On the Fractional Matching Extendability of Cayley Graphs of Abelian Groups

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Abstract

Fractional matching extendability is a concept that brings together two widely studied topics in graph theory, namely that of fractional matchings and that of matching extendability. A *fractional matching* of a graph Γ with edge set E is a function f from E to the real interval $[0,1]$ with the property that for each vertex v of Γ, the sum of f-values of all the edges incident to v is at most 1. When this sum equals 1 for each vertex v, the fractional matching is *perfect*. A graph of order at least $2t+1$ is *fractional t-extendable* if it contains a matching of size t and if each such matching M can be extended to a fractional perfect matching in the sense that the corresponding function f assigns value 1 to each edge of M.

In this paper, we study fractional matching extendability of Cayley graphs of Abelian groups. We show that, except for the odd cycles, all connected Cayley graphs of Abelian groups are fractional 1-extendable and we classify the fractional 2-extendable Cayley graphs of Abelian groups. This extends the classification of 2-extendable (in the classical sense) connected Cayley graphs of Abelian groups of even order from 1995, obtained by Chan, Chen and Yu.

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1 Introduction

In [23], Scheinerman and Ullman presented various fascinating aspects of fractional graph theory and showed that by allowing certain parameters to be real numbers and not only integers, one obtains interesting generalizations of classical concepts such as the independence number, minimum vertex cover, packing number, etc. The corresponding results generalize their classical counterparts and can thereby give bounds on certain well-studied parameters of graphs. The fractional parameters are often much easier to compute. This

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way, efficient approximation algorithms for some classical NP-hard or NP-complete problems can be obtained.

One example, where the fractional generalization is in particular fruitful, is that of matchings in graphs. Instead of choosing a set of independent edges, in the fractional setting one assigns certain values from the real interval $[0, 1]$ to the edges of the graph in such a way that the sum of the values at each vertex is at most 1 (see Section 2 for a precise definition and for the definitions of all other terms not defined in the Introduction). When this sum equals 1 for all the vertices, one obtains what is called a fractional perfect matching. Fractional matchings in (hyper)graphs have been studied extensively for more than four decades (see [2, 14, 17, 22, 25] for a few examples). In particular, the problem of existence of fractional perfect matchings in (hyper)graphs is a common topic (see [7, 18] for two of the more recent papers on the subject).

A related concept is that of matching extendability, where one is interested in extending a given matching to a perfect matching. Introduced by Plummer in 1980 [19], the concept has attracted much attention since (see for instance the surveys [20, 21] and [5, 6, 10, 15, 28] for some more recent results). We mention that the corresponding matching extension problem in general graphs is also difficult, as proven recently in [8] (see also [21]).

It is thus natural to investigate the concepts at the intersection of these two topics - the fractional (perfect) matchings and extendability. This leads to the so-called fractional extendability of matchings, first considered by Ma [12] and later also investigated in [13, 27]. Here the question is, whether in a given graph, one can arbitrarily choose a prescribed number of independent edges and extend the chosen set to a fractional perfect matching (see Section 2 for a precise definition). This concept is the main theme of the paper. However, certain restrictions are needed in order to obtain some classification results. In this paper, we focus on a well-studied class of regular graphs, namely, Cayley graphs of Abelian groups. We mention that for this class of graphs, extendability of matchings in the classical sense has been previously investigated (see [3, 15]).

The paper is organized as follows. In Section 2, we gather several definitions and known results that we will need in the subsequent sections. In Section 3, we study the fractional extendability for Cayley graphs of Abelian groups. In Subsection 3.1, we classify the fractional 1-extendable ones (Corollary 6). The main result of Section 3, however, is the complete classification of all fractional 2-extendable connected Cayley graphs of Abelian groups (Theorem 19). In Subsection 3.2, the case of graphs of even order is settled using results from [3]. There are no analogous results for graphs of odd order, and so the analysis of these is much more complex. The series of intermediate results that altogether lead to our main result, is carried out in Subsection 3.3. Finally, some interesting questions and possible directions for future research are given in Section 4.

2 Preliminaries

Throughout this paper all graphs are assumed to be simple and undirected. Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. An edge $e \in E$ is an unordered pair of distinct (adjacent) vertices of Γ. Therefore, for an edge $e \in E$ and a vertex $v \in V$, notation $v \in e$ stands for "v is an endvertex of e". We also use $N(v)$ for the neighbourhood of a vertex $v \in V$. For a subset $U \subset V$, we denote by $\Gamma[U]$ the subgraph of Γ induced on U, and by $\Gamma - U$ the subgraph of Γ induced on $V \setminus U$ (that is, $\Gamma - U = \Gamma[V \setminus U]$).

A *fractional matching* of a graph $\Gamma = (V, E)$ is a function f from E to the closed interval $[0, 1] \subset \mathbb{R}$ such that $\sum_{v \in e} f(e) \leqslant 1$ for each vertex v of Γ. In the case that $\sum_{v \in e} f(e) = 1$ holds for each $v \in V$, the function f is said to be a *fractional perfect matching* of Γ. Note that each matching of Γ is a fractional matching of Γ and each perfect matching of Γ is a fractional perfect matching of Γ. However, while an odd cycle (a cycle of odd length) does not have a perfect matching, it does have a fractional perfect matching (one simply assigns $1/2$ to each edge).

It follows from [23, Theorem 2.1.5], that if a graph Γ admits a fractional perfect matching, then one can in fact find a fractional perfect matching f of Γ, for which $f(e) \in \{0, 1/2, 1\}$ holds for each edge e of Γ. Clearly, the set of edges having nonzero value of f then consists of a set of independent edges (with f-value 1) and a set of cycles (whose edges have f-value $1/2$). This gives the following useful corollary (see [23, Proposition 2.2.2]).

Proposition 1. [23] *Let $\Gamma = (V, E)$ be a graph. Then Γ admits a fractional perfect matching if and only if it has a spanning subgraph whose connected components are either edges or odd cycles.*

The following nice generalization of the celebrated result of Tutte [24] on the existence of perfect matchings in graphs gives a necessary and sufficient condition for the existence of a fractional perfect matching in a graph. For our purposes, a straightforward corollary of this result will be very useful.

Proposition 2. [23, Theorem 2.2.4] *Let $\Gamma = (V, E)$ be a graph. Then Γ has a fractional perfect matching if and only if for each subset $U \subseteq V$ the number of isolated vertices of the graph $\Gamma - U$ is at most $|U|$.*

Corollary 3. *Let $\Gamma = (V, E)$ be a graph containing an independent set $U \subset V$ such that $|U| > |V \setminus U|$. Then Γ has no fractional perfect matching.*

We now review the concept of fractional extendability of matchings, which was introduced in [12] (but see also [13, 27]). For a graph Γ and a non-negative integer t, the graph Γ is said to be *fractional t-extendable*, if it is of order at least $2t + 1$, admits a matching of size t, and if each matching M of Γ of size t can be extended to a fractional perfect matching of Γ in the sense that the corresponding function f satisfies $f(e) = 1$ for each $e \in M$.

The following corollary of Proposition 1 characterizes fractional t-extendable graphs and will be used throughout Section 3.

Proposition 4. *Let $t \geqslant 1$ be an integer and let Γ be a graph of order at least $2t + 1$ which admits a matching of size t. Then Γ is fractional t-extendable if and only if for each matching M of size t, the subgraph of Γ obtained by removing the endvertices of the edges of M, has a spanning subgraph whose connected components are either edges or odd cycles.*

Let us point out that while the classical concept of extendability of matchings is only interesting for graphs of even orders, the concept of fractional extendability can be studied for all graphs. Moreover, when considering only graphs of even order, it is of course clear that each t-extendable graph (a graph in which each matching of size t can be extended to a perfect matching) is fractional t-extendable. But the converse does not hold. For example, let Γ be the graph obtained by taking two copies of the complete graph K_4 and adding an edge e, joining one vertex from one copy to one vertex from the other copy. The matching $M = \{e\}$ clearly cannot be extended to a perfect matching of Γ. On the other hand, Proposition 4 implies that it can be extended to a fractional perfect matching of Γ. It is not difficult to see that all other matchings of size 1 can also be extended to a fractional perfect matching of Γ, and so Γ is fractional 1-extendable. This shows that the condition of fractional t-extendability is weaker than that of t-extendability.

We conclude this section by recalling the well-known notion of a Cayley graph. Let G be a group and let $S \subset G$ be an inverse-closed subset not containing the identity. Then the *Cayley graph* $\mathrm{Cay}(G; S)$ of G with respect to the *connection set* S is the graph with vertex set G, in which $N(g) = \{gs \colon s \in S\}$ for each $g \in G$. Note that the graph $\mathrm{Cay}(G; S)$ is regular of valency $|S|$ and is connected if and only if $\langle S \rangle = G$. In the case that G is the (additive) cyclic group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ (with addition modulo n), we denote $\mathrm{Cay}(\mathbb{Z}_n; S)$ by $\mathrm{Circ}(n; S)$ and say that the graph is a *circulant*.

Throughout the next section we will be working with Cayley graphs of Abelian groups. When working with specific Abelian groups (like cyclic groups or their direct products), we will assume that the corresponding operation is additive (and denote the corresponding groups by $\mathbb{Z}_n$, $\mathbb{Z}_n \times \mathbb{Z}_m$, etc.). However, when working with a general Abelian group A, it will be more convenient to assume that the corresponding operation is multiplicative. This should cause no confusion.

3 Cayley graphs of Abelian groups

As mentioned in the Introduction, the classical extendability of matchings for Cayley graphs of Abelian groups was first studied in [3], where the 2-extendable Cayley graphs of Abelian groups (of even order) were classified. In this section, we consider fractional extendability of connected Cayley graphs of Abelian groups. We first show that, with the exception of odd cycles, all such graphs are fractional 1-extendable (see Corollary 6). We then classify the fractional 2-extendable examples (see Theorem 19).

