{US_{3_1}}\} \cup \{v_{T_1 S_{1_1}}, v_{T_1 S_{1_2}}, w_1\} \cup \{v_{T_2 S_{2_1}}, v_{T_2 S_{2_2}}, w_2\} \cup \{v_{T_3 S_{3_1}}, v_{T_3 S_{3_2}}, w_3\} \rangle_{cl(G)}$ is exactly an induced $N_{3,3,3}$ in $cl(G)$. On the other hand, since G is $\{K_{1,3}, N_{1,3,3}\}$-free, $cl(G)$ is $N_{1,3,3}$-free by Theorem 5, a contradiction.

In the following, we shall find an induced $B_{2,4}$ in G to obtain a contradiction. We first have the following fact.

Claim 13. *For $i \in \{1, 2, 3\}$ and $j \in \{1, \cdots t_i\}$, $dist_{\langle S_{i_j} \rangle_G}(v_{US_{i_j}}, v_{S_{i_j} T_i}) \geqslant 2$.*

Proof. Otherwise, we may assume that $v_{US_{1_1}} v_{S_{1_1} T_1}$ is an edge in G without loss of generality. Let $t \in T_1$ and $u \in U$ such that $v_{S_{1_1} T_1} t, v_{US_{1_1}} u \in E(G)$. Probably $t = v_{T_1 S_{1_2}}, u = v_{S_{1_2} U}$. Considering the graph $\langle S_{1_1} \cup \{u, t\} \rangle_G$, which is a connected claw-free graph with two pendant edges $v_{S_{1_1} T_1} t, v_{US_{1_1}} u$. By Lemma 7, $\langle S_{1_1} \cup \{u, t_1\} \rangle_G$ contains an induced $B_{1,1}$ containing u and t, say $B_{1,1}(u, t)$. Recall that $w_j \in S_{j_2} \setminus (U \cup T_j)$ for $j = 1, 2, 3$. Let $P(t, w_1)$ be a shortest path connecting t and w_1 in $\langle T_1 \cup S_{1_2} \rangle_G$, and let $P(u, w_2)$ be a longest induced path connecting u and w_2 in $\langle \{U, S_{2_1}, T_2, S_{2_2}\} \rangle_G$. It is easy to see the lengths of $P(t, w_1)$ and $P(u, w_2)$ are at least one and three, respectively. Together these two induced paths with $B_{1,1}(u, t)$ can yield an induced $B_{2,4}$ in G, a contradiction. This proves Claim 13. □

Recall that $w_j \in S_{j_2} \setminus (U \cup T_j)$ for $j = 1, 2, 3$, and let $P(v_{US_{j_1}}, w_j)$ be a shortest path connecting $v_{US_{j_1}}$ and w_j in $\langle S_{j_1} \cup T_j \cup S_{j_2} \rangle_G$. For each $j = 1, 2, 3$, $P(v_{US_{j_1}}, w_j)$ has length at least four by Claim 13, and let $v^+_{US_{j_1}}$ denote the the successor of $v_{US_{j_1}}$ on the path $P(v_{US_{j_1}}, w_j)$. Consider the graph $\langle U \cup \{v^+_{US_{1_1}}, v^+_{US_{2_1}}, v^+_{US_{3_1}}\} \rangle_G$, which is an induced subgraph of G containing three pendant vertices $v^+_{US_{1_1}}, v^+_{US_{2_1}}, v^+_{US_{3_1}}$. By Lemma 6, $\langle U \cup \{v^+_{US_{1_1}}, v^+_{US_{2_1}}, v^+_{US_{3_1}}\} \rangle_G$ has an induced $B_{l,k} (l, k \geqslant 1)$ containing $\{v^+_{US_{1_1}}, v^+_{US_{2_1}}\}$, together with two paths $P(v_{US_{1_1}}, w_1), P(v_{US_{2_1}}, w_2)$, is an induced $B_{4,4}$ in G, a contradiction. The proof is complete. □

4 Proofs of Theorems 11 and 2

We start with the following notation. The *core* of a graph G, denoted by G_0, is obtained by recursive deleting all pendant vertices of G. We define $\Lambda(G)$ to be the set of the vertices in G which is incident with at least one pendant vertex. An edge cut X of a graph G is *essential* if $G \setminus X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge-cut X with $|X| < k$. Note that if G is essentially 2-edge-connected then its core G_0 is 2-edge-connected. Let G be an essentially 2-edge-connected graph and let G_0 be the core of G. For any block B of G_0, let $\mathfrak{B} = B \cup \{e : e \text{ is a pendant edge of } G \text{ and has at least one end in } V(B) \cap \Lambda(G)\}$. We then call $\mathfrak{B}$ a *super-block* of G. Furthermore, if $\mathfrak{B} \cap G_0$ contains at least two cut vertices of G_0 then $\mathfrak{B}$ is called an *inner-super-block* of G, otherwise $\mathfrak{B}$ is called *outer-super-block* of G. For integer $i, j, k \geqslant 0$, let $T_{i,j,k}$ be the graph obtained three paths P_{i+1}, P_{j+1} and P_{k+1} by identifying exactly one end vertex of each path. The resulting special vertex of $T_{i,j,k}$ is called the *root*. We call the other end of P_{k+1} in $T_{i,j,k}$ the *special leaf*.

Let G be a claw-free graph. Then $cl(G)$ is the line graph of a triangle-free graph by Theorem 3. Note that by Theorems 4 and 5 that Theorems 11 and 2 can be equivalently expressed as follows:

- every 2-connected $N_{1,1,5}$-free line graph $L(H)$ with $g(H) \geqslant 4$ is traceable,
- every 2-connected $N_{1,3,4}$-free line graph $L(H)$ with $g(H) \geqslant 4$ is traceable or $L(H)$ is the line graph of a member in $\mathcal{F}$.

Observe that $L(H)$ is $N_{i,j,k}$-free if and only if H has no subgraph isomorphic to $T_{i+1,j+1,k+1}$. Note that $L(H)$ is k-connected if and only if H is essentially k-edge-connected or complete. If $L(H)$ is complete, then $L(H)$ is traceable. Therefore, by Theorem 8, Theorems 11 and 2 can be equivalently expressed as the following two theorems.

Theorem 14. *Let G be an essentially 2-edge-connected triangle-free graph without subgraphs isomorphic to $T_{2,2,6}$. Then G has a dominating trail.*

Theorem 15. *Let G be an essentially 2-edge-connected triangle-free graph without subgraphs isomorphic to $T_{2,4,5}$. Then either G has a dominating trail or $G \in \mathcal{F}$.*

Before to prove theorems 14 and 15 we need the following lemmas. For a subgraph H of G and for $v \in V(H)$, we denote by $P_v(H)$ a longest path in H starting from v, and use $|P_v(H)|$ to denote the order of $P_v(H)$. Note that if G is triangle-free then $|P_v(\mathfrak{B})| \geqslant 4$.

