Pattern-avoiding modified ascent sequences

Giulio Cerbai

Abstract

We initiate an in-depth study of pattern avoidance on modified ascent sequences. Our main technique consists in using Stanley's standardization to obtain a transport theorem between primitive modified ascent sequences and permutations avoiding a bivincular pattern of length three. We enumerate some patterns via bijections with other combinatorial structures such as Fishburn permutations, lattice paths and set partitions. We settle the last remaining case of a conjecture by Duncan and Steingrímsson by proving that modified ascent sequences avoiding 2321 are counted by the Bell numbers.

Mathematics Subject Classifications: 05A19, 05A05

1 Introduction

Modified ascent sequences have recently assumed a central role in the study of Fishburn structures. Originally [5], they were defined as the bijective image of (plain) ascent sequences under a certain hat map, with the primary role of making their relation with $(\mathbf{2}+\mathbf{2})$-free posets more transparent. More recently, Claesson and the current author [11] introduced the Burge transpose to develop a theory of transport of patterns between modified ascent sequences and Fishburn permutations, defined as those avoiding a certain bivincular pattern of length three. They also characterized modified ascent sequences as Cayley permutations where each entry is a leftmost copy if and only if it sits at an ascent top (see also Proposition 1). This alternative description—not relying on the hat map—opened the door for a study of modified ascent sequences as independent objects, under both a geometrical and enumerative perspective. Ultimately, it led to the introduction by the same authors of Fishburn trees [12]. This class of binary, labeled trees originates from the max-decomposition of modified ascent sequences. Conversely, modified ascent sequences are obtained by reading the labels of Fishburn trees with the in-order traversal. The relation between Fishburn trees and other Fishburn structures, namely Fishburn matrices and $(\mathbf{2}+\mathbf{2})$-free posets, is extremely transparent. For instance, Fishburn matrices arise by decomposing a Fishburn tree with respect to its maximal

Department of Mathematics, University of Iceland, Reykjavik, Iceland (`giulio@hi.is`).

right-paths. The reader who is interested in the state of the art on Fishburn structures is referred to the same paper [12].

Motivated by all the above reasons, we conduct a more systematic study of pattern avoidance on modified ascent sequences, using a variety of combinatorial tools and methods. Our investigation is parallel to the one by Duncan and Steingrímsson on plain ascent sequences [20]. Given a pattern y, our goal is to "solve" it by counting the number of modified ascent sequences of given length that avoid y. Here, to count means to obtain an explicit formula, when possible, a generating function, or a bijection with another combinatorial structure whose enumeration is known. An overview of our results can be found in Table 1. Our main technique relies on what could be merely regarded as a "trick"—one that is unexpectedly effective in practical terms. Namely, we study primitive ascent sequences first, defined in Section 2.2 as those with no pairs of consecutive equal entries. We show that Stanley's standardization [28] maps bijectively primitive modified ascent sequences to the set Ω of permutations that start with 1 and avoid the bivincular pattern ω, defined in Section 3. As a result, we obtain in Theorem 11 a mechanism to transport patterns between primitive modified ascent sequences and Ω. The main advantage of this approach is that it often allows us to work with permutations, a task that is much easier due to the arsenal of tools at our disposal. Finally, as showed in Proposition 2, by applying a simple binomial transform to the counting sequence of primitive words we immediately obtain the enumeration of the general case.

Let us end this preamble with a more detailed presentation of our paper.

In Section 2, we give a short introduction to permutation patterns and define (primitive) modified ascent sequences. Then, we prove in Proposition 2 that if y is a primitive pattern, then modified ascent sequences avoiding y are counted by a binomial transform of their primitive counterpart.

In Section 3, we recall the definition of Stanley's standardization and prove some related properties. The main result of this section, Theorem 11, is the theorem of transport between Ω and the set of primitive modified ascent sequences mentioned previously.

In Section 4, we enumerate modified ascent sequences avoiding any pattern of length two, as well as a couple of simple patterns of length three. We give a bijection between modified ascent sequences avoiding 122 and set partitions whose minima of blocks form an interval, computing some related generating functions in the process.

In Section 5, we solve several primitive patterns with the machinery of Proposition 2 and Theorem 11. The hardest one is 312, which we settle by showing a bijection with Dyck paths avoiding the consecutive subpath `dudu`. Our construction is based on a geometric decomposition of Dyck paths that leads to a generating function first discovered by Sapounakis, Tasoulas and Tsikouras [25].

In Section 6, we slightly tweak Proposition 2 to solve the pattern 221, which is not primitive. En passant, we prove in Proposition 23 that modified ascent sequences avoiding 2321 are enumerated by the Bell numbers, settling the last remaining case of a conjecture first proposed by Duncan and Steingrímsson [20] and solved only partially by the current author [10].

In Section 7, we provide some data for the unsolved patterns and leave some sugges-

y	$|\hat{A}_n(y)|$	$|\hat{A}_n^{\mathrm{pr}}(y)|$	Reference
11	$1, 1, 1, \ldots$	$1, 1, 1, \ldots$	Section 4.1
12	$1, 1, 1, \ldots$	$1, 0, 0, \ldots$	Section 4.1
21,121	2^{n-1}	$1, 1, 1, \ldots$	Section 4.1
112	2^{n-1}	Fibonacci	Section 4.2
122	A026898	A229046	Section 4.3
123	2^{n-1}	$1, 1, 1, \ldots$	Section 5.3
132	Odd Fibonacci	Fibonacci	Section 5.4
212,1212	Bell	Bell (shifted)	[10]
213,1213	Catalan	Motzkin	Sections 5.2, 5.6
221	New	Bell (shifted)	Section 6
231	Catalan	Motzkin	Section 5.2
312,1312	New	A102407	Sections 5.5, 5.6
321	A007317	Catalan	Section 5.2
1123	Catalan	A082582?	[11]
1232	A047970	A229046	Section 5.1
1234	Catalan	Motzkin	Section 5.3
2132	Bell	Bell (shifted)	[10]
2213	Bell	?	[10]
2231	Bell	?	[10]
2321	Bell	Bell (shifted)	Section 6

Table 1: Enumeration of modified ascent sequences avoiding a single pattern y. The counting sequences start from $n = 1$. Patterns in the same row determine the same set of sequences, while a question mark denotes numerical data that we were not able to confirm.

tions for future work.

2 Preliminaries

Given a natural number $n \geqslant 0$, let $[n] = \{1, 2, \ldots, n\}$. An *endofunction* of size n is a map $x : [n] \to [n]$. We shall identify x with the word $x = x_1 \ldots x_n$, where $x_i = x(i)$ for each $i \in [n]$. When $n = 0$, we identify the empty endofunction with the empty word. A *Cayley permutation* [8, 23] is an endofunction $x : [n] \to [n]$ whose image is $\mathrm{Im}(x) = [k]$, for some $k \leqslant n$. In other words, an endofunction x is a Cayley permutation if it contains at least a copy of every integer between 1 and its maximum value. For the rest of this paper, if A is a set whose elements are equipped with a notion of size, we will denote with A_n the set of elements in A of size n. Conversely, given a definition of A_n (of elements of size n) we assume $A = \cup_{n \geqslant 0} A_n$. As an example, we define the set of Cayley permutations of size n as Cay_n and let $\mathrm{Cay} = \cup_{n \geqslant 0} \mathrm{Cay}_n$. A Cayley permutation $x = x_1 \cdots x_n$ with $\max(x) = k$ encodes the ordered set partition $B_1 \ldots B_k$, where $i \in B_{x_i}$. The map defined this way is bijective, and for this reason Cayley permutations are counted by the Fubini numbers (listed as sequence A000670 in the OEIS [27]).

A *left-to-right minimum* (briefly, lrmin) of $x = x_1 \cdots x_n$ is a pair (i, x_i) such that $x_i < \min(x_1 \cdots x_{i-1})$. If we replace the strict inequality with a weak one, i.e. if $x_i \leqslant \min(x_1 \cdots x_{i-1})$, then (i, x_i) is said to be a *weak left-to-right minimum* (briefly, wlrmin). We denote the set of lrmin and wlrmin of x respectively by $\mathrm{lrmin}(x)$ and $\mathrm{wlrmin}(x)$. Left-to-right maxima, right-to-left minima and maxima, as well as their weak counterparts, are defined analogously. When there is no ambiguity, we omit the index i from the pair (i, x_i). For instance, we sometimes write $\mathrm{wrlmax}(x) = \{x_i : x_i \geqslant x_j$ for each $j > i\}$.

An comprehensive introduction to permutation patterns can be found in the book by Kitaev [22]. Bevan's note [4] contains a brief presentation of the most used notions and definitions in the permutation patterns field. Below, we quickly recall those that are necessary in this paper.

Let $x \in \mathrm{Cay}_n$ and $y \in \mathrm{Cay}_k$ be two Cayley permutations, with $k \leqslant n$. We say that x *contains* y if there is a subsequence $x_{i_1} x_{i_2} \cdots x_{i_k}$, with $i_1 < i_2 < \cdots < i_k$, that is *order isomorphic* to y. Here, order isomorphic means that $x_{i_s} < x_{i_t}$ if and only if $y_s < y_t$, and $x_{i_s} = x_{i_t}$ if and only if $y_s = y_t$. In this case, we write $x \geqslant y$ and $x_{i_1} x_{i_2} \cdots x_{i_k} \simeq y$; further, the subsequence $x_{i_1} x_{i_2} \cdots x_{i_k}$ is an *occurrence* of the *pattern* y in x. If no subsequence of x is order isomorphic to y, we say that x *avoids* y. Given a pattern y, we let $\mathrm{Cay}(y)$ be the set of Cayley permutations that avoid y. More in general, when B is a set of patterns, $\mathrm{Cay}(B)$ shall denote the set of Cayley permutations avoiding every pattern in B. We use analogous notations for subsets of Cay, as well as for other types of pattern. For instance, $\hat{A}(112)$ denotes the set of modified ascent sequences (defined in Section 2.1) avoiding the pattern 112. The set of *permutations* (i.e. bijective endofunctions) is defined via pattern avoidance as $\mathrm{Sym} = \mathrm{Cay}(11)$.

Classical patterns are generalized by mesh patterns and Cayley-mesh patterns. A *mesh pattern* [15] is a pair (y, R), where $y \in \mathrm{Sym}_k$ is a permutation (classical pattern) and $R \subseteq [0, k] \times [0, k]$ is a set of pairs of integers. The pairs in R identify the lower left
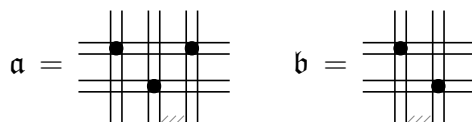
Figure 1: Cayley-mesh patterns such that $\hat{A} = \text{Cay}(\mathfrak{a}, \mathfrak{b})$.

corners of unit squares in the plot of x which specify forbidden regions. An occurrence of the mesh pattern (y, R) in the permutation x is an occurrence of the classical pattern y such that no other points of x occur in the forbidden regions specified by R. By allowing additional regions for repeated entries, we arrive at *Cayley-mesh patterns* [9]; that is, mesh patterns on Cayley permutations. To ease notation, we often define a (Cayley-)mesh pattern (y, R) by simply plotting the underlying classical pattern y, with the forbidden regions determined by R shaded. An interesting example is the following. Claesson and the current author [11] characterized the set $\hat{A}$ of *modified ascent sequences* as $\hat{A} = \text{Cay}(\mathfrak{a}, \mathfrak{b})$, where $\mathfrak{a}$ and $\mathfrak{b}$ are defined by Figure 1.

2.1 Modified ascent sequences

Recall from the end of the previous section that the set $\hat{A}$ of modified ascent sequences is $\hat{A} = \text{Cay}(\mathfrak{a}, \mathfrak{b})$, where $\mathfrak{a}$ and $\mathfrak{b}$ are depicted in Figure 1. Let us point out that this definition departs slightly from the original one [5]: our sequences are 1-based instead of being 0-based. Below we recall two useful alternative definitions of $\hat{A}$. Given a Cayley permutation x of length n, let

$$\text{top}(x) = \{(1, x_1)\} \cup \{(i, x_i) : 1 < i \leqslant n, \ x_{i-1} < x_i\}$$

be the set of *ascent tops* and their indices, including the first element; further, let

$$\text{nub}(x) = \{(\min x^{-1}(j), j) : 1 \leqslant j \leqslant \max(x)\}$$

be the set of *leftmost copies* and their indices. When there is no ambiguity, we will sometimes abuse notation and simply write $x_i \in \text{nub}(x)$ or $x_i \in \text{top}(x)$. If $x_i \in \text{nub}(x)$ and $x_i = a$, we say that x_i is the leftmost copy of a in x; or, that x_i is a leftmost copy in x. It is easy to see [11] that x avoids $\mathfrak{a}$ if and only if $\text{top}(x) \subseteq \text{nub}(x)$; similarly, x avoids $\mathfrak{b}$ if and only if $\text{top}(x) \supseteq \text{nub}(x)$. The next proposition, which will be repeatedly used throughout the whole paper, follows immediately.