3.1 Fractional 1-extendability

Clearly, odd cycles are not fractional 1-extendable. In this subsection, we show that all other connected Cayley graphs of Abelian groups of order at least 3 are fractional 1-extendable. Our proof is based on the following result from [16] on extendability of paths to Hamilton cycles (cycles containing all vertices of the graph).

Proposition 5. [16, Theorem 4.1] *Let Γ be a connected Cayley graph of an Abelian group of order at least 4. Then each path of length 3 of Γ lies in a Hamilton cycle of Γ if and* *only if* Γ *is not isomorphic to one of the following graphs:*

(i) $\mathrm{Circ}(4m; \{\pm 1, 2m\})$, $m \geqslant 2$,

(ii) $\mathrm{Circ}(4m+2; \{\pm 2, 2m+1\})$, $m \geqslant 1$,

(iii) $\mathrm{Circ}(4m+2; \{\pm 1, \pm 2m\})$, $m \geqslant 1$,

(iv) $\mathrm{Circ}(2m+1; \{\pm 1, \pm 3\})$, $m \geqslant 3$,

(v) $\mathrm{Circ}(m; \{\pm 1, \pm 2\})$, $m \geqslant 6$.

We mention that the graphs from the first two families of Proposition 5, are the well-known Möbius ladders and prisms, respectively. The ones from family (iii) are often called the wreath graphs.

Corollary 6. *Let* Γ *be a connected Cayley graph of an Abelian group of order at least* 3*. Then* Γ *is fractional* 1*-extendable if and only if it is not an odd cycle.*

Proof. As already mentioned, odd cycles are not fractional 1-extendable. The even cycles clearly are (they are in fact 1-extendable). We can thus assume that Γ is of valency at least 3.

Denote the corresponding Abelian group (with multiplicative operation) and connection set by A and S, respectively, that is, $\Gamma = \mathrm{Cay}(A; S)$. Suppose first that Γ is not isomorphic to a member of one of the five families of circulants from Proposition 5. Recall that each Cayley graph is vertex-transitive, meaning that the automorphism group is transitive on the vertex set. To show that Γ is fractional 1-extendable, it thus suffices to verify that for each $s \in S$, a fractional perfect matching f of Γ exists for which $f(\{1, s\}) = 1$. In fact, since A is Abelian, the map sending each $a \in A$ to a^{-1} is an automorphism of Γ, and so it suffices to take just one of s and s^{-1} for each pair of an element s and its inverse from S. So pick any $s \in S$. Since Γ is not a cycle, there exists an element $s' \in S \setminus \{s, s^{-1}\}$. By Proposition 5, there exists a Hamilton cycle C of Γ containing the path $P = (s', 1, s, ss')$. Deleting the edges of P from C and adding the edge $\{s', ss'\}$ (note that $ss' = s's$ since A is Abelian), we obtain a cycle meeting all the vertices of Γ except 1 and s. We can now assign the value of $1/2$ to all the edges of this cycle and 1 to $\{1, s\}$ to obtain a desired fractional perfect matching of Γ.

To complete the proof, we only need to show that the graphs from the five families from Proposition 5 are fractional 1-extendable. To see that this holds for the examples of even order, we can refer to [3]. However, we can also use the result of [4] that each edge of a connected Cayley graph of an Abelian group of order at least 3 is contained in a Hamilton cycle. The graph is thus 1-extendable (just take every other edge of such a Hamilton cycle). To verify that the graphs $\mathrm{Circ}(2m+1; \{\pm 1, \pm 3\})$, $m \geqslant 3$, are fractional 1-extendable, let Γ' be the subgraph obtained by removing the endvertices of the edge $\{0, 1\}$ or $\{0, 3\}$. Since the cycle $(2, 3, 4, \ldots, 2m)$, or $(2, 1, 4, 5, \ldots, 2m)$, respectively, is a spanning subgraph of Γ', we can apply Proposition 4. Similarly, for $\mathrm{Circ}(m; \{\pm 1, \pm 2\})$, with $m \geqslant 7$ odd, removing the endvertices of the edge $\{0, 1\}$ or $\{0, 2\}$ results in a subgraph, spanned by the 3-cycle $(4, 5, 6)$ and $(m-5)/2$ independent edges. $\square$

3.2 Fractional 2-extendability – the case of even order

The problem of classifying the fractional 2-extendable Cayley graphs of Abelian groups is considerably more difficult. However, in the case of graphs of even order, the classification of connected 2-extendable Cayley graphs of Abelian groups of even order from [3] is of great help. It states that, besides the obvious examples of cycles of length at least 6, the only connected Cayley graphs of Abelian groups of even order which are not 2-extendable, are the examples from items (i), (ii), (iii) and (v) of Proposition 5, where m is even in case (v). To classify the fractional 2-extendable Cayley graphs of Abelian groups of even order we thus only need to see if any of these graphs are fractional 2-extendable.

Proposition 7. *Let* Γ *be a connected Cayley graph of an Abelian group of even order at least* 6*. Then* Γ *is fractional* 2*-extendable if and only if it is not isomorphic to one of the following circulants:*

(i) $\mathrm{Circ}(2m; \{\pm 1\})$, $m \geqslant 3$,

(ii) $\mathrm{Circ}(4m; \{\pm 1, 2m\})$, $m \geqslant 2$,

(iii) $\mathrm{Circ}(4m+2; \{\pm 2, 2m+1\})$, $m \geqslant 1$,

(iv) $\mathrm{Circ}(4m+2; \{\pm 1, \pm 2m\})$, $m \geqslant 1$,

(v) $\mathrm{Circ}(2m; \{\pm 1, \pm 2\})$, $m \geqslant 3$.

Proof. As observed above, we only need to prove that none of the examples from the proposition is fractional 2-extendable. Clearly, this holds for the cycles $\mathrm{Circ}(2m; \{\pm 1\})$, $m \geqslant 3$. Similarly, removing the endvertices of the edges $\{0, 1\}$ and $\{3, 4\}$ of a graph Γ from family (v) results in a graph with an isolated vertex. Proposition 1 now implies that Γ is not fractional 2-extendable.

For each of the graphs Γ from families (ii)–(iv) we exhibit a pair of independent edges e and e' of Γ such that the subgraph Γ', obtained from Γ by removing the endvertices of these two edges, has an independent set consisting of more than half of the vertices of Γ'. Then Corollary 3 ensures that Γ is not fractional 2-extendable.

For the examples from family (ii), one can take $e = \{0, 1\}$ and $e' = \{2m-1, 2m\}$, and observe that the resulting subgraph Γ' is bipartite with bipartition parts of different sizes (namely, $2m-1$ and $2m-3$). A similar conclusion can be made by taking $e = \{0, 2\}$ and $e' = \{2m-1, 2m+1\}$ for the examples from family (iii), and by taking $e = \{0, 1\}$ and $e' = \{2m+1, 2m+2\}$ for the examples from family (iv). □

3.3 Fractional 2-extendability – the case of odd order

We now embark on the project of classifying the fractional 2-extendable connected Cayley graphs of Abelian groups of odd orders. We omit the trivial case of odd cycles from our considerations. Let $\Gamma = \mathrm{Cay}(A; S)$ be a connected Cayley graph, where A is an Abelian group of odd order at least 5 and $|S| > 2$. Suppose that Γ is not fractional 2-extendable. Since Γ is vertex-transitive, Proposition 4 implies existence of disjoint edges $e = \{1, s_1\}$ and $e' = \{a, as_2\}$ for some $a \in A$ and $s_1, s_2 \in S$ such that the subgraph

$\Gamma' = \Gamma - \{1, s_1, a, as_2\}$ of Γ cannot be covered by a set of pairwise disjoint edges and odd cycles. For further reference we record these assumptions.

Assumption 8. Let A be a (multiplicative) Abelian group of odd order at least 5 and $S \subset A$ be such that $\langle S \rangle = A$, $S^{-1} = S$, $1 \notin S$ and $|S| \geqslant 3$. Let $\Gamma = \mathrm{Cay}(A; S)$. We assume that Γ is not fractional 2-extendable and we let $s_1, s_2 \in S$ and $a \in A$ be such that the edges $e = \{1, s_1\}$ and $e' = \{a, as_2\}$ are disjoint and the subgraph $\Gamma' = \Gamma - \{1, s_1, a, as_2\}$ does not have a fractional perfect matching. In other words, it does not have a spanning subgraph consisting of pairwise disjoint edges and odd cycles.

In the next series of lemmas, we determine the consequences of Assumption 8 on the group A, the set S and the graph Γ. As the analysis is rather complex, we divide it into the following three main steps. We first analyze the case that the subgroup $H = \langle s_1, s_2 \rangle$ is a proper subgroup of A (Lemma 9). We then focus on the case that $H = A$, but none of s_1 and s_2 generates A (Lemma 10). The remaining case when at least one of s_1 and s_2 generates A (in which case A is cyclic), is the most involved. We therefore divide it into several subcases (Lemma 12–Lemma 18).

Throughout this section, for any $s \in S$ the edges of the form $\{b, bs\}$ will be called *s-edges*. Moreover, for $s \in S$ and a vertex $b \in A$, the unique cycle through b consisting of s-edges will be called the *s-cycle through b*.

Lemma 9. *With reference to Assumption 8, if $\langle s_1, s_2 \rangle$ is a proper subgroup of the group A, then $|A| = 3n$ for some odd integer $n \geqslant 3$, and one of the following holds:*

(i) $\Gamma \cong \mathrm{Circ}(3n; \{\pm 1, \pm 3\})$.

(ii) $\Gamma \cong \mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_3; \{\pm(1,0), \pm(1,1)\})$.

(iii) $\Gamma \cong \mathrm{Circ}(3n; \{\pm 1, \pm(n-1), \pm(n+1)\})$.