Lemma 16. *Let G be an essentially 2-edge-connected triangle-free graph, and let $\mathfrak{B}$ be an outer-super-block of G containing the cut vertex v of G_0. Suppose that $\mathfrak{B}$ has no dominating cycle containing v. Then each of the following holds:*

(i) if $c(\mathfrak{B}) \geqslant 6$, then $|P_v(\mathfrak{B})| \geqslant 6$ and $\mathfrak{B}$ contains a subgraph $T_{2,2,k} (k \geqslant 1)$ with the special leaf v, furthermore, either $\mathfrak{B}$ contains a subgraph $T_{2,4,0}$ with the root v or $\mathfrak{B}$ has a dominating trail starting from v,

(ii) if $c(\mathfrak{B}) = 5$, then $\mathfrak{B}$ contains a subgraph $T_{2,4,0}$ with the root v, and contains a subgraph $T_{2,2,k} (k \geqslant 2)$ with the special leaf v and $|P_v(\mathfrak{B})| \geqslant 6$,

(iii) if $c(\mathfrak{B}) = 4$, then either $\mathfrak{B}$ contains a subgraph $T_{2,2,k} (k \geqslant 2)$ with the special leaf v and contains a subgraph $T_{2,4,0}$ with the root v and $|P_v(\mathfrak{B})| \geqslant 5$, or $|P_v(\mathfrak{B})| \geqslant 6$ and $\mathfrak{B}$ contains a subgraph $T_{2,2,k} (k \geqslant 1)$ with the special leaf v.

Proof. Suppose that $c(\mathfrak{B}) \geqslant 6$. It is easy to see that $|P_v(\mathfrak{B})| \geqslant 6$. Let C be a longest cycle of $\mathfrak{B}$. If $v \in V(C)$, then $E(\mathfrak{B} - V(C)) \neq \emptyset$, for otherwise C is a dominating cycle of $\mathfrak{B}$ containing v. Let D be a nontrivial component of $\mathfrak{B} - V(C)$. Note that $|N_{\mathfrak{B}}(D) \cap V(C)| \geqslant 2$. It is easy to deduce that $\langle V(C) \cup V(D) \rangle_G$ contains a subgraph $T_{2,4,0}$ with the root v and also contains a subgraph $T_{2,2,k} (k \geqslant 1)$ with the special leaf v. Hence we may assume that $v \notin V(C)$. Since $\mathfrak{B} \cap G_0$ is 2-connected, there exist two internally disjoint paths Q_1, Q_2 in $\mathfrak{B} \cap G_0$ joining v and C. If some Q_i has length at least two, then $Q_1 \cup Q_2 \cup C$ contains a subgraph $T_{2,4,0}$ with the root v and also contains a subgraph $T_{2,2,k} (k \geqslant 1)$ the special leaf v. Hence we may assume that each $Q_i (i = 1, 2)$ is an edge, implying that $\langle V(C) \cup \{v\} \rangle_G$ is connected. Assume that $Q_1 = vv', Q_2 = vv''$. If $c(\mathfrak{B}) \geqslant 7$ or $dist_C(v', v'') = 2$, then one can easily check that $\langle V(C) \cup \{v\} \rangle_G$ contains a subgraph $T_{2,4,0}$ with the root v and also contains a subgraph $T_{2,2,k} (k \geqslant 1)$ with the special leaf v. So we have $c(\mathfrak{B}) = 6$ and $dist_C(v', v'') = 3$. Clearly $\langle V(C) \cup \{v\} \rangle_G$ contains a subgraph $T_{2,2,1}$ with the special leaf v. If $\mathfrak{B} - V(C) \cup \{v\}$ is edgeless, then $\langle V(C) \cup \{v\} \rangle_G$ contains a dominating trail of $\mathfrak{B}$ starting from v. So we assume that $E(\mathfrak{B} - V(C) \cup \{v\}) \neq \emptyset$. Then there exists an edge u_1u_2 in $\mathfrak{B} - V(C) \cup \{v\}$ incident with some vertex in $V(C) \cup \{v\}$, one can easily check that $\langle V(C) \cup \{v, u_1, u_2\} \rangle_G$ contains a subgraph $T_{2,4,0}$ with the root v.

Now suppose that $c(\mathfrak{B}) \leqslant 5$. Since $\mathfrak{B} \cap G_0$ is 2-connected and by Lemma 9(iii), there is a cycle C of length $c(\mathfrak{B})$ in $\mathfrak{B} \cap G_0$ containing v. Choose C such that the number of vertices in $V(\mathfrak{B}) \cap \Lambda(G)$ is maximized. Since C is a longest cycle of $\mathfrak{B} \cap G_0$ and by Lemma 9(i), $D \cap G_0$ is a 1-component of $\mathfrak{B} \cap G_0 - V(C)$, say u_1. Since D is nontrivial, $u_1 \in V(\mathfrak{B}) \cap \Lambda(G)$ and let u_1u_2 be a pendant edge of G. Since $\mathfrak{B} \cap G_0$ is 2-connected and C is a longest cycle of $\mathfrak{B} \cap G_0$, it follows that $|N_{\mathfrak{B}}(u_1) \cap V(C)| = 2$. Suppose further that $c(\mathfrak{B}) = 5$. Let $C = vv_1v_2v_3v_4v$. If $v \in N_{\mathfrak{B}}(u_1) \cap V(C)$, then $N_{\mathfrak{B}}(u_1) \cap V(C) = \{v, v_2\}$ or $N_{\mathfrak{B}}(u_1) \cap V(C) = \{v, v_3\}$. By symmetry, we may assume that $N_{\mathfrak{B}}(u_1) \cap V(C) = \{v, v_2\}$, then $v_1 \in V(H) \cap \Lambda(G)$; otherwise the cycle $vu_1v_2v_3v_4v$ has more vertices in $V(\mathfrak{B}) \cap \Lambda(G)$. Let v_1z be a pendant edge of G. Then $v_2v_1z \cup v_2u_1u_2 \cup v_2v_3v_4v$ is a $T_{2,2,3}$ with the special leaf v and $vv_4v_3v_2u_1u_2$ is a path of order 6. If $v \notin N_{\mathfrak{B}}(u_1) \cap V(C)$, then $N_{\mathfrak{B}}(u_1) \cap V(C) = \{v_1, v_3\}$ or $N_{\mathfrak{B}}(u_1) \cap V(C) = \{v_2, v_4\}$. By symmetry, we may assume that $N_{\mathfrak{B}}(u_1) \cap V(C) = \{v_1, v_3\}$, then $v_2 \in V(H) \cap \Lambda(G)$; otherwise the cycle $vv_1u_1v_3v_4v$ has more vertices in $V(\mathfrak{B}) \cap \Lambda(G)$. Let v_2z be a pendant edge of G. Then $v_3u_1u_2 \cup v_3v_2z \cup v_3v_4v$ is a $T_{2,2,2}$ with v as its leaf and $vv_4v_3v_2v_1u_1u_2$ path of order 7.

