A Note on the Structure of Locally Finite Planar Quasi-Transitive Graphs

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Abstract

In an early work from 1896, Maschke established the complete list of all finite planar Cayley graphs. This result initiated a long line of research over the next century, aiming at characterizing in a similar way all planar infinite Cayley graphs. Droms (2006) proved a structure theorem for finitely generated *planar groups*, i.e., finitely generated groups admitting a planar Cayley graph, in terms of Bass-Serre decompositions. As a byproduct of his structure theorem, Droms proved that such groups are finitely presented. More recently, Hamann (2018) gave a graph theoretical proof that every planar quasi-transitive graph G admits a generating $\mathrm{Aut}(G)$-invariant set of closed walks with only finitely many orbits, and showed that a consequence is an alternative proof of Droms' result. Based on the work of Hamann, we show in this note that we can also obtain a general structure theorem for 3-connected locally finite planar quasi-transitive graphs, namely that every such graph admits a canonical tree-decomposition whose edge-separations correspond to cycle-separations in the (unique) embedding of G, and in which every part is still quasi-transitive and admits a vertex-accumulation free embedding. This result can be seen as a version of Droms' structure theorem for quasi-transitive planar graphs. As a corollary, we obtain an alternative proof of a result of Hamann, Lehner, Miraftab and Rühmann (2022) that every locally finite quasi-transitive planar graph admits a canonical tree-decomposition, whose parts are either 1-ended or finite planar graphs.

Mathematics Subject Classifications: 05C10, 05C63, 05C75, 68R10

1 Introduction

In his seminal work, Maschke [Mas96] gave the full list of all finite planar Cayley graphs. A group admitting a planar Cayley graph is called a *planar group*, and Maschke showed in the same paper that the finite planar groups are exactly the countable groups of isometries of the 2-dimensional sphere $\mathbb{S}^2$. Based on the works of Wilkie [Wil66] and MacBeath

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[Mac67], Zieschang, Volgt and Coldewey [ZVC80] established the complete list of *planar discontinuous groups*, which are exactly the countable groups for which there exists a planar Cayley graph with a *vertex-accumulation-free* planar embedding[1]. We also refer to [MS83, Section III.5] for a complementary work on such groups. It is worth mentioning that every one-ended group is planar discontinuous, hence all the planar groups which do not enter into the scope of the aforementioned characterisation are the multi-ended ones, i.e., the groups whose number of ends is 2 or ∞.

In [Dro06], Droms proved that finitely generated planar groups are finitely presented, and thus, by a result of Dunwoody [Dun85], they are also accessible. In order to do this, he proved a decomposition theorem for such groups, inspired by Stallings' ends theorem [Sta68]. In group theoretic terms, this result states that every such group admits a finite Bass-Serre decomposition in which all operations involved are special kind of HNN-extensions and free amalgamations, where at each step the parts which are amalgamated correspond to finite subgroups, which are faces of the planar Cayley graphs drawn. From a graph theoretic perspective, this theorem intuitively states that every locally finite planar Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ admits a tree-decomposition which is invariant under the action of Γ, whose parts correspond to planar Cayley graphs of finitely generated subgroups of Γ, and whose edge-separations correspond to separations induced by cycles in some planar drawing of G. See Figures 1 and 2 for an illustration of such a decomposition. We show in this note that a decomposition with such properties still exists if we simply assume that G is 3-connected locally finite quasi-transitive.

Theorem 1 (Theorem 18)**.** *Every planar locally finite 3-connected quasi-transitive graph G admits a canonical tree-decomposition whose edge-separations correspond to cycle-separations in the (unique) embedding of G, and where every part is a quasi-transitive subgraph of G admitting a vertex-accumulation-free planar embedding.*

From a metric perspective, quasi-transitivity is known to be more general than than the property of being a Cayley graph: in [EFW12], the authors exhibited a construction of a quasi-transitive graph which is not quasi-isometric to any Cayley graph, answering an initial question of Woess [Woe91]. However, MacManus recently proved that this is not true anymore if we restrict to the class of quasi-transitive graphs which are quasi-isometric to some planar graph [Mac24]; namely, every locally finite quasi-transitive graph which is quasi-isometric to a planar graph is quasi-isometric to some planar Cayley graph. In light of this result, it is then not surprising that planar quasi-transitive graphs should satisfy similar properties as planar Cayley graphs. Nevertheless, the proof we give here offers the advantage to be based on purely graph-theoretic arguments, building on the papers [Ham15, Ham18]. Combining Theorem 1 with Tutte's canonical decomposition of 2-connected graphs, we obtain as a corollary the following, which was already proved in [HLMR22, Theorem 7.6], using the machinery of tree-amalgamations.

[1]Note that the basic definition of planar discontinuous from Zieschang, Volgt and Coldewey [ZVC80] differs from the one we gave, however it is shown in [ZVC80, Theorems 4.13.11, 6.4.7 and Corrolary 4.13.15] that both definitions are equivalent.

Corollary 2 (Corollary 21)**.** *Let G be a locally finite quasi-transitive planar graph. Then G admits a canonical tree-decomposition of bounded finite adhesion, whose parts are quasi-transitive planar graphs with at most one end.*

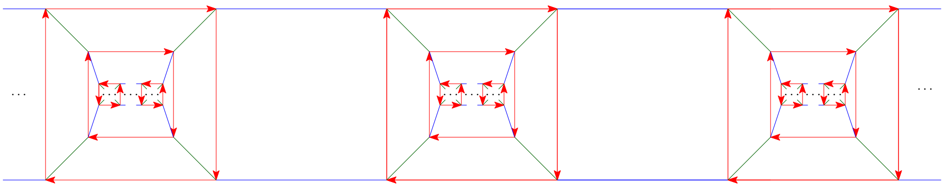

Figure 1: A section of the planar Cayley graph of the group $\langle a, b, c \mid a^4, b^2, c^2, abab, acac\rangle$. The edges associated to a, b and c are respectively colored in red, blue and green. This planar drawing is obtained by drawing a first bi-infinite ladder that corresponds to the Cayley graph of the subgroup generated by a and b, and then, in each square face delimited by a cycle C labeled by a^4, we draw a copy of the cube, which correponds to the Cayley graph of the subgroup generated by a and c, and which we attach along C. We repeat this construction an infinite number of steps by alternatively drawing a new ladder or a cube in each new facial cycle delimited by a square labeled by a^4. The obtained graph then admits a (canonical) tree-decomposition, whose decomposition tree T is the barycentric subdivision of the regular tree with infinite countable degree ω. Each bag corresponding to a node of infinite degree in T contains a bi-infinite ladder, and each bag corresponding to a node of degree 2 contains a cube. The adhesion sets of this tree-decomposition are cycles of size 4.

Related work. A huge amount of work has been done related to the structure of planar quasi-transitive and Cayley graphs. We refer to [Bab97, Ren03, Moh06, Dun09, Geo14, GH15, Geo17b, Geo17a, Geo20, MS22, GH23] for more details on the topic.