Moreover, the above graphs are indeed not fractional 2-extendable.

Proof. Suppose that $H = \langle s_1, s_2 \rangle$ is a proper subgroup of A and let m denote its index $[A : H]$ in A. The graph Γ thus contains a spanning subgraph consisting of m disjoint copies of the connected graph $\Gamma_0 = \mathrm{Cay}(H; H \cap S)$, one for each coset of H in A. Note that Γ_0 is of odd order and (being a connected Cayley graph of an Abelian group) contains a Hamilton cycle. Moreover, the following easy observation will play a crucial role throughout the proof. For any $b \in A$, $s \in S$ and $s' \in S \setminus H$, since A is Abelian, $bss' = bs's$, and so the s-edges $\{b, bs\}$ and $\{bs', bs's\}$ are linked by the s'-edges $\{b, bs'\}$ and $\{bs, bss'\}$. Observe also that e is an edge of Γ_0 and that $m \geqslant 3$ since m is odd. We first prove that Γ_0 is a cycle and $m = 3$ by a series of three claims.

Claim 1: e' is not an edge of Γ_0.
Suppose on the contrary that e' is an edge of Γ_0. We can then find a suitable spanning subgraph of Γ' to contradict Assumption 8 as follows. Since H is a proper subgroup of A and $\langle S \rangle = A$, there exists an $s \in S \setminus H$. To cover the vertices of $H \cup sH$, we can take the edges $\{s, ss_1\}$, $\{sa, sas_2\}$ and all of the edges of the form $\{h, hs\}$, $h \in H \setminus \{1, s_1, a, as_2\}$

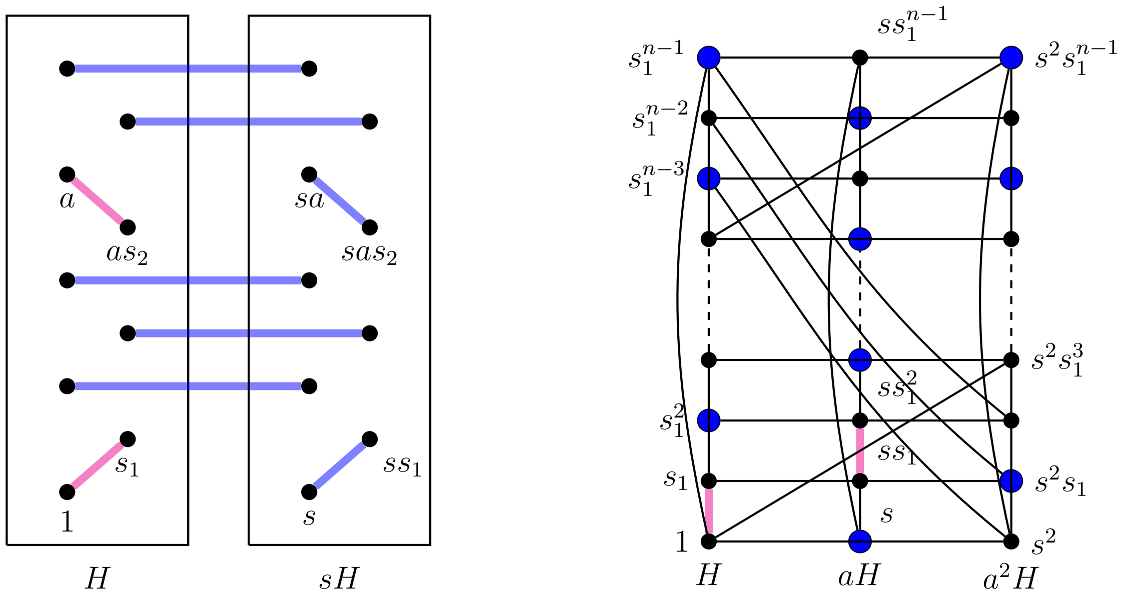

Figure 1: Two situations from the proof of Lemma 9.

(see the left part of Figure 1). For each of the remaining copies of Γ_0 we can simply take a Hamilton cycle. This contradicts Assumption 8, proving our claim.

Claim 2: $aH \neq H$, $S \cap H = \{s_1, s_1^{-1}\}$ and Γ_0 is a cycle.
Since $s_2 \in H$, the edge e' is contained in the copy of Γ_0 corresponding to the coset aH, and so Claim 1 implies that $aH \neq H$. Corollary 6 ensures that unless Γ_0 is a cycle of odd length, we can extend each of e and e' to a fractional perfect matching of the corresponding copy of Γ_0. Since we can take a Hamilton cycle in each of the remaining copies of Γ_0, this would contradict Assumption 8. Therefore, $S \cap H = \{s_1, s_1^{-1}\}$, implying that Γ_0 is a cycle consisting of s_1-edges.

Claim 3: $m = 3$.
Suppose on the contrary that $m > 3$. Note that for each $b \in A$ and each $s \in S \setminus \{s_1, s_1^{-1}\}$, the fact that s is of odd order implies that $s^{-1}bH$, bH and sbH are three different cosets. It is thus easy to see that since $m > 3$, we can find $s, s' \in S \setminus \{s_1, s_1^{-1}\}$ such that $(H \cup sH) \cap (aH \cup s'aH) = \emptyset$. We can then again find a suitable spanning subgraph of Γ' to contradict Assumption 8. Namely, take the edges $\{s, ss_1\}$ and $\{s'a, s'as_2\}$, cover all of the remaining vertices of $H \cup sH$ (in Γ') by s-edges, all of the remaining vertices of $aH \cup s'aH$ by s'-edges, and take the s_1-cycles in each of the remaining $m-4$ copies of Γ_0.

Note that Claim 2 implies that $s_2 \in \{s_1, s_1^{-1}\}$. Since we can replace a by as_1^{-1} if needed, we can thus assume that $s_2 = s_1$. Denote the order of s_1 by n and note that Claims 2 and 3 imply that Γ is of order $3n$. Moreover, Γ consists of the three n-cycles corresponding to the s_1-edges, which are linked by the s-edges corresponding to all $s \in S \setminus \{s_1, s_1^{-1}\}$. We next determine the set $S \setminus \{s_1, s_1^{-1}\}$.

Claim 4: $S \cap aH \subset \{as_1, as_1^{-1}\}$ and $|S| \in \{4, 6\}$.
As in the proof of Claim 3 we see that for each $s \in S \setminus \{s_1, s_1^{-1}\}$, one of s and s^{-1} is in aH and the other is in a^2H. Take any $s \in S \cap aH$ and let $i \in \{0, 1, \ldots, n-1\}$ be such that $a = ss_1^i$. Suppose that $i \notin \{1, n-1\}$. Depending on whether i is odd or even, we can take the edge $\{s_1^{i-1}, ss_1^{i-1}\}$ or $\{s_1^{i+2}, ss_1^{i+2}\}$, respectively, cover all of the remaining vertices of H and aH by independent s_1-edges, and take the s_1-cycle in a^2H to contradict Assumption 8. This shows that $S \cap aH \subseteq \{as_1, as_1^{-1}\}$. Since $|S \cap H| = 2$ and $s^{-1} \in S \cap aH$ for each $s \in S \cap a^2H$, this also implies that $|S| \in \{4, 6\}$, as claimed.

We now analyze the two possibilities, depending on whether $|S| = 4$ or $|S| = 6$.

Case $|S| = 4$:
By Claim 4, $S \cap aH$ consists of one of as_1 and as_1^{-1}. We only consider the possibility that $s = as_1^{-1} \in S \cap aH$ in all detail (the other one is completely analogous). Let $\ell \in \{0, 1, \ldots, n-1\}$ be such that $s^3 = s_1^\ell$. We claim that $\ell \in \{1, n-3, n-1\}$. Suppose on the contrary that $\ell \notin \{1, n-3, n-1\}$ and note that this implies $n > 3$. If ℓ is even (in which case $\ell \leqslant n-5$), consider the odd cycle $C = (s_1^4, ss_1^4, s^2s_1^4, s_1^{\ell+4}, s_1^{\ell+3}, \ldots, s_1^5)$. It covers the vertices ss_1^4 and $s^2s_1^4$ from aH and a^2H, respectively, and all vertices of H, except for the endvertices of e, the consecutive vertices s_1^2 and s_1^3, and an even number of consecutive vertices $s_1^{\ell+5}, s_1^{\ell+6}, \ldots, s_1^{-1}$. Since C does not meet e or e', we can thus take the cycle C, the edge $\{s^2s_1, s^2s_1^2\}$, and cover all of the remaining vertices of H by independent s_1-edges and the remaining vertices of aH and a^2H by s-edges to contradict Assumption 8. Similarly, if ℓ is odd (in which case $\ell \geqslant 3$), take the cycle $(s_1^{-1}, ss_1^{-1}, s^2s_1^{-1}, s_1^{\ell-1}, s_1^\ell, \ldots, s_1^{-2})$, the edge $\{s^2s_1, s^2s_1^2\}$, and cover the remaining vertices by independent s_1-edges and s-edges, again contradicting Assumption 8. Therefore, $\ell \in \{1, n-3, n-1\}$, as claimed.

