A Linear Lower Bound for the Square Energy of Graphs

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Abstract

Let G be a graph of order n with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. Let

$$s^+(G) = \sum_{\lambda_i > 0} \lambda_i^2, \qquad s^-(G) = \sum_{\lambda_i < 0} \lambda_i^2.$$

The smaller value, $s(G) = \min\{s^+(G), s^-(G)\}$ is called the *square energy* of G. In 2016, Elphick, Farber, Goldberg, and Wocjan conjectured that for every connected graph G of order n, $s(G) \geqslant n - 1$. No linear bound for $s(G)$ in terms of n is known. Let $H_1, \ldots, H_k$ be disjoint induced subgraphs of G. In this note, we prove that

$$s^+(G) \geqslant \sum_{i=1}^{k} s^+(H_i) \quad \text{and} \quad s^-(G) \geqslant \sum_{i=1}^{k} s^-(H_i),$$

and then use this result to prove that $s(G) \geqslant \frac{3n}{4}$ for every connected graph G of order $n \geqslant 4$.

Mathematics Subject Classifications: 05C50

1 Introduction

We use standard graph theory notation throughout the paper. All graphs are simple, i.e. with no loops or multiple edges. Let $G = (V, E)$ be a graph of order n and size m. The *adjacency matrix* of G is an $n \times n$ matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. The *eigenvalues* of G are the eigenvalues of

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$A(G)$. Since $A(G)$ is a real symmetric matrix, all eigenvalues of $A(G)$ are real and can be listed as

$$\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G).$$

Let $\mathcal{B} = \{x_1, \ldots, x_n\}$ be an orthonormal basis for $\mathbb{R}^n$ that contains the eigenvectors of $A(G)$, where x_i is the eigenvector corresponding to the eigenvalue λ_i for $i = 1, \ldots, n$. By the spectral decomposition (see [6, Theorem 4.1.5]), we have $A(G) = \sum_{i=1}^{n} \lambda_i x_i x_i^T$. Define

$$A_+ = \sum_{\lambda_i > 0} \lambda_i x_i x_i^T, \; A_- = -\sum_{\lambda_i < 0} \lambda_i x_i x_i^T.$$

Both A_+ and A_- are positive semidefinite matrices, and the following equalities hold:

$$A_+ A_- = A_- A_+ = 0, \quad A(G) = A_+ - A_-.$$

The *energy* of a graph G, $\mathcal{E}(G)$, is defined to be the sum of absolute values of all eigenvalues of G, i.e.

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

Define

$$s^+(G) = \sum_{\lambda_i > 0} \lambda_i^2(G) \quad \text{and} \quad s^-(G) = \sum_{\lambda_i < 0} \lambda_i^2(G).$$

The parameters $s^+(G)$ and $s^-(G)$ are called the *positive square energy* and the *negative square energy* of G, respectively. Define $s(G) = \min\{s^+(G), s^-(G)\}$ and call it *square energy* of G. Clearly, $s^+(G) = \operatorname{tr}((A_+)^2)$ and $s^-(G) = \operatorname{tr}((A_-)^2)$.

Based on the fact that $s(G) = |E(G)|$ for every bipartite graph, Elphick, Farber, Goldberg and Wocjan [4] proposed the following conjecture.

Conjecture 1 ([4])**.** For every connected graph G of order n,

$$s(G) \geqslant n - 1.$$

The above conjecture has been verified for several graph classes, including all regular graphs, but the general case is wide open (see [5, 1, 7] for partial results). The best known general lower bound for $s(G)$ is $\sqrt{n}$ as observed by Elphick and Linz [5], and is a consequence of a result on the chromatic number of graphs by Ando and Lin [3]. In this paper, our main result is a linear lower bound for the square energy of connected graphs. In particular, we show that for any connected graph G of order $n \geqslant 4$, $s(G) \geqslant \frac{3n}{4}$.

We believe that the above bound can be improved to $\frac{4n}{5}$ using a more intricate partitioning of the graph G and applying Theorem 3. We avoid doing this and content ourselves with the slightly weaker $\frac{3n}{4}$ bound because we believe that more ideas are needed to resolve Conjecture 1.

2 Main Results

An important result concerning the energy of graphs is the following.

Theorem 2 ([2]). *Let $H_1, \ldots, H_k$ be disjoint induced subgraphs of a graph G. Then*

$$\mathcal{E}(G) \geqslant \sum_{i=1}^{k} \mathcal{E}(H_i).$$

We prove a similar result for the square energy of graphs.

