Cubic Edge-Transitive Graphs of Order $2p^4$

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Abstract

A graph Γ is *edge-transitive* (*s-arc-transitive*, respectively) if its full automorphism group Aut (Γ) acts transitively on the set of edges (the set of s-arcs in Γ for an integer $s \geqslant 0$, respectively). A 1-arc-transitive graph is called an *arc-transitive graph* or a *symmetric graph*. In this paper, we construct cubic symmetric bi-Cayley graphs over some groups of order p^4, where $p \geqslant 7$ is a prime. Using these constructions, we classify the connected cubic edge-transitive graphs of order $2p^4$ for each prime p and we also show that all these graphs are symmetric.

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1 Introduction

Throughout this paper, we consider finite groups and finite connected simple graphs. For a graph Γ, we denote by $V(\Gamma)$, $E(\Gamma)$ and Aut (Γ) the vertex set, the edge set and the full automorphism group of Γ, respectively. For a non-negative integer s, an *s-arc* in a graph Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices in Γ such that v_{i-1} is adjacent to v_i for $1 \leqslant i \leqslant s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant s-1$. Usually, a 1-arc is called an *arc*. A graph Γ is said to be *s-arc-transitive* if Aut (Γ) is transitive on the set of s-arcs in Γ. Note that a 0-arc-transitive graph means vertex-transitive graph, and a 1-arc-transitive graph is called an *arc-transitive graph* or a *symmetric graph*. An s-arc-transitive graph Γ is called *s-arc-regular* if for every two s-arcs in Γ there exists a unique automorphism in Aut (Γ) sending one to the other. A graph Γ is *edge-transitive* if Aut (Γ) is transitive on $E(\Gamma)$, and *semisymmetric* if Γ is edge-transitive but not vertex-transitive with regular valency. A graph Γ is *edge-regular* if for any two edges in Γ there exists a unique automorphism in Aut (Γ) sending one to the other. We say that a graph is *half-arc-transitive* if it is vertex-transitive and edge-transitive but not arc-transitive. Note that every half-arc-transitive graph has even valency (see [17]).

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In this paper, we are interested in regular edge-transitive graphs with a given order. By definition, it is easily seen that edge-transitive graphs with regular valency can be partitioned into three classes: symmetric graphs, half-arc-transitive graphs and semisymmetric graphs.

Let p be a prime. In the literature, a fair amount of work has been done on edge-transitive graphs of order kp^ℓ for some integers k and ℓ. It is easy to show that edge-transitive graphs of order p are also vertex-transitive. In 1971, Chao [2] classified the vertex- and edge-transitive graphs of order p. For the case of edge-transitive graphs of order $2p$, in 1967, Folkman [11] showed that a regular edge-transitive graph of order $2p$ is also vertex-transitive, and in 1987, Cheng and Oxley [3] classified the vertex- and edge-transitive graphs of order $2p$ by using deep group theory. Edge-transitive graphs of order $2p^2$ have also been studied in [11, 28]. Folkman [11] showed that a regular edge-transitive graph of order $2p^2$ is vertex-transitive, and Zhou and Zhang [28] gave a classification of the vertex- and edge-transitive graphs of order $2p^2$. It is worth mentioning that all regular edge-transitive graphs of order p, $2p$ or $2p^2$ are symmetric (see [2, 3, 11, 28]). For the case of order $2p^3$, the situation becomes much more complicated. Unlike in the case of $p, 2p, 2p^2$, there exists a semisymmetric graph of order $2p^3$ for some p, for example the Gray graph. Actually, Malnič at el. [14] proved that the Gray graph is the only cubic semisymmetric graph of order $2p^3$. Following their work, Du et al. started the project of classification of the semisymmetric graphs of order $2p^3$, and they have been working a big progress on this project, see [8, 18, 19, 20, 21, 22]. On the other hand, there are also symmetric graphs of order $2p^3$ and half-arc-transitive graphs of order $2p^3$. In 2006, Feng and Kwak [10] classified all cubic symmetric graphs of order $2p^3$, and in 2019, Zhang and Zhou [25] classified all tetravalent half-arc-transitive graphs of order $2p^3$. Note here that all cubic edge-transitive graphs of order kp^ℓ for $k \leqslant 2$ and $\ell \leqslant 3$ are classified.

Motivated by this, we shall focus on cubic edge-transitive graphs of order $2p^4$. In the following main theorem, we classify the connected cubic edge-transitive graphs of order $2p^4$ for each prime $p \geqslant 2$ and we also determine their s-arc-regular property for each case.

Theorem 1. *Let p be a prime. Let Γ be a connected cubic edge-transitive graph of order $2p^4$. Then Γ satisfies one of the following:*

(i) *Γ is 1-arc-regular and Γ is isomorphic to one of graphs in* $\{\mathrm{F}162A, \Upsilon_{4,0,i}, \Upsilon_{3,1,i}, \Gamma_{H_3,p,i}, \Gamma_{H_9,7,i,k}\ (k \neq 3), \Gamma_{H_9,p,i,k}\ (p > 7), \Gamma_{H_{10},p,i}\}$,

(ii) *Γ is 2-arc-regular and Γ is isomorphic to* F32, F162B, F1250A *or* $\Upsilon_{2,2,0}$,

(iii) *Γ is 3-arc-regular and Γ is isomorphic to* F162C *or* F1250B,

(iv) *Γ is 4-arc-regular and Γ is isomorphic to* $\Gamma_{H_9,7,i,3}$,

where we refer to [6] *and Equations (11)-(12) for notations.*

Moreover, the following interesting corollary is obtained immediately from Theorem 1.

Corollary 2. *Let p be a prime. Every connected cubic edge-transitive graph of order $2p^4$ is symmetric.*

This paper is organized as follows. In Section 2, we set up notations and preliminary results about groups of order p^4, bi-Cayley graphs and cubic edge-transitive graphs. In Section 3, we construct cubic symmetric bi-Cayley graphs of order $2p^4$ with $p \geqslant 7$ a prime and we determine their s-arc-regular property ($s \geqslant 1$). In Subsection 3.1 (3.2, respectively), we construct such bi-Cayley graphs over abelian (non-abelian, respectively) groups of order p^4. Using these constructions, we complete the proof of Theorem 1 in Section 4.

2 Preliminaries

In this section, we set up notations and preliminary results which will be used in this paper. In Subsections 2.1, we consider groups of order p^4 and we recall bi-Cayley graphs and cubic edge-transitive graphs in Subsection 2.2.

Let G be a group. We write $\mathrm{Aut}(G)$, $Z(G)$, G' and $\Phi(G)$ for the automorphism group, the center, the derived subgroup and the Frattini subgroup of G, respectively. For any elements $x,\ y \in G$, we denote by $o(x)$ the order of x and by $[x, y]$ the commutator $x^{-1}y^{-1}xy$. For a subgroup $H \leqslant G$ and for a normal subgroup $N \trianglelefteq G$, denote by $C_G(H)$, $N_G(H)$ and G/N the centralizer of H in G, the normalizer of H in G and the quotient group, respectively. For a positive integer n, we denote by $\mathbb{Z}_n$ and $\mathbb{Z}_n^*$ the cyclic group of order n and the multiplicative group of integers modulo n, respectively. For two groups G_1 and G_2, we write $G_1 \times G_2$ for the direct product of G_1 and G_2, and $G_1 \rtimes G_2$ for a semidirect product of G_1 by G_2.