We now show that in this case the graph Γ' indeed does not have a fractional perfect matching. To see this take

$$\begin{aligned} U = &\{s_1^{2i} \colon 1 \leqslant i \leqslant (n-1)/2\} \cup \{ss_1^{2i+1} \colon 1 \leqslant i \leqslant (n-3)/2\} \\ &\cup \{s^2s_1^{2i} \colon 2 \leqslant i \leqslant (n-1)/2\} \cup \{s, s^2s_1^\delta\}, \end{aligned}$$

where $\delta = 1$ if $\ell = n-3$ and $\delta = 2$ otherwise (see the right part of Figure 1, where the possibility $\ell = n-3$ is depicted). It is not difficult to see that U is an independent set of vertices in Γ' containing more than half of its vertices, and so we can apply Corollary 3. To conclude this case we determine the group A and the connection set S. If $\ell \in \{1, n-1\}$, then clearly $|s| = 3n$ and $\Gamma \cong \mathrm{Circ}(3n; \{\pm1, \pm3\})$. If however $\ell = n-3$, then $s^3 = s_1^{-3}$ implies that $a^3 = 1$. Since $\langle s_1 \rangle$ and $\langle a \rangle$ clearly intersect trivially, $A \cong \mathbb{Z}_n \times \mathbb{Z}_3$, where we can let s_1 corresponds to $(-1, 0)$ and a to $(0, 1)$. Since $s = as_1^{-1}$, it is clear that we can assume that $S = \{\pm(1, 0), \pm(1, 1)\}$.

Case $|S| = 6$:
By Claim 4, $S \cap aH = \{as_1, as_1^{-1}\}$. Denote $s = as_1^{-1}$ and $s' = as_1 = ss_1^2$. Again, set $\ell \in \{0, 1, \ldots, n-1\}$ be such that $s^3 = s_1^\ell$, and note that the above proof still applies. Thus, $\ell \in \{1, n-3, n-1\}$. The claims of the lemma in the case of $n = 3$ can be verified

easily, so we assume $n > 3$. We claim that $\ell = n-3$. If $\ell = 1$, then $s^2 s_1^{-1} s' = s_1^2$. Consider the odd cycle $C = (s_1^2, s^2 s_1^{-1}, s^2 s_1^{-2}, s s_1^{-2}, s s_1^{-1}, s_1^{-1}, s_1^{-2}, \dots, s_1^3)$. It covers all the vertices of H (except the endvertices of e) and the vertices $s s_1^{n-2}, s s_1^{n-1}$ of aH and $s^2 s_1^{n-2}, s^2 s_1^{n-1}$ of $a^2 H$. We can thus take the cycle C, the edge $\{s^2 s_1, s^2 s_1^2\}$, and cover the remaining vertices of aH and $a^2 H$ by s-edges to contradict Assumption 8. Similarly, if $\ell = n-1$, we can take the odd cycle $(s_1^2, s s_1^4, s s_1^5, \dots, s s_1^{-1}, s^2 s_1)$, the edges $\{s, s^2 s_1^2\}$ and $\{s s_1^3, s^2 s_1^5\}$, and cover all of the remaining vertices by independent s_1-edges, a contradiction. Therefore, $\ell = n-3$, as claimed. It is not difficult to see that in this case the above defined set U is still an independent set of vertices of Γ'. Therefore, Corollary 3 implies that Γ' has no fractional perfect matching.

Since $\ell = n-3$, the argument from the above case shows that $A \cong \mathbb{Z}_n \times \mathbb{Z}_3$, where s_1 corresponds to $(-1, 0)$ and a to $(0, 1)$. Note also that $s' = s s_1^2 = a s_1$, and so s'^{-1} corresponds to $(1, -1)$. We can thus assume that $S = \{\pm(1,0), \pm(1,1), \pm(1,-1)\}$. That $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_3; \{\pm(1,0), \pm(1,1), \pm(1,-1)\}) \cong \mathrm{Circ}(3n; \{\pm 1, \pm(n-1), \pm(n+1)\})$, is easy to verify. □

We remark that the circulant from item (iii) of the above Lemma 9 is what is known as the lexicographic product of the n-cycle by an edgeless graph of order 3 (just like the wreath graph from item (iv) of Proposition 7 is the lexicographic product of the $(2m+1)$-cycle by an edgeless graph of order 2). Moreover, we mention that if n is coprime to 3 in item (ii) of Lemma 9, then the graph is a circulant (with connection set being $\{\pm 1, \pm(n-1)\}$ or $\{\pm 1, \pm(n+1)\}$, depending on whether $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$, respectively).

Lemma 10. *With reference to Assumption 8, if $A = \langle s_1, s_2 \rangle$ but none of s_1 and s_2 generates A, then $A \cong \mathbb{Z}_n \times \mathbb{Z}_3$ for some odd integer n with $3 \mid n$. Moreover, $\Gamma \cong \mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_3; \{\pm(1,0), \pm(1,1)\})$ or $\Gamma \cong \mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_3; \{\pm(1,0), \pm(1,1), \pm(1,-1)\}) \cong \mathrm{Circ}(3n; \{\pm 1, \pm(n-1), \pm(n+1)\})$.*

Proof. Suppose that $A = \langle s_1, s_2 \rangle$ but none of s_1 and s_2 generates A. Denote the order of s_1 by n_1 and the index $[A : \langle s_1 \rangle]$ by m_1, and similarly n_2, m_2 for s_2. Then A (and thus Γ) is of order $m_1 n_1 = m_2 n_2$. Since none of s_1, s_2 generates A and m_1, m_2 are odd, we also have $m_1, m_2 \geqslant 3$. Note that the edge $e' = \{a, a s_2\}$ connects vertices from two different cosets $a\langle s_1 \rangle, a s_2 \langle s_1 \rangle$ in A, as equality would imply $s_2 \in \langle s_1 \rangle = \langle s_1, s_2 \rangle$, a contradiction. Since $\langle s_1 \rangle$ and $\langle s_1, s_2 \rangle$ are of orders n_1 and $m_1 n_1$, respectively, there exists a unique $\ell \in \{0, 1, \dots, n_1 - 1\}$ such that $s_2^{m_1} = s_1^{\ell}$. The order n_2 of s_2 thus equals $m_1 n_1 / \gcd(\ell, n_1)$. Since $m_1 n_1 = m_2 n_2$, it thus follows that $\gcd(\ell, n_1) = m_2$ (implying in particular that m_2 divides n_1). We proceed to show that $m_1 = m_2 = 3$, that $3 \mid n_1$, and that $a \in (s_2 \langle s_1 \rangle) \cap (s_1^2 \langle s_2 \rangle)$ by the following three claims.

Claim 1: $a, a s_2 \notin \langle s_1 \rangle$.

Suppose first that $a \in \langle s_1 \rangle$. Than we can simply take the edge $\{s_2, s_2 s_1\}$ (which is of course disjoint from e' since e is), all of the edges $\{s_1^i, s_1^i s_2\}$, where $s_1^i \notin \{1, s_1, a\}$, together with the s_1-cycles (of odd length n_1) on all of the remaining $m_1 - 2$ cosets of $\langle s_1 \rangle$ to contradict Assumption 8. The case when $a s_2 \in \langle s_1 \rangle$ is similar.

Claim 2: $m_1 = m_2 = 3$ and $3 \mid n_1$.
First, consider the s_2-cycle C through s_1^2. Since $\gcd(\ell, n_1) = m_2$, the set of vertices of some coset $s_2^i\langle s_1\rangle$, where $0 \leqslant i < m_1$, that are on C, is the set $\{s_2^i s_1^{2+jm_2} : 0 \leqslant j < n_1/m_2\}$. Note that $m_2 \geqslant 3$ implies that the cycle C is disjoint from e. Moreover, the set of elements of a coset of $\langle s_1\rangle$, not covered by C, consists of n_1/m_2 sets of $m_2 - 1$ consecutive vertices. Therefore, if C does not contain e', we can take the cycle C (which is of odd length n_2) and all of the remaining s_2-edges between the vertices of $a\langle s_1\rangle$ and $as_2\langle s_1\rangle$. As for the cosets of $\langle s_1\rangle$ in A except $a\langle s_1\rangle$ and $as_2\langle s_1\rangle$, we can cover their remaining vertices (those not already on C) with disjoint s_1-edges (since m_2 is odd). This contradicts Assumption 8, showing that C does contain e'. In a s similar way we see that the s_2-cycle C' containing s_1^{-1} also contains e', and thus $C' = C$. This implies that $m_2 = 3$. Exchanging the roles of s_1 and s_2, we have $m_1 = m_2 = 3$. Since m_2 divides n_1, the claim follows.

Claim 3: $a \in (s_2\langle s_1\rangle) \cap (s_1^2\langle s_2\rangle)$.
Since e' is contained in C, it clearly follows that $a \in s_1^2\langle s_2\rangle$. Moreover, since $m_2 = 3$ and e' is disjoint from $\langle s_1\rangle$, we also get $a \in s_2\langle s_1\rangle$, which proves our claim.

That the conclusion of the lemma is correct when $n_1 = 3$ (in which case $\ell = 0$ and $A \cong \mathbb{Z}_3 \times \mathbb{Z}_3$) can easily be verified. Let us thus assume that $n_1 > 3$ and note that in this case $n_1 \geqslant 9$ (since $m_2 \mid n_1$). Recall that $\gcd(\ell, n_1) = 3$. It can be verified that renaming the vertices using multiplication by s_1^{-1} and then exchanging the roles of s_1 and s_1^{-1}, the parity of the corresponding ℓ changes. This shows that we can assume that ℓ is odd, which implies that $3 \leqslant \ell \leqslant n_1 - 6$ (recall that n_1 is odd). We now prove that $a = s_1^{-1}s_2$, that $\ell = 3$ and that $A \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_3$ by the following two claims.

Claim 4: $a = s_1^{-1}s_2$.

Claim 5: $\ell = 3$ and $A \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_3$.
Suppose on the contrary that $\ell \neq 3$, consider the cycle

$$(s_1^{-3}, s_1^{-3}s_2, s_1^{-3}s_2^2, s_1^{\ell-3}, s_1^{\ell-2}, \ldots, s_1^{-4}),$$

and note that it does not meet e or e'. Similarly as above, we see that we can cover the remaining vertices of Γ' by independent s_1-edges and s_2-edges to contradict Assumption 8. Therefore, $\ell = 3$, and so $s_1^{-1}s_2$ is of order 3. Since $\langle s_1\rangle$ and $\langle s_1^{-1}s_2\rangle$ intersect trivially, $A \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_3$, where s_1 and s_2 correspond to $(1,0)$ and $(1,1)$, respectively.