Theorem 3. *Let $H_1, \ldots, H_k$ be disjoint induced subgraphs of a graph G. Then*

$$s^+(G) \geqslant \sum_{i=1}^{k} s^+(H_i) \quad \text{and} \quad s^-(G) \geqslant \sum_{i=1}^{k} s^-(H_i).$$

Equality holds in both simultaneously if and only if G is the disjoint union of $H_1, \ldots, H_k$.

Proof. It is sufficient to prove the assertion when G is partitioned into two disjoint induced subgraphs H_1 and H_2. So, let $A(G) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} and A_{22} are the adjacency matrices of the induced subgraphs H_1 and H_2, respectively. We show that

$$s^+(G) \geqslant s^+(H_1) + s^+(H_2).$$

If we apply this inequality to $-A$, we get the second inequality.

Let $A_+ = [B_{ij}]$ and $A_- = [C_{ij}]$, $1 \leqslant i, j \leqslant 2$, partitioned conformally as $A(G)$. We have $A_{ii} = B_{ii} - C_{ii}$, for $i = 1, 2$. Since A_+ and A_- are positive semidefinite matrices, both B_{ii} and C_{ii} are also positive semidefinite for $i = 1, 2$ (see [6, Theorem 7.7.7]). Now, we have

$$s^+(G) = \operatorname{tr}((A_+)^2) = \operatorname{tr}(B_{11}^2) + \operatorname{tr}(B_{22}^2) + 2\operatorname{tr}(B_{12}B_{12}^T).$$

Since $B_{12}B_{12}^T$ is a positive semidefinite matrix, we have

$$\operatorname{tr}(B_{12}B_{12}^T) \geqslant 0.$$

Since $B_{ii} = A_{ii} + C_{ii}$ and C_{ii} is a positive semidefinite matrix, $\lambda_r(B_{ii}) \geqslant \lambda_r(A_{ii})$ for $1 \leqslant r \leqslant p_i$, where p_i is the number of positive eigenvalues of A_{ii} for $i = 1, 2$. This implies

$$s^+(G) \geqslant \operatorname{tr}(B_{11}^2) + \operatorname{tr}(B_{22}^2) \geqslant s^+(H_1) + s^+(H_2).$$

Note that if equality holds simultaneously, then $\operatorname{tr}(B_{12}B_{12}^T) = 0 = \operatorname{tr}(C_{12}C_{12}^T)$ which implies $B_{12} = 0 = C_{12}$ and so $B_{21} = B_{12}^T = 0 = C_{12}^T = C_{21}$. Hence, H_1 and H_2 are disjoint. Conversely, if G is the disjoint union of H_1 and H_2, the equality is clearly satisfied. The proof is complete. □

We recall the following well-known fact (cf. [6, Theorem 4.3.17]).

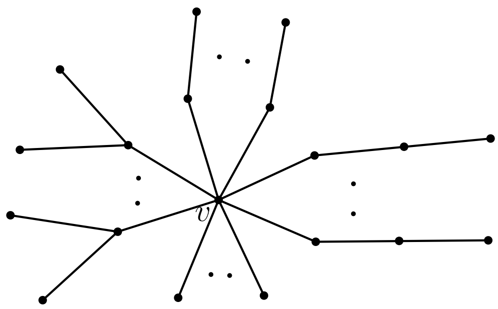

Figure 1: Spanning tree T

Theorem 4 (Interlacing Theorem)**.** *Let A be a real symmetric matrix of order n. Let B be a principal submatrix of A of order $n-1$. Then, for $1 \leqslant i \leqslant n-1$,*

$$\lambda_i(A) \geqslant \lambda_i(B) \geqslant \lambda_{i+1}(A).$$

We now give a linear lower bound for the square energy of connected graphs using Theorem 3.

Theorem 5. *For any connected graph G of order $n \geqslant 4$,*

$$s(G) \geqslant \frac{3n}{4}.$$

Proof. For $n \leqslant 10$, one can use a computer to verify the stronger claim that $s(G) \geqslant n-1$. So, assume $n \geqslant 11$. We proceed by induction on n.