Let G be a permutation group on a finite set Ω. For each element $\alpha \in \Omega$, the subgroup of G fixing α is denoted by G_α (i.e., *the stabilizer* of α in G). The group G is *semiregular* if $G_\alpha = 1$ holds for any $\alpha \in \Omega$, and G is *regular* if G is transitive and semiregular.

2.1 Groups of order p^4

To prove the main result Theorem 1, we need the classification of groups of order p^4 where $p \geqslant 2$ is a prime. In view of the results by Conder [4, 5] for $2 \leqslant p \leqslant 5$, we mainly consider the case for $p \geqslant 7$.

Theorem 3. [12, Chapter 3], [24, Section 3] *Up to isomorphism, there are fifteen groups of order p^4 with $p \geqslant 7$ a prime as follows.*

(i) *Five abelian groups:*

$$\mathbb{Z}_{p^4}, \quad \mathbb{Z}_{p^3} \times \mathbb{Z}_p, \quad \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}, \quad \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p \quad \text{and} \quad \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p;$$

(ii) *Ten non-abelian groups:*

$$
\begin{aligned}
&H_1 = \langle a, b \mid a^{p^3} = b^p = 1, b^{-1}ab = a^{1+p^2} \rangle;\\
&H_2 = \langle a, b \mid a^{p^2} = b^{p^2} = 1, b^{-1}ab = a^{1+p} \rangle;\\
&H_3 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle;\\
&H_4 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, b^{-1}ab = a^{1+p}, [a, c] = [b, c] = 1 \rangle;\\
&H_5 = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, [a, b] = c, [c, a] = [c, b] = [d, a] = [d, b] = [d, c] = 1 \rangle;\\
&H_6 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [b, c] = a^p, [a, b] = [a, c] = 1 \rangle;\\
&H_7 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = c, [c, a] = 1, [c, b] = a^p \rangle;\\
&H_8 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = c, [c, a] = 1, [c, b] = a^{\nu p} \rangle (\nu \in \mathbb{Z}_p^*, \nu^2 \not\equiv 1 (\mathrm{mod}\ p));\\
&H_9 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = c, [c, a] = a^p, [c, b] = 1 \rangle;\\
&H_{10} = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, [a, b] = c, [c, a] = 1, [c, b] = d, [d, a] = [d, b] = [d, c] = 1 \rangle.
\end{aligned}
\tag{1}
$$

Let G be a finite p-group, where p is a prime. We write $c(G)$ and $\exp(G)$ for the nilpotent class of G and the exponent of G (i.e., the largest order of the elements in G), respectively. If $(xy)^p = x^p y^p$ holds for any $x, y \in G$, then G is called *p-abelian*. The following result gives an equivalent condition for being p-abelian for some p-groups.

Proposition 4. [23, Theorem 2] *Let G be a finite p-group which is generated by two elements and whose derived subgroup G' is abelian. Then G is p-abelian if and only if $\exp(G') \leqslant p$ and $c(G) < p$.*

Using Proposition 4, we show the following result which will be used in Sections 3-4.

Lemma 5. *Referring to (1), each group $H_t (t = 7, 8, 9, 10)$ is p-abelian and it satisfies*

$$Z(H_t) = \langle [[x, y], z] \mid x, y, z \in H_t \rangle = \begin{cases} \langle a^p \rangle & \textit{if } t = 7, 8, 9 \\ \langle d \rangle & \textit{if } t = 10 \end{cases}.$$

Proof. We first show that H_t is p-abelian for each $7 \leqslant t \leqslant 10$. Put $K_t := \langle a^p \rangle \times \langle c \rangle$ $(7 \leqslant t \leqslant 9)$ and $K_{10} := \langle c \rangle \times \langle d \rangle$. Then $K_t \leqslant H_t'$, $K_t \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $|H_t/K_t| = p^2$ $(7 \leqslant t \leqslant 10)$ all hold. Since every group of order p^2 is abelian, H_t/K_t is abelian and thus $H_t' \leqslant K_t$. This shows $H_t' = K_t \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and hence H_t' is abelian with $\exp(H_t') \leqslant p$. Since $c(H_t) \leqslant 3 < p$ holds by $|H_t'| = p^2$, it follows by Proposition 4 that H_t is p-abelian.

Since any non-trivial p-group has non-trivial center, we have $|Z(H_t)| \geqslant p$. Suppose $|Z(H_t)| \geqslant p^2$. Then $H_t' \leqslant Z(H_t)$ follows by $|H_t/Z(H_t)| \leqslant p^2$. This is impossible as $c \in H_t' \setminus Z(H_t)$. This shows

$$|Z(H_t)| = p \ (7 \leqslant t \leqslant 10). \tag{2}$$

On the other hand, it follows by (2), $|H_t'| = p^2$ and $Z(H_t) \cap \langle [[x, y], z] \mid x, y, z \in H_t \rangle \neq \{1\}$ that

$$Z(H_t) = \langle [[x, y], z] \mid x, y, z \in H_t \rangle \tag{3}$$

holds. For $t = 10$, we obtain $Z(H_{10}) = \langle d \rangle$ by $d \in Z(H_{10})$, $d^p = 1$ and (2). Now we will show $Z(H_t) = \langle a^p \rangle$ $(t = 7, 8, 9)$. Since H_t is p-abelian,

$$b^{-1}a^p b = (b^{-1}ab)^p = (b^{-1})^p a^p b^p = a^p \text{ and } c^{-1}a^p c = (c^{-1}ac)^p = (c^{-1})^p a^p c^p = a^p \tag{4}$$

holds by $b^p = c^p = 1$. It follows by (4), $a^{p^2} = 1$ and (2) that $Z(H_t) = \langle a^p \rangle$ $(t = 7, 8, 9)$. This completes the proof by (3). □

2.2 Bi-Cayley graphs and cubic edge-transitive graphs

A graph Γ is called a *bi-Cayley graph over a group H* if Γ admits a group of automorphisms which is isomorphic to H and acts semiregularly on $V(\Gamma)$ with two orbits of the same size. We note that every bi-Cayley graph can be constructed as follows (see [7, 27]).

Definition 6. Let H be a finite group, let R, L and S be subsets of H satisfying $R^{-1} = R$, $L^{-1} = L$ and $1 \notin R \cup L$. Let $H_i = \{(h, i) \mid h \in H\}$ with $i \in \{0, 1\}$. For convenience, we write h_i to denote (h, i) for each $h \in H$ and $i \in \{0, 1\}$. Define the graph $\mathrm{BiCay}(H, R, L, S)$ to be a graph with vertex set $H_0 \cup H_1$ and edge set

$$\{\{h_0,\ (xh)_0\}, \{h_1, (yh)_1\}, \{h_0, (zh)_1\} \mid h \in H, x \in R, y \in L, z \in S\}.$$

We say that $\mathrm{BiCay}(H, R, L, S)$ is a *bi-Cayley graph* over H relative to (R, L, S).