If $|S| = 4$, the proof is complete. Suppose then that $|S| > 4$. If S contains some s_1^i, $1 < i < n_1 - 1$, then the subgraph of Γ induced on $\langle s_1\rangle$ is fractional 1-extendable by Corollary 6. However, this contradicts Assumption 8, since we can cover the vertices of $s_2\langle s_1\rangle$ and $s_2^2\langle s_1\rangle$ in Γ' by s_2-edges. We thus have an element $s = s_1^i s_2 \in S$ with $1 \leqslant i < n_1$. We claim that $i = n_1 - 2$. Suppose on the contrary that this is not the case. If $i = n_1 - 1$, we can take the 7-cycle $(s_1^{-1}, s_1^{-2}s_2, s_1^{-2}s_2^2, s_1^{-3}s_2^2, s_1^{-3}s_2, s_1^{-3}, s_1^{-2})$ and cover the remaining vertices of Γ' by independent s_1-edges, contradicting Assumption 8. If i is even with $i < n_1 - 1$, we can take the cycle $(s_1^{-3}, s_1^{-3}s_2, s_1^{-3}s_2^2, s_1^i, s_1^{i+1}, \ldots, s_1^{-4})$, cover the remaining vertices of $\langle s_1\rangle$ by independent s_1-edges (we can do this since i is even), and cover the remaining vertices of $s_2\langle s_1\rangle$ and $s_2^2\langle s_1\rangle$ by s_2-edges, a contradiction. Finally, if i is odd with $i < n_1 - 2$, we can take the cycle $(s_1^2, s_1^{i+2}s_2, s_1^{i+1}s_2, \ldots, s_1^2 s_2)$ and cover all of the remaining vertices by independent s_1-edges, a contradiction. This proves our claim that $i = n_1 - 2$. Note that this actually forces $S \cap s_2\langle s_1\rangle = \{s_1^{-2}s_2\}$, and so $S = \{s_1, s_1^{-1}, s_2, s_2^{-1}, s_1^{-2}s_2, s_1^{-1}s_2^2\}$. Of course, the element $s_1^{-1}s_2^2$ corresponds to $(1, -1)$ in $\mathbb{Z}_{n_1} \times \mathbb{Z}_3$. □

To conclude the analysis of fractional 2-extendability of Cayley graphs of Abelian groups, we finally consider the examples in which at least one of s_1 and s_2 from Assumption 8 generates A. In particular, Γ is a circulant in this case. We can thus assume that $A = \mathbb{Z}_n$ for some odd integer $n \geqslant 5$ (with additive operation), and that one of s_1 and s_2, say s_2, is 1. It is easy to see that none of the graphs $\mathrm{Circ}(n; \{\pm 1, \pm 2\})$ and $\mathrm{Circ}(n; \{\pm 1, \pm 3\})$ is fractional 2-extendable. For the first one we can repeat the argument from the proof of Proposition 7. For the second one, removing the endvertices of the edges $\{0, 1\}$ and $\{2, 3\}$ results in a bipartite graph of odd order, and we can apply Corollary 3. We shall thus only consider the examples $\mathrm{Circ}(n; S)$, where $1 \in S$ but either $S \cap \{2, 3\} = \emptyset$ or $|S| > 4$.

It turns out that the analysis of the situation is somewhat different if $s_1 \in \{\pm 1\}$ or $s_1 \notin \{\pm 1\}$. We first consider the former case. Observe that, replacing $e' = \{a, a+1\}$ by $\{a+1, a+2\}$ if necessary, we can assume that $s_1 = 1$. Moreover, exchanging the roles of e and e' if necessary, we can assume that $a \leqslant (n-1)/2$. In the following two lemmas we will thus be working with the following assumption.

Assumption 11. Let $n \geqslant 5$ be an odd integer, let $S \subset \mathbb{Z}_n \setminus \{0\}$ be such that $-S = S$, $1 \in S$ and that $|S| \geqslant 4$, and that either $S \cap \{2, 3\} = \emptyset$ or $|S| \geqslant 6$. Let $\Gamma = \mathrm{Circ}(n; S)$. We assume that $a \in \mathbb{Z}_n$, where $2 \leqslant a \leqslant (n-1)/2$, is such that the subgraph $\Gamma' = \Gamma - \{0, 1, a, a+1\}$ does not have a fractional perfect matching. We also denote $e = \{0, 1\}$ and $e' = \{a, a+1\}$.

Lemma 12. *With reference to Assumption 11, a is odd and there exists no odd $s \in S$ with $1 < s \leqslant (n-1)/2$.*

Proof. We first show that there is no odd $s \in S$ with $1 < s \leqslant (n-1)/2$. Suppose on the contrary that such an s exists. If a is odd, we let $j \in \{2, 3, \ldots, a-1\}$ be the smallest even integer such that $j + s \geqslant a + 2$. It is easy to see that in this case $j + s \leqslant n + 2 - s$. Consider the cycle $C = (2, 3, \ldots, j, j+s, j+s+1, \ldots, n+2-s)$ of Γ' and note that it is of odd length $n - 2s + 2$ (see the left part of Figure 2, where the case of $n = 21$, $a = 9$ and $s = 7$ is presented). The vertices of Γ', which are not covered by C, are the $a - j - 1$ consecutive vertices $j+1, j+2, \ldots, a-1$, the $j + s - a - 2$ consecutive vertices $a+2, a+3, \ldots, j+s-1$, and the $s - 3$ consecutive vertices $n+3-s, n+4-s, \ldots, n-1$. Since all of these three numbers are even, we can cover these vertices by independent 1-edges, contradicting Assumption 11 by Proposition 4.

Similarly, if a is even and $a \neq 2$, then there exists an odd $j \in \{3, 4, \ldots, a-1\}$ such that $a + 2 \leqslant j + s \leqslant n + 2 - s$. We can then again take the cycle $(2, 3, \ldots, j, j+s, j+s+1, \ldots, n+2-s)$ and cover the remaining vertices of Γ' by independent 1-edges. For the

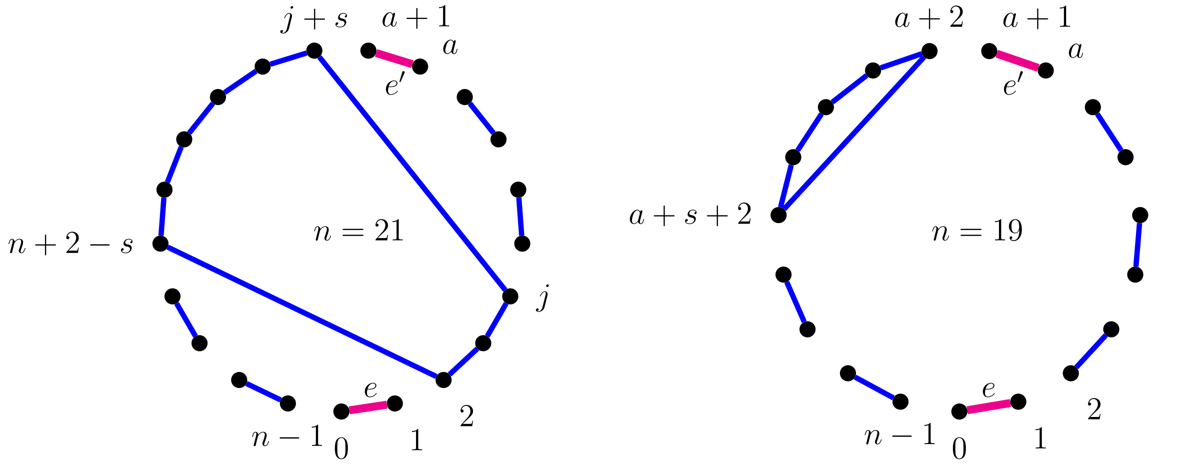

Figure 2: Two situations from the proof of Lemma 12.

case when $a = 2$, we can do the following. We take an even $s \in S$ with $s \leqslant n-5$, which exists since otherwise $S = \{\pm 1, \pm 3\}$, contradicting Assumption 11. We can then take the cycle $(4, 5, \ldots, 4+s)$ and cover the remaining vertices of Γ' by independent 1-edges. This finally shows that there is no odd $s \in S$ with $1 < s \leqslant (n-1)/2$, as claimed.

To complete the proof, we need to show that a is odd. Suppose on the contrary that a is even. Then Γ' has an even number of consecutive vertices between 2 and $a-1$. Since $s \geqslant (n+1)/2$ for each odd $s \in S$ with $s \neq 1$, it follows that $s \leqslant (n-1)/2$ for each even $s \in S$ with $s \neq n-1$. If there exists an even $s \in S$ with $s \leqslant n-a-3$, then we can take the odd cycle $(a+2, a+3, \ldots, a+s+2)$ in Γ', and cover the remaining vertices of Γ' by independent 1-edges, contradicting Assumption 11 (see the right part of Figure 2, where the case of $n = 19$, $a = 8$ and $s = 4$ is presented). Thus $s > n-a-3$ for each even $s \in S$. Since $n-a-3$ is even, this in fact implies that $s \geqslant n-a-1 \geqslant (n-1)/2$ (recall that $a \leqslant (n-1)/2$) for each even $s \in S$. But then $S = \{\pm 1, \pm(n-1)/2\}$ and $a = (n-1)/2$, and so $(n-1, a-1, a-2, \ldots, 2, a+2, a+3, \ldots, n-2)$ is a Hamilton cycle of Γ', a contradiction. $\square$

Lemma 13. *With reference to Assumption 11, Γ is isomorphic to* $\mathrm{Circ}(n; \{\pm 1, \pm 3\})$*, or $n = 3m$ for some odd $m \geqslant 3$, and Γ is isomorphic to one of the following graphs:*

(i) $\mathrm{Circ}(n; \{\pm 1, \pm(m-1)\})$,

(ii) $\mathrm{Circ}(n; \{\pm 1, \pm(m+1)\})$,

(iii) $\mathrm{Circ}(n; \{\pm 1, \pm(m-1), \pm(m+1)\})$.

