Rainbow Common Graphs Must Be Forests

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Abstract

We study the rainbow version of the graph commonness property: a graph H is *r-rainbow common* if the number of rainbow copies of H (where all edges have distinct colors) in an r-coloring of edges of K_n is maximized asymptotically by independently coloring each edge uniformly at random. H is *r-rainbow uncommon* otherwise. We show that if H has a cycle, then it is r-rainbow uncommon for every r at least the number of edges of H. This generalizes a result of Erdős and Hajnal, and proves a conjecture of De Silva, Si, Tait, Tunçbilek, Yang, and Young.

Mathematics Subject Classifications: 05C15, 05D10

1 Introduction

1.1 History

In extremal graph theory, a graph H is *common* if the number of copies of H in any n vertex graph G and its complement $\overline{G}$ is minimized asymptotically by the Erdős-Rényi random graph $\mathbb{G}(n, 1/2)$. In other words, the minimum number of monochromatic copies of H in any 2-edge coloring of K_n is asymptotically achieved by coloring each edge independently uniformly at random. Commonness is extensively studied due to its connection to other homomorphism density inequalities, including the Sidorenko conjecture. In particular, Sidorenko graphs are common [7].

A similar question is asked in anti-Ramsey theory: instead of minimizing the number of monochromatic H, we maximize the number of *rainbow* copies of H, where all edges have distinct colors [3, 1, 2].

Definition 1. For a graph H and $r \in \mathbb{N}$, we say that H is *r-rainbow common* if the maximum number of rainbow copies of H in an r-coloring of edges of K_n is achieved asymptotically by coloring each edge independently with a uniform random color. Otherwise, H is *r-rainbow uncommon*.

We say that H is *rainbow common* (resp. *rainbow uncommon*) if it is r-rainbow common (resp. r-rainbow uncommon) for all $r \geqslant e(H)$, where $e(H)$ denotes the number of edges of H.

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All asymptotics are up to a $1 + o(1)$ factor as $n \to \infty$. This definition of r-rainbow commonness is called r-anti-common in [2] and r-rainbow uncommon graphs are called not r-anti-common.

If $r < e(H)$, the condition in Definition 1 is trivial as there is no rainbow H by the pigeonhole principle. Note that H can be r-rainbow common for some r and s-rainbow uncommon for $s \neq r$, in which case it is neither rainbow common nor uncommon. However, it is conjectured that no such H exists [2, Conjecture 5.2]

It is an old result of Erdős and Hajnal [3] that K_3 is 3-rainbow uncommon: There is a coloring that beats the $2/9$ density of rainbow K_3's obtained by the uniform random 3-coloring. Erdős and Sós posed the question to determine the maximum density [9]. With a flag algebra approach, [1] settled this question and determined that the maximum density is $2/5$ asymptotically.

More recently, [2] showed that disjoint unions of stars $K_{1,s}$ are rainbow common and introduced the iterated blowup technique to show that K_s is $\binom{s}{2}$-rainbow uncommon for all $s \in \mathbb{N}$, thereby generalizing the result of Erdős and Hajnal. In [2], it is conjectured that for every $s \geqslant 3$, the cycle graph with s edges is s-rainbow uncommon and path graph with s edges is s-rainbow common.

1.2 Main results

In this paper, we extend the result of Erdős and Hajnal [3] and settle the cycle conjecture in [2] with a much stronger statement.

Theorem 2. *If graph H contains a cycle, then it is rainbow uncommon.*

Since graphs without cycles must be forests, the following result is immediate.

Corollary 3. *If graph H is r-rainbow common for any $r \geqslant e(H)$, then H is a forest.*

In Section 3, we prove Theorem 2 using a graphon perturbation technique inspired by that of [5, 6] to study local Sidorenko properties.

1.3 Notation

In this paper, we consider simple graphs G with vertex set $V(G)$ and edge set $E(G)$. Let $v(G)$ and $e(G)$ denote the size of $V(G)$ and $E(G)$, respectively. Let $[n] := \{1, 2, \ldots, n\}$. Let $(n)_k := k!\binom{n}{k}$ be the falling factorial. Let binomial coefficient $\binom{n}{k} = 0$ if $k > n$.

