Counting substructures and eigenvalues II: quadrilaterals

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Abstract

Let G be a graph with m edges and $\lambda(G)$ be the spectral radius of G. Nikiforov [Combin. Probab. Comput., 2002] proved that if $\lambda(G) > \sqrt{(1-\frac{1}{r})2m}$ then G contains a K_{r+1}. Bollobás and Nikiforov [J. Combin. Theory, Ser. B, 2007] proved some spectral counting results for cliques, which is a spectral Moon-Moser Inequality. Very recently, the present authors proved a counting result of spectral Rademacher Theorem for triangles.

It is natural to consider counting results for classes of degenerate graphs. A previous result due to Nikiforov [Linear Algebra Appl., 2009] asserted that every graph G on $m \geqslant 10$ edges contains a 4-cycle if $\lambda(G) > \sqrt{m}$. Define $f(m)$ to be the minimum number of copies of 4-cycles in such a graph. A consequence of a recent theorem due to Zhai et al. [European J. Combin., 2021] shows that $f(m) = \Omega(m)$. In this article, by somewhat different techniques, we prove that $f(m) = \Theta(m^2)$. We mention some problems for further study.

Mathematics Subject Classifications: 05C50; 05C35

1 Introduction

This is the second paper of our project [26], which aims to study the relationship between copies of a given substructure and the eigenvalues of a graph. In this article, we study the supersaturation problem of 4-cycles under the condition of spectral radius and size of a graph.

The study of 4-cycles plays an important role in the history of extremal graph theory. The extremal number of C_4 (i.e., a 4-cycle), denoted by $ex(n, C_4)$, is defined to be the

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maximum number of edges in a graph which contains no 4-cycle as a subgraph. The study of $ex(n, C_4)$ can be at least dated back to Erdős [8] eighty years ago. A longstanding conjecture of Erdős and Simonovits [9, Conjecture 1] states that every graph on n vertices and at least $ex(n, C_4) + 1$ edges contains at least $\sqrt{n} + o(\sqrt{n})$ copies of 4-cycles. This topic of counting C_4 appeared in some papers of Erdős many times. A weaker one states that there are at least two copies of C_4 under the same condition (see [5, Conjecture 42] and [6, p. 84]). It seems very uncertain and mysterious to count C_4's for graphs with size $ex(n, C_4) + 1$.

The original supersaturation problem of subgraphs in graphs focuses on the following function: for a given graph H and for integers $n, t \geqslant 1$,

$$h_H(n,t) = min\{\#H : |V(G)| = n, |E(G)| = ex(n,H) + t\},$$

where $ex(n, H)$ is the Turán function of H. Establishing a conjecture of Erdős, Lovász and Simonovits [19] proved that $h_{C_3}(n, k) \geqslant k\lfloor \frac{n}{2} \rfloor$ for all $1 \leqslant k < \lfloor \frac{n}{2} \rfloor$. But the weak version of Erdős' conjecture on copies of C_4 mentioned above tells us $h_{C_4}(n, 1) = 2$ for some positive integers n. This means that supersaturation phenomenon of C_4 is quite different from the cases of triangles [19]. On the other hand, counting the copies of 4-cycles plays a heuristic important role in measuring the quasirandom-ness of a graph (see Chap. 9 in [1]).

Throught this paper, we use $\lambda(G)$ to denote the spectral radius of a graph G. As an important case of spectral Zarankiewicz problem, Nikiforov [23] proved that every n-vertex C_4-free graph satisfies that $\lambda(G) \leqslant \frac{1}{2} + \sqrt{n - \frac{3}{4}}$, and the earlier bound of Babai and Guiduli [2] gives the correct order of the main term. As the counterpart of these results, we consider sufficient eigenvalue conditions (in terms of the size of a graph) for the existence of 4-cycles. A pioneer result can be found in [22].

Theorem 1 ([22])**.** *Let G be a graph with m edges, where $m \geqslant 10$. If $\lambda(G) \geqslant \sqrt{m}$ then G contains a 4-cycle, unless G is a star (possibly with some isolated vertices).*

Recently, Theorem 1 was extended by the following.