Next, we introduce normal bi-Cayley graphs as well as normal edge-transitive bi-Cayley graphs.

Definition 7. Let Γ be a bi-Cayley graph $\mathrm{BiCay}(H, R, L, S)$ over a group H. For each automorphism $\alpha \in \mathrm{Aut}\,(H)$ and for any elements $x, y, z \in H$, we define some permutations on $H_0 \cup H_1$ as follows:

$$\mathcal{R}(z) \ : \ h_i \mapsto (hz)_i, \quad \forall i \in \mathbb{Z}_2,\ \forall h \in H, \tag{5}$$
$$\delta_{\alpha,x,y} \ : \ h_0 \mapsto (xh^{\alpha})_1,\ h_1 \mapsto (yh^{\alpha})_0,\ \forall h \in H, \tag{6}$$
$$\sigma_{\alpha,z} \ : \ h_0 \mapsto (h^{\alpha})_0,\ h_1 \mapsto (zh^{\alpha})_1,\ \forall h \in H. \tag{7}$$

For a subset $T \subseteq H$ and for each $\alpha \in \mathrm{Aut}\,(H)$, define $T^{\alpha} := \{h^{\alpha} \mid h \in T\}$ and put

$$\mathcal{R}(H) \ := \ \{\mathcal{R}(z) \mid z \in H\}, \tag{8}$$
$$I \ := \ \{\delta_{\alpha,x,y} \mid \alpha \in \mathrm{Aut}\,(H), x, y \in H,\ R^{\alpha} = x^{-1}Lx,\ L^{\alpha} = y^{-1}Ry,\ S^{\alpha} = y^{-1}S^{-1}x\}, \tag{9}$$
$$F \ := \ \{\sigma_{\alpha,z} \mid \alpha \in \mathrm{Aut}\,(H), z \in H,\ R^{\alpha} = R,\ L^{\alpha} = z^{-1}Lz,\ S^{\alpha} = z^{-1}S\}. \tag{10}$$

Then by [27], we see that $\mathcal{R}(H) \leqslant \mathrm{Aut}\,(\Gamma)$, $I \subseteq N_{\mathrm{Aut}\,(\Gamma)}(\mathcal{R}(H))$ and $F \leqslant N_{\mathrm{Aut}\,(\Gamma)}(\mathcal{R}(H))$. Moreover, $\mathcal{R}(H)$ acts semiregularly on $V(\Gamma)$ with two orbits H_0 and H_1. If $\mathcal{R}(H)$ is normal in $\mathrm{Aut}\,(\Gamma)$, then Γ is called a *normal bi-Cayley graph over H* (see [27]). If $N_{\mathrm{Aut}\,(\Gamma)}(\mathcal{R}(H))$ is transitive on $E(\Gamma)$, then Γ is called a *normal edge-transitive bi-Cayley graph over H* (see [7]).

Now, we recall some preliminary results on bi-Cayley graphs which will be used in this paper.

Proposition 8. [27, Theorem 1.1 & Lemmas 3.1–3.2] *Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected bi-Cayley graph over a group H. Referring to Equations (5)–(10), the following hold:*

(i) *H is generated by $R \cup L \cup S$;*

(ii) *S can be chosen to contain the identity element of H;*

(iii) *For any $\alpha \in \mathrm{Aut}\,(H)$, $\mathrm{BiCay}(H,R,L,S) \cong \mathrm{BiCay}(H,R^\alpha,L^\alpha,S^\alpha)$;*

(iv) $N_{\mathrm{Aut}\,(\Gamma)}(\mathcal{R}(H)) = \begin{cases} \mathcal{R}(H) \rtimes F & \text{if } I = \emptyset \\ \mathcal{R}(H)\langle F, \delta_{\alpha,x,y}\rangle \text{ for some } \delta_{\alpha,x,y} \in I & \text{if } I \neq \emptyset \end{cases}$;

(v) *$I \subseteq Aut(\Gamma)$, and if $I \neq \emptyset$, then for each $\delta_{\alpha,x,y} \in I$, $\langle \mathcal{R}(H), \delta_{\alpha,x,y}\rangle$ acts transitively on $V(\Gamma)$;*

(vi) *If α is an automorphism of H of order 2, then $\Gamma \cong \mathrm{Cay}(\bar{H}, R \cup \alpha S)$, where $\bar{H} = H \rtimes \langle\alpha\rangle$.*

In the rest of this subsection, we review some results about cubic edge-transitive graphs.
Let Γ be a connected graph. Let G be a subgroup of $\mathrm{Aut}\,(\Gamma)$ such that G is transitive on $E(\Gamma)$, and let N be a normal subgroup of G which is intransitive on $V(\Gamma)$. The *quotient graph* of Γ relative to N, denoted by Γ_N, is the graph whose vertices are the orbits of N on $V(\Gamma)$, where two different orbits are adjacent if there is an edge in Γ between those two orbits.

Proposition 9. [13, Theorem 9] *Let Γ be a connected cubic graph. Let G be a subgroup of $\mathrm{Aut}\,(\Gamma)$ acting arc-transitively on Γ and let N be a normal subgroup of G. Then the following hold.*

(i) *G acts regularly on the s-arcs of Γ for some $s \geqslant 1$.*

(ii) *Suppose that N has more than two orbits on $V(\Gamma)$. Then N is semiregular on $V(\Gamma)$ and N is the kernel of G acting on $V(\Gamma_N)$. Moreover, Γ_N is a cubic graph, $G/N \leqslant \mathrm{Aut}\,(\Gamma_N)$ holds and G/N acts arc-transitively on Γ_N.*

Proposition 10. [15, Lemmas 17-18] *For an integer $n \geqslant 1$ and a prime number $p \geqslant 3$, let Γ be a connected cubic edge-transitive graph of order $2p^n$.*

(i) *If $p \in \{5,7\}$ and $n \geqslant 2$, then the maximal normal p-subgroup of $\mathrm{Aut}\,(\Gamma)$ has order p^n or p^{n-1}.*

(ii) *If $p \geqslant 5$ then Γ is a bi-Cayley graph over H, where H is a Sylow p-subgroup of $\mathrm{Aut}\,(\Gamma)$ with order p^n. Moreover, if $p \geqslant 11$ then Γ is a normal bi-Cayley graph over H.*

Proposition 11. [15, Theorem 1] *For a prime number $p \geqslant 3$, let H be a non-abelian metacyclic p-group. If Γ is a connected cubic edge-transitive bi-Cayley graph over H, then $p = 3$ holds and Γ is either the Gray graph or a normal bi-Cayley graph over H.*

3 Constructions of cubic symmetric bi-Cayley graphs

In this section, we consider constructions of cubic symmetric bi-Cayley graphs of order $2p^4$ with $p \geqslant 7$ a prime and their s-arc-regular property ($s \geqslant 1$). In Subsection 3.1 (3.2, respectively), we construct such bi-Cayley graphs over abelian (non-abelian, respectively) groups of order p^4. To consider their s-arc-regular property for $s \geqslant 1$, we first need the following lemma.