2 Preliminaries

2.1 Graphons

In this section, we introduce the notions of graphons and graph homomorphism densities that we need for Theorem 2. We will follow [12, 7], where complete expositions are given.

Definition 4. A *graphon* is a measurable function $W : [0, 1]^2 \to [0, 1]$, where $W(x, y) = W(y, x)$ for all $x, y \in [0, 1]$.

- We define the *associated graphon* W_G of a graph G: partition $[0,1]$ into $v(G)$ equal-length intervals $I_1, \dots, I_{v(G)}$. For $(x,y) \in I_i \times I_j$, define $W_G = \mathbf{1}\{(i,j) \in E(G)\}$.
- For a graphon W, the *W-random graph* $\mathbb{G}(n,W)$ on $[n]$ is defined by sampling independently $x_1, \dots, x_n \in [0,1]$ and putting an edge (i,j) with probability $W(x_i, x_j)$ independently.

For graphs G, H and graphon W, define the *H-homomorphism density in G and W* as

$$t(H,G) = \frac{\hom(H,G)}{v(G)^{v(H)}} \quad \text{and} \quad t(H,W) = \int_{[0,1]^{v(H)}} \prod_{(u,v)\in E(H)} W(x_u, x_v) \prod_{v \in V(H)} dx_v,$$

respectively, where $\hom(H,G)$ is the number of graph homomorphisms from H to G.

For example, the Erdős-Rényi random graph $\mathbb{G}(n,p)$ is W-random graph $\mathbb{G}(n,W)$, where W is the constant graphon $W(x,y) = p$. One can similarly view stochastic block models as graphons.

In this language, a graph H is Sidorenko if $t(H,W) \geqslant t(K_2,W)^{e(H)}$ for every graphon W and H is common if $t(H,W) + t(H, 1-W) \geqslant 2^{1-e(H)}$ for every W [12, 7]. Note that $t(H,G)$ is the probability that a uniform random map from $V(H) \to V(G)$ is a graph homomorphism. The following simple observation shows that the two definitions of homomorphism densities coincide for graphs.

Lemma 5. *$t(H, W_G) = t(H,G)$ for all graphs G, H and graphon W_G associated to G.*

The following statement captures the viewpoint that graphons are limits of graphs in the context of homomorphism densities. This uses the notion of left convergence of graphs to graphons.

Proposition 6 ([12, Theorem 4.4.2])**.** *Let W be a graphon. For each $n \in \mathbb{N}$, let $G_n \sim \mathbb{G}(n,W)$. Then, G_n left converges to W almost surely, i.e. for all graphs H, $t(H,G_n) \to t(H,W)$ as $n \to \infty$.*

An alternative viewpoint uses the notion of convergence in *cut metric* $\delta_\square$ to view graphons as the completion of the space of graphs with respect to $\delta_\square$. We refer to [12, 7, 8] for more details.

2.2 Colorings

We can view graphs as two colorings of complete graphs. This perspective allows us to naturally extend the notions of graphons and homomorphism densities to edge colorings by considering the graphons associated to the induced subgraph spanned by each color.

Definition 7. For $r \in \mathbb{N}$, we say that $W = (W_1, \dots, W_r)$ is an *r-coloring graphon* if W_i is a graphon for every $i \in [r]$ and, for every $x, y \in [0,1]$,

$$\sum_{i=1}^{r} W_i(x,y) = 1. \tag{1}$$

We define the *associated r-coloring graphon* of a coloring $\phi : E(K_n) \to [r]$ as $W_\phi = ((W_\phi)_1, \dots, (W_\phi)_r)$, where $(W_\phi)_i$ is the associated graphon of graph $G_i = (V(G), \phi^{-1}(i))$ spanned by edges of color i.

Similar to the Erdős-Rényi random graph $\mathbb{G}(n,p)$, the uniform random r-coloring can be represented by the r-coloring graphon $\mathbf{1}/r = (1/r, \dots, 1/r)$. We now define an analog of homomorphism densities to count the density of rainbow copies of H in an r-coloring graphon W with $r \geqslant e(H)$.