2 Preliminaries

2.1 Quasi-transitive graphs and quasi-isometries

Let Γ be a group acting on a graph G (by automorphisms). For every subset $X \subseteq V(G)$, we let $\mathrm{Stab}_\Gamma(X) := \{\gamma \in \Gamma : \gamma \cdot X = X\}$ denote the stabilizer of X, and for each $x \in X$, we let $\Gamma_x := \mathrm{Stab}_\Gamma(\{x\})$.

The action of Γ on G is called *quasi-transitive* if there is only a finite number of orbits in $V(G)/\Gamma$. We say that G is *quasi-transitive* if it admits a quasi-transitive group action.

For every two graphs G and H, a *quasi-isometry* is a map $f : V(G) \to V(H)$ for which there exists some constants $\varepsilon \geqslant 0$, $\lambda \geqslant 1$, and $C \geqslant 0$ such that (i) for any $y \in V(H)$ there is $x \in V(G)$ such that $d_H(y, f(x)) \leqslant C$, and (ii) for every $x, x' \in V(G)$,

$$\frac{1}{\lambda} d_G(x, x') - \varepsilon \leqslant d_H(f(x), f(x')) \leqslant \lambda d_G(x, x') + \varepsilon.$$

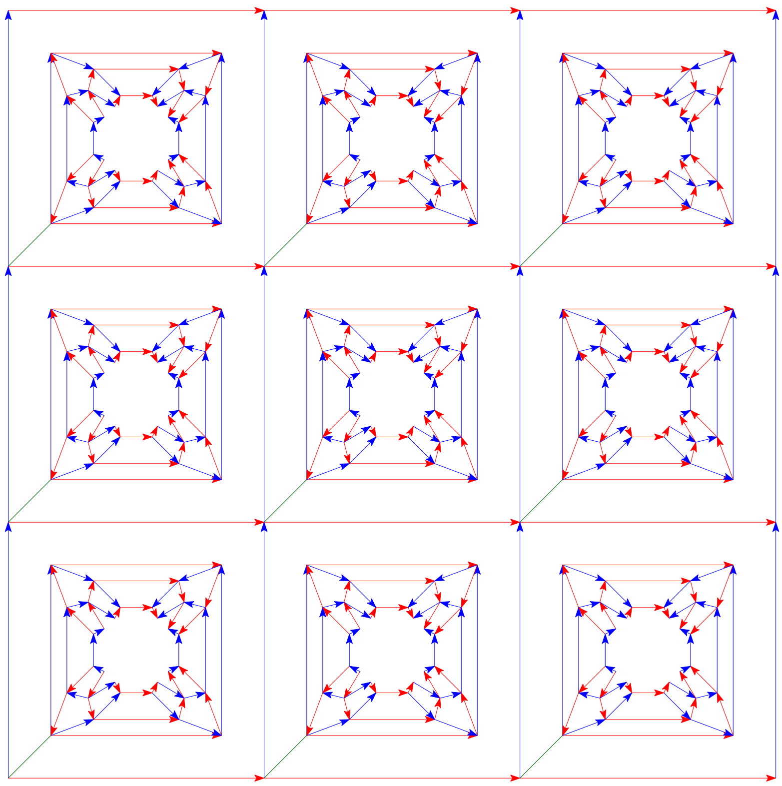

Figure 2: A section of the planar Cayley graph of the group $\langle a, b, c \mid aba^{-1}b^{-1}, c^2 \rangle$. This planar drawing is obtained by drawing a first infinite square grid, and then, in each square face, we draw another infinite square grid with a square as outer face, and connect it to the face of the initial grid with an edge. We then keep going on drawing in a similar way a new grid in every square face in the new drawing, and repeat this operation an infinite number of time. A canonical tree-decomposition of the obtained graph satisfying the properties of Corollary 2 is the one with decomposition tree T the barycentric subdivision of the regular tree with infinite countable degree ω, and where each bag associated to a node of infinite degree contains a copy of the infinite grid, while every bag associated to a node of degree 2 contains a copy of K_2.

If there exists a quasi-isometry between G and H, then we say that G and H are *quasi-isometric* to each other.

2.2 Separations

A *separation* in a graph $G = (V, E)$ is a triple (Y, S, Z) such that Y, S, Z are pairwise-disjoint subsets of $V(G)$, $V = Y \cup S \cup Z$ and there is no edge between vertices of Y and Z. A separation (Y, S, Z) is *proper* if Y and Z are nonempty. In this case, S is a *separator* of G. The *order* of a separation (Y, S, Z) is $|S|$.

The separation (Y, S, Z) is said to be *tight* if there are some components C_Y, C_Z respectively of $G[Y], G[Z]$ such that $N_G(C_Y) = N_G(C_Z) = S$.

The following lemma was originally stated in [TW93] for transitive graphs, and the same proof immediately implies that the result also holds for quasi-transitive graphs.

Lemma 3 (Proposition 4.2 and Corollary 4.3 in [TW93])**.** *Let G be a locally finite graph. Then for every $v \in V(G)$ and $k \geqslant 1$, there is only a finite number of tight separations (Y, S, Z) of order k in G such that $v \in S$. Moreover, for any group Γ acting quasi-transitively on G and any $k \geqslant 1$, there is only a finite number of Γ-orbits of tight separations of order at most k in G.*

2.3 Canonical tree-decompositions

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{V})$ where T is a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ is a family of subsets V_t of $V(G)$ such that:

- $V(G) = \bigcup_{t \in V(T)} V_t$;
- for every nodes t, t', t'' such that t' is on the unique path of T from t to t'', $V_t \cap V_{t''} \subseteq V_{t'}$;
- every edge $e \in E(G)$ is contained in an induced subgraph $G[V_t]$ for some $t \in V(T)$.

The sets V_t for every $t \in V(T)$ are called the *bags* of $(T, \mathcal{V})$, and the induced subgraphs $G[V_t]$ the *parts* of $(T, \mathcal{V})$. The *width* of $(T, \mathcal{V})$ is the supremum of $|V_t| - 1$ (possibly infinite), for $t \in V(T)$. The sets $V_t \cap V_{t'}$ for every $tt' \in E(T)$ are called the *adhesion sets* of $(T, \mathcal{V})$ and the *adhesion* of $(T, \mathcal{V})$ is the supremum of the sizes of its adhesion sets (possibly infinite). The *treewidth* of a graph G is the infimum of the width of $(T, \mathcal{V})$, among all tree-decompositions $(T, \mathcal{V})$ of G.

The *torsos* of $(T, \mathcal{V})$ are the graphs $G[\![V_t]\!]$ for $t \in V(T)$, with vertex set V_t and edge set $E(G[V_t])$ together with the edges xy such that x and y belong to a common adhesion set of $(T, \mathcal{V})$.

Let A be the set of all the orientations of the edges of $E(T)$, i.e. A contains the pairs (t_1, t_2), (t_2, t_1) for every edge t_1t_2 of T. For an arbitrary pair $(t_1, t_2) \in A$, and for each $i \in \{1, 2\}$, let T_i denote the component of $T - \{t_1t_2\}$ containing t_i. Then the *edge-separation* of G associated to (t_1, t_2) is (Y_1, S, Y_2) with $S := V_{t_1} \cap V_{t_2}$ and $Y_i := \bigcup_{s \in V(T_i)} V_s \setminus S$ for $i \in \{1, 2\}$.