Lemma 12. *Let H be a group of order p^n, where $p > 7$ is a prime and $n \geqslant 1$ is an integer. Let Γ be a cubic symmetric bi-Cayley graph over H.*

(i) *If there exists a characteristic subgroup N of H satisfying $H/N \cong \mathbb{Z}_{p^r}$ with $r \geqslant 1$, then Γ is 1-arc-regular.*

(ii) *If there exists a characteristic subgroup N of H satisfying $H/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then Γ is at most 2-arc-regular.*

Proof. Let N be a characteristic subgroup of H with $|N| \neq |H|$. Since we may identify H with $\mathcal{R}(H)$ and $H \trianglelefteq \mathrm{Aut}(\Gamma)$ holds by Proposition 10 (ii), we have $N \trianglelefteq \mathrm{Aut}(\Gamma)$. It follows by Proposition 9 that the corresponding quotient graph Γ_N is a connected cubic symmetric bi-Cayley graph over H/N satisfying $\mathrm{Aut}(\Gamma)/N \leqslant \mathrm{Aut}(\Gamma_N)$. If $H/N \cong \mathbb{Z}_{p^r}$ with $r \geqslant 1$, then Γ_N is 1-arc-regular by [9, Theorem 3.5]. This implies that Γ is also 1-arc-regular. If $H/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then Γ_N is 2-arc-regular by [9, Theorem 3.5], and hence Γ is at most 2-arc-regular. This completes the proof. □

3.1 Constructions over abelian groups of order p^4

In this subsection, we give constructions of cubic bi-Cayley graphs over abelian groups of order p^4 with $p \geqslant 7$ (i.e., $\Upsilon_{2,2,0}, \Upsilon_{4,0,i}, \Upsilon_{3,1,i}$ in Construction I) and we show in Lemma 13 that these graphs are either 1-arc-regular or 2-arc-regular.

In view of Theorem 3, there are five abelian groups of order p^4:

$$\mathbb{Z}_{p^4}, \quad \mathbb{Z}_{p^3} \times \mathbb{Z}_p, \quad \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}, \quad \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p \text{ and } \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p.$$

In the following Construction I, we construct bi-Cayley graphs over the first three groups.

Construction I. For a prime number $p \geqslant 7$ and for integers $m \geqslant n \geqslant 0$ with $m + n = 4$,

$$\text{let } H := \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \text{ and define } \Upsilon_{m,n,i} := \mathrm{BiCay}(H, \emptyset, \emptyset, \{1, a, a^i b\}) \quad (11)$$

where $i = 0$ if $m = n = 2$, and $i \in \mathbb{Z}^*_{p^{m-n}}$ satisfying $i^2 - i + 1 \equiv 0 \pmod{p^{m-n}}$ if $(m, n) = (4, 0)$ or $(3, 1)$.

In the following lemma, we consider their s-arc-regular property for $s \geqslant 1$.

Lemma 13. *Referring to (11), both* $\Upsilon_{4,0,i}$ *and* $\Upsilon_{3,1,i}$ *are* 1*-arc-regular, but* $\Upsilon_{2,2,0}$ *is* 2*-arc-regular.*

Proof. If $p = 7$, then the result follows by using MAGMA [1]. Now we assume $p > 7$ and let $i \in \mathbb{Z}^*_{p^{m-n}}$ satisfying $i^2 - i + 1 \equiv 0 \pmod{p^{m-n}}$. Then by [26, Lemma 5.1], both $\Upsilon_{4,0,i}$ and $\Upsilon_{3,1,i}$ are at least 1-arc-regular, but $\Upsilon_{2,2,0}$ is at least 2-arc-regular. For $\Upsilon_{2,2,0}$, we find that $N = \langle a^p \rangle \times \langle b^p \rangle$ is a characteristic subgroup of H satisfying $H/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and thus $\Upsilon_{2,2,0}$ is 2-arc-regular by Lemma 12 (ii). For $\Upsilon_{3,1,i}$ and $\Upsilon_{4,0,i}$, we obtain that $N = \langle a^p \rangle \times \langle b \rangle$ is a characteristic subgroup of H satisfying $H/N \cong \mathbb{Z}_{p^2}$ for $\Upsilon_{3,1,i}$ and $H/N \cong \mathbb{Z}_{p^3}$ for $\Upsilon_{4,0,i}$. By Lemma 12 (i), all they are 1-arc-regular. This completes the proof. □

3.2 Constructions over non-abelian groups of order p^4

In this subsection, we construct three kinds of cubic bi-Cayley graphs over non-abelian groups of order p^4 with $p \geqslant 7$ (see Construction II) and we show in Lemmas 14–17 that these graphs are either 1-arc-regular or 4-arc-regular.

Up to isomorphism, there are only ten non-abelian groups of order p^4, say H_i ($1 \leqslant i \leqslant 10$) (see Theorem 3 (ii)). In the following construction, we consider three of them: H_3, H_9 and H_{10}.

Construction II. For a prime number $p \geqslant 7$ and for non-abelian groups H_3, H_9, H_{10} given in (1), define

$$\begin{aligned} \Gamma_{H_3,p,i} &:= \mathrm{BiCay}(H_3, \emptyset, \emptyset, \{1, a, ba^i\}) \\ \Gamma_{H_9,p,i,k} &:= \mathrm{BiCay}(H_9, \emptyset, \emptyset, \{1, a, a^i b^k\}) \\ \Gamma_{H_{10},p,i} &:= \mathrm{BiCay}(H_{10}, \emptyset, \emptyset, \{1, b, b^i a\}) \end{aligned} \tag{12}$$

where $i,\ k \in \mathbb{Z}^*_p$ satisfying $i^2 - i + 1 \equiv 0 \pmod p$ and $k^2 - k + 1 \equiv 0 \pmod p$.

In the following three lemmas, we prove s-arc-regular property ($s \in \{1, 4\}$) for the graphs in Construction II.

Lemma 14. [16, Theorem 4.11(4)] *Referring to (12),* $\Gamma_{H_3,p,i}$ *is* 1*-arc-regular.*

Lemma 15. *Referring to (12), the following hold.*

(i) *Let* $p = 7$*. Then* $\Gamma_{H_9,7,i,3}$ *is* 4*-arc-regular and* $\Gamma_{H_9,7,i,k}$ $(k \neq 3)$ *is* 1*-arc-regular.*

(ii) *For* $p > 7$*,* $\Gamma_{H_9,p,i,k}$ *is* 1*-arc-regular.*

Proof. Let $p = 7$. With the aid of MAGMA [1], we obtain that $\Gamma_{H_9,7,i,3}$ is 4-arc-regular and $\Gamma_{H_9,7,i,k}$ is 1-arc-regular for $k \neq 3$. Now we assume $p > 7$. For given i, k as in (12), take $n \in \mathbb{Z}^*_p$ satisfying $kn \equiv 1 \pmod p$. We first prove the following claim. Recall by (1) in Theorem 3 that

$$H_9 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1,\ [a, b] = c, [c, a] = a^p,\ [c, b] = 1 \rangle. \tag{13}$$