Proposition 1. *We have*

$$\hat{A} = \{x \in \text{Cay} : \text{top}(x) = \text{nub}(x)\}.$$

In particular, in a modified ascent sequence x all the ascent tops have distinct values and $\max(x) = |\text{top}(x)|$. Furthermore, all the copies of $\max(x)$ are in consecutive positions.

Finally, a recursive definition of $\hat{A}$ goes as follows [11]. There is exactly one modified ascent sequence of length zero and one, the empty word and the single letter word 1, respectively. For $n \geqslant 1$, every $y \in \hat{A}_{n+1}$ is of one of two forms depending on whether the last letter forms an ascent with the penultimate letter:

- $y = x_1 \cdots x_n \, x_{n+1}$, with $1 \leqslant x_{n+1} \leqslant x_n$, or

- $y = \tilde{x}_1 \cdots \tilde{x}_n \, x_{n+1}$, with $x_n < x_{n+1} \leqslant 1 + \max(x_1 \cdots x_n)$,

where $x_1 \cdots x_n \in \hat{A}_n$ and, for $i \in [n]$,

$$\tilde{x}_i = \begin{cases} x_i & \text{if } x_i < x_{n+1} \\ x_i + 1 & \text{if } x_i \geqslant x_{n+1}. \end{cases}$$

Less formally, each modified ascent sequence x gives rise to $\max(x) + 1$ modified ascent sequences of length one more. These are obtained by first inserting a new rightmost entry that is less than or equal to $\max(x) + 1$; and, secondly, if the newly added entry a is an ascent top, by increasing by one all the previous entries that are greater than or equal to a.

We wrap up this section with a remark. One of the main benefits of working with modified ascent sequences is that they are Cayley permutations. This is not the case of (plain) ascent sequences, where the presence of gaps makes the study of patterns arguably less natural. A rather awkward example is the following: there are two ascent sequences of length five, namely 12123 and 12124, that contain the length five pattern 12123.

2.2 Primitive sequences

A *flat step* in a modified ascent sequence $x = x_1 \cdots x_n$ consists of two consecutive equal entries $x_i = x_{i+1}$. A modified ascent sequence is *primitive* [18] if it has no flat steps, and we let $\hat{A}^{\mathrm{pr}}$ denote the set of primitive modified ascent sequences. In the realm of Fishburn structures, primitive (modified) ascent sequences are in bijection with binary Fishburn matrices [19], ($\mathbf{2}+\mathbf{2}$)-free posets with no indistinguishable elements [18], strictly-decreasing Fishburn trees [12], and Fishburn permutations avoiding a bivincular pattern of length two. It is well known [18, Prop. 8] that any ascent sequence is uniquely obtained from a primitive ascent sequence by inserting flat steps in a suitable way. Clearly, e.g. by Proposition 1, the same property holds for modified ascent sequences too. For instance, the sequence

$$x = 1 \, \underline{11} \, 312 \, \underline{222} \, 421 \, \underline{1} \in \hat{A} \quad \text{arises from} \quad w = 1312421 \in \hat{A}^{\mathrm{pr}}$$

by inserting the underlined flat steps. More interestingly, if y is a primitive pattern and $w \in \hat{A}(y)$, then the insertion of flat steps in w does not create any occurrence of y. We state the enumerative consequences of this simple observation in the following proposition.

Proposition 2. *Let $y \in \hat{A}^{\mathrm{pr}}$. Then, for $n \geqslant 1$,*

$$|\hat{A}_n(y)| = \sum_{k=1}^{n} \binom{n-1}{k-1} |\hat{A}_k^{\mathrm{pr}}(y)|. \tag{1}$$

Proof. Any $x \in \hat{A}_n(y)$ is obtained uniquely from some $w \in \hat{A}_k^{\mathrm{pr}}(y)$, with $1 \leqslant k \leqslant n$, by inserting $n - k$ flat steps. Note that w is obtained by collapsing all the consecutive flat steps of x to a single entry. Since $x_1 = 1$ is fixed, there are $\binom{n-1}{k-1}$ positions where the remaining $k - 1$ entries of w can be placed. $\qquad\square$

A consequence of Proposition 2 is that enumerating $\hat{A}^{\mathrm{pr}}(y)$ is sufficient in order to count the whole set $\hat{A}(y)$, when y is a primitive pattern. We can also rephrase this result in terms of generating functions. From now on, given a pattern y, we let

$$\hat{A}_y(t) = \sum_{n \geqslant 0} |\hat{A}_n(y)| t^n \quad \text{and} \quad \hat{A}_y^{\mathrm{pr}}(t) = \sum_{n \geqslant 0} |\hat{A}_n^{\mathrm{pr}}(y)| t^n$$

be the OGF (ordinary generating functions) of $\hat{A}(y)$ and $\hat{A}^{\mathrm{pr}}(y)$, respectively. It is well known (see for instance Bernstein and Sloane [3]) that if

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad \text{then} \quad B(t) = \frac{1}{1-t} A\left(\frac{t}{1-t}\right),$$

where $A(t) = \sum_{n \geqslant 0} a_n t^n$ and $B(t) = \sum_{n \geqslant 0} b_n t^n$. By Proposition 2, keeping track of the shift,

$$\frac{\hat{A}_y(t) - 1}{t} = \frac{1}{1-t} \left[\frac{\hat{A}_y^{\mathrm{pr}}(s) - 1}{s}\right]_{|s=\frac{t}{1-t}}$$

$$\iff \hat{A}_y(t) = 1 + \frac{t}{1-t} \left[\frac{\hat{A}_y^{\mathrm{pr}}(s) - 1}{s}\right]_{|s=\frac{t}{1-t}} \tag{2}$$

$$= \hat{A}_y^{\mathrm{pr}}\left(\frac{t}{1-t}\right).$$

We end this section with a simple lemma.

Lemma 3. *If x is a primitive modified ascent sequence, then*

$$\mathrm{wlrmax}(x) = \mathrm{lrmax}(x) \quad \text{and} \quad \mathrm{wrlmax}(x) = \mathrm{rlmax}(x).$$

Furthermore, $\mathrm{lrmin}(x) = \{x_1\}$.

Proof. It is clear that $\mathrm{lrmin}(x) = \{x_1\}$ since $x_1 = 1$. The inclusions $\mathrm{wlrmax}(x) \supseteq \mathrm{lrmax}(x)$ and $\mathrm{wrlmax}(x) \supseteq \mathrm{rlmax}(x)$ are trivial. Let $x_i \in \mathrm{wlrmax}(x)$. Since x is primitive, we have $x_{i-1} < x_i$. Thus $x_i \in \mathrm{top}(x) = \mathrm{nub}(x)$ and x_i is a strict left-to-right maximum. We have thus proved that $\mathrm{wlrmax}(x) = \mathrm{lrmax}(x)$. Next, let $x_i \in \mathrm{wrlmax}(x)$. For a contradiction, suppose that $x_i \notin \mathrm{rlmax}(x)$; that is, there is some $x_j = x_i$, with $j > i$. Note that $x_j \in \mathrm{wrlmax}(x)$. Further, it must be $j > i + 1$ since x is primitive. Now, consider the entry x_{j-1} preceding x_j. If $x_{j-1} < x_j$, then we have a contradiction with the fact that $x_j \in \mathrm{top}(x) = \mathrm{nub}(x)$ and $x_j = x_i$. If $x_{j-1} = x_j$, then we have a flat step, which is forbidden. Finally, if $x_{j-1} > x_j$, then $x_i \notin \mathrm{wrlmax}(x)$, which is once again a contradiction. $\qquad\square$

3 Standardization of $\hat{A}$

A commonly used tool to reduce problems about multisets to sets is given by the *standardization* map, here denoted by $\mathfrak{st}$. The name standardization is due to Stanley [28, Prop. 1.7.1], but the oldest reference we could find goes back to a classic paper by Schensted [26] from 1961. Let $x = x_1 \cdots x_n$ be a Cayley permutation with $\max(x) = k$. Let a_i be the number of copies of i contained in x, for $i \in [k]$. Then $\mathfrak{st}(x)$ is the permutation obtained by replacing the a_i copies of i with

$$a_1 + \cdots + a_{i-1} + 1, \; a_1 + \cdots + a_{i-1} + 2, \; \ldots, \; a_1 + \cdots + a_{i-1} + a_i,$$

going from left to right. More informally, we replace the a_1 copies of 1 with the numbers $1, 2, \ldots, a_1$, the a_2 copies of 2 with $a_1 + 1, a_1 + 2, \ldots, a_1 + a_2$, and so on. For instance, we have $\mathfrak{st}(312112341) = 715236894$, where the 1s are replaced by $1, 2, 3, 4$, the 2s by $5, 6$, the 3s by $7, 8$, and the only 4 is replaced by 9. Some simple properties satisfied by the standardization map are listed in the following three results, where x is a Cayley permutation of length n and $p = \mathfrak{st}(x)$. The easy proofs are omitted or just sketched.

Lemma 4. *For each $i < j$,*
$$x_i \leqslant x_j \iff p_i < p_j.$$

In particular, standardization preserves (strict) descents and maps weak ascents to ascents. Further, it maps flat steps $x_i = x_{i+1}$ to ascents $p_{i+1} = p_i + 1$ that are consecutive in value.

Lemma 5. *Let $i < j$ such that $p_i = p_j + 1$. Then $x_i \in \mathrm{nub}(x)$.*

Proof. The assumption $p_i = p_j + 1$ says that the $\mathfrak{st}$ maps "reads" x_i immediately after x_j; since $i < j$, the entry x_i must be a leftmost copy in x. $\square$

In the next lemma we abuse notation by writing $\mathrm{lrmin}(x) \subseteq \mathrm{lrmin}(p)$ instead of $\{i \in [n] : x_i \in \mathrm{lrmin}(x)\} \subseteq \{i \in [n] : p_i \in \mathrm{lrmin}(p)\}$ (the same in the other items).

Lemma 6. *We have:*

$$
\begin{aligned}
&(i) && \mathrm{lrmin}(x) = \mathrm{lrmin}(p); && \mathrm{wlrmin}(x) \supseteq \mathrm{lrmin}(p). \\
&(ii) && \mathrm{lrmax}(x) \subseteq \mathrm{lrmax}(p); && \mathrm{wlrmax}(x) = \mathrm{lrmax}(p). \\
&(iii) && \mathrm{rlmin}(x) \subseteq \mathrm{rlmin}(p); && \mathrm{wrlmin}(x) = \mathrm{rlmin}(p). \\
&(iv) && \mathrm{rlmax}(x) = \mathrm{rlmax}(p); && \mathrm{wrlmax}(x) \supseteq \mathrm{rlmax}(p).
\end{aligned}
$$

From now on, we let

$$\omega = \begin{array}{c}\text{[pattern diagram]}\end{array}, \quad \zeta = \begin{array}{c}\text{[pattern diagram]}\end{array}, \quad \text{and} \quad \Omega = \mathrm{Sym}(\omega, \zeta).$$

The reader who is familiar with generalized patterns will immediately realize that ω is in fact a bivincular pattern $\omega = (321, \{1\}, \{1\})$. Further, a permutation has 1 as the

leftmost entry if and only if it avoids ζ. Indeed, the set Ω could be alternatively defined as the direct sum $\Omega = 1 \oplus \mathrm{Sym}(\omega)$.