For group Γ acting on G, we say that a tree-decomposition $(T, \mathcal{V})$ is *canonical with respect to* Γ, or simply Γ-*canonical*, if Γ induces a group action on T such that for every $\gamma \in \Gamma$ and $t \in V(T)$, $\gamma \cdot V_t = V_{\gamma \cdot t}$. In particular, for every $\gamma \in \Gamma$, note that γ sends bags of $(T, \mathcal{V})$ to bags, and adhesion sets to adhesion sets. When $(T, \mathcal{V})$ is $\mathrm{Aut}(G)$-canonical, we simply say that it is *canonical.*

If Γ acts on G and $\mathcal{N}$ is a family of separations of G, we say that $\mathcal{N}$ is Γ-*invariant* if for every $(Y, S, Z) \in \mathcal{N}$ and $\gamma \in \Gamma$, we have $\gamma \cdot (Y, S, Z) \in \mathcal{N}$. Note that if $(T, \mathcal{V})$ is Γ-canonical, then the associated set of edge-separations is Γ-invariant.

Remark 4. If $(T, \mathcal{V})$ is a Γ-canonical tree-decomposition of a locally finite graph G on which Γ acts quasi-transitively, whose edge-separations are tight, with finite bounded order, then by Lemma 3 the action of Γ on $E(T)$ must induce a finite number of orbits. In particular, Γ must also act quasi-transitively on $V(T)$.

The following two lemmas are folklore results about canonical tree-decompositions.

Lemma 5. *Let G be a locally finite Γ-quasi-transitive graph and $(T, (V_t)_{t \in V(T)})$ be a Γ-canonical tree-decomposition of G with finite adhesion whose parts are connected subgraphs of G and such that $E(T)$ admits only finitely many Γ orbits. Then for every $t \in V(T)$, $G[\![V_t]\!]$ is quasi-isometric to $G[V_t]$. Moreover, the constants ε, λ, C of the quasi-isometries can be chosen independently of $t \in V(T)$.*

Proof. We will show that the identity on V_t induces a quasi-isometry between $G[V_t]$ and $G[\![V_t]\!]$. Let $t \in V(T)$. As $G[V_t]$ is a subgraph of $G[\![V_t]\!]$, for every $u, v \in V_t$ we have $d_{G[\![V_t]\!]}(u, v) \leqslant d_{G[V_t]}(u, v)$. Moreover, as $E(T)$ has finitely many Γ-orbits and as each part is connected, the set $\{d_{G[V_t]}(u, v), \exists s \in N_T(t), u, v \in V_s \cap V_t\}$ of values admits a maximum C_t. As $V(T)$ has finitely many Γ-orbits, the set $\{C_t : t \in V(T)\}$ of values also admits a maximum $C \in \mathbb{N}$. In particular we have $d_{G[V_t]}(u, v) \leqslant C \cdot d_{G[\![V_t]\!]}(u, v)$. □

Lemma 6. *Let G be a connected locally finite Γ-quasi-transitive graph and $(T, (V_t)_{t \in V(T)})$ be a Γ-canonical tree-decomposition of G with finite adhesion, such that $E(T)$ admits only finitely many Γ orbits. Then there exists a Γ-canonical tree-decomposition $(T, (V'_t)_{t \in V(T)})$ of G with finite adhesion, with the same and Γ-action of Γ on T, and such that for each $t \in V(T)$, $G[V'_t]$ is connected and quasi-isometric to $G[\![V_t]\!]$.*

Proof. As $(T, (V_t)_{t \in (T)})$ has finite adhesion and as $E(T)$ has finitely many Γ-orbits, note that the set $\{d_G(u, v) : \exists t \in V(T), uv \in E(G[\![V_t]\!]) \setminus E(G)\}$ is bounded and thus admits a maximum, say $k \in \mathbb{N}$. We let $\mathcal{V}' := (V'_t)_{t \in V(T)}$ be defined by $V'_t := B_k(V_t) = \{v \in V(G) : \exists u \in V_t, d_G(u, v) \leqslant k\}$. It is not hard to check that $(T, \mathcal{V}')$ is also a Γ-canonical tree-decomposition of G with connected parts. As G has bounded degree, and as $V(T)$ has finitely many Γ-orbits, $(T, \mathcal{V}')$ moreover has finite adhesion.

It remains to show that for each $t \in V(T)$, $G[\![V_t]\!]$ is quasi-isometric to $G[V_t]$. For this, we let $t \in V(T)$, and fix any projection $\pi : V'_t \to V_t$ such that $\pi|_{V_t} = \mathrm{id}_{V_t}$ and such that for each $v \in V'_t$, $d_G(\pi(v), v) = d_G(V_t, v) = \min\{d_G(v, u) : u \in V_t\}$.

We show that π defines a quasi-isometry between $G[V'_t]$ and $G[\![V_t]\!]$. First, note that for every $u, v \in V'_t$ and every $\pi(u)\pi(v)$-path P in $G[\![V_t]\!]$ of length d, there exists a uv-path P' in

$G[V_t']$ of length at most $kd+2k$, obtained after replacing each edge of $E(G[\![V_t]\!]) \setminus E(G[V_t])$ in P by a path of size at most k, and connecting u to $\pi(u)$ and v to $\pi(v)$ with paths of size at most k. Conversely, for every uv-path P' of length d in $G[V_t']$, there exists some $\pi(u)\pi(v)$-path P in $G[\![V_t]\!]$ of length at most $d \cdot (2k+1)$. This follows from the fact that for every edge $xy \in E(G[V_t'])$, we have $d_{G[\![V_t]\!]}(\pi(x), \pi(y)) \leqslant 2k+1$.

Hence, for every $u, v \in V_t$ we have

$$\frac{1}{k} d_{G[V_t']}(u,v) - 2 \leqslant d_{G[\![V_t]\!]}(\pi(u), \pi(v)) \leqslant (2k+1) d_{G[V_t']}(u,v),$$

thus, as π is surjective, it indeed defines a quasi-isometry. □

2.4 Rays and ends

A *ray* in a graph G is an infinite simple one-way path $P = (v_1, v_2, \ldots)$. A *subray* P' of P is ray of the form $P' = (v_i, v_{i+1}, \ldots)$ for some $i \geqslant 1$. We define an equivalence relation $\sim$ over the set $\mathcal{R}(G)$ of rays by letting $P \sim P'$ if and only if for every finite set $S \subseteq V(G)$ of vertices, there is a component of $G - S$ that contains infinitely many vertices from both P and P'. The *ends* of G are the elements of $\mathcal{R}(G)/\sim$, the equivalence classes of rays under $\sim$. The *degree* of an end ω is the supremum number $k \in \mathbb{N} \cup \{\infty\}$ of pairwise-disjoint rays that belong to ω. An end is *thin* if it has finite degree, and *thick* otherwise.

2.5 Decompositions of quasi-transitive graphs of finite treewidth

Without being always explicitly named, bounded treewidth quasi-transitive graphs have attracted a lot of attention and admit many interesting characterisations. Below are a few of them; for more, see for example [KPS73, MS83, Woe89, TW93, Ant11].