Claim 16. *For each $j \in \{1,2\}$, H_9 has an automorphism θ_j mapping a, b, c to a_j, b_j, c_j, where*

$$a_1 := a^{i-1}b^k, b_1 := ((a^{i-1}b^k)^{-i}a^{-1})^n, c_1 := ca^{-p},$$
$$a_2 := a^{-1}, b_2 := (a^i b^{-k} a^{-i})^n, c_2 := ca^{-(i+1)p}.$$

Proof of Claim 16: Note that group H_9 satisfies the following (see Lemma 5 and (12)):

$$p\text{-abelian}, \ a^p \in Z(H_9) = \langle [[x,y],z] \mid x,y,z \in H_9 \rangle \text{ and } i^2 - i + 1 \equiv 0, \ kn \equiv 1 \pmod{p}. \tag{14}$$

Using (13)–(14), we obtain the following:

$$\begin{aligned}
a_1^p &= (a^{i-1}b^k)^p = a^{(i-1)p}(b^p)^k = a^{(i-1)p} \neq 1 \quad \text{and thus} \quad a_1^{p^2} = (a^{p^2})^{i-1} = 1; \\
a_2^p &= a^{-p} \neq 1 \quad \text{and} \quad (a_2)^{p^2} = (a^{p^2})^{-1} = 1; \\
b_1^p &= ((a^{i-1}b^k)^{-i}a^{-1})^{np} = a^{-(i^2-i+1)np} = 1 \quad \text{and} \quad b_2^p = (a^i b^{-k} a^{-i})^{np} = (a^{ip} b^{-kp} a^{-ip})^n = 1; \\
c_1^p &= (ca^{-p})^p = c^p a^{-p^2} = 1 \quad \text{and} \quad c_2^p = c^p a^{-(i+1)p^2} = 1.
\end{aligned}$$

Now, we need to obtain Equations (15)–(20) which will be used for calculating the relations a_j, b_j, c_j $(j = 1,2)$. Using (13)–(14), we obtain the following calculations for $\ell \geqslant 1$:

$$\begin{aligned}
[c, a^\ell] &= [c,a]^\ell = a^{p\ell} \ ; && (15) \\
[c^\ell, (a^{i-1}b^k)^{-i}a^{-1}] &= [c^\ell, a^{-1}][c^\ell, (a^{i-1}b^k)^{-i}] = [c^\ell, a]^{-(i^2-i+1)} = a^{-\ell(i^2-i+1)p} = 1; && (16) \\
[a^{i-1}b^k, (a^{i-1}b^k)^{-i}a^{-1}] &= [b^k, a^{-1}] = [b, a^{-1}]^k = (ca^{-p})^k; && (17) \\
[a^{-1}, (a^i b^{-k} a^{-i})^n] &= [a^{-1}, a^i b^{-k} a^{-i}]^n; && (18) \\
[a^{-1}, a^i b^{-k} a^{-i}] &= [a^{-1}, b^{-k}a^{-i}] = (a^{i+1}ca^{-i-1})^k = (ca^{-(i+1)p})^k; && (19) \\
[ca^{-(i+1)p}, (a^i b^{-k} a^{-i})^n] &= [c, (a^i b^{-k} a^{-i})^n] = [c, a^i b^{-k} a^{-i}]^n. && (20)
\end{aligned}$$

Using (14) and applying some of (15)–(20), we obtain the following:

$$\begin{aligned}
&[c_1, a_1] = [ca^{-p}, a^{i-1}b^k] = [c, a^{i-1}b^k] = [c, b^k][c, a^{i-1}] = [c, a^{i-1}] = a^{(i-1)p} = a_1^p \text{ (by (15))}; \\
&[c_1, b_1] = [ca^{-p}, ((a^{i-1}b^k)^{-i}a^{-1})^n] = [c, ((a^{i-1}b^k)^{-i}a^{-1})^n] = 1 \text{ (by (16))}; \\
&[a_1, b_1] = [a^{i-1}b^k, ((a^{i-1}b^k)^{-i}a^{-1})^n] = [a^{i-1}b^k, (a^{i-1}b^k)^{-i}a^{-1}]^n = (ca^{-p})^{kn} = ca^{-p} = c_1 \text{ (by (16), (17))}; \\
&[a_2, b_2] = [a^{-1}, (a^i b^{-k} a^{-i})^n] = [a^{-1}, a^i b^{-k} a^{-i}]^n = (ca^{-(i+1)p})^{kn} = ca^{-(i+1)p} = c_2 \text{ (by (18), (19))}; \\
&[c_2, a_2] = [ca^{-(i+1)p}, a^{-1}] = [c, a^{-1}] = a^{-p} = a_2^p \text{ (by (15))}; \\
&[c_2, b_2] = [ca^{-(i+1)p}, (a^i b^{-k} a^{-i})^n] = [c, a^i b^{-k} a^{-i}]^n = ([c, a^{-i}][c, b^{-k}][c, a^i])^n = 1 \text{ (by (15), (20))}.
\end{aligned}$$

It follows that a_j, b_j, c_j satisfy the same relations as a, b, c in (13) for each $j \in \{1,2\}$. Moreover, the following hold:

$$\begin{aligned}
a_1^i b_1^k &= (a^{i-1}b^k)^i((a^{i-1}b^k)^{-i}a^{-1})^{kn} = (a^{i-1}b^k)^i(a^{i-1}b^k)^{-i}a^{-1} = a^{-1}; \\
((a_1^i b_1^k)^{i-1}a_1)^n &= (a^{-(i-1)}a^{i-1}b^k)^n = b^{kn} = b; \\
a_1^{ip}c_1^{-1} &= (a^{i-1}b^k)^{ip}a^p c^{-1} = a^{(i-1)ip}a^p c^{-1} = a^{(i^2-i+1)p}c^{-1} = c^{-1}; \\
a_2 &= a^{-1} \ ; \\
(a_2^i b_2^k a_2^{-i})^n &= (a^{-i}a^i b^{-k} a^{-i} a^i)^n = b^{-kn} = b^{-1}; \\
c_2 a_2^{-(i+1)p} &= ca^{-(i+1)p}a^{(i+1)p} = c,
\end{aligned}$$

and hence $\langle a, b, c\rangle = \langle a_j, b_j, c_j\rangle$ for each $j \in \{1, 2\}$. Thus, the map

$$a \mapsto a_j,\ b \mapsto b_j,\ c \mapsto c_j$$

induces an automorphism θ_j $(j = 1, 2)$ of H_9. This completes the proof of Claim 16. □

Proof Lemma 15 (cont.) To complete the proof of Lemma 15, we will use Lemma 12 (i). We first show that the cubic bi-Cayley graph $\Gamma_{H9,p,i,k}$ is symmetric. By Claim 16, there exists $\theta_j \in \mathrm{Aut}\,(H_9)$ such that $a^{\theta_j} = a_j,\ b^{\theta_j} = b_j,\ c^{\theta_j} = c_j$ for each $j \in \{1, 2\}$. It is easy to verify that

$$\{1, a, a^i b^k\}^{\theta_1} = \{1, a^{i-1}b^k, a^{-1}\} = a^{-1}\{1, a, a^i b^k\},$$
$$\{1, a, a^i b^k\}^{\theta_2} = \{1, a^{-1}, b^{-k}a^{-i}\} = \{1, a, a^i b^k\}^{-1}.$$