The main goal of this section is to prove that standardization maps bijectively the set $\hat{A}^{\mathrm{pr}}$ of primitive modified ascent sequences to Ω. We shall proceed as follows. First, we show that $\mathfrak{st}(\hat{A}) \subseteq \Omega$. Then, we prove that every permutation in Ω is the standardization of a primitive modified ascent sequence. Since $\hat{A}^{\mathrm{pr}} \subseteq \hat{A}$, we get

$$\mathfrak{st}(\hat{A}^{\mathrm{pr}}) \subseteq \mathfrak{st}(\hat{A}) \subseteq \Omega \subseteq \mathfrak{st}(\hat{A}^{\mathrm{pr}}),$$

from which $\mathfrak{st}(\hat{A}^{\mathrm{pr}}) = \mathfrak{st}(\hat{A}) = \Omega$ is obtained immediately. Finally, that $\mathfrak{st}$ maps bijectively $\hat{A}^{\mathrm{pr}}$ to Ω follows since Parviainen [24, Section 5.4] proved that primitive (modified) ascent sequences and permutations in Ω are equinumerous. Let us expand and clarify a bit on this last part. Parviainen showed that $|\Omega_n| = |\hat{A}_n^{\mathrm{pr}}|$ by slightly tweaking a bijection f claimed to be defined from ascent sequences to Fishburn permutations. In fact, the map f should be defined on modified ascent sequences [5]. Specifically, f is a special instance of the *Burge transpose* [6, 11]. The Burge transpose acts on biwords (u, x) as follows. It flips the columns of (u, x) upside down; then, it sorts the columns of the resulting biword in ascending order with respect to the top entry, breaking ties by sorting in descending order with respect to the bottom entry. When $x \in \hat{A}$ and $u = 12\cdots n$, the bottom row of the transpose of (u, x) is the Fishburn permutation associated with x. If we break ties in the opposite way, i.e. by sorting in ascending order with respect to the bottom entry, and we restrict the transpose to primitive sequences, then we end up with the desired bijection between $\hat{A}_n^{\mathrm{pr}}$ and Ω_n. For instance, the sequence $x = 1312 \in \hat{A}^{\mathrm{pr}}$ is mapped to the permutation $1342 \in \Omega$ since the transpose of the biword

$$\begin{pmatrix} 1\ 2\ 3\ 4 \\ 1\ 3\ 1\ 2 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 1\ 1\ 2\ 3 \\ 1\ 3\ 4\ 2 \end{pmatrix}.$$

Note also that $\mathfrak{st}(1312) = 1423 \neq 1342$.

Proposition 7. *We have $\mathfrak{st}(\hat{A}) \subseteq \Omega$.*

Proof. Let $x \in \hat{A}$ and let $p = \mathfrak{st}(x)$. For a contradiction, suppose that $p \notin \Omega$. Note that $p_1 = 1$ since $x_1 = 1$. Thus, since $\Omega = 1 \oplus \mathrm{Sym}(\omega)$, it must be that p contains an occurrence $p_i p_{i+1} p_j$ of ω, where $p_i > p_{i+1} = p_j + 1$ and $i + 1 < j$. By Lemma 4, we have $x_i > x_{i+1}$, hence $x_{i+1} \notin \mathrm{top}(x)$. On the other hand, we have $x_{i+1} \in \mathrm{nub}(x)$ by Lemma 5. Thus $\mathrm{nub}(x) \neq \mathrm{top}(x)$, which is a contradiction with x being a modified ascent sequence. $\square$

The proof that $\Omega \subseteq \mathfrak{st}(\hat{A}^{\mathrm{pr}})$ relies upon a geometric decomposition of permutations in Ω that stems from the next lemma.

Lemma 8. *Let $p \in \Omega$. If $p_i \notin \mathrm{top}(p)$ and $p_j = p_i - 1$, then $j < i$.*

Proof. If it were $j > i$, then $p_{i-1} p_i p_j$ would be an occurrence of ω. $\square$

Let $p \in \Omega$ and let $\mathrm{top}(p) = \{p_{k_1}, \ldots, p_{k_m}\}$, where $m \geqslant 1$ and $p_{k_1} < p_{k_2} < \cdots < p_{k_m}$. By Lemma 8, every entry that is not an ascent top is located to the right of the next smaller entry in p. More specifically, all the entries whose value is included between two consecutive ascent tops, say p_{k_i} and $p_{k_{i+1}}$, appear in increasing order from left to right in p, and to the right of p_{k_i}. This property allows us to partition p in m *chains* of the form

$$(p_{k_i}, p_{k_i} + 1, p_{k_i} + 2, \ldots, p_{k_{i+1}} - 1),$$

where the only ascent top in every chain is the first (and smallest) element, and all the elements are consecutive in value and appear in increasing order in p. An example of this construction is depicted in Figure 2.

Proposition 9. *For each $p \in \Omega$, there is a primitive modified ascent sequence x such that $\mathfrak{st}(x) = p$. In other words, we have $\Omega \subseteq \mathfrak{st}(\hat{A}^{\mathrm{pr}})$.*

Proof. Let $p \in \Omega$. We determine a modified ascent sequence x such that $\mathfrak{st}(x) = p$. First, we define x with a geometric construction illustrated in Figure 2. For each ascent top $p_i \in \mathrm{top}(p)$, draw a horizontal half-line starting from p_i and going to the right. Then, let each other entry of p fall under the action of gravity until it hits one of the horizontal lines defined before. Finally, rescale the resulting word (by ignoring eventual vertical gaps created at the previous step) in order to obtain a Cayley permutation x. More formally, let y be the string obtained from p by letting

$$y_i = \max(U_i), \quad \text{where} \quad U_i = \{p_j : \ j < i, \ p_j < p_i, \ p_j \in \mathrm{top}(p)\},$$

for each $p_i \notin \mathrm{top}(p)$, and $y_i = p_i$ otherwise. Finally, let x be the only Cayley permutation order isomorphic to y. Note that every $p_i \notin \mathrm{top}(p)$ necessarily hits some half-line since there is a half-line starting from $p_1 = 1$; equivalently, the set U_i is not empty since $p_1 = 1 \in \mathrm{top}(p)$. The construction of x can be alternatively described in terms of the chains of p (defined just before this proposition): all the entries in the same chain fall at the same level as the leftmost element of the chain, which is the only ascent top of the chain, as well as its smallest entry. The equivalence of the two definitions is omitted. To complete the proof, we need to show that $x \in \hat{A}$, x contains no flat steps, and $\mathfrak{st}(x) = p$. We just sketch the proof of these claims, leaving some technicalities to the reader.

- To see that $x \in \hat{A}$, observe that $p_i \in \mathrm{top}(p)$ if and only if $x_i \in \mathrm{top}(x)$. Now, if $p_i \in \mathrm{top}(p)$, then $x_i \in \mathrm{nub}(x)$ as well. On the other hand, if $p_i \notin \mathrm{top}(p)$, then p_i falls at the same level as some $p_j \in \mathrm{top}(p)$, with $j < i$, and thus $x_i \notin \mathrm{nub}(x)$. Hence, we have $\mathrm{top}(x) = \mathrm{nub}(x)$, and $x \in \hat{A}$ follows. Note that we did not use that p avoids ω here.

- Next, we show that the avoidance of ω guarantees that x contains no flat steps. For a contradiction, suppose that $x_i = x_{i+1}$ is a flat step in x. Note that it must be $p_{i+1} \notin \mathrm{top}(p)$, or else p_{i+1} would not fall. Thus we have $p_i > p_{i+1}$ and, since $x_i = x_{i+1}$, we have $p_i \notin \mathrm{top}(p)$ as well. Since p_i and p_{i+1} fall at the same level, they must belong to the same chain. But this is impossible since entries in a chain appear in increasing order in p and $p_i > p_{i+1}$.
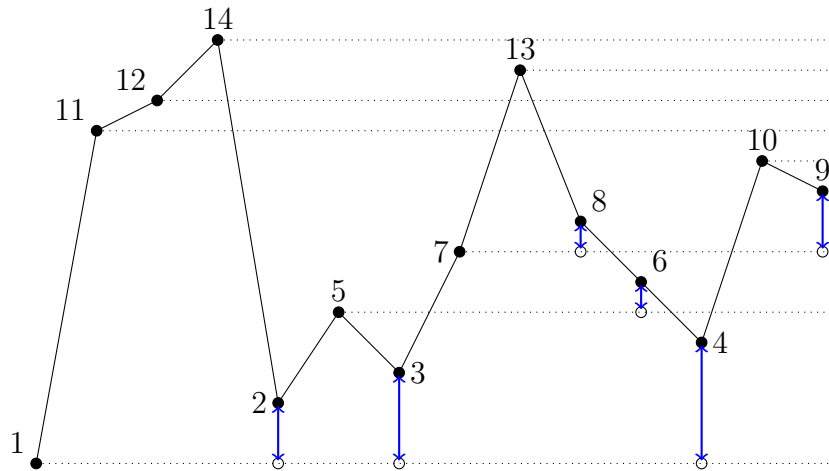
Figure 2: The primitive modified ascent sequence $x = 15681213732143$ associated with the permutation $p = 1, 11, 12, 14, 2, 5, 3, 7, 13, 8, 6, 4, 10, 9$ in Ω. Note that $\mathfrak{st}(x) = p$. The chains of p of length two or more are $(1, 2, 3, 4)$, $(5, 6)$, $(7, 8, 9)$.

- To see that $\mathfrak{st}(x) = p$, observe that the entries of x that are equal to 1 correspond to the chain of p whose smallest entry is $p_1 = 1$. As observed after Lemma 8, such chain is $(p_1, 2, 3, \ldots, \ell_1)$, for some $\ell_1 \geqslant 1$. Further, standardization sets the ith copy of 1 equal to i, matching the desired value of each entry in p. The same argument holds for the remaining chains of p, and our claim follows.

This concludes the proof. □

Corollary 10. *Standardization is a size-preserving bijection from $\hat{A}^{\mathrm{pr}}$ to Ω.*

Schensted [26] observed that the decreasing subsequences of x and $p = \mathfrak{st}(x)$ are in one-to-one correspondence, while the increasing subsequences of p are in one-to-one correspondence with the weakly increasing subsequences of x. Roughly speaking, the reason is that the behavior of the standaridization map on any subsequence of x is not affected by the remaining entries of x. Specifically, if $x_{i_1} \cdots x_{i_k}$ is an occurrence of y in x, then $p_{i_1} \cdots p_{i_k}$ is an occurrence of $\mathfrak{st}(y)$ in p. Conversely, if $p_{i_1} \cdots p_{i_k} \simeq q$, then $\mathfrak{st}(x_{i_1} \cdots x_{i_k}) = q$ as well. The following theorem of transport of patterns from Ω to $\hat{A}^{\mathrm{pr}}$ is obtained immediately.

Theorem 11. *Given $p \in \mathrm{Sym}$, let $[p] = \{x \in \mathrm{Cay} : \mathfrak{st}(x) = p\}$. Then standardization is a size-preserving bijection from $\hat{A}^{\mathrm{pr}}[p]$ to $\Omega(p)$.*

Theorem 11 is analogous to the transport theorem between Fishburn permutations and modified ascent sequences [11, Thm. 5.1]. Standardization plays the role of the Burge transpose, and the set $[p]$ replaces the Fishburn basis. As we will see later, by pairing Theorem 11 with Proposition 2 we are sometimes able to rephrase the original problem of counting $\hat{A}(y)$ in terms of permutations, making our task much easier. Examples where this approach is fruitful can be found in Section 5.2 and Section 6.

4 Easy patterns

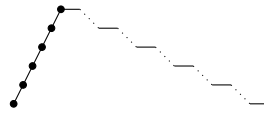As a warm up for the next sections, we solve some simple patterns of short length.

4.1 Patterns of length two

The only modified ascent sequence of length n that avoids 11 is the strictly increasing sequence $x = 12 \ldots n$. Similarly, there is only one sequence that avoids 12, namely the sequence containing all ones $x = 11 \cdots 1$.

A modified ascent sequence avoids 21 if and only if it is a weakly increasing Cayley permutation [11], and the number of such sequences of length n is 2^{n-1}. Further [10], we have $\hat{A}(21) = \hat{A}(121)$.

4.2 Pattern 112

Let $x \in \hat{A}(112)$. Then $x = 1y1^{k_1}$, for some $k_1 \geqslant 0$, where each entry in y is strictly greater than 1. Indeed, no entry greater than or equal to two is allowed to appear to the right of the second copy of 1. Further, by Proposition 1, the subsequence y is order isomorphic to some $\tilde{y} \in \hat{A}(112)$; namely, y is obtained by increasing by one each entry of $\tilde{y}$. Iterating the same argument on y yields a "left pyramid" structure:

$$x = 12 \cdots m m^{k_m} \cdots 2^{k_2} 1^{k_1},$$

where $m = \max(x)$ and $k_i \geqslant 0$, for $i = 1, \ldots, m$. Therefore, any $x \in \hat{A}_n(112)$ is uniquely determined by a tuple $(k_1+1, k_2+1, \ldots, k_m+1)$ recording the multiplicity of its values; that is, by a composition of n (with $m = \max(x)$ parts). Finally, the number of compositions of n is well known to be equal to 2^{n-1}.

With a little more effort, we can enumerate $\hat{A}^{\mathrm{pr}}(112)$. A 112-avoiding modified ascent sequence as above is primitive if and only if $k_m = 0$ and $k_i \in \{0, 1\}$ for each $i < m$. In other words, by ignoring the last entry $k_m + 1 = 1$ in the tuple $(k_1 + 1, k_2 + 1, \ldots, k_m + 1)$, we obtain a composition of $n - 1$ with no parts greater than two. A quick look in the OEIS [27] reveals that the number of such compositions of $n - 1$ is given by the nth Fibonacci number.