Now suppose that $c(\mathfrak{B}) = 4$. Let $C = v_0v_1v_2v_3v$. If $v \in N_{\mathfrak{B}}(u_1) \cap V(C)$, then $N_{\mathfrak{B}}(u_1) \cap V(C) = \{v, v_2\}$, implying that $v_1 \in V(H) \cap \Lambda(G)$; otherwise the cycle $vu_1v_2v_3v$ has more vertices in $V(\mathfrak{B}) \cap \Lambda(G)$. Let v_1z be a pendant edge of G. Then $v_2v_1z \cup v_2u_1u_2 \cup v_2v_3v$ is a $T_{2,2,2}$ with the special leaf v and $zv_1vv_3v_2u_1u_2$ is a $T_{2,4,0}$ with the root v and $vv_3v_2u_1u_2$ is path of order 5. If $v \notin N_{\mathfrak{B}}(u_1) \cap V(C)$, then $N_{\mathfrak{B}}(u_1) \cap V(C) = \{v_1, v_3\}$, implying that $v_2 \in V(H) \cap \Lambda(G)$; otherwise the cycle $vv_1u_1v_3v$ has more vertices in $V(\mathfrak{B}) \cap \Lambda(G)$. Let v_2z be a pendant edge of G. Then $v_3u_1u_2 \cup v_3v_2z \cup v_3v$ is a $T_{2,2,1}$ with the special leaf v and $vv_3v_2v_1u_1u_2$ is a path of order 6. This completes the proof. □

Lemma 17. *Let G be an essentially 2-edge-connected graph without subgraphs isomorphic to $T_{2,2,6}$. If $\kappa(G_0) \geqslant 2$, then G has a dominating trail.*

Proof. Suppose, by contradiction, that G has no dominating trail. Let $C = v_0v_1v_2 \cdots v_{c(G)-1}\, v_0$ be a longest cycle of G. Then $E(G - V(C)) \neq \emptyset$, for otherwise C is a dominating trail of G. Thus $G - V(C)$ has a nontrivial component D. Let P be a longest 2-attaching path of C in $D \cap G_0$ with a 2-attaching pair $\{v_{i'}, v_{i''}\}$. Since C is a longest cycle of G, we have $dist_C(v_{i'}, v_{i''}) \geqslant 2$. Since D is nontrivial, there exists an edge u_1z_1 in D incident with one of $v_{i'}$ and $v_{i''}$. Then $c(G) \leqslant 8$, for otherwise $\langle V(C) \cup \{u_1, z_1\}\rangle_G$ contains a subgraph isomorphic to $T_{2,2,6}$. By Lemma 9(iv), it suffices to consider the case when $|V(C)| = 8$, then $dist_C(v_{i'}, v_{i''}) \leqslant 4$.

In the following, the subscript i of v_i is in $\{0, 1, \cdots, 7\}$.

Claim 18. *For any two independent edges u_1z_1, u_2z_2 in $G - V(C)$. If u_1z_1 is incident with v_i in C, then u_2z_2 cannot be incident with any vertex in $\{v_i, v_{i+1}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+7}\}$.*

Proof. Otherwise, we may assume that $u_1v_i \in E(G)$, then we obtain the following

$$\begin{cases} v_iu_1z_1 \cup v_iu_2z_2 \cup v_iv_{i+1}v_{i+2}v_{i+3}v_{i+4}v_{i+5}v_{i+6} \cong T_{2,2,6} & \text{if } u_2v_i \in E(G) \\ v_{i+1}v_iu_1 \cup v_{i+2}u_2z_2 \cup v_{i+1}v_{i+2}v_{i+3}v_{i+4}v_{i+5}v_{i+6}v_{i+7} \cong T_{2,2,6} & \text{if } u_2v_{i+1} \in E(G) \\ v_{i+3}v_{i+2}v_{i+1} \cup v_{i+3}u_2z_2 \cup v_{i+3}v_{i+4}v_{i+5}v_{i+6}v_{i+7}v_iu_1 \cong T_{2,2,6} & \text{if } u_2v_{i+3} \in E(G) \\ v_{i+4}v_{i+3}v_{i+2} \cup v_{i+4}u_2z_2 \cup v_{i+4}v_{i+5}v_{i+6}v_{i+7}u_1z_1 \cong T_{2,2,6} & \text{if } u_2v_{i+4} \in E(G) \\ v_{i+5}v_{i+6}v_{i+7} \cup v_{i+5}u_2z_2 \cup v_{i+5}v_{i+4}v_{i+3}v_{i+2}v_{i+1}v_iu_1 \cong T_{2,2,6} & \text{if } u_2v_{i+5} \in E(G) \\ v_{i+7}v_iu_1 \cup v_{i+7}u_2z_2 \cup v_{i+7}v_{i+6}v_{i+5}v_{i+4}v_{i+3}v_{i+2}v_{i+1} \cong T_{2,2,6} & \text{if } u_2v_{i+7} \in E(G) \end{cases}$$

a contradiction. □

Claim 19. *D is the only one nontrivial component of $G - V(C)$.*

Proof. Suppose to the contrary, and let D' be a nontrivial component of $G-V(C)$ distinct from D. Note that $|N_G(D) \cap V(C)| \geqslant 2$ and $|N_G(D') \cap V(C)| \geqslant 2$. Since $\{v_{i'}, v_{i''}\} \subseteq N_G(D) \cap V(C)$ and by Claim 18, $N_G(D') \cap V(C) \cap \{v_{i'}, v_{i'+1}, v_{i'+3}, v_{i'+4}, v_{i'+5}, v_{i'+7}\} = \emptyset$ and $N_G(D') \cap V(C) \cap \{v_{i''}, v_{i''+1}, v_{i''+3}, v_{i''+4}, v_{i''+5}, v_{i''+7}\} = \emptyset$, then $N_G(D') \cap V(C) = \{v_{i'+2}, v_{i'+6}\} = \{v_{i''+2}, v_{i''+6}\}$, implying that $i'' = i' + 4$. Without loss of generality, we may assume that $v_{i'} = v_0$ and $v_{i''} = v_4$, then $N_G(D') \cap V(C) = \{v_4, v_6\}$. Since C is a longest cycle of G, it follows that $D \cap G_0$ and $D' \cap G_0$ are both 1-components of $G_0 - V(C)$, say u_1, u_2. Clearly u_1 and u_2 dominate all edges in D and D', respectively. Since D is nontrivial, $u_1 \in \Lambda(G)$ and let u_1z_1 be a pendant edge of G. Then v_7 has no neighbors in $G - V(C) \cup V(D) \cup V(D')$. Otherwise, assume that $z_2 \in N_{G-V(C)-u_1}(v_7)$, then $v_6v_7z_2 \cup v_6v_5v_4 \cup v_6u_2v_2v_1v_0u_1z_1$ is a $T_{2,2,6}$, a contradiction. But then $v_1v_0u_1v_4v_5v_6u_2v_2v_3$ is a dominating trail of G. This proves Claim 19. □

If P is a dominating path of D, then by Claim 19, $\langle V(C) \cup V(P)\rangle_G$ contains a dominating trail of G, a contradiction. Hence P cannot be a dominating path in D. It follows that P has length at least two, for otherwise $D \cap G_0$ is a 1-component of $G_0 - V(C)$ and clearly $D \cap G_0$ dominates all edges of D. Recall that P is a longest 2-attaching path of C in $D \cap G_0$ with a 2-attaching pair $\{v_{i'}, v_{i''}\}$. Since C is a longest cycle of G, we have $3 \leqslant dist_C(v_{i'}, v_{i''}) \leqslant 4$ and P is an edge if $dist_C(v_{i'}, v_{i''}) = 3$. Suppose that $dist_C(v_{i'}, v_{i''}) = 3$. Since P is not a dominating path of D, there is an edge u_1u_2 in D incident with P, which is not dominated by P. But then $\langle V(C) \cup V(P) \cup \{u_1, u_2\}\rangle_G$ contains a subgraph isomorphic to $T_{2,2,6}$, a contradiction.