Theorem 2 ([29])**.** *Let r be a positive integer and G be a graph with m edges where $m \geqslant 16r^2$. If $\lambda(G) \geqslant \sqrt{m}$, then G contains a copy of $K_{2,r+1}$, unless G is a star (possibly with some isolated vertices).*

Let B_r be an *r-book*, that is, the graph obtained from $K_{2,r}$ by adding one edge within the partition set of two vertices. Very recently, Nikiforov [25] proved that, if $m \geqslant (12r)^4$ and $\lambda(G) \geqslant \sqrt{m}$, then a graph G contains a copy of B_{r+1}, unless G is a complete bipartite graph (possibly with some isolated vertices). This result further extends above two theorems and solves a conjecture proposed in [29].

The central topic of this article is the following spectral radius version of supersaturation problem of 4-cycles :

Problem 3. Let $f(m)$ be the minimum number of copies of 4-cycles over all labelled graphs G on m edges with $\lambda(G) > \sqrt{m}$. Give an estimate of $f(m)$.

Till now, the only counting result related to Problem 3 is a consequence of Theorem 2. Note that $K_{2,r+1}$ contains $\frac{r(r+1)}{2}$ copies of 4-cycles for $r = \frac{\sqrt{m}}{4}$. Theorem 2 implies that $f(m) \geqslant \frac{m}{32}$, unless G is a star (possibly with some isolated vertices).

One may ask for the best answer to Problem 3. In this paper, we make the first progress towards this problem.

Theorem 4. *Let $m \geqslant 3.6 \times 10^9$ be a positive integer. Let G be an m-edge graph with $\lambda(G) \geqslant \sqrt{m}$. Then G has at least $\frac{m^2}{2000}$ copies of C_4, unless G is a star (possibly with some isolated vertices).*

Throughout the left part, we also define $f(G)$ to be the number of copies of 4-cycles in a graph G.

Problem 5. $f(m) \leqslant \frac{(m-1)(m-2\sqrt{m})}{8}$.

Proof. Let $s = \sqrt{m} + 1$ and K_s^+ be the graph obtained from the complete graph K_s by adding $m - \binom{s}{2}$ pendent edges to one vertex of K_s. Clearly, $\lambda(K_s^+) \geqslant \lambda(K_s) = \sqrt{m}$. However, observe that K_s^+ contains $\binom{s}{4}$ copies of K_4 and every K_4 contains three copies of 4-cycles. Consequently, $f(K_s^+) = 3\binom{s}{4} = \frac{(m-1)(m-2\sqrt{m})}{8}$. □

Together with Theorem 4 and Proposition 5, one can easily find that $f(m) = \Theta(m^2)$.

In general, for a family $\mathcal{F}$ of graphs, a graph G is said to be *$\mathcal{F}$-free*, if it does not contain any $F \in \mathcal{F}$ as a subgraph. Let $\mathcal{G}(m, \mathcal{F})$ denote the set of $\mathcal{F}$-free graphs on m edges without isolated vertices. An interesting variation of Turán-type problem is as follows.

Problem 6. (Brualdi-Hoffman-Turán-type problem) What is the maximum spectral radius over all graphs in $\mathcal{G}(m, \mathcal{F})$?

The study of Problem 6 can be dated back to 1970, when Nosal [27] showed that $\lambda(G) \leqslant \sqrt{m}$ for every graph $G \in \mathcal{G}(m, K_3)$. Nikiforov extended Nosal's result to K_r and characterized the extremal graphs.

Theorem 7 ([20, 21])**.** *For every graph $G \in \mathcal{G}(m, K_{r+1})$, we have $\lambda(G) \leqslant \sqrt{(1 - \frac{1}{r})2m}$, with equality if and only if G is a complete bipartite graph for $r = 2$, and G is a regular complete r-partite graph for $r \geqslant 3$.*

Let $spex(m, K_3)$ denote the maximum spectral radius of an m-edge graph containing no K_3. Since $spex(m, K_3)$ can only be attained at complete bipartite graphs, one may ask what is the maximum spectral radius of a non-bipartite triangle-free graph on m edges? Let G be a non-bipartite triangle-free graph on m edges without isolated vertices. Lin, Ning, and Wu [12] showed that $\lambda(G) \leqslant \sqrt{m-1}$, with equality if and only if G is a 5-cycle. Zhai and Shu [30] extended the above result by a sharp upper bound for $m \geqslant 5$ where the number of vertices in extremal graphs can tend to be infinite. Later, Li and Peng [15], and independently, Sun and Li [28] obtained a sharp upper bound for non-bipartite $\{C_3, C_5\}$-free graphs. Lou, Lu, and Huang [18] gave a sharp upper bound for non-bipartite $\{C_3, C_5, \ldots, C_{2k+1}\}$-free graphs. For a recent work on spectral conditions for cycles with consecutive lengths, we refer the reader to the work by Li, Zhai, and Shu [16]. For more detailed description of development in this direction, we refer to introduction part of Li-Feng-Peng's work [14] and the references therein.