Computing the number of primitive sequences in the cases discussed so far is a fairly easy task. The interested reader is invited to check Table 1 to see the resulting sequences.

4.3 Pattern 122

Let $x \in \hat{A}_n(122)$. Since $x_1 = 1$, every integer between 2 and $\max(x)$ appears exactly once in x. Furthermore, all the entries between two copies of 1 appear in increasing order due to the equality $\mathrm{nub}(x) = \mathrm{top}(x)$. In other words, if x contains k copies of 1, then x decomposes as

$$x = 1B_1 \ 1B_2 \ \ldots \ 1B_k,$$

where entries in each block B_i are greater than or equal to 2, and B_i is strictly increasing (possibly empty). Thus,

$$|\{x \in \hat{A}_n(122) : x \text{ contains } k \text{ copies of } 1\}| = k^{n-k}.$$

Indeed, a sequence x as above is determined by choosing, for each of the $n - k$ entries greater than 1, the index $i \in \{1, 2, \ldots, k\}$ of its block B_i. Summing over k, we get

$$|\hat{A}_n(122)| = \sum_{k=1}^{n} k^{n-k}.$$

According to A026898 [27], the size of $\hat{A}_n(122)$ is equal to the number of set partitions of $[n]$ whose minima of blocks form an interval. A simple bijective proof goes as follows. Given $x \in \hat{A}_n(122)$, insert a block separator before every copy of 1 (ignoring the leftmost one), and compute $\mathfrak{st}(x)$ as usual. The result is a set partition whose minima of blocks correspond to the copies of 1 in x. For instance, if $x = 134112561$:

$$x = 134|1|1256|1 \longmapsto \mathfrak{st}(x) = 167|2|3589|4 = \{1, 6, 7\}\{2\}\{3, 5, 8, 9\}\{4\}.$$

We were not able to find a reference for the OGF given in A026898, and we wish to fill this gap below. Recall that

$$\hat{A}_{122}(t) = \sum_{n \geqslant 0} |\hat{A}_n(122)| t^n$$

denotes the OGF of 122-avoiding modified ascent sequences. An OGF for sequences in $\hat{A}_n(122)$ that contain exactly k copies of 1 is

$$t^k \sum_{m \geqslant 0} k^m t^m = \frac{t^k}{1 - kt},$$

where $n = m + k$. Summing over k, we obtain

$$\hat{A}_{122}(t) = \sum_{k \geqslant 0} \frac{t^k}{1 - kt}.$$

Finally, an OGF for the sequence A026898, which is shifted by one position compared to $\hat{A}_{122}(t)$, is

$$\frac{1}{t}(\hat{A}_{122}(t) - 1) = \sum_{k \geqslant 0} \frac{t^k}{1 - (k+1)t},$$

which matches the one given in the OEIS.

To end this section, we wish to enumerate $\hat{A}^{\mathrm{pr}}(122)$, something we will use in Section 5.1. Let $n \geqslant 1$ and let $x \in \hat{A}_n^{\mathrm{pr}}(122)$. Once again, we shall decompose x by highlighting the copies of 1 it contains. The only difference compared to the general case, is that only the last block is allowed to be empty since x is primitive (and any other empty block

would result in two consecutive copies of 1). Thus, if $x \in \hat{A}_n^{\mathrm{pr}}(122)$ contains k copies of 1, we have either

$$x = 1B_1 \ \ldots \ 1B_{k-1} \ 1B_k \quad \text{or} \quad x = 1B_1 \ \ldots \ 1B_{k-1} \ 1,$$

according to whether or not B_k is empty. Clearly, the former are (in bijection with) ordered set partitions of size $n - k$ with k blocks, which are counted by $k!S(n-k, k)$; the latter are ordered set partitions of size $n - k$ with $k - 1$ blocks, counted by $(k-1)!S(n-k, k-1)$. Here, we denote by $S(n, k)$ the (n, k)th Stirling number of the second kind. Finally, for $n \geqslant 1$ we obtain

$$
\begin{aligned}
|\hat{A}_n^{\mathrm{pr}}(122)| &= \sum_{k \geqslant 1} \big[k!S(n-k, k) + (k-1)!S(n-k, k-1) \big] \\
&= \sum_{k \geqslant 1} (k-1)! \big(kS(n-k, k) + S(n-k, k-1) \big) \\
&= \sum_{k \geqslant 1} (k-1)!S(n-k+1, k).
\end{aligned}
$$

For the rest of this section, let

$$F(t) = \sum_{n \geqslant 0} \sum_{k \geqslant 0} k!S(n-k, k)t^n,$$

so that

$$
\begin{aligned}
\hat{A}_{122}^{\mathrm{pr}}(t) &= 1 + \sum_{n \geqslant 1} |\hat{A}_n^{\mathrm{pr}}(122)|t^n \\
&= 1 + \sum_{n \geqslant 1} \sum_{k \geqslant 0} k!S(n-k, k)t^n + \sum_{n \geqslant 1} \sum_{k \geqslant 1} (k-1)!S(n-k, k-1)t^n \\
&= F(t) + \sum_{n \geqslant 1} \sum_{j \geqslant 0} j!S(n-j-1, j)t^n \\
&= F(t) + t \sum_{m \geqslant 0} \sum_{j \geqslant 0} j!S(m-j, j)t^m \\
&= (1+t)F(t).
\end{aligned}
$$

A shift by one position of $\hat{A}_{122}^{\mathrm{pr}}(t)$ is recorded as A229046. Cao et al. [7] showed that its n-th term—i.e. $|\hat{A}_{n+1}^{\mathrm{pr}}(122)|$—is equal to the number of inversion sequences of length n avoiding the triple of binary relations $(-, -, =)$; or, equivalently, avoiding the patterns 111, 121 and 212. A bijection between the two structures remains to be found. Similarly, a shift of $F(t)$ gives A105795. Each of these two entries in the OEIS contains (at least) an OGF for the corresponding sequence, but we could not find any formal proof. We bridge this gap below, starting from $F(t)$. Stanley [28, Eq. (1.94)] proved the following two equations

involving the Stirling numbers of the second kind:

$$k!S(n,k) = \sum_{i \geqslant 0}(-1)^{k-i}\binom{k}{i}i^n; \qquad\qquad (1.94)(a)$$

$$\sum_{m \geqslant 0} S(m,k)t^m = \frac{t^k}{(1-t)(1-2t)\cdots(1-kt)}. \qquad\qquad (1.94)(c)$$

Now,

$$F(t) = \sum_{n \geqslant 0}\sum_{k \geqslant 0} k!S(n-k,k)t^n$$

$$= \sum_{m \geqslant 0}\sum_{k \geqslant 0} k!S(m,k)t^{m+k} \qquad\qquad m = n-k$$

$$= \sum_{k \geqslant 0} k!t^k\frac{t^k}{(1-t)(1-2t)\cdots(1-kt)} \qquad\qquad \text{By } (1.94)(c)$$

$$= \sum_{k \geqslant 0}\prod_{j=1}^{k}\frac{jt^2}{1-jt}.$$

Alternatively,

$$F(t) = \sum_{m \geqslant 0}\sum_{k \geqslant 0} k!S(m,k)t^{m+k}$$

$$= \sum_{m \geqslant 0}\sum_{k \geqslant 0}\sum_{i \geqslant 0}(-1)^{k-i}\binom{k}{i}i^m t^{m+k} \qquad\qquad \text{By } (1.94)(a)$$

$$= \sum_{i \geqslant 0}\sum_{k \geqslant 0}(-1)^{k-i}\binom{k}{i}t^k\left(\sum_{m \geqslant 0}i^m t^m\right)$$

$$= \sum_{i \geqslant 0}\frac{t^i}{1-it}\left(\sum_{k \geqslant 0}(-1)^{k-i}\binom{k}{i}t^{k-i}\right)$$

$$= \sum_{i \geqslant 0}\frac{t^i}{(1-it)(1+t)^{i+1}},$$

where the last step follows from the binomial theorem:

$$
\begin{aligned}
(1+t)^{-i-1} &= \sum_{j \geqslant 0} \binom{-i-1}{j} t^j \\
&= \sum_{j \geqslant 0} (-1)^j \binom{i+1+j-1}{j} t^j \\
&= \sum_{k \geqslant 0} (-1)^{k-i} \binom{k}{k-i} t^{k-i} \qquad\qquad k = i + j \\
&= \sum_{k \geqslant 0} (-1)^{k-i} \binom{k}{i} t^{k-i}.
\end{aligned}
$$

We have thus proved the following proposition.

Proposition 12. *Let $F(t) = \sum_{n \geqslant 0} \sum_{k \geqslant 0} k! S(n-k, k) t^n$. Then*

$$
F(t) = \sum_{k \geqslant 0} \prod_{j=1}^{k} \frac{jt^2}{1-jt} \tag{3}
$$

$$
= \sum_{i \geqslant 0} \frac{t^i}{(1-it)(1+t)^{i+1}}. \tag{4}
$$

Two OGFs for $\hat{A}_{122}^{\mathrm{pr}}(t)$ are obtained immediately as $\hat{A}_{122}^{\mathrm{pr}}(t) = (1+t)F(t)$. An OGF for A229046 is

$$
G(t) = \frac{1}{t}(\hat{A}_{122}^{\mathrm{pr}}(t) - 1) = F(t) + \frac{1}{t}(F(t) - 1).
$$

Using equation (3), we compute

$$
G(t) = \sum_{k \geqslant 0} \frac{1}{1-(k+1)t} \prod_{j=1}^{k} \frac{jt^2}{1-jt},
$$

which agrees with the OGF given in A229046.

5 Primitive patterns

This whole section is devoted to the solution of primitive patterns.

5.1 Pattern 1232

We start by enumerating $\hat{A}(1232)$. The key is the following lemma.

Lemma 13. *For each $n \geqslant 0$,*

$$
\hat{A}_n^{\mathrm{pr}}(122) = \hat{A}_n^{\mathrm{pr}}(1232).
$$

Proof. Clearly, if x avoids 122 then it avoids 1232 too. The inclusion $\hat{A}^{\mathrm{pr}}(122) \subseteq \hat{A}^{\mathrm{pr}}(1232)$ follows. Conversely, if $x \in \hat{A}^{\mathrm{pr}}$ contains an occurrence $x_i x_j x_k$ of 122, then $x_k \notin \mathrm{nub}(x) = \mathrm{top}(x)$ and $x_i x_j x_{k-1} x_k$ is an occurrence of 1232. $\qquad\square$

Proposition 14. *For $n \geqslant 1$,*

$$|\hat{A}_n(1232)| = \sum_{k=1}^{n} \binom{n-1}{k-1} \sum_{j=1}^{k} (j-1)! S(k-j+1, j).$$

Furthermore,

$$\hat{A}_{1232}(t) = \sum_{i \geqslant 0} \frac{t^i (1-t)}{1 - (i+1)t}.$$

Proof. The first statement follows from Proposition 2, Lemma 13, and the equality $|\hat{A}_k^{\mathrm{pr}}(122)| = \sum_{j=1}^{k} (j-1)! S(k-j+1, j)$, proved in Section 4.3. As observed below Proposition 12, two expressions for the OGF of $\hat{A}^{\mathrm{pr}}(122)$ can be obtained as $\hat{A}_{122}^{\mathrm{pr}}(t) = (1+t)F(t)$. Since $\hat{A}_{122}^{\mathrm{pr}}(t) = \hat{A}_{1232}^{\mathrm{pr}}(t)$, the second statement follows—with a little bit of additional work—by setting $y = 1232$ in equation (2) and using equation (4). $\qquad\square$

A shift by one position of $\hat{A}_{1232}(t)$ is recorded as A047970 [27].

5.2 Patterns 213, 231 and 321

We solve the patterns $y \in \{213, 231, 321\}$ with the machinery of Theorem 11. First, we show that in each of these cases standardization maps bijectively $\hat{A}^{\mathrm{pr}}(y)$ to $\Omega(y)$. Then, we count $\Omega(y)$ and use Proposition 2 to recover the full enumeration of $\hat{A}(y)$. Let us start with a simple lemma.

Lemma 15. *We have*

$$\hat{A}^{\mathrm{pr}}(212, 213) = \hat{A}^{\mathrm{pr}}(213) \quad \text{and} \quad \hat{A}^{\mathrm{pr}}(221, 231) = \hat{A}^{\mathrm{pr}}(231).$$

Proof. Showing that $\hat{A}^{\mathrm{pr}}(213)$ is contained in $\hat{A}^{\mathrm{pr}}(212, 213)$ is sufficient to prove the first equality. Let $x \in \hat{A}^{\mathrm{pr}}(213)$. For a contradiction, suppose that x contains 212 and let $x_i x_j x_k$ be an occurrence of 212 in x. Note that $x_k \notin \mathrm{nub}(x) = \mathrm{top}(x)$. Since x is primitive, it must be $x_{k-1} > x_k$. Hence $x_i x_j x_{k-1}$ is an occurrence of 213, which is impossible. The second equality can be proved similarly. If $x_i x_j x_k \simeq 221$, then it must be $x_{j-1} > x_j$, and $x_i x_{j-1} x_k \simeq 231$. $\qquad\square$

To prove the next result, we combine the previous lemma with the transport theorem. Recall from Theorem 11 that standardization maps bijectively $\hat{A}^{\mathrm{pr}}[p]$ to $\Omega(p)$, where $[p] = \{x \in \mathrm{Cay} : \mathfrak{st}(x) = p\}$ and $\Omega = 1 \oplus \mathrm{Sym}(\omega)$.