Theorem 7. *Let G be a connected Γ-quasi-transitive locally finite graph. Then the following are equivalent:*

- (i) *G has finite treewidth;*
- (ii) *there exists a Γ-canonical tree-decomposition of G with tight edge-separations and finite width;*
- $(ii)'$ *there exists a Γ-canonical tree-decomposition $(T, \mathcal{V})$ of G with finite width, connected parts and such that $E(T)$ has finitely many Γ-orbits;*
- (iii) *the ends of G have finite uniformly bounded degree;*
- (iv) *all the ends of G are thin;*

We give a short roadmap on a possible way to find a proof of the above equivalences the way we stated them. The implications $(ii)' \Rightarrow (i)$ and $(iii) \Rightarrow (iv)$ are immediate, and $(ii) \Rightarrow (ii)'$ is an immediate consequence of Lemma 6. It is not hard to see that if a graph has an end of degree $k \geqslant 1$, then it admits the $k \times k$ grid as a minor, thus if G has ends of arbitrary large degree it has infinite treewidth, so $(i) \Rightarrow (iii)$ holds. $(iv) \Rightarrow (ii)$ follows from [HLMR22, Theorem 7.4].

2.6 Nested sets of separations

We define an order $\preceq$ on the set of separations of a graph G as follows. For any two separations $(Y, S, Z), (Y', S', Z')$, we write $(Y, S, Z) \preceq (Y', S', Z')$ if and only if $Y \subseteq Y'$ and $Z' \subseteq Z$.

Two separations $(Y, S, Z), (Y', S', Z')$ of a graph G are said to be *nested* if (Y, S, Z) is comparable either with (Y', S', Z') or with (Z', S', Y') with respect to the order $\preceq$. A set $\mathcal{N}$ of separations of G is *nested* if all its separations are pairwise nested. We say that $\mathcal{N}$ is *symmetric* if for every $(Y, S, Z) \in \mathcal{N}$, we also have $(Z, S, Y) \in \mathcal{N}$. It is not hard to observe that if $(T, \mathcal{V})$ is a tree-decomposition and $\mathcal{N}$ denotes its set of edge-separations, then $\mathcal{N}$ is symmetric and nested. Moreover, if $(T, \mathcal{V})$ is Γ-canonical, then $\mathcal{N}$ is also Γ-invariant with respect to the action of Γ on the set of separations of G.

Extending a known result [CDHS11, Theorem 4.8] for finite graphs, Elbracht, Kneip and Teegen proved in [EKT22, Lemma 2.7] that symmetry and nestedness together with a third property are sufficient conditions to obtain a tree-decomposition from a nested set of separations.

We say that a set $\mathcal{N}$ of separations has *finite intervals* if for every infinite increasing sequence $(Y_1, S_1, Z_1) \prec (Y_2, S_2, Z_2) \prec \cdots$ of separations from $\mathcal{N}$, we have

$$\bigcap_{i \geqslant 1} (S_i \cup Z_i) = \emptyset.$$

Theorem 8 (Lemma 2.7 in [EKT22])**.** *Let $\mathcal{N}$ be a symmetric nested set of separations with finite intervals in an arbitrary graph G. Then there exists a tree-decomposition $(T, \mathcal{V})$ of G such that the edge-separations of $(T, \mathcal{V})$ are exactly the separations from $\mathcal{N}$ and the correspondence is one-to-one. Moreover, if $\mathcal{N}$ is Γ-invariant with respect to some group Γ acting on G, then $(T, \mathcal{V})$ is Γ-canonical.*

Lemma 9. *If G is connected, locally finite and $\mathcal{N}$ is a nested set of separations in G such that for every $(Y, S, Z) \in \mathcal{N}$, S has uniformly bounded diameter with respect to the metric d_G, then $\mathcal{N}$ has finite intervals.*

Note that we do not require in Lemma 9 that the graphs $G[S]$ induced by the separators S are connected, even though it will be the case later as we will apply this lemma on family of separations whose separators are cycles in planar graphs.

Proof. We let $(Y_1, S_1, Z_1) \prec (Y_2, S_2, Z_2) \prec \ldots$ denote an infinite sequence of separations from $\mathcal{N}$.

As G is connected locally finite, for every finite set X of vertices, $G - X$ has only finitely many connected components, hence there is only a finite number of indices $i \geqslant 1$ such that $X = S_i$. In particular, we may assume up to taking an infinite subsequence of separations that for every two indices $i \neq j$, $S_i \neq S_j$. Let $A \in \mathbb{N}$ be an upper bound of the set $\{\text{diam}_G(S) : (Y, S, Z) \in \mathcal{N}\}$. Then for every $i \neq j$ such that $S_i \cap S_j \neq \emptyset$, the separator S_j is included in the ball of radius $2A$ around S_i. In particular, as G is locally finite, this ball is finite, hence up to taking an infinite subsequence of separations, we may assume moreover that for every $i \neq j$, $S_i \cap S_j = \emptyset$.

Observe that for every two separations $(Y,S,Z) \prec (Y',S',Z')$ such that $S \cap S' = \emptyset$, we have $S' \subseteq Z$. Note that if some vertex x belongs to $Y_i \cup S_i$ for some $i \geqslant 1$, then we also have $x \in Y_j$ for all $j > i$, hence it will be enough to prove that for every $x \in V(G)$, there exists some $i \geqslant 1$ such that $x \in Y_i \cup S_i$, in order to conclude that $\mathcal{N}$ has finite intervals. We consider $x \in Z_1$. Observe that for each $i \geqslant 1$ such that $x \in Z_{i+1}$, we have $d_G(x, S_{i+1}) < d_G(x, S_i)$. Indeed, by previous observation, we have $S_{i+1} \cup Z_{i+1} \subseteq Z_i$, hence S_{i+1} separates Z_{i+1} from S_i, and thus every shortest path from x to S_i must intersect S_{i+1}. In particular, if we set $D := d_G(x, S_1)$, we must have $x \notin Z_{D+1}$, hence $x \in Y_{D+1} \cup S_{D+1}$, as desired. □

2.7 Cycle nestedness in plane graphs

Recall that if a graph G is planar, and $\varphi : G \to \mathbb{R}^2$ is a planar embedding, then the pair (G, φ) is called a *plane graph*. We say that two cycles C, C' in a plane graph (G, φ) are *nested* if $\varphi(C')$ does not intersect both the interior and the exterior regions of C. When φ is fixed, we let $V_{\text{int}}(C)$ (respectively $V_{\text{ext}}(C)$) denote the set of vertices $v \in V(G)$ such that $\varphi(v)$ belongs to the interior (respectively exterior) of C. Then $(V_{\text{int}}(C), V(C), V_{\text{ext}}(C))$ is a separation of G, and if C and C' are nested in (G, φ), then $(V_{\text{int}}(C), V(C), V_{\text{ext}}(C))$ and $(V_{\text{int}}(C'), V(C'), V_{\text{ext}}(C'))$ are nested with respect to the definition of nestedness we gave in Section 2.6. However note that the converse is not true in general as the fact that C and C' are nested might depend of the planar embedding of G we choose.