Motivated by these papers and a previous result on triangles (see [26]), we would like to present a counting version of Problem 6 as follows.

Problem 8. Let F be a given graph and $\lambda^*(m)$ be the maximum spectral radius over all graphs in $\mathcal{G}(m, F)$. How many copies of F can have in a graph G on m edges with $\lambda(G) > \lambda^*(m)$?

Let us introduce some necessary notation and terminologies. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $u \in V(G)$, we denote by $N_G(u)$ the set of neighbors of u, and by $d_G(u)$ the degree of u. The symbol $G - v$ denotes the subgraph induced by $V(G)\backslash\{v\}$ in G.

The paper is organized as follows. In Section 2, we shall give some necessary preliminaries and prove a key lemma. We present a proof of our main theorem in Section 3. We conclude this article with one corollary and some open problems for further study.

2 Preliminaries

In this section, we introduce some lemmas, which will be used in the subsequent proof.

The first lemma is known as Cauchy's Interlace Theorem.

Lemma 9 ([4])**.** *Let A be a symmetric $n \times n$ matrix and B be an $r \times r$ principal submatrix of A for some $r < n$. If the eigenvalues of A are $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and the eigenvalues of B are $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r$, then $\lambda_i \geqslant \mu_i \geqslant \lambda_{i+n-r}$ for all $1 \leqslant i \leqslant r$.*

The following inequality is due to Hofmeistar.

Lemma 10 ([10])**.** *Let G be a graph of order n and $M(G) = \sum_{u \in V(G)} d_G^2(u)$. Then*

$$\lambda(G) \geqslant \sqrt{\frac{1}{n}M(G)}, \tag{1}$$

with equality if and only if G is either regular or bipartite semi-regular.

The following lemma is due to Liu and Liu [17].

Lemma 11 ([17])**.** *Let G be a graph of order n and size m. Then*

$$f(G) = \frac{1}{8}\sum_{i=1}^{n} \lambda_i^4 + \frac{m}{4} - \frac{1}{4}M(G), \tag{2}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A(G)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$.

The following result is well-known (see [27, 3]).

Lemma 12 ([27, 3])**.** *Let G be a bipartite graph with m edges, where $m \geqslant 1$. Then $\lambda(G) \leqslant \sqrt{m}$, with equality if and only if G is a complete bipartite graph (possibly with some isolated vertices).*

We need prove the last lemma. A proof of its one special case that $m \leqslant n-2$ can be found in [11, Lemma 2.4].

Lemma 13. *Let G be a graph with m edges. Then $M(G) \leqslant m^2 + m$.*

Proof. Let G be an extremal graph with the maximum $M(G)$ and $n := |G|$. Let $V(G) = \{u_1, \dots, u_n\}$, and $d_i := d_G(u_i)$ for each $u_i \in V(G)$. We may assume that $d_1 \geqslant \cdots \geqslant d_n \geqslant 1$.

If there exists some integer $i \geqslant 2$ such that $u_i u_1 \notin E(G)$, then choose a vertex $u_j \in N_G(u_i)$ and define $G' := G - u_i u_j + u_i u_1$. Now $d_{G'}(u_1) = d_1 + 1$, $d_{G'}(u_j) = d_j - 1$ and $d_{G'}(u_k) = d_G(u_k)$ for each $k \in \{2, \dots, n\} \setminus \{j\}$. Consequently,

$$M(G') - M(G) = (d_1+1)^2 + (d_j - 1)^2 - d_1^2 - d_j^2 = 2d_1 - 2d_j + 2 \geqslant 2,$$

a contradiction. Thus, $N_G(u_1) = V(G) \setminus \{u_1\}$, and so $d_1 = n - 1$.

Let $|E(G-u_1)| = k$. Clearly, $|E(G-u_1)| = m - d_1$. If $|E(G-u_1)| = 0$, then $G \cong K_{1,m}$, and so $\sum_{i=1}^{n} d_i^2 = m^2 + m$, as desired. In the following, we assume $|E(G-u_1)| \geqslant 1$.