So we have $dist_C(v_{i'}, v_{i''}) = 4$. We may assume that $v_{i'} = v_0, v_{i''} = v_4$ without loss of generality. Since C is a longest cycle of G, we have $2 \leqslant |V(P)| \leqslant 3$. If P is an edge and let $P = x_1x_2$, then since P is not a dominating path of D, there is an edge x_3z in D such that x_3 is adjacent to x_1 or x_2, where x_3z is not dominated by x_1x_2. We may assume that $x_1x_3 \in E(G)$ without loss of generality. Note that $D \cap G_0$ is a 2-component of $G_0 - V(C)$. By Lemma 9 (ii) $D \cap G_0$ is a star, implying that x_1 is the center and x_3 is a leaf of $D \cap G_0$. Then x_3z is a pendant edge of G and $x_3 \in \Lambda(G)$. Since G_0 is 2-connected, $N_G(x_3) \cap V(C) \neq \emptyset$. By Claim 18, $N_G(x_3) \cap \{v_0, v_1, v_3, v_4, v_5, v_7\} = \emptyset$ and therefore $N_G(x_3) \cap \{v_2, v_6\} \neq \emptyset$, implying that $x_2x_1x_3$ is a 2-attaching path of C in $D \cap G_0$. But this contradicts the fact that P is a longest 2-attaching path of C in $D \cap G_0$.

Hence we have $|V(P)| = 3$ and let $P = y_1y_2y_3$. Since P is not a dominating path of D, there is an edge y_4z in D such that y_4 is adjacent to some y_i in $\{y_1, y_2, y_3\}$, where y_4z is not dominated by $y_1y_2y_3$. Therefore, $y_1y_4, y_3y_4 \notin E(G)$, for otherwise, up to symmetry, we may assume that $y_4y_1 \in E(G)$, but then $y_1y_4z \cup y_1y_2y_3 \cup y_1v_0v_1v_2v_3v_4v_5$ is a $T_{2,2,6}$. Hence we have $y_4y_2 \in E(G)$. We further claim that z is a pendant vertex of G. If not, then $z \in V(G_0)$. Since G_0 is 2-connected, there exist two internally disjoint paths Q_1, Q_2 in G_0 from z to $\{y_1, y_3\}$. Note that $D \cap G_0$ is a 3-component of $G_0 - V(C)$. Then there is no cycle in $D \cap G_0$ containing y_3, y_2, y_4, z or y_1, y_2, y_4, z, implying that one of Q_1 and Q_2, say Q_1, joins z and C such that $y_1, y_2, y_3, y_4 \notin V(Q_1)$, then there exists a 2-attaching path of C in $D \cap G_0$ containing y_3, y_2, y_4, z or y_1, y_2, y_4, z. But this contradicts the fact that P is a longest 2-attaching path of C in $D \cap G_0$. This implies that z is a pendant vertex of G and then $y_4 \in V(G_0)$. Since G_0 is 2-connected, there exist two internally disjoint path Q', Q'' in G_0 from y_4 to $\{y_1, y_3\}$. Since $D \cap G_0$ is a 3-component of $G_0 - V(C)$, there is no cycle in $D \cap G_0$ containing y_3, y_2, y_4 or y_1, y_2, y_4, implying that one of Q' and Q'', say Q', joins y_4 and C such that $y_1, y_2, y_3 \notin V(Q')$. Therefore, since $D \cap G_0$ is a 3-component of $G_0 - V(C)$, Q' is an edge and hence $N_G(y_4) \cap V(C) \neq \emptyset$. By Claim 18, $N_G(y_4) \cap \{v_0, v_1, v_3, v_4, v_5, v_7\} = \emptyset$ and then $N_G(y_4) \cap \{v_2, v_6\} \neq \emptyset$. By symmetry, we may assume that $v_2 \in N_G(y_4)$, but then $v_0v_1v_2y_4y_2y_3v_4v_5v_6v_7v_0$ is a 10-cycle, a contradiction. This completes the proof of Lemma 17. □

Proof of Theorem 14. We argue by contradiction, and assume that G is a counter-example to Theorem 14 such that the number of super-blocks of G is minimized. By Lemma 17 we have $\kappa(G_0) = 1$.

For a super-block $\mathfrak{B}$ of G, we use $G/\mathfrak{B}$ to mean that deleting all edges between vertices of $\mathfrak{B}$ and then identifying the vertices of $\mathfrak{B}$ into a single vertex v, we call v the *concentration* in $G/\mathfrak{B}$.

Claim 20. *Every outer-super-block of G has no dominating cycle containing the cut vertex of G_0.*

Proof. Suppose not, and let $\mathfrak{B}$ be an outer-super-block of G such that $\mathfrak{B}$ has a dominating cycle C containing the cut vertex v of G_0 in $\mathfrak{B}$. Note that $G/\mathfrak{B}$ has less super-blocks than G and no subgraph isomorphic to $T_{2,2,6}$. By the minimality of the super-blocks of G, $G/\mathfrak{B}$ has a dominating trail T, but then $T \cup C$ is a dominating trail of G, contradicting the choice of G. This proves Claim 20. □

Claim 21. *G_0 has only one cut vertex v.*

Proof. Suppose otherwise. Then G_0 contains two end cut vertices v_1, v_2. Let $\mathfrak{B}_1, \mathfrak{B}_2$ be two outer-super-blocks of G such that $v_i \in V(\mathfrak{B}_i)$. Clearly $V(\mathfrak{B}_1) \cap V(\mathfrak{B}_2) = \emptyset$. By Claim 20 and Lemma 16, $\mathfrak{B}_i$ contains a subgraph $T_{2,2,k}(k \geqslant 1)$ with the special leaf v and $|P_{v_i}(\mathfrak{B}_i)| \geqslant 5$ for each $i = 1, 2$. Choose a longest path $P(v_1, v_2)$ in G joining v_1 and v_2. Since G_0 is 2-edge-connected, $P(v_1, v_2)$ passes through at least one nontrivial block (contains a cycle) of G_0, implying that $|P(v_1, v_2)| \geqslant 3$. It is easy to see that $\mathfrak{B}_1 \cup P(v_1, v_2) \cup P_{v_2}(\mathfrak{B}_2)$ contains a subgraph $T_{2,2,7}$ and clearly contains a subgraph $T_{2,2,6}$, a contradiction. This proves Claim 21. □

Let $\mathfrak{B}_1, \mathfrak{B}_2$ be two outer-super-blocks of G. By Claim 21 we have $V(\mathfrak{B}_1) \cap V(\mathfrak{B}_2) = \{v\}$. By Claim 20 and Lemma 16, for each $i = 1, 2$, $\mathfrak{B}_i$ contains a subgraph $T_{2,2,k}(k \geqslant 1)$ with the special leaf v and $|P_{v_i}(\mathfrak{B}_i)| \geqslant 5$. Since $\mathfrak{B}_1 \cup \mathfrak{B}_2$ has no subgraph isomorphic to $T_{2,2,6}$, it follows that $|P_{v_i}(\mathfrak{B}_i)| = 5$ for $i = 1, 2$. Therefore, again by Lemma 16, $\mathfrak{B}_i$ contains a subgraph $T_{2,2,k}(k \geqslant 2)$ with the special leaf v and $|P_{v_i}(\mathfrak{B}_i)| = 5$. It is easy to see that $\mathfrak{B}_1 \cup \mathfrak{B}_2$ contains a subgraph $T_{2,2,6}$, a contradiction. This completes the proof of Theorem 14. □

We now need the following lemmas to show Theorem 15.