Recall that by Whitney's theorem [Whi33], every 3-connected planar graph admits a unique embedding in the 2-dimensional sphere $\mathbb{S}^2$, up to composition with a homeomorphism of $\mathbb{S}^2$. Imrich [Imr75] moreover proved that this result also holds in infinite graphs. In particular if G is planar 3-connected, then for any cycles C, C' both the unordered pair $\{V_{\text{int}}(C), V_{\text{ext}}(C)\}$ and the property for C and C' to be nested do not depend on the choice of the planar embedding φ of G. In this case, we will then not need to precise the planar embedding of G when talking about nestedness. Note also that if G is 3-connected, then for any pair of cycles C, C' and any automorphism $\gamma \in \text{Aut}(G)$, C and C' are nested if and only if $\gamma \cdot C$ and $\gamma \cdot C'$ are nested.

We say that a set $F \subseteq E(G)$ of edges is *even* if every vertex from $V(G)$ has even degree in the graph $(V(G), F)$. If we identify a cycle with its sets of edges, then the cycles of G are exactly the inclusionwise minimal finite nonempty sets of edges that are even. If $(C_1, \ldots, C_k)$ are cycles in G, their $\mathbb{Z}_2$*-sum* $\sum_{i=1}^k C_i$ is the finite subset of $E(G)$ obtained by keeping every edge appearing in an odd number of C_i's. We let $\mathcal{C}(G)$ denote the *cycle space* of G, that is the $\mathbb{Z}_2$-vector space consisting of $\mathbb{Z}_2$-sums of cycles of G. We say that a subset $\mathcal{E}$ of $\mathcal{C}(G)$ *generates* $\mathcal{C}(G)$ if every element of $\mathcal{C}(G)$ can be written as a (finite) $\mathbb{Z}_2$-sum of elements from $\mathcal{E}$.

Remark 10. It is well known and not hard to check that elements from $\mathcal{C}(G)$ correspond exactly to the finite even subsets of $E(G)$.

2.8 VAP-free graphs

Given a plane graph (G, φ), an *accumulation point* is a point $x \in \mathbb{R}^2$ that contains infinitely many vertices of G in all its (topological) neighborhoods. A planar graph G is *vertex-accumulation-free* or simply *VAP-free* if it admits an embedding in $\mathbb{R}^2$ with no vertex accumulation point, or equivalently an embedding in $\mathbb{S}^2$ with at most one accumulation point.

A known result that can be deduced from [Bab97, Lemma 2.3] is that one-ended locally finite planar graphs are VAP-free. We will show in Theorem 18 that locally finite VAP-free quasi-transitive graphs form the base class of graphs from which we can inductively build all locally finite quasi-transitive planar graphs. The following is a folklore result about VAP-free graphs.

Proposition 11. *If G is a quasi-transitive locally finite connected VAP-free graph with at least two ends, then G has bounded treewidth.*

Proof. We let G be a locally finite VAP-free graph and $\varphi : G \to \mathbb{R}^2$ be a VAP-free planar embedding of G.

Assume that G has unbounded treewidth. Then by Theorem 7, G has a thick end, thus by a recent strengthening of Halin's grid theorem [Hal65, GH24], G contains a subdivision H of the infinite hexagonal grid $\mathbb{H}$ as a subgraph of G. By Whitney's unique embedding theorem [Whi33, Imr75], the hexagonal grid admits a unique embedding $\varphi_{\mathbb{H}}$ in the 2-dimensional sphere $\mathbb{S}^2$, up to composition with a homeomorphism of $\mathbb{S}^2$, and thus essentially one VAP-free embedding in $\mathbb{R}^2$. In particular, it implies that the faces of (H, φ_H) are bounded by subdivisions of the faces of $(\mathbb{H}, \varphi_{\mathbb{H}})$, and thus the facial cycles of (H, φ_H) are finite. We let ω_0 denote the end of H in G, i.e., the set of rays of G that are equivalent to any ray of H. Let r be a ray in G. We will show that $r \in \omega_0$, which immediately implies that G has a unique end, as desired. As G is connected, we may assume that its first vertex belongs to $V(H)$. Thus every vertex of r is either in $V(H)$ or drawn in a face of $(H, \varphi|_H)$. As φ is a VAP-free embedding, every facial cycle of $(H, \varphi|_H)$ contains only finitely vertices of G in its interior region. In particular, r intersects infinitely many times $V(H)$ so we have $r \in \omega_0$. □

3 Proof of Theorem 1 and Corollary 2.

3.1 Generating families of cycles

For every locally finite graph G and every $i \geqslant 1$, we let $\mathcal{C}_i(G)$ denote the subset of $\mathcal{C}(G)$ of cycles that can be written as $\mathbb{Z}_2$-sums of cycles of length at most i.

Theorem 12 (Theorem 3.3 in [Ham15])**.** *Let G be a 3-connected locally finite planar graph and Γ be a group acting quasi-transitively on G. Then there exists a nested Γ-invariant set of cycles generating $\mathcal{C}(G)$. Moreover, for any $i \geqslant 0$ there exists a Γ-invariant nested family $\mathcal{E}_i$ of cycles of length at most i generating $\mathcal{C}_i(G)$.*

In the same paper, the author also proved the following result, which can be seen as a generalization of the result of [Dro06] that finitely generated planar groups are finitely presented.

Theorem 13 (Theorem 7.2 in [Ham18]). *Let G be a quasi-transitive planar graph and Γ be a group acting quasi-transitively on G. Then there exists a Γ-invariant set $\mathcal{E}$ of cycles generating $\mathcal{C}(G)$ with finitely many Γ-orbits.*

Despite the fact that the proof of Theorem 13 from [Ham18] is based on Theorem 12, the family which is constructed in Theorem 13 is not necessarily nested anymore. However we will observe in Corollary 15 that combining Theorems 12 and 13, we can find in the 3-connected case a generating family of cycles which is both nested and has finitely many $\mathrm{Aut}(G)$-orbits.

The following is a basic fact in homology, and comes from a remark of Matthias Hamann (private communication).

Remark 14. Theorem 13 was stated in [Ham18] in a more general way for $\mathbb{Z}$-sums of oriented cycles, i.e., formal sums of oriented cycles with coefficients in $\mathbb{Z}$. To see that [Ham18, Theorem 7.2] implies Theorem 13 the way we stated it, observe that if a cycle can be written as a formal sum $\alpha_1\overrightarrow{C_1} + \cdots + \alpha_k\overrightarrow{C_k}$ of oriented cycles with coefficients $\alpha_i \in \mathbb{Z}$, then it can also be written in $\mathcal{C}(G)$ as the $\mathbb{Z}_2$-sum of the cycles C_i such that α_i is odd.

We observe that in the 3-connected case, one can find a generating family $\mathcal{E}$ of cycles combining both the properties of Theorems 12 and 13.

Corollary 15. *Let G be a locally finite 3-connected planar graph and Γ be a group acting quasi-transitively on G. Then there exists a Γ-invariant set of cycles generating $\mathcal{C}(G)$ which is nested and has finitely many Γ-orbits.*

An example of a family satisfying the properties of Corollary 15 is given in Figure 3 below.