If $|E(G-u_1)| \leqslant d_1 - 2$, then $d_i + d_j \leqslant |E(G-u_1)| + 3 \leqslant d_1 + 1$ for each $u_i u_j \in E(G-u_1)$. Now let $G' = G - u_i u_j + u_1 u_0$, where $u_i u_j \in E(G-u_1)$ and u_0 is a new vertex only adjacent to u_1 in G'. Then

$$\begin{aligned} M(G') - M(G) &= (d_1+1)^2 + 1 + (d_i - 1)^2 + (d_j - 1)^2 - d_1^2 - d_i^2 - d_j^2 \\ &= 2(d_1 - d_i - d_j) + 4. \end{aligned}$$

It follows that $M(G') > M(G)$, a contradiction. Therefore, $|E(G-u_1)| \geqslant d_1 - 1$.

Now define a new graph $G' := K_{1,d_1+k}$. Then $k \geqslant d_1 - 1$ and $|E(G')| = d_1 + k = |E(G)| = m$. Note that $n = d_1 + 1$ and $2k = 2|E(G-u_1)| = \sum_{i=2}^{n}(d_i - 1)$. Hence, $2kd_1 \geqslant \sum_{i=2}^{n} d_i^2 - d_1^2 = M(G) - 2d_1^2$. It follows that

$$M(G') - M(G) = (k+d_1)^2 + (k+d_1) - M(G) \geqslant k^2 - d_1^2 + (k+d_1) \geqslant 0,$$

as $k \geqslant d_1 - 1$. Thus, $M(G) \leqslant M(G') = m^2 + m$. This proves Lemma 13. □

3 Proof of Theorem 4

In this section, we give a proof of Theorem 4. We would like to point out that the techniques used in the left part are completely different from [26].

3.1 A key lemma

We first prove a key lemma.

Lemma 14. *Let G be a graph of size $m \geqslant 1.8 \times 10^9$ and X be the Perron vector of G with component x_u corresponding to $u \in V(G)$. If $\lambda(G) \geqslant \sqrt{m}$ and $x_u x_v > \frac{1}{9\sqrt{m}}$ for any $uv \in E(G)$, then $f(G) \geqslant \frac{m^2}{500}$ unless G is a star (possibly with some isolated vertices).*

Proof. We may assume that $\delta(G) \geqslant 1$, where $\delta(G)$ is the minimum degree of G. Since $x_u x_v > \frac{1}{9\sqrt{m}}$ for any $uv \in E(G)$, we can see that X is a positive vector, and hence G is connected. Let $A = \{u \in V(G) : x_u > \frac{1}{3\sqrt[4]{m}}\}$ and $B = V(G) \setminus A$. Clearly, B is an independent set. Now suppose that $f(G) < \frac{m^2}{500}$ and set $\lambda := \lambda(G)$. We will prove a series of claims.

Claim 15. *We have $\delta(G) \geqslant 2$ unless $G \cong K_{1,m}$.*

Proof. Assume that there exists a vertex $u \in V(G)$ with $d_G(u) = 1$ and $N_G(u) = \{\bar{u}\}$. Then $x_u x_{\bar{u}} = \frac{x_{\bar{u}}^2}{\lambda} \leqslant \frac{x_{\bar{u}}^2}{\sqrt{m}}$. Since $x_u x_{\bar{u}} > \frac{1}{9\sqrt{m}}$, we have $x_{\bar{u}} > \frac{1}{3}$. Let $u^* \in V(G)$ with $x_{u^*} = \max_{v \in V(G)} x_v$. Then $x_{u^*} > \frac{1}{3}$.

Now let $S := N_G(u^*)$, $T := V(G) \setminus (S \cup \{u^*\})$, and $N_S(v) = N_G(v) \cap S$ for a vertex $v \in V(G)$. Moreover, we partite S into three subsets S_1, S_2 and S_3, where $S_1 = \{v \in S : \frac{1}{4} < x_v \leqslant x_{u^*}\}$, $S_2 = \{v \in S : \frac{1}{6} < x_v \leqslant \frac{1}{4}\}$, and $S_3 = \{v \in S : 0 < x_v \leqslant \frac{1}{6}\}$.