The assertion is true if G is a bipartite graph or a cycle (see [1]). So, assume G is not bipartite and has maximum degree $\Delta \geqslant 3$. Let T be a spanning tree of G rooted at a vertex v, where $\deg_T(v) = \Delta$. If $\Delta = 3$, we can find an edge e in T such that $T - e$ has two components T_1 and T_2, both of order at least 4 since $n \geqslant 11$. Using Theorem 3 and the induction hypothesis,

$$s(G) \geqslant s(G[V(T_1)]) + s(G[V(T_2)]) \geqslant \frac{3|V(T_1)|}{4} + \frac{3|V(T_2)|}{4} = \frac{3n}{4}.$$

Now, let $\Delta \geqslant 4$. Suppose $T - v$ has a component C of order at least 4. Since C and $T - V(C)$ are connected and have order at least 4, we are done by the induction hypothesis. So, we may assume that every component of $T - v$ has at most 3 vertices. Hence, T is a tree, as shown in Figure 1.

First, suppose that $G - v$ has a subgraph $H \cong P_4$. Let e be an edge in H. Suppose $T - v + e$ has a component C of order at least 4. Since $|V(C)| \leqslant 6$, $G - V(C)$ is connected and has order at least 5. Thus, by the induction hypothesis and Theorem 3, $s(G) \geqslant \frac{3n}{4}$. So we may assume that every component of $T - v + e$ has order at most 3. This implies that the component (say C) of the graph $T - v + E(H)$ (i.e., the graph $T - v$ with extra edges $E(H)$), which contains H, has order at most 6. Again, applying the induction hypothesis to C and $G - V(C)$ gives $s(G) \geqslant \frac{3n}{4}$.

Now, consider the case that $G-v$ has no P_4 as a subgraph. Then $G-v$ is the disjoint union of some K_3, P_3, K_2 and K_1. Let the number of K_3, P_3, K_2 and K_1 in $G-v$ be ℓ_1, ℓ_2, ℓ_3 and ℓ_4, respectively. Then $n = 3\ell_1 + 3\ell_2 + 2\ell_3 + \ell_4 + 1$. Moreover, the spectrum of $G-v$ is

$$\{2^{(\ell_1)}, \sqrt{2}^{(\ell_2)}, 1^{(\ell_3)}, 0^{(\ell_2+\ell_4)}, (-1)^{(2\ell_1+\ell_3)}, (-\sqrt{2})^{(\ell_2)}\}.$$

Note that G is not bipartite, so it contains an odd cycle. Since $G-v$ does not have P_4 as a subgraph, all odd cycles in G are triangles.

Now, if $n = 11$, then we can find an edge uw (where $u, w \neq v$) in G such that $G-u-w$ is connected. Then, using the stronger claim for $n \leqslant 10$, we have

$$s(G) \geqslant s(K_2) + s(G-u-w) \geqslant n-2 \geqslant \frac{3n}{4}.$$

Now assume $n \geqslant 12$. Note that $\min\{\lambda_1(G), |\lambda_n(G)|\} \geqslant \sqrt{\ell_1+\ell_2+\ell_3+\ell_4}$ since G has an induced star of order $\ell_1+\ell_2+\ell_3+\ell_4+1$. Using the Interlacing Theorem,

$$\begin{aligned} s^-(G) &\geqslant \lambda_n^2(G) + s^-(G-v) - \lambda_{n-1}^2(G-v) \\ &\geqslant (\ell_1+\ell_2+\ell_3+\ell_4) + (2\ell_1+\ell_3+2\ell_2) - 2 \\ &\geqslant n-3 \geqslant \frac{3n}{4}. \end{aligned}$$

This proves the assertion for $s^-(G)$.

Now, if $\ell_1 \geqslant 1$, then $\lambda_1(G-v) = 2$. Using the Interlacing Theorem,

$$\begin{aligned} s^+(G) &\geqslant \lambda_1^2(G) + s^+(G-v) - \lambda_1^2(G-v) \\ &\geqslant (\ell_1+\ell_2+\ell_3+\ell_4) + (4\ell_1+2\ell_2+\ell_3) - 4 \\ &= n-3+(2\ell_1-2) \geqslant n-3 \geqslant \frac{3n}{4}. \end{aligned}$$

On the other hand, if $\ell_1 = 0$, then $\lambda_1(G-v) \leqslant \sqrt{2}$. Again,

$$\begin{aligned} s^+(G) &\geqslant \lambda_1^2(G) + s^+(G-v) - \lambda_1^2(G-v) \\ &\geqslant (\ell_2+\ell_3+\ell_4) + (2\ell_2+\ell_3) - 2 \\ &= n-3 \geqslant \frac{3n}{4}. \end{aligned}$$

This proves the assertion for $s^+(G)$, and the proof is complete. □

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