Definition 8. Given a graph H and an r-coloring graphon $W = (W_1, \dots, W_r)$, let $\mathcal{H}$ be the set of injections from $E(H)$ to $[r]$. We define the *rainbow homomorphism density of H in W* as

$$t(\mathrm{rb}\, H, W) := \sum_{h \in \mathcal{H}} \int_{[0,1]^{v(H)}} \left(\prod_{(u,v) \in E(H)} W_{h(u,v)}(x_u, x_v) \right) \prod_{v \in V(H)} dx_v. \qquad (2)$$

Note that $h(u,v)$ is the color of edge (u,v), which occurs with probability W_h. The condition of $\mathcal{H}$ that h is injective corresponds exactly to the copy we picked out being rainbow. The analogs of Lemma 5 and Proposition 6 hold for rainbow graph density. For brevity, we only state and prove what we need for Theorem 2.

Lemma 9. *For any $r, n \in \mathbb{N}$, given an r-coloring graphon $W = (W_1, \dots, W_r)$, we define a random r-coloring $\phi \sim \mathbb{G}(n, W)$ of edges of K_n with vertex set $[n]$ as follows.*

1. *Independently sample $x_1, x_2, \dots, x_n$ uniformly from $[0,1]$.*
2. *For every edge $(u,v) \in E(K_n)$, independently sample $\phi(u,v) \in [r]$, where $\phi(u,v) = i$ with probability $W_i(x_u, x_v)$. Color edge (u,v) by color $\phi(u,v)$.*

Then, the probability that a uniform random copy of H is rainbow under ϕ is $t(\mathrm{rb}\, H, W)$.

Proof. Fix a uniform random copy H. We compute the probability that it is rainbow. The rainbow edge colorings of H are precisely $\mathcal{H}$. For each $h \in \mathcal{H}$, the probability that ϕ colors edge (u,v) of H by color $h(u,v)$ is precisely $W_{h(u,v)}(x_u, x_v)$. By independence of edges (u,v), we take the product and take the expectation over $x_1, \dots, x_n$ independently uniform on $[0,1]$ to prove Lemma 9. □

Corollary 10. *For a graph H and $r \geqslant e(H)$, H is r-rainbow uncommon if there exists an r-coloring graphon $W = (W_1, \dots, W_r)$ such $t(\mathrm{rb}\, H, W) > t(\mathrm{rb}\, H, \mathbf{1}/r) = r^{-e(H)}(r)_{e(H)}$.*

Proof. By (2), we compute that $t(\mathrm{rb}\, H, \mathbf{1}/r) = r^{-e(H)}(r)_{e(H)}$. This is equal to the probability that a uniform random r-coloring of edges of H is rainbow. The total number of copies of H in K_n is

$$\frac{(n)_{v(H)}}{|\mathrm{Aut}(H)|} = \Theta\left(n^{v(H)}\right),$$

where $|\mathrm{Aut}(H)| = \hom(H,H)$ is the number of automorphisms of H. By linearity of expectation and Lemma 9, if $t(\mathrm{rb}\, H, W) > r^{-e(H)}(r)_{e(H)}$, then the expected number of rainbow H under $\phi \sim \mathbb{G}(n, W)$ is asymptotically greater than that of $\mathbf{1}/r$. By the probabilistic method, there exists such a coloring ϕ of $E(K_n)$, so H is r-rainbow uncommon. □

We can now pass from asymptotics in n to coloring graphons W.

3 Graphon Perturbation

3.1 Some lemmas

We prove Theorem 2 by constructing a graphon coloring W in Corollary 10 as a perturbation of the uniform random coloring graphon $\mathbf{1}/r$. To do so, we need a few lemmas.

Lemma 11. *Define $f : [0,1]^2 \to \mathbb{R}$ to be 1 on $[0,1/2]^2 \cup [1/2,1]^2$ and -1 otherwise.*

1. *For every $x, y \in [0,1]$, $f(x,y) = f(y,x)$.*
2. *For every $t \in [0,1]$,*
$$\int_0^1 f(x,t)dx = \int_0^1 f(t,x)dx = 0.$$
3. *For every $s \in \mathbb{N}$, let $x_{s+1} = x_1$. Then,*
$$\int_{[0,1]^s} \prod_{i=1}^{s} f(x_i, x_{i+1})dx_1 \dots dx_s = 1.$$

Proof. (1) and (2) are clear.