Choose a vertex $u \in S_1$ arbitrarily. By Cauchy-Schwarz inequality,

$$(\lambda x_u)^2 = \Big(\sum_{v \in N_G(u)} x_v \Big)^2 \leqslant d_G(u) \sum_{v \in N_G(u)} x_v^2 \leqslant d_G(u)(1 - x_u^2). \tag{3}$$

Since $x_u > \frac{1}{4}$ and $\lambda \geqslant \sqrt{m}$, we have $d_G(u) \geqslant \frac{m}{15}$. If $|N_T(u)| \leqslant \frac{m}{450}$, then $|N_S(u)| \geqslant \frac{m}{15} - \frac{m}{450} - 1 \geqslant \frac{m}{15.6}$, and thus G contains a copy of $K_{2,\lceil \frac{m}{15.6} \rceil}$. Hence, G contains at least $\binom{\lceil \frac{m}{15.6} \rceil}{2}$ ($\geqslant \frac{m^2}{500}$) quadrilaterals, a contradiction. Therefore, $|N_T(u)| \geqslant \frac{m}{450}$ and $|N_S(u)| < \frac{m}{15.6}$. Now let $S^* = \{v \in S : x_v < \frac{1}{108}\}$, $T^* = \{v \in T : x_v < \frac{1}{108}\}$ and $V' = (S \setminus S^*) \cup (T \setminus T^*)$. Since X is a unit vector, we have $|V'| \leqslant 108^2$. By Cauchy-Schwarz inequality,

$$\sum_{v \in V'} x_v \leqslant \sqrt{|V'| \sum_{v \in V'} x_v^2} \leqslant \sqrt{|V'|} \leqslant 108. \tag{4}$$

Consequently,

$$\sum_{v \in N_S(u)} x_v = \sum_{v \in N_{S \setminus S^*}(u)} x_v + \sum_{v \in N_{S^*}(u)} x_v \leqslant 108 + \frac{1}{108} |N_{S^*}(u)|.$$

Recall that $|N_{S^*}(u)| \leqslant |N_S(u)| \leqslant \frac{m}{15.6}$ and $x_{u^*} > \frac{1}{3}$. It follows that

$$\sum_{v \in N_S(u)} x_v \leqslant (324 + \frac{1}{36} |N_{S^*}(u)|) x_{u^*} < \frac{1}{36} \cdot \frac{m}{15} x_{u^*}.$$

On the other hand, note that $|N_{T^*}(u)| \geqslant |N_T(u)| - 108^2 \geqslant \frac{m}{525}$ and $x_v < \frac{1}{108} < \frac{1}{36}x_{u^*}$ for any $v \in T^*$. Then

$$\begin{aligned}\sum_{v\in N_T(u)} x_v &< \sum_{v\in N_{T\setminus T^*}(u)} x_{u^*} + \sum_{v\in N_{T^*}(u)} \frac{1}{36}x_{u^*} \leqslant |N_T(u)|x_{u^*} - \frac{35}{36}\cdot\frac{m}{525}x_{u^*} \\ &= |N_T(u)|x_{u^*} - \frac{1}{36}\cdot\frac{m}{15}x_{u^*}.\end{aligned}$$

It follows that $\sum_{v\in N_{S\cup T}(u)} x_v < |N_T(u)|x_{u^*}$. Let $e(S,T)$ be the number of edges from S to T, and $e(S)$ be the number of edges within S. Then

$$\sum_{u\in S_1}\sum_{v\in N_{S\cup T}(u)} x_v < e(S_1,T)x_{u^*}. \tag{5}$$

Secondly, consider a vertex $u \in S_2$ arbitrarily. Note that $x_u > \frac{1}{6}$ and $\lambda \geqslant \sqrt{m}$. Then (3) gives $d_G(u) \geqslant \frac{m}{35}$. Since $S^* \subseteq S_3$ and $x_{u^*} - x_u > \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, we have

$$\sum_{v\in N_{S^*}(u)} x_v \leqslant \frac{1}{108}|N_{S_3}(u)| \leqslant \frac{1}{9}|N_{S_3}(u)|(x_{u^*} - x_u),$$

and by (4) we have $\sum_{v\in N_{S\setminus S^*}(u)} x_v \leqslant \sum_{v\in V'} x_v \leqslant 108$. Then

$$\sum_{v\in N_S(u)} x_v = \sum_{v\in N_{S\setminus S^*}(u)} x_v + \sum_{v\in N_{S^*}(u)} x_v \leqslant 108 + \frac{1}{9}|N_{S_3}(u)|(x_{u^*} - x_u). \tag{6}$$