Lemma 22. *Let G be an essentially 2-edge-connected trangle-free graph that has no subgraph isomorphic to $T_{2,4,5}$. If $\kappa(G_0) \geqslant 2$, then G has a dominating trail.*

Proof. Suppose, by way contradiction, that G has no dominating trail. Let $C = v_0v_1v_2 \cdots v_{c(G)-1}v_0$ be a longest cycle of G. Then $E(G-V(C)) \neq \emptyset$, for otherwise C is a dominating trail of G. Thus $G-V(C)$ has a nontrivial component D. Let P be a longest 2-attaching path of C in $D \cap G_0$ with a 2-attaching pair $\{v_{i'}, v_{i''}\}$. Since C is a longest cycle of G, we have $dist_C(v_{i'}, v_{i''}) \geqslant 2$. Since D is nontrivial, there exists an edge u_1z_1 in D incident with one of $v_{i'}$ and $v_{i''}$. Then $c(G) \leqslant 9$, for otherwise $\langle V(C) \cup \{u_1, z_1\}\rangle_G$ contains a subgraph isomorphic to $T_{2,4,5}$. By Lemma 9(iv), it suffices to consider the case when $|V(C)| = 8, 9$, then $dist_C(v_{i'}, v_{i''}) \leqslant 4$.

Claim 23. *For any nontrivial component D of $G-V(C)$, every longest 2-attaching path of C in $D \cap G_0$ is a dominating path of D.*

Proof. Suppose not, and let D be a nontrivial component of $G-V(C)$ such that C has a longest 2-attaching path in $D \cap G_0$ which is not a dominating path of D. It follows that D contains a path P of length three such that its end-vertex adjacent to some vertex of C, but then $\langle V(C) \cup V(P)\rangle_G$ contains a subgraph isomorphic to $T_{2,4,5}$, a contradiction. This proves Claim 23. □

Since P is a longest 2-attaching path of C in $D \cap G_0$, by Claim 23, P dominates all edges of D. We now distinguish two cases.

Case 1. $c(G) = 9$.

Claim 24. *D is the only one nontrivial component of $G-V(C)$.*

Proof. Suppose not the contrary that that $G-V(C)$ has a nontrivial component D' distinct from D. Then there exist two edges x_1x_2, y_1y_2 in D and D', respectively, such that x_1, y_1 are incident with C. We may assume that $v_0x_1 \in E(G)$ without loss of generality. Then y_1 cannot be adjacent to any vertex in $\{v_2, v_3, v_4, v_5, v_6, v_7\}$. Otherwise we obtain the following

$$\begin{cases} v_0x_1x_2 \cup v_0v_1v_2y_1y_2 \cup v_0v_8v_7v_6v_5v_4 \cong T_{2,4,5} & \text{if } y_1v_2 \in E(G) \\ v_0x_1x_2 \cup v_0v_1v_2v_3y_1 \cup v_0v_8v_7v_6v_5v_4 \cong T_{2,4,5} & \text{if } y_1v_3 \in E(G) \\ v_0x_1x_2 \cup v_0v_8v_7v_6v_5 \cup v_0v_1v_2v_3v_4y_1 \cong T_{2,4,5} & \text{if } y_1v_4 \in E(G) \\ v_5y_1y_2 \cup v_5v_4v_3v_2v_1 \cup v_5v_6v_7v_8v_0x_1 \cong T_{2,4,5} & \text{if } y_1v_5 \in E(G) \\ v_6y_1y_2 \cup v_6v_7v_8v_0x_1 \cup v_6v_5v_4v_3v_2v_1 \cong T_{2,4,5} & \text{if } y_1v_6 \in E(G) \\ v_7y_1y_2 \cup v_7v_8v_0x_1x_2 \cup v_7v_6v_5v_4v_3v_2 \cong T_{2,4,5} & \text{if } y_1v_7 \in E(G) \end{cases}$$

a contradiction. It follows that $N_G(D') \cap V(C) \subseteq \{v_0, v_1, v_8\}$, but this yield a cycle with length more than C in G, a contradiction. This proves Claim 24. □

By Claims 23 and 24, $\langle V(C) \cup V(P)\rangle_G$ contains a dominating trail of G, a contradiction. In the following the subscript i of v_i is in $\{0, 1, \cdots, c(G)-1\}$.

Case 2. $c(G) = 8$.

Claim 25. *For any two independent edges x_1x_2, y_1y_2 in $G - V(C)$ such that x_1, y_1 are incident with C. If $x_1v_i \in E(G)$, then y_1 cannot be incident with any vertex in $\{v_{i+2}, v_{i+3}, v_{i+5}, v_{i+6}\}$.*

Proof. Otherwise, we obtain the following

$$\begin{cases} v_{i+2}y_1y_2 \cup v_{i+2}v_{i+1}v_ix_1x_2 \cup v_{i+2}v_{i+3}v_{i+4}v_{i+5}v_{i+6}v_{i+7} \cong T_{2,4,5} & \text{if } y_1v_{i+2} \in E(G) \\ v_{i+3}y_1y_2 \cup v_{i+3}v_{i+4}v_{i+5}v_{i+6}v_{i+7} \cup v_{i+3}v_{i+2}v_{i+1}v_ix_1x_2 \cong T_{2,4,5} & \text{if } y_1v_{i+3} \in E(G) \\ v_ix_1x_2 \cup v_iv_{i+1}v_{i+2}v_{i+3}v_{i+4} \cup v_iv_{i+7}v_{i+6}v_{i+5}y_1y_2 \cong T_{2,4,5} & \text{if } y_1v_{i+5} \in E(G) \\ v_ix_1x_2 \cup v_iv_{i+7}v_{i+6}y_1y_2 \cup v_iv_{i+1}v_{i+2}v_{i+3}v_{i+4}v_{i+5} \cong T_{2,4,5} & \text{if } y_1v_{i+6} \in E(G) \end{cases}$$

a contradiction. This proves Claim 25. □

Note that $2 \leqslant dist_C(v_{i'}, v_{i''}) \leqslant 4$. Suppose that $dist_C(v_{i'}, v_{i''}) = 2$. We may assume that $v_{i'} = v_0, v_{i''} = v_2$ without loss of generality. Then D is the only one nontrivial component of $G - V(C)$. If not, and let D' be a nontrivial component of $G - V(C)$ distinct from D. By Claim 25, $N_G(D') \cap V(C) \cap \{v_2, v_3, v_5, v_6, v_4, v_7, v_0\} = \emptyset$, but then $N_G(D') \cap V(C) \subseteq \{v_1\}$, a contradiction. By Claim 23, $\langle V(C) \cup V(P)\rangle_G$ contains a dominating trail of G, a contradiction.

Suppose next that $dist_C(v_{i'}, v_{i''}) = 3$. We may assume that $v_{i'} = v_0, v_{i''} = v_3$ without loss of generality. If D is the only one nontrivial component of $G - V(C)$, then by Claim 23 P is a dominating path of D, but now $v_0Pv_3 \cdots v_7v_0v_1v_2$ is a dominating trail of G, a contradiction. Hence $G - V(C)$ has a nontrivial component D' distinct from D. By Claim 25, $N_G(D') \cap V(C) \cap \{v_2, v_3, v_5, v_6, v_0, v_1\} = \emptyset$, implying that $N_G(D') \cap V(C) \subseteq \{v_4, v_7\}$. Let Q be a longest 2-attaching path of C in $D \cap G_0$. By Claim 23 Q dominates all edges of D'. Similarly we can again apply Claim 25 to obtain D, D' are all nontrivial components $G - V(C)$, but then $v_2v_1v_0Pv_3v_4Qv_7v_6v_5$ is a dominating trail of G, a contradiction.