Corollary 16. *For $n \geqslant 1$, standardization maps bijectively:*

$$\hat{A}_n^{\mathrm{pr}}(213) \longrightarrow \Omega_n(213);$$
$$\hat{A}_n^{\mathrm{pr}}(231) \longrightarrow \Omega_n(231);$$
$$\hat{A}_n^{\mathrm{pr}}(321) \longrightarrow 1 \oplus \mathrm{Sym}_{n-1}(321).$$

Proof. Observe that $[213] = \{212, 213\}$ and $[231] = \{221, 231\}$. By Lemma 15, $\hat{A}^{\mathrm{pr}}(213) = \hat{A}^{\mathrm{pr}}[213]$ and $\hat{A}^{\mathrm{pr}}(231) = \hat{A}^{\mathrm{pr}}[231]$. The first two items follow immediately by Theorem 11. The last item follows as well since $[321] = \{321\}$ and 321 is the classical pattern underlying ω. $\square$

Now, it is easy to prove that $\hat{A}(321)$ is counted by the binomial transform of the Catalan numbers, shifted by one position (A007317 in the OEIS [27]).

Proposition 17. *For $n \geqslant 1$, we have*

$$|\hat{A}_n(321)| = \sum_{j=0}^{n-1} \binom{n-1}{j} c_j,$$

where $c_j = \frac{1}{j+1}\binom{2j}{j}$ is the jth Catalan number.

Proof. We have:

$$
\begin{aligned}
|\hat{A}_n(321)| &= \sum_{k=1}^{n} \binom{n-1}{k-1} |\hat{A}_k^{\mathrm{pr}}(321)| && \text{by Proposition 2} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} |1 \oplus \mathrm{Sym}_{k-1}(321)| && \text{by Corollary 16} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} c_{k-1} && \text{since } |\mathrm{Sym}_{k-1}(321)| = c_{k-1} \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} c_j.
\end{aligned}
$$

$\square$

Remark. The set $\mathrm{RGF}(321)$ of restricted growth functions avoiding 321 is equinumerous [14] with $\hat{A}(321)$. Note that RGF encodes set partitions in the same way as Cay encodes ordered set partitions (and $\hat{A} \subseteq \mathrm{Cay}$). Is there any other example of Wilf-equivalence between pattern-avoiding RGFs and modified ascent sequences?

Let us take care of the patterns 213 and 231 next. For $n \geqslant 0$, denote by m_n the nth *Motzkin number* (see also A001006 [27]).

Proposition 18. *For $y \in \{213, 231\}$ and $n \geqslant 0$, we have*

$$|\mathrm{Sym}_n(\omega, y)| = m_n.$$

Proof. Let $M(t) = \sum_{n \geqslant 0} |\mathrm{Sym}_n(\omega, y)| t^n$. We show that $M = M(t)$ satisfies

$$M = 1 + tM + t^2 M^2, \tag{5}$$

a combinatorial equation defining the Motzkin numbers. Let us start from the pattern $y = 213$. Let $p \in \mathrm{Sym}(\omega, 213)$. If p is not the empty permutation, then p decomposes as $p = L1R$, where L and R are possibily empty. Since p avoids 213, we have $L > R$, i.e. each entry in the prefix L is greater than each entry in the suffix R. Also, each of L and R is (order isomorphic to) a permutation avoiding ω and 213. Now, there are exactly two possibilites:

- $L = \emptyset$. Then $p = 1R$, which gives the tM term in equation (5).

- $L \neq \emptyset$. In this case, the smallest entry of L is forced to be in the leftmost position of L; indeed, let $p_i = \min(L)$ and let j be such that $p_j = p_i - 1$. Note that either $p_j = 1$ or $p_j \in R$. In any case, it must be $j > i$. Thus, if it were $i \geqslant 2$, then we would have an occurrence $p_{i-1}p_ip_j$ of ω in p, which is impossible. We have thus showed that the position of the smallest entry of L is forced. On the other hand, the remaining entries of L (and R) are allowed to form any $(\omega, 213)$-avoiding permutation. This contributes with the t^2M^2 term in equation (5).

In the end,

$$M = \underbrace{1}_{\text{empty}} + \underbrace{t \cdot M}_{L=\emptyset} + \underbrace{t^2 \cdot M^2}_{L\neq\emptyset}.$$

The equation for $y = 231$ is obtained similarly. Any $p \in \mathrm{Sym}(\omega, 231)$ decomposes as $p = LnR$, with $L < R$, and the smallest entry of R is forced to be in the leftmost position of R by (the avoidance of) ω. $\qquad\square$

Corollary 19. *Let $y \in \{213, 231\}$. Then $|\hat{A}_n(y)|$ is equal to the nth Catalan number.*

Proof. The case $n = 0$ is trivial. For $n \geqslant 1$,

$$
\begin{aligned}
|\hat{A}_n(y)| &= \sum_{k=1}^{n} \binom{n-1}{k-1} |\hat{A}_k^{\mathrm{pr}}(y)| && \text{by Proposition 2} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} |\Omega_k(y)| && \text{by Corollary 16} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} |1 \oplus \mathrm{Sym}_{k-1}(\omega, y)| && \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} m_{k-1} && \text{by Proposition 18} \\
&= c_n,
\end{aligned}
$$

where the last one is a well known equality relating the Motzkin and the Catalan numbers (see Donaghey [17, Eq. (2)]). $\qquad\square$

5.3 Patterns 123 and 1234

The enumeration of modified ascent sequences avoiding $y \in \{123, 1234\}$ can be obtained as a consequence of the transport of patterns developed by Claesson and the current author [11]. Indeed, for every $k \geqslant 1$, the Burge transpose maps bijectively $\hat{A}_n(12 \cdots k)$ to $F_n(12 \cdots k)$, where $F(12 \cdots k)$ is the set of Fishburn permutations [5] avoiding $12 \cdots k$; further, Gil and Weiner [21] proved that

$$|F_n(123)| = 2^{n-1} \quad \text{and} \quad |F_n(1234)| = c_n.$$

An alternative and arguably more direct approach for $y = 123$ consists in counting $\hat{A}^{\mathrm{pr}}(123)$ and using our favourite Proposition 2. Indeed, we have

$$\hat{A}^{\mathrm{pr}}(123) = \{\epsilon, 1, 12, 121, 1312, 13121, 141312, 1413121, \dots\}$$

where ϵ denotes the empty sequence. In other words, there is only one primitive, 123-avoiding modified ascent sequence of length n; namely, the sequence

$$\begin{array}{ll} 1k1(k-1)1(k-2)1 \cdots 121 & \text{if } n = 2k-1 \text{ is odd}; \\ 1k1(k-1)1(k-2)1 \cdots 12 & \text{if } n = 2(k-1) \text{ is even}. \end{array}$$

Finally,

$$|\hat{A}_n(123)| = \sum_{k=1}^{n} \binom{n-1}{k-1} |\hat{A}_k^{\mathrm{pr}}(123)| = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}.$$

Note also that (again by Proposition 2) primitive modified ascent sequences avoiding 1234 are enumerated by the Motzkin numbers

5.4 Pattern 132

We prove that 132-avoiding modified ascent sequences are counted by the odd Fibonacci numbers (A001519 [27]). As usual, let us count the primitive sequences first.

Let $n \geqslant 1$ and let $x \in \hat{A}_n^{\mathrm{pr}}(132)$. Recall from Proposition 1 that all the copies of $\max(x)$ are in consecutive positions. Since x is primitive, it contains only one copy of its maximum value. Let $m \in [n]$ denote the index of the only entry $x_m = \max(x)$. We show that either $m = n-1$ or $m = n$. There is nothing to prove if $n \leqslant 3$. Otherwise, let $n > 3$. For a contradiction, assume $m \leqslant n-2$. Since x is primitive, at least one of the last two entries, say x_i, $i \in \{n-1, n\}$, is not equal to 1; hence $x_1 x_m x_i$ is an occurrence of 132, which is impossible. Consequently, any $x \in \hat{A}^{\mathrm{pr}}(132)$ falls in exactly one of the following two cases:

- $m = n$. In this case, $x_1 \cdots x_{n-1} \in \hat{A}_{n-1}^{\mathrm{pr}}(123)$ and $x_n = \max(x_1 \cdots x_{n-1}) + 1$.

- $m = n-1$. In this case, it must be $x_n = 1$, or else we would have $x_1 x_m x_n \hat{A}_{122}^{\mathrm{pr}} \simeq 132$. Specifically, we have $x_1 \cdots x_{n-2} \in \hat{A}_{n-2}^{\mathrm{pr}}(123)$, $x_{n-1} = \max(x_1 \cdots x_{n-2}) + 1$, and $x_n = 1$.

Conversely, it is easy to see that inserting a suffix m or $m1$ to any $y \in \hat{A}^{\mathrm{pr}}(132)$, where $m = \max(y) + 1$, yields a primitive, 132-avoiding modified ascent sequence. Therefore,

$$|\hat{A}_n^{\mathrm{pr}}(132)| = \underbrace{|\hat{A}_{n-1}^{\mathrm{pr}}(132)|}_{m=n} + \underbrace{|\hat{A}_{n-2}^{\mathrm{pr}}(132)|}_{m=n-1}.$$

Since $|\hat{A}_0^{\mathrm{pr}}(132)| = |\hat{A}_1^{\mathrm{pr}}(132)| = 1$, it follows that $|\hat{A}_n^{\mathrm{pr}}(132)|$ is equal to the nth Fibonacci number f_n. In the end, a well known formula for the odd-indexed Fibonacci numbers gives

$$|\hat{A}_n(132)| = \sum_{k=1}^{n} \binom{n-1}{k-1} |\hat{A}_k^{\mathrm{pr}}(132)|$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} f_k = f_{2n-1}.$$

5.5 Pattern 312

In this section, we give a bijection between $\hat{A}_{n+1}^{\mathrm{pr}}(312)$ and the set of Dyck paths of semilength n that avoid the (consecutive) subpath `dudu`. Sapounakis et al. [25] proved that an OGF for these paths is

$$D(t) = \frac{1 + t - t^2 - \sqrt{t^4 - 2t^3 - 5t^2 - 2t + 1}}{2t}. \tag{6}$$

To do so, they found two equations relating them with Dyck paths that start with a low peak `ud`. Using Lagrange's inversion formula, they also computed the number of `dudu`-avoiding Dyck paths of semilength n (see also A102407)

$$d_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n-j} \binom{n-j}{j} \sum_{i=0}^{n-2j} \binom{n-2j}{i} \binom{j+i}{n-2j-i+1},$$

where $n \geqslant 1$. Letting $d_0 = 1$ and applying Proposition 2, we obtain

$$|\hat{A}_n(312)| = \sum_{k=1}^{n} \binom{n-1}{k-1} |\hat{A}_k^{\mathrm{pr}}(312)|$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} d_{k-1},$$

which gives the sequence

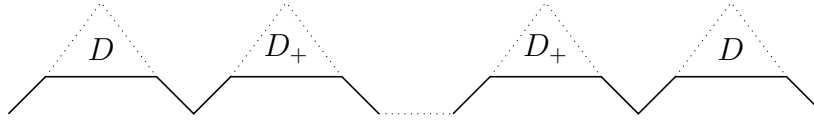$$\left(|\hat{A}_n(312)| \right)_{n \geqslant 0} = 1, 1, 2, 5, 14, 43, 142, 495, 1796, 6715, 25692, \ldots.$$

Figure 3: Decomposition of a `dudu`-avoiding Dyck path that hits the x-axis at least twice. Here, D denotes a generic `dudu`-avoiding path, while D_+ denotes a nonempty one.