Proof. We let $\mathcal{E}$ be a Γ-invariant family of cycles generating $\mathcal{C}(G)$ with finitely many Γ-orbits given by Theorem 13. Then in particular there is a bound $K \geqslant 0$ on the size of the cycles from $\mathcal{E}$. By Theorem 12, there exists a nested Γ-invariant family $\mathcal{E}'$ of cycles of lenth at most K in G generating the set $\mathcal{C}_K(G)$. In particular, $\mathcal{E}'$ also generates the whole cycle space $\mathcal{C}(G)$. As G has bounded degree, every vertex $v \in V(G)$ belongs to only finitely many cycles of size at most K. In particular, as Γ acts quasi-transitively on $V(G)$, it implies that Γ also acts quasi-transitively on the set of cycles of size at most K in G. Thus $\mathcal{E}'$ has finitely many Γ-orbits and satisfies the desired properties. $\square$

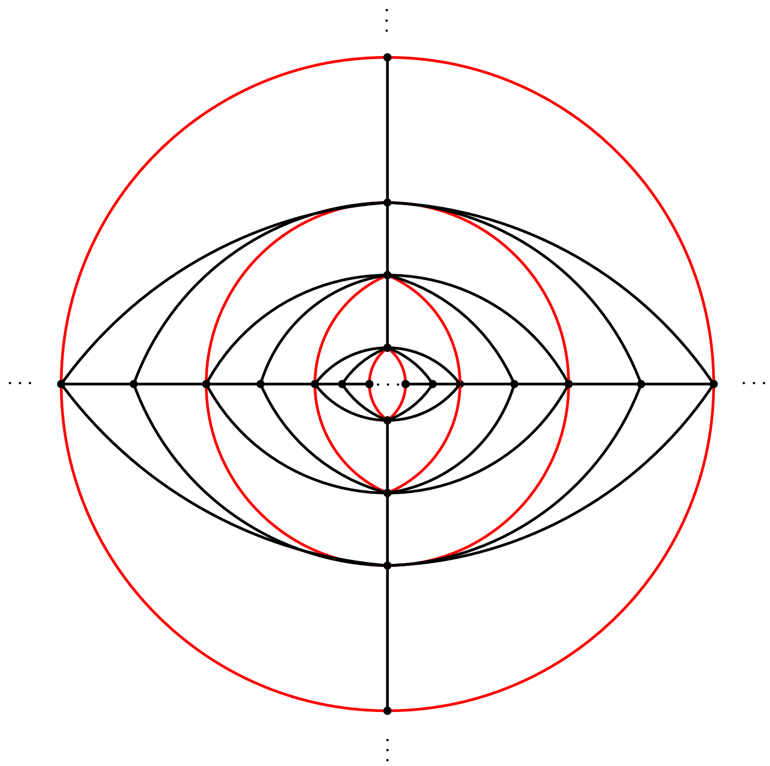

Figure 3: A 2-ended locally finite quasi-transitive 3-connected planar graph G. The set $\mathcal{E}$ of cycles formed by the union of the red cycles together with the set of facial cycles of G forms a nested $\mathrm{Aut}(G)$-invariant generating family of the cycle space of G. The subgraphs induced by the $\mathcal{E}$-blocks are the subgraphs obtained after taking two consecutive red cycles, together with the vertices and edges lying between them.

3.2 Structure of locally finite quasi-transitive planar graphs

If $\mathcal{N}$ is a set of separations, an $\mathcal{N}$-*block* is a maximal set $X \subseteq V(G)$ such that for each $(Y, S, Z) \in \mathcal{N}$, either $X \cap Y = \emptyset$ or $X \cap Z = \emptyset$. If (G, φ) is a plane graph and $\mathcal{E}$ is a set of cycles, then an $\mathcal{E}$-*block* of (G, φ) is a set of vertices which is an $\mathcal{N}$-block, where $\mathcal{N}$ denotes the symmetric set of separations induced by $\mathcal{E}$ in (G, φ).

Lemma 16. *Let (G, φ) be a 3-connected locally finite plane graph, Γ be a group acting quasi-transitively on G and $\mathcal{E}$ be a Γ-invariant nested family of cycles of bounded length generating the cycle space $\mathcal{C}(G)$. Then for each $\mathcal{E}$-block X, the family $\mathcal{E}_X := \{C \in \mathcal{E} : V(C) \subseteq X\}$ generates the cycle space $\mathcal{C}(G[X])$.*

Proof. In this proof, we will identify every cycle of G with its even set of edges.

We let C be a cycle of $\mathcal{C}(G[X])$ and $C_1, \ldots, C_k \in \mathcal{E}$ be such that C equals to the $\mathbb{Z}_2$-sum $\sum_{i=1}^{k} C_i$. Choose $C_1, \ldots, C_k$ that minimize the number k of cycles from $\mathcal{E}$ required to write C as a $\mathbb{Z}_2$-sum $\sum_{i=1}^{k} C_i$. As X is an $\mathcal{E}$-block of (G, φ), every cycle from $\mathcal{E}_X$ must be facial in the plane graph $(G[X], \varphi|_{G[X]})$, and C is nested with every cycle from $\mathcal{E}$. We will show that $C_i \in \mathcal{E}_X$ for all $i \in [k]$, implying the desired result.

Assume for a contradiction that $C_i \notin \mathcal{E}_X$ for some $i \in [k]$. Then as X is an $\mathcal{E}$-block, there exists some cycle $C^* \in \mathcal{E}$ separating X from C_i, and one of its two associated separations (Y, S, Z) is such that $X \subseteq Y \cup S$ and $V(C_i) \subseteq S \cup Z$. In particular, as $V(C) \subseteq X$, it implies that the cycles C_i and C are not drawn in the same face of C^*, hence there are two separations $(Y_1, S_1, Z_1), (Y_2, S_2, Z_2)$ respectively associated to C and C_i such that

$$(Y_1, S_1, Z_1) \preceq (Y, S, Z) \preceq (Y_2, S_2, Z_2).$$

As cycles from $\mathcal{E}$ have bounded length, by Lemma 9, the set $\mathcal{N}$ of separations induced by $\mathcal{E} \cup \{C\}$ in (G, φ) has finite intervals. In particular, there are only finitely many cycles C^* separating X from C_i and we can choose such a cycle C^* which is minimal with respect to $\preceq$.

Claim 17. *We have $C^* \in \mathcal{E}_X$.*

Proof of Claim. Assume for a contradiction that $C^* \notin \mathcal{E}_X$. Then $V(C^*) \not\subseteq X$, and as X is an $\mathcal{E}$-block, there exists some cycle $C' \in \mathcal{E}$ separating X from C^*. In particular, C' also separates X from C_i and one of its two associated separations (Y', S', Z') satisfies

$$(Y_1, S_1, Z_1) \preceq (Y', S', Z') \preceq (Y, S, Z),$$

contradicting the minimality of (Y, S, Z). □

We now let D be the cycle associated to the maximal separation $(Y', S', Z') \in \mathcal{E}$ such that $(Y, S, Z) \preceq (Y', S', Z') \preceq (Y_2, S_2, Z_2)$ and such that $D \in \mathcal{E}_X$. In particular, by previous observation D is facial in $(G[X], \varphi|_{G[X]})$ so $\varphi(C_i)$ must be drawn in the closure of the face Λ of $(G[X], \varphi|_{G[X]})$ which is delimited by D. We let $I \subseteq [k]$ be the set of indices $j \in [k]$ such that $\varphi(C_j)$ is contained in the closure of Λ. In particular, $i \in I$ so $I \neq \emptyset$. As $\mathcal{E}$ is nested, for every $j \in [k] \setminus I$, $\varphi(C_j)$ does not intersect Λ. Let C' be the $\mathbb{Z}_2$-sum $\sum_{j \in I} C_j$.