Finally suppose that $dist_C(v_{i'}, v_{i''}) = 4$. We may assume that $v_{i'} = v_0, v_{i''} = v_4$ without loss of generality. Then D is the only one nontrivial component of $G - V(C)$. If not, and let D' be a nontrivial component of $G - V(C)$ distinct from D. By Claim 25 $N_G(D') \cap V(C) \cap \{v_2, v_3, v_5, v_6, v_7, v_0, v_1, v_4\} = \emptyset$, a contradiction. By Claim 23, $\langle V(C) \cup V(P)\rangle_G$ is a dominating trail of G, a contradiction. This completes the proof of Lemma 22. □

Lemma 26. *Let G be an essentially 2-edge-connected graph without subgraphs isomorphic to $T_{2,4,5}$, and let $\mathcal{B}$ be a super-block of G. Assume $G/\mathcal{B}$ has a dominating closed trail T which contains the concentration vertex v. Then G has a dominating trail.*

Proof. Let $S = \{v_1, \cdots, v_t\}$ be the set of cut vertcies of G_0 in $\mathcal{B}$. We constructs the graph $\mathcal{B}'$ obtained from $\mathcal{B}$ by adding t pendant edges $v_1w_1, \cdots, v_tw_t$. Note that $\mathcal{B}'$ has no subgraph isomorphic to $T_{2,4,5}$. Applying Lemma 22 to $\mathcal{B}'$, we obtain that $\mathcal{B}'$ has a dominating trail T'. Clearly $S \subset V(T')$. Since $G/\mathcal{B}$ has a dominating closed trail T which contains the concentration vertex v, it follows that $T \cup T'$ is a dominating trail of G. The proof is complete. □

Lemma 27. *Let G be an essentially 2-edge-connected triangle-free graph without subgraphs isomorphic to $T_{2,4,5}$. If G has an inner-super-block $\mathfrak{B}$, then G has a dominating trail.*

Proof. We argue by contradiction, and assume that G is a counter-example to Lemma 27 such that the number of super-blocks of G is minimized.

Claim 28. *Every inner-super-block of G contains exactly two cut vertices of the core G_0 of G.*

Proof. Suppose not. We may assume, without loss of generality, that $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ are three super-block of G such that $\mathfrak{B} \cap \mathfrak{B}_i = \{v_i\}$ where v_i is a cut vertex of G_0 in $\mathfrak{B}$. Since $\mathfrak{B} \cap G_0$ is 2-connected and G is triangle-free, there exists a cycle C of length at least four in $\mathfrak{B} \cap G_0$ containing v_1 and v_3. Note that $|P_{v_i}(\mathfrak{B}_i)| \geqslant 4$ for $i = 1, 2, 3$. If $v_2 \in V(C)$, then $P_{v_1}(\mathfrak{B}_1) \cup P_{v_2}(\mathfrak{B}_2) \cup P_{v_3}(\mathfrak{B}_3) \cup C$ contains a subgraph isomorphic to $T_{2,4,5}$ with the root v_1 or v_2, a contradiction. Hence we have $v_2 \notin V(C)$. Then there exists a path P in $\mathfrak{B} \cap G_0$ joining v_2 and C since $\mathfrak{B} \cap G_0$ is connected, but now $P_{v_1}(\mathfrak{B}_1) \cup P_{v_2}(\mathfrak{B}_2) \cup P_{v_3}(\mathfrak{B}_3) \cup C \cup P$ contains a subgraph isomorphic to $T_{2,4,5}$ with the root v_1 or v_2, a contradiction. This proves Claim 28. □

Claim 29. *There is no triple of super-blocks of G such that they have a common cut vertex of G_0.*

Proof. Suppose otherwise. Then exist three super-block of G such that they have a common cut vertex v of G_0. It follows that G has at least three outer-super-blocks. We may assume, without loss of generality, that $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ are three outer-super-block of G such that $\mathfrak{B} \cap \mathfrak{B}_1 \cap \mathfrak{B}_2 = \{v\}$ and $\mathfrak{B} \cap \mathfrak{B}_3 = \{u\}$. Note that if G has more than one inner-super-block then it is more easier than this case to get a contradiction. Note that $|P_v(\mathfrak{B} \cup \mathfrak{B}_3)| \geqslant 6$. Since G has no subgraph isomorphic to $T_{2,4,5}$, it follows that for each $i = 1, 2$, $|P_v(\mathfrak{B}_i)| \leqslant 5$, implying that $c(\mathfrak{B}_i) \leqslant 5$. If some $\mathfrak{B}_i$ has a dominating cycle C_v containing v, say $\mathfrak{B}_1$, then $G/\mathfrak{B}_2$ has super-blocks less than G and has an inner-super-block $\mathfrak{B}$ and no subgraph isomorphic to $T_{2,4,5}$. By the choice of G, $G/\mathfrak{B}_2$ has a dominating trail T, but then $T \cup C_v$ is a dominating trail of G, a contradiction. Hence for each $i \in \{1, 2\}$, $\mathfrak{B}_i$ has no dominating cycle containing v and since $c(\mathfrak{B}_i) \leqslant 5$, by Lemma 16(ii)-(iii) $\mathfrak{B}_i$ contains a subgraph $T_{2,4,0}$ with the root v implying that $\mathfrak{B}_i \cup P_v(\mathfrak{B} \cup \mathfrak{B}_4)$ contains a subgraph isomorphic to $T_{2,4,5}$ with the root v, a contradiction. This proves Claim 29. □

By Claims 28 and 29, G has exactly two outer-super-blocks $\mathfrak{B}_1, \mathfrak{B}_2$. We may assume, without loss of generality, that $\mathfrak{B}, \mathfrak{B}_1, \mathfrak{B}_2$ are all three super-blocks of G three super-blocks if G has more than three super-blocks then it is more easier than this case to obtain a contradiction. Let $\mathfrak{B} \cap \mathfrak{B}_1 = \{v_1\}$ and $\mathfrak{B} \cap \mathfrak{B}_2 = \{v_2\}$. Note that $|P_{v_2}(\mathfrak{B}_1 \cup \mathfrak{B})| \geqslant 6$ and $|P_{v_1}(\mathfrak{B}_2 \cup \mathfrak{B})| \geqslant 6$. Since G has no subgraph isomorphic to $T_{2,4,5}$, it follows that each $\mathfrak{B}_i (i = 1, 2)$ contains no subgraph $T_{2,4,0}$ with the root v_i. If each $\mathfrak{B}_i (i = 1, 2)$ has a dominating cycle containing v_i, then these two cycle form a dominating closed trial of $G/\mathfrak{B}$ and containing the concentration vertex v_i, by Lemma 26 G has a dominating trail of G.