For (3), let Z_i be 0 if $x_i \in [0,1/2]$ and 1 otherwise, so $f(x_i, x_{i+1}) = (-1)^{Z_i + Z_{i+1}}$. Hence, with expectation taken over independent $Z_i \sim \mathrm{Unif}(\{0,1\})$, we have that

$$\int_{[0,1]^s} \prod_{i=1}^{s} f(x_i, x_{i+1})dx_1 \dots dx_s = \mathbb{E}\left[\prod_{i=1}^{s}(-1)^{Z_i+Z_{i+1}}\right] = \mathbb{E}\left[(-1)^{2\sum_{i=1}^{s} Z_i}\right] = 1. \qquad \square$$

Let f be as in Lemma 11. For any graph G, define

$$\mathcal{I}_f(G) := \int_{[0,1]^{v(G)}} \left(\prod_{(x_u,x_v)\in E(G)} f(x_u,x_v)\right) \prod_{v\in V(G)} dx_v. \tag{3}$$

Then, Lemma 11(3) says that $\mathcal{I}_f(G) = 1$ if G is a cycle graph.

Lemma 12. *For f defined in Lemma 11, $\mathcal{I}_f(G) = 0$ if G has a leaf.*

Proof. If leaf ℓ is connected to vertex k, then $\int_0^1 f(x_\ell, x_k)dx_\ell = 0$ by Lemma 11(2). Now,

$$\mathcal{I}_f(G) = \int_{[0,1]^{v(G)-1}} \left(\int_0^1 f(x_\ell,x_k)dx_\ell\right) \prod_{(x_u,x_v)\in E(G)\setminus\{(x_k,x_\ell)\}} f(x_u,x_v) \prod_{v\in V(G)\setminus\{\ell\}} dx_v = 0. \quad \square$$

Lemma 13. *For every $r \geqslant s \geqslant 3$, there exists some $k := k(r,s) \in [r-1]$ such that*

$$F(r,s,k) := \sum_{i=0}^{s}(-1)^{s-i}\binom{k}{i}\binom{r-k}{s-i}\left(\frac{r-k}{k}\right)^i > 0. \tag{4}$$

Proof. Let $\ell = r - s \geqslant 0$. For $s \geqslant 4$, we have that

$$\begin{aligned}
F(r,s,r-2) &= \binom{r-2}{s}\left(\frac{2}{r-2}\right)^{s} - 2\binom{r-2}{s-1}\left(\frac{2}{r-2}\right)^{s-1} + \binom{r-2}{s-2}\left(\frac{2}{r-2}\right)^{s-2} \\
&= \left(\frac{2}{r-2}\right)^{s}\binom{r-2}{s-2}\left(\frac{(r-s)(r-s-1)}{s(s-1)} - \frac{(r-s)(r-2)}{s-1} + \frac{1}{4}(r-2)^2\right) \\
&= \left(\frac{2}{r-2}\right)^{s}\binom{r-2}{s-2}\left(\frac{\ell(\ell-1)}{s(s-1)} - \frac{\ell(s+\ell-2)}{s-1} + \frac{1}{4}(s+\ell-2)^2\right) \\
&= \left(\frac{2}{r-2}\right)^{s}\binom{r-2}{s-2}\left(\frac{s+\ell}{4s}\right)\left((s-4)\ell + (s-2)^2\right) \\
&> 0.
\end{aligned}$$

For $s = 3$, we know $r \geqslant 3$, so

$$F(r,3,1) = \binom{r-1}{2}(r-1) - \binom{r-1}{3} = \frac{r(r-1)(r-2)}{3} > 0. \qquad \square$$

3.2 Proof of Theorem 2

Fix any graph H with girth $s \in \mathbb{N}$ and positive integer $r \geqslant e(H)$,

$$r \geqslant e(H) \geqslant s \geqslant 3. \tag{5}$$

Hence, we can apply Lemma 13 to define $k = k(r,s) \in [r-1]$. Define $\sigma : [r] \to \mathbb{R}$ by

$$\sigma(i) = \begin{cases} \frac{1}{k} & \text{if } i \in [k] \\ -\frac{1}{r-k} & \text{if } i \in [r] \setminus [k] \end{cases}. \tag{6}$$