Note that the way we defined it, C' is a finite subset of edges of $E(G)$ but not necessarily a cycle of G.

First, note that for each $uv \in E(G) \setminus E(G[X])$, as $uv \notin C$, it must appear in an even number of C_j's. In particular, as we assumed that $I \neq \emptyset$, we must have $|I| \geqslant 2$. In particular, for every $uv \in E(G)$ such that $\varphi(uv)$ intersects Λ, uv can only appear in cycles C_j such that $j \in I$, and its total number of occurences in $(C_j)_{j\in[k]}$ is even, so $uv \notin C'$. It implies that $C' \subseteq D$. By Remark 10, C' is even so as D is a cycle, we have either $C' = \emptyset$ or $C' = D$. According to whether $C' = \emptyset$ or $C' = D$, we consider the decomposition of C as a sum of cycles from $\mathcal{E}$ obtained after either removing the sum $\sum_{j\in I} C_j$ in the decomposition of C or replacing it by the cycle $D \in \mathcal{E}_X$. In both cases it gives a decomposition of C involving at most $k - |I| + 1 < k$ cycles from $\mathcal{E}$, and thus contradicting the minimality of k. □

We are now ready to give a proof of Theorem 1.

Theorem 18. *Let G be a locally finite 3-connected quasi-transitive planar graph and Γ be a group acting quasi-transitively on G. Then there exists a Γ-canonical tree-decomposition $(T, \mathcal{V})$ of G of finite adhesion whose edge-separations correspond to separations associated to cycles of G and whose parts are 2-connected VAP-free quasi-transitive graphs. Moreover $E(T)$ has finitely many Γ-orbits.*

Proof. We let $\mathcal{E}$ be a nested Γ-invariant family of cycles of G generating $\mathcal{C}(G)$ with finitely many Γ-orbits given by Corollary 15. We consider the associated symmetric family $\mathcal{N}$ of separations of G of the form $(V_{\text{int}}(C), V(C), V_{\text{ext}}(C))$ and $(V_{\text{ext}}(C), V(C), V_{\text{int}}(C))$ for each $C \in \mathcal{E}$. As G is 3-connected, our previous remarks imply that $\mathcal{N}$ is a Γ-invariant nested family of separations. Moreover, as $\mathcal{E}$ has finitely many Γ-orbits, separations in $\mathcal{N}$ must have finite bounded order so Lemma 9 implies that $\mathcal{N}$ has finite intervals. We thus can apply Theorem 8 and find a Γ-canonical tree-decomposition $(T, \mathcal{V})$ whose edge-separations are in one-to-one correspondence with the different separations of $\mathcal{N}$. In particular, each adhesion set of $(T, \mathcal{V})$ admits a spanning cycle from $\mathcal{N}$ and thus is finite. As $\mathcal{N}$ has finitely many Γ-orbits, Γ acts quasi-transitively on $E(T)$. By [EGLD24, Lemma 3.13], Γ_t induces a quasi-transitive action on the part $G[V_t]$ of $(T, \mathcal{V})$ for every $t \in V(T)$.

Note that as G is connected, the torsos of $(T, \mathcal{V})$ must be connected. Moreover, as the adhesion sets of $(T, \mathcal{V})$ are connected, each part $G[V_t]$ must also be connected. Moreover, note that as adhesion sets of $(T, \mathcal{V})$ contain spanning cycles, then for every $t \in V(T)$, $|V_t| \geqslant 3$ and for any three different vertices $u, v, w \in V_t$, any path in G from u to v avoiding w can be modified to a path in $G[V_t]$ from u to v avoiding w. Hence each part of $(T, \mathcal{V})$ is 2-connected.

It remains to show that each part of $(T, \mathcal{V})$ is VAP-free. By [CDHS11, Theorem 4.8], parts of $(T, \mathcal{V})$ are either "hubs", i.e., vertex sets of cycles from $\mathcal{E}$, or $\mathcal{N}$-blocks[2] (and

[2] Note that [CDHS11, Theorem 4.8] only deals with finite graphs. However, the tree-decomposition given by [EKT22, Lemma 2.7] generalizes the construction from [CDHS11] when one considers nested sets of separations having finite intervals, and the proof that its bags are either hubs or blocks extends in this case.

equivalently $\mathcal{E}$-blocks). Hubs parts are finite and thus obviously VAP-free. Assume now that $G[V_t]$ is an $\mathcal{E}$-block for some $t \in V(T)$. Then by Lemma 16, $\mathcal{E}_{V_t}$ generates the cycle space $\mathcal{C}(G[V_t])$. In particular, note that cycles from $\mathcal{E}_{V_t}$ must be facial in the plane graph $(G[V_t], \varphi|_{G[V_t]})$. The plane graph $(G[V_t], \varphi|_{G[V_t]})$ is thus 2-connected and its cycle space is generated by a family of facial walks, so by [Tho80, Theorem 7.4] it must be a VAP-free graph. □

Combining Theorem 18 with results from [CHM22] allowing to combine canonical tree-decompositions, we obtain the following result for 3-connected planar graphs.

Corollary 19. *For every locally finite 3-connected quasi-transitive planar graph G, and every group Γ acting quasi-transitively on G, there exists a Γ-canonical tree-decomposition $(T, \mathcal{V})$ of G of finite adhesion whose parts are connected and either finite or quasi-transitive one-ended, and such that $E(T)$ has finitely many Γ-orbits.*

Proof. Let $(T, (V_t)_{t \in V(T)})$ be the Γ-canonical tree-decomposition of G given by Theorem 18. We let $t \in V(T)$ be such that V_t is infinite. If $G[V_t]$ has at least 2 ends, then Proposition 11 implies that $G[V_t]$ has bounded treewidth. By Lemma 5, $G[\![V_t]\!]$ is quasi-isometric to $G[V_t]$, so as the property of having finite treewidth in bounded degree graphs is invariant under taking quasi-isometries, $G[\![V_t]\!]$ also has bounded treewidth. By Theorem 7 $(ii)'$, there exists a Γ_t-canonical tree-decomposition $(T_t, \mathcal{V}_t)$ of $G[\![V_t]\!]$ of finite width whose parts are connected, and such that $E(T_t)$ has finitely many Γ_t-orbits. Then by [CHM22, Proposition 7.2] (see [EGLD24, Proposition 3.10, Remark 3.11] for a statement of this result closer to the one we use here), there exists a Γ-canonical tree-decomposition $(T', \mathcal{V}')$ of G *refining* $(T, \mathcal{V})$, whose torsos are connected with at most one end, and whose adhesion sets are either adhesion sets of $(T, \mathcal{V})$ or adhesion sets of some $(T_t, \mathcal{V}_t)$. In particular, as G is locally finite quasi-transitive, every finite set is the separator of a finite bounded number of separations, hence $E(T')$ must have only finitely many Γ-orbits. Finally, we find a tree-decomposition of G with the desired properties by applying Lemma 6 to $(T', \mathcal{V}')$. The fact that its parts are quasi-transitive follows from [EGLD24, Lemma 3.13]. □

See Figure 4 below for an illustration of the tree-decomposition obtained (which turns out to be a path-decomposition in this specific example) when applying the proof of Theorem 18 with respect to the family of cycles from Figure 3.