In view of Proposition 8, $\sigma_{\theta_1,a}$ and $\delta_{\theta_2,1,1}$ are automorphisms of $\Gamma_{H9,p,i,k}$. Moreover, $\sigma_{\theta_1,a}$ fixes 1_0 and it permutates the three neighbors of 1_0 cyclically. This implies that $\mathcal{R}(H_9) \rtimes \langle\sigma_{\theta_1,a}\rangle$ acts regularly on $E(\Gamma_{H9,p,i,k})$. On the other hand, $\delta_{\theta_2,1,1}$ swaps 1_0 and 1_1. This implies that $\Gamma_{H9,p,i,k}$ is vertex-transitive and thus it is symmetric. Since $C_{H_9}(H_9')$ is a characteristic subgroup of H_9' and H_9' is a characteristic subgroup of H_9, we obtain that $C_{H_9}(H_9')$ is a characteristic subgroup of H_9. We find by $C_{H_9}(H_9') = \langle a^p, b, c\rangle \cong \mathbb{Z}_p^3$ that $H_9/(C_{H_9}(H_9')) \cong \mathbb{Z}_p$ holds. Therefore, $\Gamma_{H9,p,i,k}$ is 1-arc-regular by Lemma 12 (i). This completes the proof of Lemma 15. □

Lemma 17. *Referring to (12), $\Gamma_{H_{10},p,i}$ for $p \geqslant 7$ is 1-arc-regular.*

Proof. If $p = 7$ then we find by applying MAGMA [1] that $\Gamma_{H_{10},7,i}$ is 1-arc-regular. Now we assume $p > 7$. We first prove the following claim. Recall by (1) in Theorem 3 that

$$H_{10} = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, [a, b] = c, [c, a] = 1, [c, b] = d, [d, a] = [d, b] = [d, c] = 1\rangle. \tag{21}$$

Claim 18. *For each $j \in \{1, 2\}$, a_j, b_j, c_j, d_j have the same relations as a, b, c, d in (21), where*

$$a_1 := (b^{i-1}a)^{-i}b^{-1}, b_1 := b^{i-1}a, c_1 := cd^{-1}, d_1 := d^{i-1},$$
$$a_2 := b^i a^{-1} b^{-i}, b_2 := b^{-1}, c_2 := cd^{-(1+i)}, d_2 := d^{-1}.$$

Proof of Claim 18. Note that group H_{10} satisfies the following (see Lemma 5 and (12)):

$$p\text{-abelian},\ Z(H_{10}) = \langle d\rangle \text{ and } i^2 - i + 1 \equiv 0 \pmod p. \tag{22}$$

Using (21) and the p-abelian property of H_{10}, we find $o(a_j) = o(b_j) = o(c_j) = o(d_j) = p$ for each $j = 1, 2$. It follows by $d_1, d_2 \in Z(H_{10}) = \langle d\rangle$ (see (22)) that $[d_j, a_j] = [d_j, b_j] =$

$[d_j, c_j] = 1$ for each $j = 1, 2$. Using (21)–(22), we obtain the following relations:

$$\begin{aligned}
[a_1, b_1] &= [(b^{i-1}a)^{-i}b^{-1}, b^{i-1}a] = [b^{-1}, b^{i-1}a] = [b^{-1}, a] = cd^{-1} = c_1;\\
[c_1, a_1] &= [cd^{-1}, (b^{i-1}a)^{-i}b^{-1}] = [c, (b^{i-1}a)^{-i}b^{-1}] = [c, b^{-1}][c, (b^{i-1}a)^{-i}]\\
&= [c, b]^{-1}[c, b^{i-1}a]^{-i} = [c, b]^{-1}[c, b]^{-i(i-1)} = [c, b]^{-(i^2-i+1)} = d^{-(i^2-i+1)} = 1;\\
[c_1, b_1] &= [cd^{-1}, b^{i-1}a] = [c, b^{i-1}a] = [c, b^{i-1}] = d^{i-1} = d_1;\\
[a_2, b_2] &= [b^i a^{-1} b^{-i}, b^{-1}] = [a^{-1}b^{-i}, b^{-1}] = cd^{-(1+i)} = c_2;\\
[c_2, a_2] &= [cd^{-(1+i)}, b^i a^{-1} b^{-i}] = [c, b^i a^{-1} b^{-i}] = [c, b^{-i}][c, a^{-1}][c, b^i] = [c, b^{-i}][c, b^i] = 1;\\
[c_2, b_2] &= [cd^{-(1+i)}, b^{-1}] = [c, b^{-1}] = d^{-1} = d_2.
\end{aligned}$$

This completes the proof of Claim 18. □

To complete the proof of Lemma 17, we will use Lemma 12 (i). We first show that given cubic bi-Cayley graph $\Gamma_{H_{10},p,i}$ is symmetric. By Claim 18, the map

$$a \mapsto a_j,\ b \mapsto b_j,\ c \mapsto c_j,\ d \mapsto d_j$$

induces an automorphism of H_{10}, say θ_j $(j = 1, 2)$, respectively. It is easy to verify that

$$\begin{aligned}
\{1, b, b^i a\}^{\theta_1} &= \{1, b^{i-1}a, b^{-1}\} = b^{-1}\{1, b, b^i a\},\\
\{1, b, b^i a\}^{\theta_2} &= \{1, b^{-1}, (b^i a)^{-1}\} = \{1, b, b^i a\}^{-1}.
\end{aligned}$$

In view of Proposition 8, $\sigma_{\theta_1,b}$ and $\delta_{\theta_2,1,1}$ are automorphisms of $\Gamma_{H_{10},p,i}$. Moreover, $\sigma_{\theta_1,b}$ fixes 1_0 and it permutates the three neighbors of 1_0 cyclically. This implies that $\mathcal{R}(H_{10}) \rtimes \langle\sigma_{\theta_1,b}\rangle$ acts regularly on $E(\Gamma_{H_{10},p,i})$. On the other hand, $\delta_{\theta_2,1,1}$ swaps 1_0 and 1_1. This implies that $\Gamma_{H_{10},p,i}$ is vertex-transitive and thus it is symmetric. Since $C_{H_{10}}(H'_{10})$ is a characteristic subgroup of H'_{10} and H'_{10} is a characteristic subgroup of H_{10}, we obtain that $C_{H_{10}}(H'_{10})$ is a characteristic subgroup of H_{10}. We find by $C_{H_{10}}(H'_{10}) = \langle a, c, d\rangle \cong \mathbb{Z}_p^3$ that $H_{10}/(C_{H_{10}}(H'_{10})) \cong \mathbb{Z}_p$ holds. Therefore, $\Gamma_{H_{10},p,i}$ is 1-arc-regular by Lemma 12 (i). This completes the proof of Lemma 17. □

4 Proof of Theorem 1

In this section, we prove the main result of this paper. We first consider the following lemma.