Recall f from Lemma 11. For $\varepsilon \in (0, 1/r]$ chosen later, define $W_i : [0,1]^2 \to [0,1]$ by

$$W_i(x,y) = \frac{1}{r} + \varepsilon\sigma(i) f(x,y) \tag{7}$$

for each $i \in [r]$. Clearly, W_i is measurable for each i. For every $x, y \in [0,1]$, $W_i(x,y) = W_i(y,x)$ as $f(x,y) = f(y,x)$. Since $\sigma_i \in [-1,1]$ and $f(x,y) \in [-1,1]$, choosing any $\varepsilon \in [0,1/r]$ satisfies $W_i(x,y) \in [0,1]$. Hence, W_i is a graphon for each i. Now, for any $x, y \in [0,1]$, we compute that

$$\sum_{i=1}^{r} W_i(x,y) = 1 + \varepsilon f(x,y) \sum_{i=1}^{r} \sigma(i) = 1,$$

so $W = (W_1, \ldots, W_r)$ is an r-coloring graphon. Recall that $\mathcal{H}$ is the set of injections from $E(H)$ to $[r]$. By Corollary 10, it suffices to show that $t(\mathrm{rb}\, H, W) > t(\mathrm{rb}\, H, \mathbf{1}/r) = r^{-e(H)}|\mathcal{H}|$. Note that

$$t(\mathrm{rb}\, H, W) = \sum_{h \in \mathcal{H}} \int_{[0,1]^{v(G)}} \prod_{(u,v) \in E(H)} \left[\frac{1}{r} + \varepsilon\sigma(h(x_u, x_v)) f(x_u, x_v)\right] \prod_{v \in V(H)} dx_v$$

$$= \sum_{h \in \mathcal{H}} \int_{[0,1]^{v(G)}} \sum_{G \subset H} r^{e(G)-e(H)} \varepsilon^{e(G)} \prod_{(u,v) \in E(G)} \sigma(h(u,v)) f(x_u, x_v) \prod_{v \in V(H)} dx_v$$
$$= \sum_{G \subset H} r^{e(G)-e(H)} \varepsilon^{e(G)} \sum_{h \in \mathcal{H}} \int_{[0,1]^{v(G)}} \prod_{(u,v) \in E(G)} \sigma(h(u,v)) f(x_u, x_v) \prod_{v \in V(H)} dx_v,$$

where in the second step we expand the first product and index the terms, where we pick out the second term in the square brackets by subgraph $G \subset H$. The summand corresponding to $G = \varnothing$ is exactly $r^{-e(H)}|\mathcal{H}| = r^{-e(H)}(r)_{e(H)} = t(\mathrm{rb}\, H, \mathbf{1}/r)$. Since f does not depend on h and σ does not depend on x_v, we can factor out the terms to rewrite

$$t(\mathrm{rb}\, H, W) - t(\mathrm{rb}\, H, \mathbf{1}/r)$$
$$= \sum_{\varnothing \neq G \subset H} r^{e(G)-e(H)} \varepsilon^{e(G)} \sum_{h \in \mathcal{H}} \prod_{(u,v) \in E(G)} \sigma(h(u,v)) \left(\int_{[0,1]^{v(G)}} \prod_{(u,v) \in E(G)} f(x_u, x_v) \prod_{v \in V(G)} dx_v \right)$$
$$= \sum_{\varnothing \neq G \subset H} r^{e(G)-e(H)} \varepsilon^{e(G)} \left(\int_{[0,1]^{v(G)}} \prod_{(u,v) \in E(G)} f(x_u, x_v) \prod_{v \in V(G)} dx_v \right) \left(\sum_{h \in \mathcal{H}} \prod_{(u,v) \in E(G)} \sigma(h(u,v)) \right)$$
$$= \sum_{\varnothing \neq G \subset H} r^{e(G)-e(H)} \varepsilon^{e(G)} \mathcal{I}_f(G) \left(\sum_{h \in \mathcal{H}} \prod_{(u,v) \in E(G)} \sigma(h(u,v)) \right)$$

by (3). Now, by Lemma 12, $\mathcal{I}_f(G)$ is zero unless $G \subset H$ has no leaves. Since H has girth s, the only subgraphs $G \subset H$ such that $\mathcal{I}_f(G) \neq 0$ and $e(G) \leqslant s$ are those isomorphic to the cycle graph on s vertices, written $G \simeq C_s$. There, $\mathcal{I}_f(G) = 1$ by Lemma 11(3). We make the following claim.