Moreover, none of these graphs is fractional 2-extendable.

Proof. Lemma 12 implies that a is odd (and thus $a \geqslant 3$) and that there exists no odd $s \in S$ with $1 < s \leqslant (n-1)/2$. Pick any $s \in S \setminus \{1\}$ with $s \leqslant (n-1)/2$, which thus must be even. We proceed by proving two claims.

Claim 1: $s \in \{a-1, a+1\}$.
Suppose on the contrary that this does not hold. If $s < a-1$ then $s \leqslant a-3$, and so $(2, 3, \ldots, s+2)$ is a an odd cycle in Γ'. Since we can clearly cover the remaining vertices of Γ' by independent 1-edges, this contradicts Assumption 11. Thus $s \geqslant a-1$. Furthermore, if $s > a+1$, then $s \geqslant a+3$, and so the fact that $s \leqslant (n-1)/2$ implies that

$$a+2 \leqslant s-1 < s+2 < n-s+2 < n+a-s+2 \leqslant n-1.$$

We can thus take the cycle $(2, s+2, s+3, \ldots, n-s+2)$ of odd length $n-2s+2$, every other edge of the cycle $(n-1, s-1, s-2, \ldots, a+2, n+a-s+2, n+a-s+3, \ldots, n-2)$ of even length $2(s-a-2)$, and cover all of the remaining vertices of Γ' by independent 1-edges, contradicting Assumption 11. Thus $s \in \{a-1, a+1\}$, as claimed.

Claim 2: $2s \in \{n-a-2, n-a, n-a+2\}$ or $s = 2$.
If $2s \geqslant n-a+4$, then $a+s-1 > n+2-s$. The fact that $s+a < n-1$ (and thus $n+2-s > a+3$) then implies that $(2, 3, \ldots, a-1, a+s-1, a+s-2, \ldots, n+2-s)$ is a cycle of odd length $2s+2a-n-4 \geqslant 3$ in Γ'. Since this leaves us with $n-s-a$ (an even number) consecutive vertices $a+2, a+3, \ldots, n+1-s$ and the same number of consecutive vertices $a+s, a+s+1, \ldots, n-1$, this contradicts Assumption 11. Thus $2s \leqslant n-a+2$. Next, if $2s \leqslant n-a-4$ and $s \neq 2$, then

$$a+2 < a+s-1 < a+s+2 < n-s-1 < n-s+2 < n-1,$$

and so $(2, 3, \ldots, a-1, a+s-1, a+s-2, \ldots, a+2, a+s+2, a+s+3, \ldots, n-s-1, n-1, n-2, \ldots, n-s+2)$ is a cycle of Γ', containing all the vertices of Γ' except for the endvertices of the two edges $\{a+s, a+s+1\}$ and $\{n-s, n-s+1\}$. Since this again contradicts Assumption 11, our claim is proven.

We are now ready to determine the set S. Suppose first that $2 \in S$. Then Claim 1 implies that $a = 3$. Since Assumption 11 guarantees that $|S| \geqslant 6$, we have an even $s \in S$ with $2 < s \leqslant (n-1)/2$. But then Claim 1 implies that $s = 4$, while Claim 2 implies that $8 \in \{n-5, n-3, n-1\}$, forcing $n \in \{9, 11, 13\}$. In the case that $n \in \{11, 13\}$, we have a Hamilton cycle $(2, 6, 5, 7, 8, \ldots, n-3, n-1, n-2)$ of Γ', a contradiction. Thus, $n = 9$ and $S = \{\pm 1, \pm 2, \pm 4\}$. By Lemma 9, the graph $\mathrm{Circ}(9; \{\pm 1, \pm 2, \pm 4\}$ is indeed not fractional 2-extendable.

We are left with the possibility that $2 \notin S$. Claim 2 implies that $|S| \leqslant 6$. Suppose first that $|S| = 6$. Then $S = \{\pm 1, \pm(a-1), \pm(a+1)\}$, and so Claim 2 implies that $2(a-1), 2(a+1) \in \{n-a-2, n-a, n-a+2\}$. Therefore, $2(a-1) = n-a-2$, and thus $n = 3a$. Consequently, $\Gamma \cong \mathrm{Circ}(3a; \{\pm 1, \pm(a-1), \pm(a+1)\})$, which by Lemma 9 is indeed not fractional 2-extendable.

Suppose finally, that $|S| = 4$, and let $s \in S$ be the unique element with $1 < s \leqslant (n-1)/2$. Then Claims 1 and 2 imply that either $s = a-1$ and $n \in \{3a-4, 3a-2, 3a\}$, or $s = a+1$ and $n \in \{3a, 3a+2, 3a+4\}$. If $n = 3a$, then S is one of $\{\pm 1, \pm(a-1)\}$ and $\{\pm 1, \pm(a+1)\}$. In any of these two cases, consider the set

$$U = \{2, 4, \ldots, a-1\} \cup \{a+2, a+4, \ldots, 2a-1\} \cup \{2a+2, 2a+4, \ldots, 3a-1\}.$$

One can verify that U is an independent set of vertices in Γ' and that it contains more than half of the vertices of Γ'. Corollary 3 thus implies that Γ is indeed not fractional 2-extendable. If however, n is coprime to 3, then multiplication by 3 is an isomorphism from Γ to $\mathrm{Circ}(n; 3S)$, where $3S = \{3s \colon s \in S\}$. It is easy to see that in each of the four cases $3S = \{\pm 1, \pm 3\}$. Therefore, $\Gamma \cong \mathrm{Circ}(n; \{\pm 1, \pm 3\})$, which we already know is not fractional 2-extendable. □

We finally analyze the situation in which $s_1 \notin \{\pm 1\}$. Observe that the permutation, mapping each $i \in \mathbb{Z}_n$ to $-i$, is an automorphism of a circulant $\mathrm{Circ}(n; S)$. Changing a if necessary, we can thus assume that $s_1 \leqslant (n-1)/2$. Our assumption throughout the rest of this section will thus be as follows.

Assumption 14. Let $n \geqslant 5$ be an odd integer, let $s_1 \in \mathbb{Z}_n$ be such that $1 < s_1 \leqslant (n-1)/2$, and let $S \subset \mathbb{Z}_n \setminus \{0\}$ be such that $-S = S$, that $1, s_1 \in S$, and that either $S \cap \{2, 3\} = \emptyset$ or $|S| \geqslant 6$. Let $\Gamma = \mathrm{Circ}(n; S)$. We assume that $a \in \mathbb{Z}_n \setminus \{n-1, 0, s_1 - 1, s_1\}$ is such that the subgraph $\Gamma' = \Gamma - \{0, s_1, a, a+1\}$ does not have a fractional perfect matching. We also denote $e = \{0, s_1\}$ and $e' = \{a, a+1\}$.

We point out that, unlike in Assumption 11, we now make no assumption on whether $a \leqslant (n-1)/2$ or not. Just like in all of the previous cases, we now determine the consequences of Assumption 14, in this case on the parameters n, s_1 and a. In the next two lemmas we first prove that $a > s_1$ and that s_1 is even. We then complete the analysis in Lemmas 17 and 18.

Lemma 15. *With reference to Assumption 14, $a > s_1$.*

Proof. Suppose on the contrary that $a < s_1$. Let $V_1 = \{1, 2, \ldots, a-1\}$, $V_2 = \{a+2, a+3, \ldots, s_1 - 1\}$ and $V_3 = \{s_1 + 1, s_1 + 2, \ldots, n-1\}$. Observe that renaming the vertices via the map $i \mapsto s_1 - i$ exchanges the names of the endvertices of e, as well as the sets V_1 and V_2. We will refer to this renaming as "exchanging the roles of 0 and s_1". We consider two cases depending on the parity of s_1.

Case s_1 is even:
In this case $s_1 \geqslant 4$ and $|V_3|$ is even, while precisely one of $|V_1|$ and $|V_2|$ is odd. Exchanging the roles of 0 and s_1 if necessary, we can assume that $|V_2|$ is odd. Note that this implies that a is odd and that V_2 contains at least the vertex $s_1 - 1$. Since $2s_1 \leqslant n-1$, we can take the odd cycle $(n-1, s_1 - 1, 2s_1 - 1, 2s_1, \ldots, n-2)$, which covers the "last" vertex of V_2 and all of the vertices of V_3, except the "first" $s_1 - 2$ of them. Since this is an even number, we can clearly cover all of the remaining vertices of Γ' by independent 1-edges, contradicting Assumption 14.