At present, these numbers do not appear in the OEIS [27]. Combining equation (2) with equation (6), we obtain an OGF for $\hat{A}_{312}$:

$$\hat{A}_{312}(t) = \frac{1}{2}\left(3 + \frac{t}{1-t} - \frac{t^2}{(1-t)^2} - \sqrt{\frac{1 - 6t + 7t^2 - 2t^3 + t^4}{(1-t)^4}}\right).$$

A more direct method to determine $D(t)$ is illustrated below. The main advantage of our construction is that it relies on a combinatorial decomposition of `dudu`-avoiding Dyck paths which we can replicate on $\hat{A}^{\mathrm{pr}}(312)$ to define a bijection between these two structures. Any nonempty `dudu`-avoiding Dyck path P that hits the x-axis k times, $k \geqslant 1$, decomposes as

$$P = \mathtt{u}Q_1\mathtt{d}\ \mathtt{u}Q_2^+\mathtt{d}\ \cdots\ \mathtt{u}Q_{k-1}^+\mathtt{d}\ \mathtt{u}Q_k\mathtt{d}, \tag{7}$$

where each factor Q_i is a `dudu`-avoiding Dyck path; further, all the factors except for Q_1 and Q_k must be nonempty, as denoted by the superscript "+". Hence $D = D(t)$ satisfies the combinatorial equation

$$D = \overbrace{1}^{\text{empty path}} + \overbrace{tD}^{k=1} + \overbrace{t^2D^2\sum_{k\geqslant 0}[t(D-1)]^k}^{k\geqslant 2}$$

$$= 1 + tD + \frac{t^2D^2}{1 - t(D-1)},$$

whose solution is given by the OGF of equation 6.

To obtain an analogous decomposition on $\hat{A}^{\mathrm{pr}}(312)$, we shall decompose primitive, 312-avoiding modified ascent sequences by highlighting their copies of 1—something we have already done for the pattern 122 in Section 4.3. First, we collect some geometric properties of $\hat{A}^{\mathrm{pr}}(312)$ in the next proposition.

Proposition 20. *Let $x \in \hat{A}_n^{\mathrm{pr}}(312)$, with $n \geqslant 1$. Write*

$$x = 1B_1\ 1B_2\ \cdots\ 1B_{k-1}\ 1B_k,$$

where $k \geqslant 1$ is the number of copies of 1 contained in x. For $i \in [k]$, let $m_i = \max(B_i)$ and denote by ℓ_i the leftmost entry in B_i. Then, for each $i \leqslant k-1$:

1. *$B_i \neq \emptyset$.*

2. *$B_{i+1} \geqslant m_i$; that is, $a \geqslant m_i$ for each $a \in B_{i+1}$.*

3. $\ell_{i+1} = 1 + m_i$.

4. Let $\bar{B}_1$ be obtained by subtracting 1 to each entry of B_1. Then $\bar{B}_1 \in \hat{A}^{\mathrm{pr}}(312)$.

5. Let $\tilde{B}_i$ be obtained by subtracting $m_{i-1} - 1$ to each entry of B_i, for $i = 2, \ldots, k$. Then $1\tilde{B}_i \in \hat{A}^{\mathrm{pr}}(312)$.

Proof. 1. This claim follows immediately since we are assuming x to be primitive.

2. An entry $a \in B_{i+1}$, $a < m_i$, would realize an occurrence $m_i 1a$ of 312, which is impossible.

3. In a 312-avoiding modified ascent sequence, all the ascent tops must be in (strictly) increasing order from left to right. Indeed, if $x_{j_1} > x_{j_2}$ were ascent tops with $j_1 < j_2$, then $x_{j_1} x_{j_2-1} x_{j_2}$ would be an occurrence of 312. Now, recall that the set $\mathrm{top}(x) = \mathrm{nub}(x)$ contains exactly one copy of each integer from 1 to $\max(x)$. Further, m_i is the rightmost (and thus largest) ascent top in B_i, while ℓ_{i+1} is the leftmost (and thus smallest) ascent top in B_{i+1}. The desired claim follows immediately.

4. All the values between 2 and $\max(B_1)$ appear in B_1 due to what proved in Item 3. Note also that the leftmost entry of $\bar{B}_1$ is equal to $x_2 - 1 = 2 - 1 = 1$. Thus $\bar{B}_1$ is a Cayley permutation on $[\max(B_1) - 1]$ that starts with 1. Since $\mathrm{nub}(\bar{B}_1) = \mathrm{top}(\bar{B}_1)$ and $\bar{B}_1$ avoids 312, the word $\bar{B}_1$ is a primitive, 312-avoiding modified ascent sequence.

5. The proof of this item is analogous to the previous one. The only difference is that, the correct quantity to subtract in order to rescale the entries of B_i properly is $m_{i-1} - 1 = \ell_i - 2$. Indeed, let $a \in B_i$. Then $a \geqslant m_{i-1}$ due to Item 2, and

$$a - (m_{i-1} - 1) \geqslant m_{i-1} - m_{i-1} + 1 = 1.$$

Similarly, $\ell_i = m_{i-1} + 1$ due to Item 3, and

$$\ell_i - (m_{i-1} - 1) = m_{i-1} + 1 - m_{i-1} + 1 = 2.$$

As a result, all the values between 1 and $m_i - m_{i-1} + 1$ appear in $1\tilde{B}_i$; that is, the word $1\tilde{B}_i$ is a Cayley permutation on $[m_i - m_{i-1} + 1]$. More specifically, in analogy with what observed for B_1, it is a primitive, 312-avoiding modified ascent sequence. $\square$

Keeping the same notations of Proposition 20, every nonempty $x \in \hat{A}^{\mathrm{pr}}(312)$ that contains $k \geqslant 1$ copies of 1 decomposes as

$$x = 1B_1 \, 1B_2 \, \cdots \, 1B_{k-1} \, 1B_k,$$

where B_i is nonempty for $i \leqslant k-1$, the leftmost block B_1 satisfies $\bar{B}_1 \in \hat{A}^{\mathrm{pr}}(312)$, and $1\tilde{B}_i \in \hat{A}^{\mathrm{pr}}(312)$ for each $i \geqslant 2$. As an example, let $x = 123432561761897$. Note that $x \in \hat{A}^{\mathrm{pr}}(312)$. Then x decomposes as

$$x \;=\; 1 \underbrace{2343256}_{B_1}\, 1 \underbrace{76}_{B_2}\, 1 \underbrace{897}_{B_3},$$

where

$$\bar{B}_1 = 1232145,$$
$$1\tilde{B}_2 = 121,$$
$$1\tilde{B}_3 = 1231.$$

On the other hand, given any such sequence $\bar{B}_1, 1\tilde{B}_2, \ldots, 1\tilde{B}_k$ of primitive, 312-avoiding modified ascent sequences, one uniquely reconstruct $x = 1B_1\, 1B_2\, \cdots\, 1B_k$ by suitably rescaling the entries of the blocks $\bar{B}_1, \tilde{B}_2, \ldots, \tilde{B}_k$ as in the last two items of Proposition 20; that is, by adding 1 to each entry of $\bar{B}_1$, and $m_{i-1} - 1$ to each entry of $\tilde{B}_i$, where m_{i-1} is the maximum of B_{i-1} and $i \geqslant 2$.

We now have all the ingredients to define a bijection between $\hat{A}^{\mathrm{pr}}_{n+1}(312)$ and the set of $\mathtt{dudu}$-avoiding Dyck paths of semilength n. Given $x \in \hat{A}^{\mathrm{pr}}(312)$, we define the path $\phi(x) = P$ recursively by letting $\phi(\emptyset) = \phi(1) = \emptyset$ and, if $x = 1B_1 1B_2 \cdots 1B_{k-1} 1B_k$ has length two or more,

$$
\begin{aligned}
\phi(1B_1 &\qquad 1B_2 \qquad \ldots \qquad 1B_{k-1} \qquad 1B_k) \\
&= \mathtt{u}\phi(\bar{B}_1)\mathtt{d}\;\; \mathtt{u}\phi(1\tilde{B}_2)\mathtt{d}\;\; \ldots \;\; \mathtt{u}\phi(1\tilde{B}_{k-1})\mathtt{d}\;\; \mathtt{u}\phi(1\tilde{B}_k)\mathtt{d}.
\end{aligned}
$$

For instance, we have

$$\phi(12) = \mathtt{u}\phi(\bar{2})\mathtt{d} = \mathtt{u}\phi(1)\mathtt{d} = \mathtt{ud};$$
$$\phi(121) = \mathtt{u}\phi(\bar{2})\mathtt{d}\mathtt{u}\phi(\emptyset)\mathtt{d} = \mathtt{udud};$$
$$\phi(123) = \mathtt{u}\phi(\bar{2}\bar{3})\mathtt{d} = \mathtt{u}\phi(12)\mathtt{d} = \mathtt{uudd}.$$

The Dyck path (of semilength 14) associated with the sequence $x = 123432561761897$ (of semilength 15) of the previous example is depicted in Figure 4. The leftmost factor of the path is obtained from $1B_1 = 12343256$, recursively, as

$$
\begin{aligned}
\mathtt{u}\phi(\bar{B}_1)\mathtt{d} &= \mathtt{u}\left(\phi(1232145)\right)\mathtt{d} \\
&= \mathtt{u}\left((\mathtt{u}\phi(121)\mathtt{d})\,(\mathtt{u}\phi(123)\mathtt{d})\right)\mathtt{d} \\
&= \mathtt{u}\left((\mathtt{uududd})\,(\mathtt{uuuddd})\right)\mathtt{d} \\
&= \mathtt{u}^3\mathtt{dud}^2\mathtt{u}^3\mathtt{d}^4.
\end{aligned}
$$

By Proposition 20, the path P satisfies the decomposition determined by equation 7. In particular, for $2 \leqslant i < k$ the path $\phi(1\tilde{B}_i)$ is not empty since B_i is nonempty. Hence P is a $\mathtt{dudu}$-avoiding Dyck path. Note also that the semilength of $P = \phi(x)$ is equal to one less than the length of x; the shift in length is a result of having mapped the leftmost
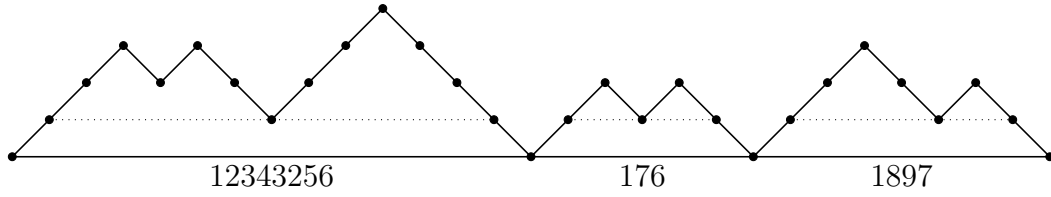
Figure 4: Dyck path associated with the sequence $x = 123432561761897$.

block $1B_1$ to $\mathtt{u}\phi(\bar{B}_1)\mathtt{d}$; on the other hand, the semilength of every other factor $\mathtt{u}\phi(1\tilde{B}_i)\mathtt{d}$ matches the length of the corresponding block $1B_i$ of x. Due to the above discussion, the map ϕ defined this way is a bijection between $\hat{A}^{\mathrm{pr}}_{n+1}(312)$ and the set of $\mathtt{dudu}$-avoiding Dyck paths of semilength n.

Remark. We end this section with a numerical remark. Bao et al. [2] have recently showed a bijection between $\mathtt{dudu}$-avoiding Dyck paths and the set of permutations that are sorted by the $(132, 321)$-machine. They also characterized these permutations as

$$\mathrm{Sort}(132, 321) = \mathrm{Sym}\left(132, \begin{array}{c}\begin{array}{|c|c|}\hline \bullet & \\\hline & \bullet \\\hline \bullet & \\\hline\end{array}\end{array}\right).$$

Using the BiSC algorithm [1] and Theorem 11, we can conjecture that

$$\mathfrak{st}\left(\hat{A}^{\mathrm{pr}}(312)\right) = \mathrm{Sym}\left(2413, \begin{array}{c}\begin{array}{|c|c|}\hline \bullet & \\\hline & \bullet \\\hline & \bullet \\\hline\end{array}\end{array}, \begin{array}{c}\begin{array}{|c|c|}\hline \bullet & \\\hline & \bullet \\\hline \bullet & \\\hline\end{array}\end{array}\right).$$

Can the Wilf-equivalence between $\mathrm{Sort}_n(132, 321)$ and $\mathfrak{st}\left(\hat{A}^{\mathrm{pr}}_{n+1}(312)\right)$ be explained more transparently?

5.6 Patterns 1213 and 1312

Modified ascent sequences are subject to the geometric constraints established by Proposition 1. This explains the presence of patterns $x, y \in \mathrm{Cay}$, $x \neq y$, equivalent in the sense that $\hat{A}(x) = \hat{A}(y)$. For instance, we [10] have proved that

$$\hat{A}(212) = \hat{A}(1212) = \hat{A}(2132) = \hat{A}(12132).$$

The following result has the same flavor.