Claim 14. $Q(G) := \sum_{h \in \mathcal{H}} \prod_{(u,v) \in E(G)} \sigma(h(u,v)) > 0$ *for all* $G \subset H$ *with* $G \simeq C_s$.

We first show how Claim 14 finishes the proof. For all other $G \not\simeq C_s$ such that $\mathcal{I}_f(G) \neq 0$, $e(G) > s$. Once we fixed r and H, everything is fixed except for $\varepsilon \in (0, 1/r]$. Then, by taking ε sufficiently small, the term with $G \simeq C_s$ dominates and is positive, i.e.

$$t(\mathrm{rb}\, H, W) - t(\mathrm{rb}\, H, \mathbf{1}/r) = \sum_{\varnothing \neq G \subset H} r^{e(G)-e(H)} \varepsilon^{e(G)} \mathcal{I}_f(G) \left(\sum_{h \in \mathcal{H}} \prod_{(u,v) \in E(G)} \sigma(h(u,v)) \right)$$
$$= \varepsilon^s \left(\sum_{G \subset H : G \simeq C_s} r^{s-e(H)} Q(G) \right) + O_{\varepsilon \to 0}(\varepsilon^{s+1})$$
$$\geqslant \frac{1}{2} \varepsilon^s \sum_{G \subset H : G \simeq C_s} r^{s-e(H)} Q(G)$$
$$> 0,$$

where the second to last step holds by choosing sufficiently small ε with respect to r and H, and the last step holds by Claim 14. By Corollary 10, it remains to prove Claim 14.

Proof of Claim 14. Recall that $h(u,v)$ specifies the color of edge (u,v). By (6), $\sigma(h(u,v))$ depends only on whether $h(u,v) \in [k]$ or $h(u,v) \in [r] \setminus [k]$. Suppose h assigns i edges of $G \simeq C_s$ to have colors in $[k]$, then it assigns colors among $[r] \setminus [k]$ to the other $s-i$ edges of G. For each such h

$$\prod_{(u,v)\in E(G)} \sigma(h(u,v)) = \left(\frac{1}{k}\right)^i \left(-\frac{1}{r-k}\right)^{s-i}.$$

We count the number of such $h \in \mathcal{H}$. Since h is injective, there are $\binom{k}{i}$ ways to pick out i colors in $[k]$ and $\binom{r-k}{s-i}$ ways to pick out $s-i$ colors in $[r] \setminus [k]$. Then, there are $s!$ ways to assign these s total colors to edges of G. Now, it remains to assign colors to $E(H) \setminus E(G)$. There are $(r-s)_{e(H)-e(G)}$ ways. In total, the number of such $h \in \mathcal{H}$ is

$$\binom{k}{i}\binom{r-k}{s-i} \cdot s! \cdot (r-s)_{e(H)-s}.$$

Therefore, for $G \simeq C_s$, we have that

$$\begin{aligned} Q(G) &= \sum_{i=1}^{s} \left(\frac{1}{k}\right)^i \left(-\frac{1}{r-k}\right)^{s-i} \binom{k}{i}\binom{r-k}{s-i} \cdot s! \cdot (r-s)_{e(H)-s} \\ &= (r-k)^{-s} \cdot s! \cdot (r-s)_{e(H)-s} F(r,s,k), \end{aligned}$$

where we recall F from (4). By (5), the conditions of Lemma 13 hold, so $k \in [r-1]$ satisfies $F(r,s,k) > 0$. Clearly, k^s, $(r-k)^s$ and $s!$ are all strictly positive. We show that falling factorial $(r-s)_{e(H)-s} > 0$. This is clear combinatorially since it is the number of ways to extend h from $E(G)$ to $E(H)$. More carefully, it is obvious if $r > s$. If $r = s$, then as $r \geqslant e(H) \geqslant s$ by (5), so $(r-s)_{e(H)-s} = (0)_0 = 1$. Therefore, $Q(G) > 0$ for $G \simeq C_s$. $\square$

This concludes the proof of Theorem 2.

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