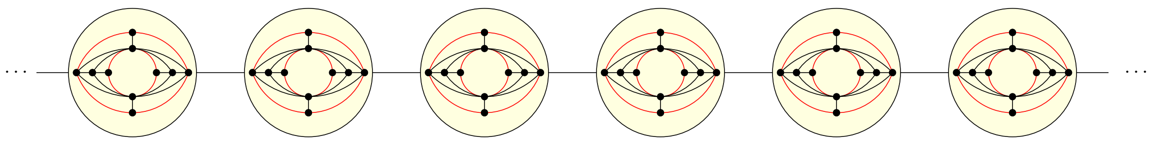

Figure 4: The path-decomposition of the graph from Figure 3 obtained after applying Theorem 8 to the nested family of red cycles. The red cycles form the adhesion sets of the path-decomposition.

To derive a structure theorem for general planar graphs, we will need to decompose graphs with connectivity at most 2 into parts of larger connectivity. To do this, we will use a general decomposition theorem, initally proved by Tutte [Tut84] in the finite case, and generalized to infinite graphs in [DSS98]. See also [CK23, Theorem 1.6.1] for a more precise version.

Theorem 20 ([DSS98])**.** *Every locally finite graph G has a canonical tree-decomposition of adhesion at most 2 with tight edge-separations, whose torsos are minors of G and are complete graphs of order at most 2, cycles, or 3-connected graphs.*

Our proof of Corollary 2 will now simply follow from a combination of Corollary 19 with Theorem 20.

Corollary 21. *For every connected planar locally finite graph G, and every group Γ acting quasi-transitively on G, there exists a Γ-canonical tree-decomposition $(T, \mathcal{V})$ of finite adhesion whose parts $G[V_t]$ are connected, either finite or one-ended planar graphs, on which Γ_t acts quasi-transitively for each $t \in V(T)$, and such that $E(T)$ has finitely many Γ-orbits.*

Proof. We first consider Tutte's canonical tree-decomposition $(T_0, \mathcal{V}_0)$ of G given by Theorem 20. We let G^+ be the supergraph obtained from G after adding an edge uv for each pair of vertices u, v belonging to a common adhesion set of $(T_0, \mathcal{V}_0)$. In particular for each $t \in V(T_0)$, $G[\![V_t]\!] = G^+[V_t]$. As the edge-separations of $(T_0, \mathcal{V}_0)$ are tight, Lemma 3 implies that for each $v \in V(G)$, there is only a finite bounded number of edges $tt' \in E(T_0)$ such that $v \in V_t \cap V_{t'}$. In particular, for each $v \in V(G)$, there is only a finite bounded number of $t \in V(T_0)$ such that $v \in V_t$. Thus G^+ is also locally finite, and as $(T_0, \mathcal{V}_0)$ is Γ-canonical, Γ also acts quasi-transitively on G^+. We will now show that G^+ is planar. Note that $(T_0, \mathcal{V}_0)$ also corresponds to Tutte's decomposition of G^+, and as every torso $G[\![V_t]\!]$ of $(T_0, \mathcal{V}_0)$ is a minor of G, every torso $G[\![V_t]\!]$ is planar. By a result attributed to Erdős (see for example [Tho80]), a countable graph is planar if and only it excludes $K_{3,3}$ and K_5 as minors. In particular it implies that a countable graph is planar if and only if all its finite subgraphs are planar, so it is enough to check that every finite subgraph of G^+ is planar. As the adhesion sets of $(T_0, \mathcal{V}_0)$ induce complete graphs in G^+ and have size at most 2, note that any finite subgraph of G^+ is obtained after perfoming the following operation a finite number of times: taking two disjoint planar graphs G_1, G_2 with two edges $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$, and gluing them by identifying u_1 with u_2 and v_1 with v_2. Note that such an operation does preserve planarity, hence every finite subgraph of G^+ must be planar and we deduce that G^+ is also planar.

Claim 22. *If G^+ admits a Γ-canonical tree-decomposition $(T, \mathcal{V})$ of finite adhesion whose parts are connected and finite or one-ended and such that $E(T)$ has finitely many Γ-orbits, then G also admits a Γ-canonical tree-decomposition with the same properties.*

Proof of Claim. We write $\mathcal{V} = (V_t)_{t \in V(T)}$. Note that $(T, \mathcal{V})$ is also a Γ-canonical tree-decomposition of G. As $E(T_0)$ has finitely many Γ-orbits, note that the set $\{d(u, v) : uv \in E(G^+) \setminus E(G)\}$ admits some maximum $k_1 \in \mathbb{N}$. As $E(T)$ has finitely many Γ-orbits, the set $\{d_G(u, v) : \exists t \in V(T), uv \in E(G[\![V_t]\!]) \setminus E(G[V_t])\}$ also admits a maximum $k_2 \in \mathbb{N}$. We set $k := \max(k_1, k_2)$ and let $\mathcal{V}' := (V'_t)_{t \in V(T)}$ be defined by $V'_t := B_k(V_t) = \{v \in V(G) : \exists u \in$

$V_t, d_G(u,v) \leqslant k\}$. We claim that the proof of Lemma 6 still works here and implies that $(T, \mathcal{V}')$ is a Γ-canonical tree-decomposition of G^+ of finite adhesion, such that for every $t \in V(T)$, $G^+[\![V_t]\!]$ is quasi-isometric to $G[V'_t]$. More precisely, every mapping $\pi : V'_t \to V_t$ such that for all $v \in V'_t$, $d_G(\pi(v), v) = d_G(V_t, v)$ defines a quasi-isometry between $G^+[V_t]$ and $G[V'_t]$. In particular, for each $t \in V(T)$, $G[V'_t]$ has at most one end. □

Claim 22 allows us to assume without loss of generality that $G^+ = G$, i.e., that for each $t \in V(T_0)$ we have $G[\![V_t]\!] = G[V_t]$. For each $t \in V(T_0)$ such that V_t is infinite, $G[\![V_t]\!] = G[V_t]$ is 3-connected, thus we can apply Corollary 19 to find a Γ_t-canonical tree-decomposition $(T_t, \mathcal{V}_t)$ of $G[V_t]$ with finite adhesion whose parts are connected and have at most one end, and such that $E(T_t)$ has finitely many Γ_t-orbits. Eventually, [CHM22, Proposition 7.2] (or [EGLD24, Proposition 3.10, Remark 3.11]) implies that there exists some canonical tree-decomposition $(\widetilde{T}, \widetilde{V})$ refining $(T_0, \mathcal{V}_0)$, whose parts are connected and either finite or one-ended, and such that $E(T)$ has finitely many Γ-orbits. The fact that for each $t \in V(\widetilde{T})$, Γ_t acts quasi-transitively on $G[V_t]$ follows from [EGLD24, Lemma 3.13]. □

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