Lemma 19. *For a prime number $p \geqslant 7$, let Γ be a connected cubic edge-transitive graph of order $2p^4$. Then Γ is a normal edge-transitive bi-Cayley graph over H, where H is a Sylow p-subgroup of* Aut (Γ) *with order p^4. Furthermore, if $p > 7$ then Γ is a normal Cayley graph over H.*

Proof. By Proposition 10 (ii), if $p > 7$ then the result follows; and if $p = 7$ then Γ is a bi-Cayley graph over a Sylow p-subgroup H of Aut (Γ) with order p^4. To complete the proof, it is enough to show that if $p = 7$ then the bi-Cayley graph Γ is normal edge-transitive. Let Q be the maximal normal p-subgroup of $\mathrm{Aut}\,(\Gamma)$. Then $|Q| \in \{p^3, p^4\}$ holds by Proposition 10 (i). We first consider $|Q| = p^4$. As Q is a Sylow p-subgroup of $\mathrm{Aut}\,(\Gamma)$, $H = Q^\alpha = Q$ holds for some $\alpha \in \mathrm{Aut}\,(\Gamma)$ by the Sylow theorems. This shows $H \trianglelefteq \mathrm{Aut}\,(\Gamma)$ (i.e., Γ is a normal bi-Cayley graph over H). Now we consider $|Q| = p^3$. Then the quotient graph Γ_Q of Γ relative to Q is isomorphic to the Heawood graph. With the aid of Magma [1], there is a subgroup $\overline{B} := B/Q$ in $\mathrm{Aut}\,(\Gamma)/Q$ such that $\overline{B}$ acts regularly on the edge set $E(\Gamma_Q)$ satisfying $\overline{B} \cong \mathbb{Z}_p \rtimes \mathbb{Z}_3$. Let $\overline{P}$ be a Sylow p-subgroup of $\overline{B}$. Then $\overline{P} \trianglelefteq \overline{B}$ and thus $QP \trianglelefteq B$ follows by $\overline{P} = PQ/Q = QP/Q$ for some $P \leqslant \mathrm{Aut}\,(\Gamma)$. As $|QP| = p^4$, QP is a Sylow p-subgroup of $\mathrm{Aut}\,(\Gamma)$. By the Sylow theorems, we have $H = (QP)^\alpha \trianglelefteq B^\alpha$ for some $\alpha \in \mathrm{Aut}\,(\Gamma)$. Clearly, B^α acts edge-transitively on Γ and thus Γ is a normal edge-transitive bi-Cayley graph over H. □

Now we are ready to prove the main result.

Proof of Theorem 1. For a prime $p \geqslant 2$, let Γ be a connected cubic edge-transitive graph of order $2p^4$. We first consider $2 \leqslant p \leqslant 5$. By Conder [4, 5], one of the following holds:

(i) if $p = 2$, then Γ is 2-arc-regular and $\Gamma \cong$ F32 ;

(ii) if $p = 3$, then Γ is s-arc-regular and $\Gamma \cong$ F162S, where $(s, S) \in \{(1, A), (2, B), (3, C)\}$;

(iii) if $p = 5$, then Γ is s-arc-regular and $\Gamma \cong$ F1250S, where $(s, S) \in \{(2, A), (3, B)\}$.

In the rest of the proof, we assume $p \geqslant 7$. It follows by Lemma 19 that Γ is a normal edge-transitive bi-Cayley graph over H, where H is a Sylow p-subgroup of $\mathrm{Aut}\,(\Gamma)$ with order p^4. Without loss of generality, put $\Gamma := \mathrm{BiCay}(H, R, L, S)$. Since each orbit H_i $(i = 0, 1)$ has no edges of Γ, $R = L = \emptyset$ holds. By Proposition 8, we may assume $S = \{1, x, y\} \subseteq H$ and thus

$$H = \langle S \rangle = \langle x, y \rangle \quad \text{and} \quad \Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, \{1, x, y\}). \tag{23}$$

As Γ is normal edge-transitive, there exists $\alpha \in \mathrm{Aut}\,(H)$ satisfying

$$x^\alpha = x^{-1}y,\ y^\alpha = x^{-1} \text{ and } o(\alpha) = 3. \tag{24}$$

Now we divide the rest of the proof into two cases: H is abelian or not. We first show that H satisfies one of the following (iv)–(vii). It follows by [7, Proposition 5.2] and Construction I (see (11)) that

(iv) if H is abelian with $p \geqslant 7$ then $\Gamma \cong \Upsilon_{2,2,0}$ or $\Gamma \cong \Upsilon_{m,n,i}$ holds, where $(m, n) \in \{(4, 0), (3, 1)\}$ and $i \in \mathbb{Z}^*_{p^{m-n}}$ satisfying $i^2 - i + 1 \equiv 0 \pmod{p^{m-n}}$.

Now we assume that H is non-abelian with $p \geqslant 7$. By Theorem 3, $H \cong H_t$ for some $1 \leqslant t \leqslant 10$. Since H_1 and H_2 are metacyclic p-groups, $p = 3$ follows by Proposition 11. As $p \geqslant 7$, $H \ncong H_i$ $(i = 1, 2)$. Since H_3 is inner-abelian and non-metacyclic, it follows by [16, Theorem 1.1] and (12) that

(v) if $H \cong H_3$ then $\Gamma \cong \Gamma_{H_3,p,i}$ holds, where $i \in \mathbb{Z}_p^*$ satisfying $i^2 - i + 1 \equiv 0 \pmod p$.

Since H is generated by two elements by (23), $H \not\cong H_t$ $(t = 4, 5, 6)$. To consider $H \cong H_t$ $(t = 7, 8, 9, 10)$, we need the following claim.

Claim 20. *Let $H \cong H_t$ $(7 \leqslant t \leqslant 10)$ and $z := [x, y]$ (see (23)). Then the following hold.*

(a) *$z \neq 1$, $z \in H'$, $z \notin Z(H)$.*

(b) *$Z(H) = \langle \xi \rangle$ for some $\xi \in \{[z, x], [z, y]\}$ (i.e., $[z, x] \neq 1$ or $[z, y] \neq 1$ holds).*

(c) *There exists $\lambda \in \mathbb{Z}_p^*$ satisfying $[z, y] = [z, x]^{\lambda+1}$, $\lambda + 1 \not\equiv 0 \pmod p$ and $\lambda^2 + \lambda + 1 \equiv 0 \pmod p$.*

(d) *For λ in (c), let $b_1 := x^{-\lambda-1}y$. Then $o(b_1) = p$, $[z, b_1] = 1$ and $b_1 \in C_H(H')$.*