Hence we may assume that $\mathfrak{B}_1$ has no dominating cycle containing v_1 without loss of generality. Since $\mathfrak{B}_1$ contains no subgraph $T_{2,4,0}$ with the root v_1, by Lemma 16 we have $|P_{v_1}(\mathfrak{B}_1)| \geqslant 6$. Let $C(v_1, v_2)$ be a cycle in $\mathfrak{B}$ containing v_1 and v_2. Then $C(v_1, v_2)$ has length exactly four, for otherwise $C(v_1, v_2)$ has length at least five and then $C(v_1, v_2) \cup \mathfrak{B}_2$ contains a subgraph $T_{2,4,0}$ with the root v_1, but this together with $P_{v_1}(\mathfrak{B}_1)$ yield a subgraph isomorphic to $T_{2,4,5}$. Similarly one can show that $C(v_1, v_2) \setminus \{v_1, v_2\}$ has no neighbor outside in $\mathfrak{B}$, for otherwise, $\mathfrak{B} \cup \mathfrak{B}_2$ contains a subgraph $T_{2,4,k}(k \geqslant 0)$ with the root v_1, but $\mathfrak{B}_1 \cup \mathfrak{B} \cup \mathfrak{B}_2$ contains a subgraph isomorphic to $T_{2,4,5}$. Thus $C(v_1, v_2)$ is a dominating cycle of $\mathfrak{B}$ and clearly $C(v_1, v_2)$ contains a dominating path $P(v_1, v_2)$ in $\mathfrak{B}$. If $\mathfrak{B}_2$ has a dominating cycle C_{v_2} containing v_2, then $C_{v_2} \cup C(v_1, v_2)$ is a dominating closed trail of $G/\mathfrak{B}_1$ which contains the concentration vertex v_1, by Lemma 26 G has a dominating trail of G. Hence for each $i = 1, 2$, $\mathfrak{B}_i$ has no dominating cycle containing v_i. If $c(\mathfrak{B}_i) \geqslant 6$ then since $\mathfrak{B}_i$ contains no subgraph $T_{2,4,0}$ with the root v_i, by Lemma 16(i) $\mathfrak{B}_i$ has a dominating trail starting from v_i, and if $c(\mathfrak{B}_i) \leqslant 5$ then by Lemma 9(iii), $\mathfrak{B}_i \cap G_0$ has a spanning trail starting from v_i. In any cases we can obtain that each $\mathfrak{B}_i(i = 1, 2)$ has a dominating trail T_i starting from v_i, but then $T_1 \cup P(v_1, v_2) \cup T_2$ is a dominating trail of G, a contradiction. This completes the proof of Lemma 27. □

Proof of Theorem 15. Suppose that G has no dominating trail. Since G has no subgraph isomorphic to $T_{2,4,5}$, by Lemma 22 we have $\kappa(G_0) = 1$. By Lemma 27, G_0 has only one cut vertex v. Let $\mathfrak{B}_1, \cdots, \mathfrak{B}_t(t \geqslant 2)$ be all super-blocks of G.

Claim 30. $t \geqslant 3$.

Proof. Suppose not. Then $t = 2$. If some $\mathfrak{B}_i$ has a dominating cycle containing v, then since $\mathfrak{B}_1 \cup \mathfrak{B}_2 = G$, by Lemma 26 G has a dominating trail, a contradiction. Hence both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have no dominating cycle containing v. If each $c(\mathfrak{B}_i) \geqslant 6$ for $i = 1, 2$, then by Lemma 16(i), $\mathfrak{B}_i$ contains a subgraph $T_{2,4,0}$ with the root v or $\mathfrak{B}_i$ has a dominating trail T_i starting from v. Note that $|P_v(\mathfrak{B}_i)| \geqslant 6$ for $i = 1, 2$. But then $\mathcal{B}_1 \cup \mathcal{B}_2$ contains a subgraph isomorphic to $T_{2,4,5}$ or $T_1 \cup T_2$ is a dominating trail of G. If $c(\mathfrak{B}_i) \leqslant 5$ for each $i = 1, 2$, then by Lemma 9(iii) $\mathcal{B}_i \cap G_0$ has a spanning trail T^i starting from v, but now $T^1 \cup T^2$ is a dominating trail of G, a contradiction.

Hence we may assume that $c(\mathfrak{B}_1) \leqslant 5$ and $c(\mathfrak{B}_2) \geqslant 6$ without loss of generality. Since each $\mathcal{B}_i(i = 1, 2)$ has no dominating cycle containing v. Applying Lemma 16 to $\mathfrak{B}_1$, either $\mathfrak{B}_1$ contains a subgraph $T_{2,4,0}$ with the root v or $|P_v(\mathfrak{B}_1)| \geqslant 6$ holds. Again applying Lemma 16 to $\mathfrak{B}_2$, either $\mathfrak{B}_2$ contains a subgraph $T_{2,4,0}$ with the root v or has a dominating trail starting from v holds.

Note that by Lemma 9(iii) that $\mathfrak{B}_1 \cap G_0$ has a spanning trail starting from v. Since G has no dominating trail, it follows that $\mathfrak{B}_2$ has no dominating trail starting from v, implying that $\mathfrak{B}_2$ contains a subgraph $T_{2,4,0}$ with the root v. Since G has no subgraph isomorphic to $T_{2,4,5}$, it follows that $|P_v(\mathfrak{B}_1)| \leqslant 5$, implying that $\mathfrak{B}_1$ contains a subgraph $T_{2,4,0}$ with the root v. Note that $|P_v(\mathfrak{B}_2)| \geqslant 6$. But then $\mathfrak{B}_1 \cup \mathfrak{B}_2$ contains a subgraph isomorphic to $T_{2,4,5}$, a contradiction. This proves Claim 30. □

Claim 31. *For each $i = 1, \cdots, t$, $|P_v(\mathcal{B}_i)| \leqslant 5$.*

Proof. Suppose otherwise. We may assume that $|P_v(\mathcal{B}_1)| \geqslant 6$ without loss of generality. If each $\mathcal{B}_i (i = 2, \cdots, t)$ has a dominating cycle containing v, then these dominating cycles form a dominating closed trail of $G/\mathcal{B}_1$ and containing the concentration vertex v, by Lemma 26 G has a dominating trail, a contradiction. Hence we may assume that $\mathcal{B}_2$ has no dominating cycle containing v without loss of generality. By Lemma 16, $|P_v(\mathcal{B}_2)| \geqslant 5$. Note that $|P_v(\mathcal{B}_3)| \geqslant 3$. But then $P_v(\mathcal{B}_1) \cup P_v(\mathcal{B}_2) \cup P_v(\mathcal{B}_3)$ contains a subgraph isomorphic to $T_{2,4,5}$, a contradiction. This proves Claim 31. □

By Claim 31, it follows that $c(\mathcal{B}_i) \leqslant 5$ for each $i = 1, 2, \cdots, t$. Let $\mathbf{H} = \{\mathfrak{B}_1, \cdots, \mathfrak{B}_t\}$. Let