Proposition 21. *We have*

$$\hat{A}(213) = \hat{A}(1213) \quad and \quad \hat{A}(312) = \hat{A}(1312).$$

Proof. Clearly, $\hat{A}(213) \subseteq \hat{A}(1213)$. Conversely, let $x \in \hat{A}$ and suppose that x contains 213. Let $x_i x_j x_k$ be an occurrence of 213 in x. Without losing generality, we can assume that x_j is the smallest entry between x_i and x_k, taking the leftmost one in case of ties. Due to our choice, we have $x_{j-1} > x_j$. Hence $x_j \notin \mathrm{top}(x) = \mathrm{nub}(x)$. Further, if x_ℓ is the leftmost copy of x_j in x, then it must be $\ell < i$. Finally, we obtain the desired occurrence $x_\ell x_i x_j x_k$ of 1213. To prove the remaining equality, simply replace 213 with 312 and use the same argument. □

6 Patterns 221 and 2321

Recall from Section 2.2 that any $x \in \hat{A}$ is obtained (uniquely) from a primitive modified ascent sequence w by suitably inserting some flat steps. If y is primitive and w avoids y, then inserting flat steps does not create occurrences of y in x. In other words, all the positions between two consecutive entries of w are active sites in this sense. It is clear that the same mechanic fails if y is not primitive. For instance, the primitive sequence $w = 12$ avoids 122, but the insertion of a flat step at the end gives $x = 122$ (which contains 122). In this section, however, we are able to slightly tweak this approach by computing the distribution of active sites on $\hat{A}^{\mathrm{pr}}(221)$. En passant, we enumerate $\hat{A}(2321)$, finally settling the remaining case of a conjecture by Duncan and Steingrímsson [20].

Proposition 22. *For each $n \geqslant 0$, we have $\hat{A}_n^{\mathrm{pr}}(221) = \hat{A}_n^{\mathrm{pr}}(2321)$. Furthermore,*

$$\mathfrak{st}\left(\hat{A}_n^{\mathrm{pr}}(221)\right) = 1 \oplus \mathrm{Sym}_{n-1}(\text{32-1}), \quad where \quad \text{32-1} = \begin{array}{c}\end{array}.$$

Proof. Let us start with the equality $\hat{A}_n^{\mathrm{pr}}(221) = \hat{A}_n^{\mathrm{pr}}(2321)$. The inclusion $\hat{A}_n^{\mathrm{pr}}(221) \subseteq \hat{A}_n^{\mathrm{pr}}(2321)$ is trivial. To prove the other inclusion, suppose that x contains 221 and let $x_i x_j x_k$ be an occurrence of 221 in x. Note that $x_j \notin \mathrm{nub}(x) = \mathrm{top}(x)$. Since x is primitive, it must be $x_{j-1} > x_j$. Thus $x_i x_{j-1} x_j x_k \simeq 2321$, as wanted.
Next we prove that $\mathfrak{st}\left(\hat{A}_n^{\mathrm{pr}}(221)\right) = 1 \oplus \mathrm{Sym}_{n-1}(\text{32-1})$. Let $x \in \hat{A}_n^{\mathrm{pr}}$ and let $p = \mathfrak{st}(x)$. We show that $x \geqslant 221$ if and only if $p \geqslant \text{32-1}$. Initially, suppose that $x \geqslant 221$. As showed above, x contains an occurrence $x_i x_{j-1} x_j x_k$ of 2321. Then, by Lemma 4, we have $p_{j-1} p_j p_k \simeq \text{32-1}$. Conversely, suppose that p contains an occurrence $p_{j-1} p_j p_k$ of 32-1. By the same lemma, it must be $x_{j-1} > x_j > x_k$, and thus $x_j \notin \mathrm{top}(x) = \mathrm{nub}(x)$. By taking the leftmost copy of x_j in x, say x_i, we get the desired occurrence $x_i x_j x_k$ of 221. □

Duncan and Steingrímsson [20] conjectured that modified ascent sequences avoiding any of the patterns 212, 1212, 2132, 2213, 2231 and 2321 are counted by the Bell numbers. More specifically, they suggested that the distribution of the number of ascents was given by the reverse of the distribution of blocks on set partitions. The current author [10] settled this conjecture for all the patterns except for 2321, which we are finally able to solve here.

Proposition 23. *The cardinality of $\hat{A}_n(2321)$ is equal to the nth Bell number.*

Proof. Claesson [15, Prop. 2] showed that $|\mathrm{Sym}_n(\text{32-1})| = b_n$, where b_n is the nth Bell number. Here, we have:

$$
\begin{aligned}
|\hat{A}_n(2321)| &= \sum_{k=1}^{n} \binom{n-1}{k-1} |\hat{A}_k^{\mathrm{pr}}(2321)| && \text{by Proposition 2} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} |1 \oplus \mathrm{Sym}_{k-1}(\text{32-1})| && \text{by Proposition 22} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} b_{k-1} && \text{by Claesson [15]} \\
&= b_n,
\end{aligned}
$$

where the last equality is a well known recurrence for the Bell numbers. $\qquad\square$

Next, we recall a useful bijection[1] between set partitions of $[n]$ and $\mathrm{Sym}_n(\text{32-1})$ originally discovered by Claesson [15, Prop. 2]. Given a partition β of $[n]$, the *standard representation* of β is obtained by writing

(i) each block with its least element last, and the other elements in increasing order;

(ii) blocks in increasing order of their least element, with dashes separating two consecutive blocks.

For instance, the standard representation of

$$\beta = \{\{1,3,6\}, \{2,7\}, \{4\}, \{5,8,9\}\} \quad \text{is} \quad \beta = 361\text{-}72\text{-}4\text{-}895.$$

Then β is associated with the (32-1)-avoiding permutation p obtained by writing β in standard representation, and erasing the dashes. The set partition β in the previous example is associated with $p = 361724895$. Claesson [15, Prop. 3] showed that the number of (32-1)-avoiding permutations of length n with j right-to-left minima is equal to the (n,j)th Stirling number of the second kind $S(n,j)$. The next lemma follows in a similar fashion.

Lemma 24. *Let $p \in \mathrm{Sym}(\text{32-1})$ be associated with the set partition β via Claesson's bijection. Then $\mathrm{des}(p)$ is equal to the number of blocks of β that are not singletons.*

Proof. There is a descent $p_{i-1} > p_i$ in p if and only if p_i is the minimum of a block of β that has size two or more. $\qquad\square$

From now on, let $\mathrm{Par}[n]$ denote the set of set partitions over $[n]$ and let

$$p_{n,i} = |\{\beta \in \mathrm{Par}[n] : \beta \text{ has } i \text{ blocks that are not singletons}\}|.$$

The coefficients $p_{n,i}$ (see also A124324) are related to the Stirling numbers of the second kind by the following proposition.

---

[1]Claesson's map is defined on permutations avoiding the reverse of 32-1.

Proposition 25. *We have*

$$S(n, n-h) = \sum_{i=h+1}^{n} \binom{n-1}{n-i} p_{i-1,i-1-h}.$$

Proof. Let $\beta \in \mathrm{Par}[n]$ be a set partition with $n-h$ blocks. Then β consists of

- a block A that contains 1;
- some singletons $\{s_1\}, \ldots, \{s_{n-i}\}$;
- some blocks $B_1, \ldots, B_{i-1-h}$ of size at least two,

where $n-i \leqslant n-h-1 \iff i \geqslant h+1$. Alternatively, β is uniquely determined by choosing

- the singletons $\{s_1\}, \ldots, \{s_{n-i}\}$, which can be done in $\binom{n-1}{n-i}$ ways;
- a set partition α of the remaining $n-1-(n-i) = i-1$ elements, excluding 1, with $i-1-h$ blocks that are not singletons; here, the singletons of α shall form the block A, together with 1, while the $i-1-h$ non-singletons block are $B_1, \ldots, B_{i-1-h}$.

More schematically,

$$\beta = \{\overbrace{\{1, a_1, \ldots, a_\ell\}}^{A}, \{s_1\}, \ldots, \{s_{n-i}\}, B_1, \ldots, B_{i-1-h}\};$$
$$\alpha = \{\{a_1\}, \ldots, \{a_\ell\}, B_1, \ldots, B_{i-1-h}\}.$$

Since there are exactly $p_{i-1,i-1-h}$ partitions α as above, our claim follows. $\qquad\square$

Remark 26. A weighted exponential generating function for the coefficients $p_{n,i}$ is

$$P_s(t) = \sum_{n \geqslant 0} \left(\sum_{i \geqslant 0} p_{n,i} s^i \right) \frac{t^n}{n!} = \exp(s(e^t - t - 1) + t),$$

obtained my marking every non-singleton block with s. Proposition 25 could be established algebraically by observing that

$$\sum_{n \geqslant 0} \left(\sum_{i \geqslant 0} S(n,i) s^i \right) \frac{t^n}{n!} = \exp(s(e^t - 1))$$
$$= P_s(t) \cdot \exp\big(t(s-1)\big).$$

The proof is rather technical, and it can be found in the Appendix.

Now, our goal is to prove that the number of 2321-avoiding modified ascent sequences with h ascents is equal to $S(n, n-h)$. By Proposition 22 and Lemma 4,

$$
\begin{aligned}
|\{x \in \hat{A}_n^{\mathrm{pr}}(2321) : \mathrm{asc}(x) = h\}| &= |\{p \in 1 \oplus \mathrm{Sym}_{n-1}(\text{32-1}) : \mathrm{asc}(p) = h\}| \\
&= |\{p \in \mathrm{Sym}_{n-1}(\text{32-1}) : \mathrm{asc}(p) = h-1\}| \\
&= |\{p \in \mathrm{Sym}_{n-1}(\text{32-1}) : \mathrm{des}(p) = n-h-1\}| \\
&= p_{n-1, n-h-1},
\end{aligned}
$$

where the last step follows by Lemma 24. Finally, since the insertion of any number of flat steps preserves the number of ascents, by Proposition 2 we have

$$
\begin{aligned}
|\{x \in \hat{A}_n(2321) : \mathrm{asc}(x) = h\}| &= \sum_{i=h+1}^{n} \binom{n-1}{i-1} |\{x \in \hat{A}_i^{\mathrm{pr}}(2321) : \mathrm{asc}(x) = h\}| \\
&= \sum_{i=h+1}^{n} \binom{n-1}{i-1} p_{i-1, i-h-1} \\
&= S(n, n-h),
\end{aligned}
$$

where the last equality is Proposition 25.

Let us now go back to the pattern 221.

Proposition 27. *We have*

$$
|\hat{A}_n(221)| = \sum_{k=1}^{n} \sum_{i=1}^{k} S(k-1, i-1) \binom{n-1-k+i}{i-1}.
$$

Proof. Let $w \in \hat{A}_k^{\mathrm{pr}}(221)$. For $i = 1, 2, \ldots, k$, we say that i is an *active site* if inserting a flat step $a = w_i$ in the position between w_i and w_{i+1} (or after w_k, if $i = k$) does not create an occurrence of 221; that is, if

$$
w_1 \cdots w_i \, w_i \, w_{i+1} \cdots w_k \quad \text{avoids 221.}
$$

It is easy to see that i is active if and only if w_i is a weak right-to-left minimum. Specifically, if $w \in \hat{A}_k^{\mathrm{pr}}(221)$ has i weak right-to-left minima, then w has $k-i$ sites that are not active. Now, any sequence $x \in \hat{A}_n(221)$ is obtained from some $w \in \hat{A}_k^{\mathrm{pr}}(221)$, with $1 \leqslant k \leqslant n$, by inserting $n-k$ flat steps among a total of $n-1$ positions (recall that $x_1 = 1$ is forced), minus the $k - |\mathrm{wrlmin}(w)|$ sites that are not active. Thus, we can adapt the formula of Proposition 2 accordingly to obtain

$$
\begin{aligned}
|\hat{A}_n(221)| &= \sum_{k=1}^{n} \sum_{i=1}^{k} |\{w \in \hat{A}_k^{\mathrm{pr}}(221) : \#\mathrm{wrlmin}(w) = i\}| \binom{n-1-k+i}{n-k} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{k} |\{w \in \hat{A}_k^{\mathrm{pr}}(221) : \#\mathrm{wrlmin}(w) = i\}| \binom{n-1-k+i}{i-1}.
\end{aligned}
$$

y	$|\hat{A}_n(y)|$	$|\hat{A}_n^{\mathrm{pr}}(y)|$
111	1, 2, 4, 10, 29, 97, 367, 1550	1, 1, 2, 5, 14, 46, 172, 718, 3317, 16796
211,1223	A047970?	1, 1, 2, 5, 14, 44, 153, 581, 2385
1324,1342	A007317?	Catalan?
4321	1, 2, 5, 15, 53, 217, 1008, 5188	1, 1, 2, 5, 16, 61, 265, 1267

Table 2: Unsolved patterns.