Proof of Claim 20. (a)-(b): As H is non-abelian, $z \neq 1$. By $z \in H'$, $H' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and Lemma 5, we have $o(z) = p$ and thus $z \notin Z(H)$ (i.e., $[z, x] \neq 1$ or $[z, y] \neq 1$ holds).
(c): By Claim 20(b) and Lemma 5, we may assume $[z, x] \neq 1$ and $Z(H) = \langle [z, x] \rangle$. Since $Z(H)^\alpha = Z(H)$ holds for α in (24), there exists $\lambda \in \mathbb{Z}_p^*$ satisfying $[z, x]^\alpha = [z, x]^\lambda$. If $\lambda + 1 \equiv 0 \pmod p$, then $[z, x]^\alpha = [z, x]^\lambda = [z, x]^{-1}$ and thus we find $[z, x]^{\alpha^3} = [z, x]^{-3}$. On the other hand, we obtain $[z, x]^{\alpha^3} = [z, x]$ by $o(\alpha) = 3$. This shows $[z, x]^4 = 1$, which contradicts to $p \geqslant 7$. Thus, $\lambda + 1 \not\equiv 0 \pmod p$ (i.e., $\lambda + 1 \in \mathbb{Z}_p^*$). By using $[x^{-1}, z^{-1}] \in Z(H)$ (by Lemma 5) and the following:

$$z^\alpha = [x^\alpha, y^\alpha] = [x^{-1}y, x^{-1}] = [y, x^{-1}] = [x^{-1}, z^{-1}]z \;;$$
$$[z, x]^\lambda = [z, x]^\alpha = [z^\alpha, x^{-1}y] = [[x^{-1}, z^{-1}]z, x^{-1}y] = [z, x^{-1}y] = [z, x^{-1}][z, y] = [z, x]^{-1}[z, y],$$

we find $[z, y] = [z, x]^{\lambda+1}$ and thus

$$[z, x]^{\lambda^2+\lambda} = [z, y]^\lambda = [z, y]^\alpha = [z^\alpha, y^\alpha] = [[x^{-1}, z^{-1}]z, x^{-1}] = [z, x^{-1}] = [z, x]^{-1}$$

follows. This shows $[z, x]^{\lambda^2+\lambda} = [z, x]^{-1}$ and we obtain $\lambda^2 + \lambda + 1 \equiv 0 \pmod p$ by $o([z, x]) = p$. Moreover, we obtain $[z, y] \neq 1$ by $[z, x] \neq 1$ and $\lambda + 1 \not\equiv 0 \pmod p$.
(d): Since $y^p = x^{p(\lambda+1)}$ and H is p-abelian, we have $b_1^p = 1$. Note that $[x, b_1] = [x, y] = z$ and $[z, b_1] = [z, x]^{-\lambda-1}[z, y] = [z, x]^{-\lambda-1}[z, x]^{\lambda+1} = 1$. It follows by $H' = \langle z, [z, x] \rangle$ and $[z, x] \in Z(H)$ that $b_1 \in C_H(H')$. This completes the proof of Claim 20. □

We first assume the case $H \cong H_t$ $(t = 7, 8)$. Then

$$H' = \langle a^{ip},\ c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p,\ C_H(H') = \langle a \rangle \times \langle c \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \text{ and } b_1 \in \langle a^{ip}, c \rangle \leqslant \Phi(H)$$

all hold, where $i = 1$ for $t = 7$ and $i = \nu$ for $t = 8$ (see (1) and Claim 20(d)). However, this is impossible as $\Phi(H)$ is the Frattini subgroup of H, where $H = \langle b_1, x \rangle$. Hence, $H \not\cong H_t$ $(t = 7, 8)$.

Now, we consider the case $H \cong H_9$. In view of Claim 20(b) and Lemma 5, we may assume $\langle [z, x] \rangle = \langle x^p \rangle = Z(H)$. Take $k, n \in \mathbb{Z}_p^*$ satisfying $kn \equiv 1 \pmod p$ and

$[z,x]=x^{kp}$. For $a_2:=x$, $b_2:=b_1^n$ and $c_2:=z^n$, we can check by using Claim 20(d) and (1) that they have the same relations as a,b and c in H_9 as follows:

$$\begin{aligned} a_2^{p^2}&=b_2^p=c_2^p=1\ ;\\ [a_2,b_2]&=[x,b_1^n]=[x,b_1]^n=[x,y]^n=z^n=c_2\ ;\\ [c_2,a_2]&=[z^n,x]=[z,x]^n=x^{knp}=x^p=a_2^p\ ;\\ [c_2,b_2]&=[z^n,b_1^n]=1\ .\end{aligned}$$

So, there exists $\beta\in\mathrm{Aut}\,(H)$ satisfying $a_2^\beta=a$ and $b_2^\beta=b$. As $x=a_2$ and $y=x^{\lambda+1}b_1=x^{\lambda+1}b_2^k$,

$$\{1,x,y\}^\beta=\{1,a,a^{\lambda+1}b^k\}$$

holds. This shows $\Gamma\cong\Gamma^\beta=\mathrm{BiCay}(H,\emptyset,\emptyset,\{1,a,a^{\lambda+1}b^k\})$ by Proposition 8. Using Claim 20(c) with $i=\lambda+1$, we find $i\in\mathbb{Z}_p^*$ satisfying $i^2-i+1\equiv 0 \pmod p$. Now we obtain the following:

(vi) If $H\cong H_9$ then $\Gamma\cong\Gamma_{H_9,p,i,k}$ (see Construction II).

Finally, we consider the case $H\cong H_{10}$. In view of Claim 20(b) and Lemma 5, we may assume $Z(H)=\langle[z,x]\rangle$, where $[z,x]\neq 1$. Put $a_2:=x^{-\lambda-1}y, b_2:=x, c_2:=[a_2,b_2]$ and $d_2:=[c_2,b_2]$. Using

$$c_2=[a_2,b_2]=[x^{-\lambda-1}y,x]=[y,x]=z^{-1}\neq 1 \tag{25}$$

and $[z,x]\neq 1$, we find $d_2=[c_2,b_2]=[z^{-1},x]=[[y,x],x]\neq 1$. By Lemma 5,

$$\langle d_2\rangle=Z(H) \tag{26}$$

holds. Since H is p-abelian with $\exp(H)=p$ and all a_2,b_2,c_2,d_2 are non-identity,

$$a_2^p=b_2^p=c_2^p=d_2^p=1\ .$$

Moreover, we can check that they have the same relations as a,b,c and d in H_{10} as follows (see also (1)):

$$[c_2,a_2]=[z^{-1},x^{-\lambda-1}y]=[z,x^{-\lambda-1}y]^{-1}=([z,x]^{-\lambda-1}[z,y])^{-1}=1$$
$$\text{(by (25), Claim 20(c) and Lemma 5);}$$
$$[d_2,a_2]=[d_2,b_2]=[d_2,c_2]=1\quad\text{(by (26)).}$$

So, there exists $\beta\in\mathrm{Aut}\,(H)$ satisfying $a_2^\beta=a$ and $b_2^\beta=b$. By $x^\beta=b_2^\beta=b$, $y=b_2^{\lambda+1}a_2$ and $y^\beta=b^{\lambda+1}a$,

$$\{1,x,y\}^\beta=\{1,b,b^{\lambda+1}a\}$$

holds. This shows $\Gamma\cong\Gamma^\beta=\mathrm{BiCay}(H,\emptyset,\emptyset,\{1,b,b^{\lambda+1}a\})$ by Proposition 8. By Claim 20(c) with $i=\lambda+1$, we find $i\in\mathbb{Z}_p^*$ satisfying $i^2-i+1\equiv 0 \pmod p$. Now we obtain the following:

(vii) If $H\cong H_{10}$ then $\Gamma\cong\Gamma_{H_{10},p,i}$ (see Construction II).

Now the result follows by (i)–(vii) and Lemmas 13–17. This completes the proof of Theorem 1. □

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