- $\mathbf{H}_1 = \{\mathfrak{B} \in \mathbf{H} : c(\mathfrak{B}) = 5\}$;
- $\mathbf{H}_2 = \{\mathfrak{B} \in \mathbf{H} : c(\mathfrak{B}) = 4$ and $d_{\mathfrak{B}\cap G_0}(v) = 2$ and $\Delta(\mathfrak{B} \cap G_0)$ is odd$\}$;
- $\mathbf{H}_3 = \{\mathfrak{B} \in \mathbf{H} : c(\mathfrak{B}) = 4$ and $d_{\mathfrak{B}\cap G_0}(v) = 2$ and $\Delta(\mathfrak{B} \cap G_0)$ is even$\}$;
- $\mathbf{H}_4 = \{\mathfrak{B} \in \mathbf{H} : c(\mathfrak{B}) = 4$ and $d_{\mathfrak{B}\cap G_0}(v) = \Delta(\mathfrak{B} \cap G_0) \geqslant 4$ and $\Delta(\mathfrak{B} \cap G_0)$ is even$\}$;
- $\mathbf{H}_5 = \{\mathfrak{B} \in \mathbf{H} : c(\mathfrak{B}) = 4$ and $d_{\mathfrak{B}\cap G_0}(v) = \Delta(\mathfrak{B} \cap G_0) \geqslant 3$ and $\Delta(\mathfrak{B} \cap G_0)$ is odd and there exists a vertex u adjacent to v in $\mathfrak{B} \cap G_0$ such that $u \notin \Lambda(G)\}$;
- $\mathbf{H}_6 = \{\mathfrak{B} \in \mathbf{H} : c(\mathfrak{B}) = 4$ and $d_{\mathfrak{B}\cap G_0}(v) = \Delta(\mathfrak{B} \cap G_0) \geqslant 3$ and $\Delta(\mathfrak{B} \cap G_0)$ is odd and any vertex u is adjacent to v in $\mathfrak{B} \cap G_0$ such that $u \in \Lambda(G)\}$.

Clearly $\{\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_4, \mathbf{H}_5\}$ is a partition of $\mathbf{H}$.

Claim 32. *For each $i = 1, 2, 3, 4, 5, 6$, every $\mathfrak{B}$ in $\mathbf{H}_i$ is isomorphic F_i where F_i is defined in Section 1 and shown in Figure 2. Consequently, each member of $\mathbf{H}_i$ has a dominating closed trail containing v for $1 \leqslant i \leqslant 5$.*

Proof. Let $\mathfrak{B} \in \mathbf{H}$. Since $c(\mathfrak{B}) \leqslant 5$, by Lemma 9(iii) there is a cycle C of length $c(G)$ containing v. We may assume that $C = vv_1 \cdots v_{c(G)-1}v$. If $\mathfrak{B} \cap G_0 - V(C) = \emptyset$, then $\mathfrak{B} \in \mathbf{H}_1 \cup \mathbf{H}_3$, and clearly C is a dominating cycle of $\mathfrak{B}$ containing v. Hence we assume that $\mathfrak{B} \cap G_0 - V(C) \neq \emptyset$. By Lemma 9(i), every component of $\mathfrak{B} \cap G_0 - V(C)$ is a 1-component, and let $u_1, u_2, \cdots, u_t$ be all components of $\mathfrak{B} \cap G_0 - V(C)$. Then $V(\mathfrak{B} \cap G_0) = V(C) \cup \{u_1, u_2, \cdots, u_t\}$. Suppose that $c(\mathfrak{B}) = 5$. Then $\mathfrak{B} \in \mathbf{H}_1$. Since C is a longest cycle of $\mathfrak{B}$ and u_i is a 1-component to $\mathfrak{B} \cap G_0 - V(C)$, we have $|N_{\mathfrak{B}}(u_i) \cap V(C)| = 2$, implying that $N_{\mathfrak{B}}(u_i) \cap V(C) = \{v, v_2\}$ or $N_{\mathfrak{B}}(u_i) \cap V(C) = \{v, v_3\}$ for $1 \leqslant i \leqslant t$. By Claim 31, $\{v_1, v_4, u_1, \cdots, u_t\} \cap \Lambda(G) = \emptyset$, implying that $\mathfrak{B}$ is isomorphic to F_1. Clearly C is a dominating cycle containing v.

Hence we assume that $c(\mathfrak{B}) = 4$. Since C is a longest cycle of $\mathfrak{B}$ and u_i is a 1-component of $\mathfrak{B} \cap G_0 - V(C)$ for $1 \leqslant i \leqslant t$. Then $N_{\mathfrak{B}}(u_i) \cap V(C) = \{v, v_2\}$ or $N_{\mathfrak{B}}(u_i) \cap V(C) = \{v_1, v_3\}$ for $1 \leqslant i \leqslant t$. Suppose that $\mathfrak{B} \in \mathbf{H}_2 \cup \mathbf{H}_3$. Since $d_{\mathfrak{B}\cap G_0}(v) = 2$, it follows that $N_{\mathfrak{B}}(u_i) \cap V(C) = \{v_1, v_3\}$ for $1 \leqslant i \leqslant t$. By Claim 31, $\{v_2, u_1, \cdots, u_t\} \cap \Lambda(G) = \emptyset$. Clearly C is a dominating cycle containing v. If $\Delta(\mathfrak{B} \cap G_0)$ is odd then $\mathfrak{B}$ is isomorphic to F_2, if $\Delta(\mathfrak{B} \cap G_0)$ is even then $\mathfrak{B}$ is isomorphic to F_3.

Now suppose that $\mathfrak{B} \in \mathbf{H}_4$. Then $N_{\mathfrak{B}}(u_i) \cap V(C) = \{v, v_2\}$ for $1 \leqslant i \leqslant t$ and t is even. Hence $\mathfrak{B}$ is isomorphic to F_4, and clearly $\mathfrak{B} \cap G_0$ is an even graph.

Finally suppose that $\mathfrak{B} \in \mathbf{H}_5 \cup \mathbf{H}_6$. Then $N_{\mathfrak{B}}(u_i) \cap V(C) = \{v, v_2\}$ for $1 \leqslant i \leqslant t$ and t is odd. If there exists a vertex $u \in \{v_1, v_3, u_1, \cdots, u_4\}$ such that $u \notin \Lambda(G)$, then $\mathfrak{B}$ is isomorphic to F_5, clearly $\mathfrak{B} \cap G_0 - u$ is an even graph. Hence we assume that $\{v_1, v_3, u_1, \cdots, u_4\} \subseteq \Lambda(G)$, implying that $\mathfrak{B}$ is isomorphic to F_6. This proves Claim 32. □

Note that $\mathbf{H}_1 \cup \cdots \cup \mathbf{H}_6 = G$. If $\mathbf{H}_1 \cup \cdots \cup \mathbf{H}_5 \neq \emptyset$, then by Claim 32, the set $\mathbf{H}_1 \cup \cdots \cup \mathbf{H}_5$ of graphs contains a dominating closed trail containing v. Since each $\mathfrak{B}$ in $\mathbf{H}_6$ has circumference four, it follows Lemma 9(iv) that $\mathfrak{B} \cap G_0$ has a spanning trail starting at v. If $|\mathbf{H}_6| \leqslant 2$, then clearly G has a dominating trail, a contradiction. Hence $|\mathbf{H}_6| \geqslant 3$ and $G \in \mathcal{F}$. The proof is complete. □

5 Concluding remarks

In 1966, Gallai asked whether all longest paths in a connected graph have a nonempty intersection. The answer to this question is not true in general and various counterexamples have been found. However, Gallai's question has a positive solution for many well-known classes of graphs such as split graphs, series-parallel graphs, and $2K_2$- free graphs. Recently, Gao and Shan [21] proved that Gallai's question has an affirmative answer for connected $\{K_{1,3}, S\}$-free graphs where $S \in \{P_6, Z_3, B_{1,2}\}$. We think that one may extend this result to $\{K_{1,3}, S\}$-free graphs where $S \in \{N_{1,1,5}, N_{1,3,3}, B_{2,4}\}$ by applying our result (Theorem 1.1).

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