Case s_1 is odd:
If $s_1 = 3$ (in which case $a = 1$ is forced), then Assumption 14 guarantees that S contains some even s with $s < n-4$. We can then take the cycle $(4, 5, \ldots, s+4)$ of odd length $s+1$, and cover the remaining vertices of Γ' by independent 1-edges, a contradiction. Therefore, $s_1 \geqslant 5$. Exchanging the roles of 0 and s_1 if necessary, we can assume that

$V_2 \neq \emptyset$. Since this time $|V_3|$ is odd, $|V_1|$ and $|V_2|$ are of the same parity. If $|V_2|$ is even, we can take the cycle $(n-1, s_1-1, s_1-2, 2s_1-2, 2s_1-1, \dots, n-2)$, which covers the "last" two vertices of V_2 and all but the "first" s_1-3 of V_3. Since this is an even number, we can cover the remaining vertices of Γ' by independent 1-edges, a contradiction. If however $|V_2|$ is odd, we take the odd cycle $(n-1, s_1-1, 2s_1-1, 2s_1, \dots, n-2)$ and the edge $\{1, s_1+1\}$, which together cover the "first" and the "last" vertex of V_1 and V_2, respectively, and all but the s_1-3 consecutive vertices $s_1+2, s_1+3, \dots, 2s_1-2$ of V_3. We can thus cover the remaining vertices of Γ' by independent 1-edges, a contradiction. □

Lemma 16. *With reference to Assumption 14, s_1 is even.*

Proof. By way of contradiction, suppose that s_1 is odd. By Lemma 15, $a > s_1$. Let $V_1 = \{1, 2, \dots, s_1-1\}$, $V_2 = \{s_1+1, s_1+2, \dots, a-1\}$ and $V_3 = \{a+2, a+3, \dots, n-1\}$. Then $|V_1|$ is even and precisely one of $|V_2|$ and $|V_3|$ is odd. Exchanging the roles of the vertices 0 and s_1 (which this time exchanges the roles of the sets V_2 and V_3) if necessary, we can assume that $|V_2|$ is odd. Then a is also odd. We first prove the following claim.

Claim: At least one of $n = 2s_1+1$ and $V_3 = \emptyset$ holds.
Suppose that there exists an even integer j with $s_1 < j < a$ such that $a+2 \leqslant s_1+j < n$. We can then take the cycle $C = (1, s_1+1, s_1+2, \dots, j, s_1+j, s_1+j+1, \dots, n-1, s_1-1, s_1-2, \dots, 2)$. The only vertices of Γ' which are not covered by C, are the "last" $a-j-1$ (which is an even number) vertices of V_2 and the "first" $s_1+j-a-2$ (also an even number) vertices of V_3. These can be covered by independent 1-edges, contradicting Assumption 14. Therefore, for each even j with $s_1 < j < a$, at least one of $s_1+j < a+2$ and $s_1+j \geqslant n$ holds. Since s_1+1 is even and $2s_1+1 \leqslant n$, this proves our claim.

To finish the proof, we first consider the possibility that $s_1 > 3$. If $V_3 = \emptyset$ (which occurs if and only if $a = n-2$), we can take the cycle $(1, s_1+1, s_1+2, \dots, n-3, s_1-3, s_1-4, \dots, 2)$ and the edge $\{s_1-2, s_1-1\}$ to cover all the vertices of Γ', a contradiction. Hence $V_3 \neq \emptyset$, and so $n = 2s_1+1$. If $a \neq s_1+2$, we can take the 5-cycle $(1, s_1+1, s_1+2, s_1+3, 2)$ and cover all of the remaining vertices of Γ' by independent 1-edges, a contradiction. Therefore, $a = s_1+2$. But in this case we can take the 3-cycle $(n-1, s_1-1, n-2)$ and the edge $\{1, s_1+1\}$, which leaves us with s_1-3 consecutive vertices of V_1 and s_1-5 consecutive vertices of V_3. Since these are even numbers, we can cover these vertices by independent 1-edges, a contradiction.

This leaves us with the possibility that $s_1 = 3$. Note that $V_3 = \emptyset$ if and only if $a = n-2$. Moreover, if $n = 2s_1+1 = 7$, we must have that $a = 5$, again implying that $a = n-2$. By Assumption 14 there exists some even $s \in S$ with $2 \leqslant s \leqslant n-5$. We can thus take the cycle $(1, 4, 5, \dots, s+2, 2)$ and cover the remaining consecutive $n-s-5$ (which is an even number) vertices of V_2 by independent 1-edges. This contradicts Assumption 14, thus proving that s_1 is even, as claimed. □

The above two lemmas thus imply that s_1 is even and $a > s_1$. We now show that s_1 is roughly $n/3$ and a is roughly $2n/3$.

Lemma 17. *With reference to Assumption 14, one of the following holds:*

- $n = 3s_1 - 3$ *and* $a = 2s_1 - 2$,
- $n = 3s_1 - 1$ *and* $a \in \{2s_1 - 2, 2s_1 - 1, 2s_1\}$,
- $n = 3s_1 + 1$ *and* $a \in \{2s_1 - 1, 2s_1, 2s_1 + 1\}$,
- $n = 3s_1 + 3$ *and* $a = 2s_1 + 1$.

Proof. Recall that s_1 is even and $a > s_1$. We let V_1, V_2 and V_3 be as in the proof of Lemma 16. Note that this time $|V_1|$ is odd and $|V_2|$ and $|V_3|$ are of the same parity. Exchanging the roles of the vertices 0 and s_1 if necessary, we can assume that $|V_2| \geqslant |V_3|$ (but see the last paragraph of the proof). Therefore, $a - s_1 \geqslant n - a - 1$, which is equivalent to $2a - s_1 + 1 \geqslant n$. We distinguish two cases depending on the parity of $|V_2|$.

Case 1: $|V_2|$ is even.
Observe that in this case a is odd. If $2s_1 + 1 < a$, then $a \geqslant 2s_1 + 3$. We can then take the cycle $(s_1 + 2, s_1 + 3, \ldots, 2s_1 + 2)$ of odd length $s_1 + 1$, the edge $\{1, s_1 + 1\}$, and cover all of the remaining vertices of Γ' by independent 1-edges, contradicting Assumption 14. Similarly, if $a - 1 + s_1 > n$, then $j = a - 1 + s_1 - n$ is odd with $1 \leqslant j \leqslant s_1 - 3$ (note that $a + 2 \leqslant n$). We can thus take the cycle $(s_1 + 1, s_1 + 2, \ldots, a - 1, j, j - 1, \ldots, 2, 1)$ of odd length $a - s_1 + j - 1$. This leaves us with the "last" $s_1 - j - 1$ (an even number) vertices of V_1 and the whole V_3, a contradiction. We therefore find that $2s_1 + 1 \geqslant a$ and $a - 1 + s_1 \leqslant n - 1$. Together with the inequality $2a - s_1 + 1 \geqslant n$, the first of these yields $n \leqslant 3s_1 + 3$, while the second yields $n \geqslant 3s_1 - 1$. Therefore, $n \in \{3s_1 - 1, 3s_1 + 1, 3s_1 + 3\}$. Moreover, if $n = 3s_1 - 1$ then $a = 2s_1 - 1$, while if $n \in \{3s_1 + 1, 3s_1 + 3\}$ then we must have $a = 2s_1 + 1$.

Case 2: $|V_2|$ is odd.
In this case a is even and V_3 (being of odd size) is nonempty. If $2s_1 + 1 < a$, we can take the cycle $(s_1 + 1, s_1 + 2, \ldots, 2s_1 + 1)$ of odd length $s_1 + 1$, the edge $\{n - 1, s_1 - 1\}$, and cover the remaining vertices of Γ' with independent 1-edges, a contradiction. Similarly, if $a - 1 + s_1 > n$, then $j = a - 1 + s_1 - n$ is even with $2 \leqslant j \leqslant s_1 - 2$. We can then take the cycle $(1, s_1 + 1, s_1 + 2, \ldots, a - 1, j, j - 1, \ldots, 2)$, the edge $\{s_1 - 1, n - 1\}$, and cover the remaining vertices of Γ' by independent 1-edges, a contradiction. We therefore find that $2s_1 \geqslant a$ and $a - 1 + s_1 \leqslant n$. Together with $2a - s_1 + 1 \geqslant n$, these two inequalities yield $3s_1 - 3 \leqslant n \leqslant 3s_1 + 1$, implying that $n \in \{3s_1 - 3, 3s_1 - 1, 3s_1 + 1\}$. Moreover, if $n = 3s_1 - 3$ then $a = 2s_1 - 2$, while if $n \in \{3s_1 - 1, 3s_1 + 1\}$ then $a = 2s_1$.

Recall that we assumed that $|V_2| \geqslant |V_3|$, with the idea that if this is not the case we can replace the roles of the vertices 0 and s_1. One can verify that there are only two situations, which are not symmetric in the sense that exchanging the roles of the vertices 0 and s_1 changes the corresponding a. The first is when $n = 3s_1 - 1$ and $a = 2s_1$ (where the exchange yields $a = 2s_1 - 2$), and the other is when $n = 3s_1 + 1$ and $a = 2s_1 + 1$ (where the exchange yields $a = 2s_1 - 1$). ☐

We finally determine the connection set S and the relationship between n and s_1.

Lemma 18. *With reference to Assumption 14,* $n \in \{3s_1 - 3, 3s_1 - 1, 3s_1 + 1, 3s_1 + 3\}$ *and one of the following holds:*

- $S = \{\pm 1, \pm s_1\}$,
- $n = 3s_1 - 3$ *with* $s_1 \geqslant 4$ *and* $S = \{\pm 1, \pm(s_1 - 2), \pm s_1\}$,
- $n = 3s_1 + 3$ *and* $S = \{\pm 1, \pm s_1, \pm(s_1 + 2)\}$.

Moreover, in each of these three cases the graph Γ *is not fractional* 2*-extendable.*

Proof. By Lemmas 15 and 16, s_1 is even and $a > s_1$. Moreover, Lemma 17 implies that $n \in \{3s_1 - 3, 3s_1 - 1, 3s_1 + 1, 3s_1 + 3\}$ and that one of the items from that lemma holds. Suppose first that $n \in \{3s_1 - 1, 3s_1 + 1\}$ and note that in this case 3 is coprime to n. Multiplication by 3 is then an isomorphism from Γ to the circulant $\mathrm{Circ}(n; 3S)$. Note that the images of the edges e and e' are the edges corresponding to the elements 1 and 3 from $3S$. If $|S| > 4$ we therefore contradict Lemma 16. It thus follows that $|S| = 4$, that is, $S = \{\pm 1, \pm s_1\}$. Recall that we already know that $\mathrm{Circ}(n; \{\pm 1, \pm 3\})$ is not fractional 2-extendable, implying that Γ is also not fractional-2-extendable in this case.