Finally, by Proposition 22 and Lemma 6,

$$|\{w \in \hat{A}_k^{\mathrm{pr}}(221) : \#\mathrm{wrlmin}(w) = i\}| = |\{p \in \mathrm{Sym}_{k-1}(32\text{-}1) : \#\mathrm{rlmin}(p) = i - 1\}|$$
$$= S(k - 1, i - 1),$$

where the last equality is once again due to Claesson [15, Prop. 3]. $\square$

For $n \geqslant 0$, the sequence $|\hat{A}_n(221)|$ starts with $1, 1, 2, 5, 14, 44, 155, 607, 2617$ and does not appear in the OEIS [27].

7 Final remarks and future directions

In this paper, we enumerated the sets $\hat{A}(y)$ for every pattern y of length at most three, except for $y \in \{111, 211\}$. Interestingly, both patterns are currently open on plain ascent sequences too. We have reported the corresponding data in Table 2, together with longer patterns we were not able to solve despite the promising evidence. We end with a list of suggestions for future work.

- In Section 5.6, we proved that $\hat{A}(213) = \hat{A}(1213)$ and $\hat{A}(312) = \hat{A}(1312)$. Are there any other examples of patterns that are equivalent in this sense? More in general, can we characterize all the sets of equivalent patterns?

- There is only one Cayley permutation x whose standardization is the decreasing permutation $p = k \cdots 21$, namely $x = p$. Thus, by Theorem 11,

$$\mathfrak{st}\big(\hat{A}^{\mathrm{pr}}(k \cdots 21)\big) = \Omega(k \cdots 21).$$

We solved the case $k = 3$ in Section 5.2. Can we tackle the general case with the same approach? In a similar fashion, can we generalize what we proved in Section 5.3 for $\hat{A}^{\mathrm{pr}}(123)$ to $\hat{A}^{\mathrm{pr}}(12 \cdots k)$?

- Among the unsolved patterns, it appears that

$$\hat{A}_n(211) = \hat{A}_n(1223) \quad \text{and} \quad \hat{A}_n^{\mathrm{pr}}(211) = \hat{A}_n^{\mathrm{pr}}(1223)$$

at least up to $n = 10$. Can we prove that the equalities hold for every n?

- Note the following two, rather curious, chains of inclusions:

$$\begin{array}{ccccc}
\hat{A}^{\mathrm{pr}}(213) & \subseteq & \hat{A}(213) & \subseteq & \hat{A}(1324); \\
\hat{A}^{\mathrm{pr}}(231) & \subseteq & \hat{A}(231) & \subseteq & \hat{A}(1342), \\
\text{(Motzkin)} & & \text{(Catalan)} & & \text{(A007317?)}
\end{array}$$

  where each term is (counted by) the binomial transform of the term to its left. Can we use this to count $\hat{A}(1324)$ and $\hat{A}(1342)$? Is this phenomenon more general?

- In Section 4.3, we found an OGF for $\hat{A}(122)$. It appears that

$$\hat{A}_{122}(t) = (1 - t)\hat{A}_{211}(t),$$

  where 211 is one of the patterns we could not solve. Why?

- We have decided to leave the study of modified ascent sequences avoiding pairs (or sets) of patterns for a future investigation. An example that is particularly dear to us is the following. The Burge transpose [11] maps bijectively $\hat{A}(2312, 3412)$ to the set $F(3412)$ of Fishburn permutations avoiding 3412. A numerical analysis suggests that the pair of statistics right-to-left maxima and right-to-left minima on $F(3412)$ is equidistributed with the pair left-to-right maxima and right-to-left maxima over the set of 312-sortable permutations [13]. The first terms of the arising counting sequence match A202062 [27]. Currently, no formula or generating function for A202062 is known. An asymptotic analysis of this sequence has been conducted recently by Conway et al. [16].

Acknowledgements

References

[1] R. P. Ardal, T. K. Magnusson, É. Nadeau, B. J. Kristinsson, B. A. Gudmundsson, C. Bean, H. Ulfarsson, J. S. Eliasson, M. Tannock, A. B. Bjarnason, J. Pantone, A B. Arnarson, S. I. Jónsson. *PermutaTriangle/Permuta: Version 2.1.0*, https://doi.org/10.5281/zenodo.4945792.

[2] C. Bao, G. Cerbai, Y. Choi, K. Gan, O. Zhang, *On a conjecture on pattern-avoiding machines*, Annals of Combinatorics, Vol. 29, pp. 123-141, 2025.

[3] M. Bernstein, N. J. A. Sloane, *Some canonical sequences of integers*, Linear Algebra and its Applications, Vol. 226-228, pp. 57–72, 1995.

[4] D. Bevan, *Permutation patterns: basic definitions and notations*, arXiv:1506.06673, 2015.

[5] M. Bousquet-Mélou, A. Claesson, M. Dukes, S. Kitaev, *(2+2)-free posets, ascent sequences and pattern avoiding permutations*, Journal of Combinatorial Theory, Series A, Vol. 117, pp. 884–909, 2010.

[6] W. H. Burge, *Four correspondences between graphs and generalized Young tableaux*, Journal of Combinatorial Theory, Series A, Vol. 17, pp. 12–30, 1974.

[7] W. Cao, E. Y. Jin, Z. Lin, *Enumeration of inversion sequences avoiding triples of relations*, Discrete Applied Mathematics, Vol. 260, pp. 86–97, 2019.

[8] A. Cayley, *On the analytical forms called trees*, Collected Mathematical Papers, Vol. 4, Cambridge University Press, pp. 112–115, 1891.

[9] G. Cerbai, *Sorting Cayley permutations with pattern-avoiding machines*, Australasian Journal of Combinatorics, Vol. 80(3), 2021.

[10] G. Cerbai, *Modified ascent sequences and Bell numbers*, The Electronic Journal of Combinatorics, Vol. 31(4):#P4.31, 2024.

[11] G. Cerbai, A. Claesson, *Transport of patterns by Burge transpose*, European Journal of Combinatorics, Vol. 108, 2023.

[12] G. Cerbai, A. Claesson, *Fishburn trees*, Advances in Applied Mathematics, Vol. 151, 2023.

[13] G. Cerbai, A. Claesson, L. Ferrari, *Stack sorting with restricted stacks*, Journal of Combinatorial Theory, Series A, Vol. 173, 2020.

[14] G. Cerbai, A. Claesson, L. Ferrari, E. Steingrímsson, *Sorting with pattern-avoiding stacks: the 132-machine*, The Electronic Journal of Combinatorics, Vol. 27(3), #P3.32, 2020.

[15] A. Claesson, *Generalized pattern avoidance*, European Journal of Combinatorics, Vol. 22, pp. 961–971, 2001.

[16] A. R. Conway, M. Conway, A. Elvey Price, A. J. Guttmann, *Pattern-avoiding ascent sequences of length 3*, The Electronic Journal of Combinatorics, Vol. 29(4), #P4.25, 2022.

[17] R. Donaghey, *Restricted plane tree representations of four Motzkin-Catalan equations*, Journal of Combinatorial Theory, Series B, Vol. 22(2), pp. 114–121, 1977.

[18] M. Dukes, S. Kitaev, J. Remmel, E. Steíngrimsson, *Enumerating (2+2)-free posets by indistinguishable elements*, Journal of Combinatorics, Vol. 2, pp. 139-163, 2011.

[19] M. Dukes and R. Parviainen, *Ascent sequences and upper triangular matrices containing non-negative integers*, The Electronic Journal of Combinatorics, Vol. 17, #R53, 2010.

[20] P. Duncan, E. Steingrímsson, *Pattern avoidance in ascent sequences*, The Electronic Journal of Combinatorics, Vol. 18(1), #P226, 2011.

[21] J. B. Gil, M. D. Weiner, *On pattern-avoiding Fishburn permutations*, Annals of Combinatorics, Vol. 23, pp. 785–800, 2019.

[22] S. Kitaev, *Patterns in permutations and words*, Monographs in Theoretical Computer Science. An EATCS Series, Springer, Heidelberg, 2011.

[23] M. Mor, A. S. Fraenkel, *Cayley permutations*, Discrete mathematics, Vol. 48(1), pp. 101–112, 1984.

[24] R. Parviainen, *Wilf classification of bi-vincular patterns*, arXiv:0910.5103, 2009.

[25] A. Sapounakis, I. Tasoulas, P. Tsikouras, *Counting strings in Dyck paths*, Discrete Mathematics, Vol. 307, pp- 2909–2924, 2007.

[26] C. Schensted, *Longest Increasing and Decreasing Subsequences*, Canadian Journal of Mathematics, Vol. 13, pp. 179–191, 1961.

[27] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, at oeis.org.

[28] R. P. Stanley, *Enumerative combinatorics*, Vol. I, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks, 1986.

Appendix

Let

$$P_s(t) = \sum_{n \geqslant 0} \left(\sum_{i \geqslant 0} p_{n,i} s^i \right) \frac{t^n}{n!} \quad \text{and} \quad Q_s(t) = \sum_{n \geqslant 0} \left(\sum_{i \geqslant 0} S(n,i) s^i \right) \frac{t^n}{n!}$$

be the (weighted) exponential generating functions of the coefficients $p_{n,i}$, defined in Section 6, and the Stirling numbers of the second kind $S(n,i)$, respectively. We give an algebraic proof of the formula

$$S(n, n-h) = \sum_{i=h+1}^{n} \binom{n-1}{n-i} p_{i-1,i-1-h},$$

which we proved combinatorially in Proposition 25. Recall from Remark 26 that

$$Q_s(t) = P_s(t) \cdot \exp\big(t(s-1)\big)$$
$$\iff Q_s(t) \cdot \exp(t) = P_s(t) \cdot \exp(st).$$

Let us expand both sides of the latter equation. First,

$$Q_s(t) \cdot \exp(t) = \left(\sum_{n \geqslant 0} \left(\sum_{k \geqslant 0} S(n,k) s^k \right) \frac{t^n}{n!} \right) \cdot \left(\sum_{n \geqslant 0} \frac{t^n}{n!} \right)$$
$$= \sum_{n \geqslant 0} \left(\sum_{j \geqslant 0} \binom{n}{j} \sum_{k \geqslant 0} S(j,k) s^k \right) \frac{t^n}{n!}$$
$$= \sum_{n \geqslant 0} \left(\sum_{j \geqslant 0} \sum_{k \geqslant 0} \binom{n}{j} S(j,k) s^k \right) \frac{t^n}{n!}$$
$$= \sum_{n \geqslant 0} \left(\sum_{k \geqslant 0} S(n+1, k+1) s^k \right) \frac{t^n}{n!},$$

where at the last step we used that

$$S(n+1, k+1) = \sum_{j \geqslant 0} \binom{n}{j} S(j, k).$$

Secondly,

$$P_s(t) \cdot \exp(st) = \left(\sum_{n \geqslant 0} \left(\sum_{k \geqslant 0} p_{n,k} s^k \right) \frac{t^n}{n!} \right) \cdot \left(\sum_{n \geqslant 0} s^n \frac{t^n}{n!} \right)$$

$$= \sum_{n \geqslant 0} \left(\sum_{j \geqslant 0} \binom{n}{j} \sum_{k \geqslant 0} p_{j,k} s^k s^{n-j} \right) \frac{t^n}{n!}$$

$$= \sum_{n \geqslant 0} \left(\sum_{j \geqslant 0} \sum_{k \geqslant 0} \binom{n}{j} p_{j,k} s^{n+k-j} \right) \frac{t^n}{n!}.$$

By comparing the coefficients in front of $x^n/n!$ (and using ℓ instead of k in the left-hand sum), we obtain

$$\sum_{\ell \geqslant 0} S(n+1, \ell+1) s^\ell = \sum_{j \geqslant 0} \left(\sum_{k \geqslant 0} \binom{n}{j} p_{j,k} \right) s^{n+k-j}.$$

Finally, since $\ell = n + k - j \iff k = \ell + j - n$, we have

$$S(n+1, \ell+1) = \sum_{j \geqslant 0} \binom{n}{j} p_{j,\ell+j-n}$$

$$= \sum_{j=n-\ell}^{n} \binom{n}{j} p_{j,\ell+j-n}$$

$$\iff S(n,\ell) = \sum_{j=n-\ell}^{n-1} \binom{n-1}{j} p_{j,(\ell-1)+j-(n-1)}$$

$$\iff S(n,\ell) = \sum_{i=n-\ell+1}^{n} \binom{n-1}{i-1} p_{i-1,\ell+(i-1)-n}$$

$$\iff S(n, n-h) = \sum_{i=h+1}^{n} \binom{n-1}{i-1} p_{i-1,i-1-h}$$

$$= \sum_{i=h+1}^{n} \binom{n-1}{n-i} p_{i-1,i-1-h}.$$