From now on we can thus assume that either $n = 3s_1 - 3$ and $a = 2s_1 - 2$, or $n = 3s_1 + 3$ and $a = 2s_1 + 1$. Note that if $s_1 = 2$, then $n = 3s_1 + 3 = 9$ must hold, and consequently $a = 2s_1 + 1 = 5$. In this case, $3 \notin S$ since otherwise $(1, 3, 4, 7, 8)$ is a spanning 5-cycle of Γ', a contradiction. By Assumption 14 it thus follows that $S = \{\pm 1, \pm 2, \pm 4\}$. By Lemma 9, Γ is indeed not fractional 2-extendable in this case. We can thus assume that $s_1 \geqslant 4$. Let V_1, V_2 and V_3 be as in the proof of Lemma 16. Note that this time V_2 and V_3 are of equal size $s_1 - 3$ or s_1, depending on whether $n = 3s_1 - 3$ or $n = 3s_1 + 3$, respectively. We consider the two possibilities separately.

Case 1: $n = 3s_1 - 3$.
As already mentioned, $a = 2s_1 - 2$ and $|V_2| = |V_3| = s_1 - 3$, which is an odd number. For $s_1 = 4$, we get $n = 9$, in which case multiplication by 2 gives an isomorphic circulant where the edges e and e' are mapped to edges corresponding to 1 and 2 from the new connection set. Since we already settled this possibility, we can assume that $s_1 \geqslant 6$.

We claim that there exists no even $s \in S \setminus \{s_1, s_1 - 2\}$ with $s < n/2$. Suppose on the contrary that such an s exists. If $s \leqslant s_1 - 4$, we can take the cycles of (odd) length $s + 1$ consisting of the "first" $s + 1$ vertices in each of V_1, V_2 and V_3, and cover the remaining vertices of Γ' by independent 1-edges, a contradiction. If however $s \geqslant s_1 + 2$, then $2 \leqslant 2s_1 - s \leqslant s_1 - 2$, and we can take the cycle $(n - 1, s_1 - 1, s_1 - 2, \ldots, 2s_1 - s, 2s_1, 2s_1 + 1, \ldots, n - 2)$, the edge $\{1, s_1 + 1\}$ and cover the remaining vertices of Γ' by independent 1-edges, a contradiction. This proves our claim.

We now show there is also no odd $s \in S$ with $1 < s < n/2$. If such an s exists, then $3 \leqslant s \leqslant (3s_1 - 4)/2 = 2s_1 - (s_1 + 4)/2 \leqslant 2s_1 - 5$ (recall that $s_1 \geqslant 6$). But then there exists an odd $j \in V_2$ such that $j + s \in V_3$. We can therefore take the cycle $(1, s_1 + 1, s_1 + 2, \ldots, j, j + s, j + s + 1, \ldots, n - 1, s_1 - 1, s_1 - 2, \ldots, 2)$, which leaves us with $2s_1 - 3 - j$ consecutive vertices of V_2 and $j + s - 2s_1$ consecutive vertices of V_3, which can all be covered by independent 1-edges, a contradiction.

This shows that $S \subseteq \{\pm 1, \pm(s_1 - 2), \pm s_1\}$. It is easy to see that in this case the graph obtained from Γ' by removing its potential edge $\{1, s_1 - 1\}$ (but leaving its endvertices) is bipartite with 1 and $s_1 - 1$ belonging to the smaller set of the bipartition. Therefore, Corollary 3 implies that Γ' indeed has no fractional perfect matching.

Case 2: $n = 3s_1 + 3$.
In this case $a = 2s_1 + 1$ and $|V_2| = |V_3| = s_1$ is even. The argument is very similar to the one in Case 1, so we leave some details to the reader.

We first verify that there exists no even $s \in S \setminus \{s_1, s_1 + 2\}$ with $s < n/2$. For, if such an s exists, then if $s < s_1$, we can take the cycle containing the "first" $s+1$ vertices of V_1 and cover the remaining vertices of Γ' with independent 1-edges. If however, $s_1+4 \leqslant s < n/2$, then $1 \leqslant 2s_1 + 3 - s \leqslant s_1 - 1$, and we can take the cycle $(n-1, s_1-1, s_1-2, \ldots, 2s_1 + 3 - s, 2s_1+3, 2s_1+4, \ldots, n-2)$ and cover the remaining vertices of Γ' with independent 1-edges. In both cases we contradict Assumption 14.

Similarly, there exists no odd $s \in S$ with $1 < s < n/2$. For, if it exists, then $3 \leqslant s \leqslant 2s_1 - 1$ (recall that $s_1 \geqslant 4$), and so there exists an even $j \in V_2$ with $j + s \in V_3$. We then take the cycle $(1, s_1+1, s_1+2, \ldots, j, j+s, j+s+1, \ldots, n-1, s_1-1, s_1-2, \ldots, 2)$ and cover the remaining vertices of Γ' with independent 1-edges, again contradicting Assumption 14.

Therefore, $S \subseteq \{\pm 1, \pm s_1, \pm(s_1+2)\}$. It is again easy to see that the graph obtained from Γ' by removing its potential edge $\{s_1+1, n-1\}$ is bipartite with s_1+1 and $n-1$ belonging to the smaller set of the bipartition. Therefore, Γ' has no fractional perfect matching by Corollary 3. $\square$

Combining together Proposition 7 and all of the above lemmas, we finally obtain the following classification of fractional 2-extendable connected Cayley graphs of Abelian groups.

Theorem 19. *Let $\Gamma = \mathrm{Cay}(A; S)$ be a connected Cayley graph of an Abelian group of order $n \geqslant 5$. Then Γ is fractional 2-extendable if and only if it is not isomorphic to one of the following graphs:*

(i) $\mathrm{Circ}(n; \{\pm 1\})$,

(ii) $\mathrm{Circ}(n; \{\pm 1, 2m\})$, *where* $n = 4m \geqslant 8$,

(iii) $\mathrm{Circ}(n; \{\pm 2, 2m+1\})$, *where* $n = 4m + 2 \geqslant 6$,

(iv) $\mathrm{Circ}(n; \{\pm 1, \pm 2\})$,

(v) $\mathrm{Circ}(n; \{\pm 1, \pm 3\})$, *where n is odd,*

(vi) $\mathrm{Circ}(n; \{\pm 1, \pm 2m\})$, *where* $n = 4m + 2 \geqslant 6$,

(vii) $\mathrm{Circ}(n; \{\pm 1, \pm(m-1)\})$, *where $n = 3m \geqslant 9$ with m odd.*

(viii) $\mathrm{Circ}(n; \{\pm 1, \pm(m+1)\})$, *where $n = 3m \geqslant 9$ with m odd.*

(ix) $\mathrm{Circ}(n; \{\pm 1, \pm(m-1), \pm(m+1)\})$, *where $n = 3m \geqslant 9$ with m odd.*

(x) $\mathrm{Cay}(\mathbb{Z}_m \times \mathbb{Z}_3; \{\pm(1,0), \pm(1,1)\})$, *where $n = 3m \geqslant 9$ with m odd.*

4 Concluding remarks

In Subsection 3.2, we saw that when considering only graphs of even order, the families of fractional and classical 2-extendable connected Cayley graphs of Abelian groups coincide. Therefore, relaxing the condition of 2-extendability to that of fractional 2-extendability does not reduce the class of "non-examples". The following question thus arises naturally.

Question 20. Does there exist a Cayley graph of an Abelian group of even order such that for some integer t, this graph is fractional t-extendable but is not t-extendable? If so, what is the smallest t for which such a graph exists?

There are other interesting questions concerning (fractional) extendability in Cayley graphs of Abelian groups. Let us mention just one more. In 1993, Yu [26] introduced a generalization of the concept of t-extendability (in the classical sense) for graphs of even order to those of odd order in the following way. A graph Γ of odd order at least $2t+3$ is said to be $t\frac{1}{2}$-extendable (called t-near-extendable in [6]), if for each vertex v of Γ the graph $\Gamma - v$ is t-extendable. Using Proposition 2, one can show that a graph which is not fractional t-extendable, is also not $t\frac{1}{2}$-extendable, and so each $t\frac{1}{2}$-extendable graph is fractional t-extendable. It is thus interesting to investigate the difference between these two concepts, in general, but also in the context of Cayley graphs of Abelian groups of odd order. In [15], it was shown that, with the exception of cycles of odd length, all connected Cayley graphs of Abelian groups of odd order are $1\frac{1}{2}$-extendable. Corollary 6 thus implies that within the family of connected Cayley graphs of Abelian groups of odd order, the class of fractional 1-extendable examples coincides with the class of $1\frac{1}{2}$-extendable ones. The first next step would be to see if all connected Cayley graphs of Abelian groups of odd order, other than the ones from the ten families from Theorem 19, are $2\frac{1}{2}$-extendable. More generally, the following question arises naturally.

Question 21. Does there exist a positive integer $t \geqslant 2$ and a connected Cayley graph Γ of an Abelian group of odd order, such that Γ is fractional t-extendable, but is not $t\frac{1}{2}$-extendable? If so, what is the smallest t for which such a graph exists?

Finally, it would be interesting to study fractional extendability (and the corresponding analogues of Questions 20 and 21) for some other nice families of graphs. One could consider Cayley graphs of other groups or vertex-transitive graphs in general. However, other families of graphs with a large degree of regularity could also be of interest. Examples of such families, where investigations of classical extendability has been considered, are edge-regular graphs [10], quasi-strongly regular graphs [1], strongly regular graphs [6, 9, 11], and more generally distance regular graphs [5], to mention just a few. The topic thus offers many possible directions for future investigations.

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