If $|N_{S^*}(u)| \geqslant \frac{m}{72}$, then $|N_{S_3}(u)| \geqslant |N_{S^*}(u)| \geqslant \frac{m}{72}$. Since $x_{u^*} - x_u > \frac{1}{12}$, it follows from (6) that $\sum_{v\in N_S(u)} x_v < |N_{S_3}(u)|(x_{u^*} - x_u)$, and thus

$$\sum_{v\in N_{S\cup T}(u)} x_v < |N_{S_3}(u)|(x_{u^*} - x_u) + |N_T(u)|x_{u^*}. \tag{7}$$

If $|N_{S^*}(u)| \leqslant \frac{m}{72}$, then

$$|N_{T^*}(u)| \geqslant d_G(u) - |N_{S^*}(u)| - 108^2 - 1 > \frac{m}{72}.$$

Hence,

$$\sum_{v\in N_{T^*}(u)} x_v \leqslant |N_{T^*}(u)|\cdot\frac{1}{108} < |N_{T^*}(u)|x_{u^*} - 108,$$

as $x_{u^*} > \frac{1}{3}$. It follows that $\sum_{v\in N_T(u)} x_v < |N_T(u)|x_{u^*} - 108$. Combining with (6), we can also obtain (7). Therefore, in both cases we have

$$\sum_{u\in S_2}\sum_{v\in N_{S\cup T}(u)} x_v < e(S_2,S_3)x_{u^*} + e(S_2,T)x_{u^*} - \sum_{u\in S_2}|N_{S_3}(u)|x_u. \tag{8}$$

Thirdly, we consider an arbitrary vertex $u \in S_3$. Since $x_{u^*} > \frac{1}{3}$, we have $x_v \leqslant \frac{1}{6} < \frac{1}{2}x_{u^*}$ for each $v \in N_{S_3}(u)$. Thus, $\sum_{u\in S_3}\sum_{v\in N_{S_3}(u)} x_v \leqslant e(S_3)x_{u^*}$, with equality if and only if $e(S_3) = 0$. Therefore,

$$\sum_{u\in S_3}\sum_{v\in N_{S\cup T}(u)} x_v \leqslant e(S_3, S_1)x_{u^*} + e(S_3)x_{u^*} + e(S_3, T)x_{u^*} + \sum_{u\in S_3}\sum_{v\in N_{S_2}(u)} x_v. \tag{9}$$

Notice that

$$\sum_{u\in S_2} |N_{S_3}(u)|x_u = \sum_{u\in S_3}\sum_{v\in N_{S_2}(u)} x_v.$$

Combining with (5), (8) and (9), we have

$$\sum_{u\in S}\sum_{v\in N_{S\cup T}(u)} x_v \leqslant (e(S) + e(S, T))x_{u^*}, \tag{10}$$

where if equality holds then $S_1 \cup S_2 = \varnothing$ and $e(S_3) = 0$, that is, $e(S) = 0$. Furthermore, we can see that

$$\lambda^2 x_{u^*} = \sum_{u\in S}\sum_{v\in N_G(u)} x_v = |S|x_{u^*} + \sum_{u\in S}\sum_{v\in N_{S\cup T}(u)} x_v \leqslant (|S| + e(S) + e(S, T))x_{u^*} \leqslant m x_{u^*}.$$

Since $\lambda \geqslant \sqrt{m}$, the above inequality holds in equality, that is, $\lambda = \sqrt{m}$. Therefore, $m = |S| + e(S) + e(S, T)$, and (10) holds in equality (hence $e(S) = 0$). This implies that G is a bipartite graph. By Lemma 12, G is a complete bipartite graph. Since $f(G) < \frac{m^2}{500}$, G can only be a star. This completes the proof. □

In the following, we may assume that $G \ncong K_{1,m}$. Then by Claim 15, $\delta(G) \geqslant 2$.

Claim 16. $|A| \leqslant 9\sqrt{m}$.

Proof. Recall that $x_u > \frac{1}{3\sqrt[4]{m}}$ for each $u \in A$. Thus $\sum_{u\in A} x_u^2 > \frac{|A|}{9\sqrt{m}}$, and hence $|A| \leqslant 9\sqrt{m}\sum_{u\in A} x_u^2 \leqslant 9\sqrt{m}$. □

Claim 17. *Let* $|G| = \frac{m}{2} + b$*. Then* $-\frac{m}{125} \leqslant b \leqslant |A|$.

Proof. Set $\lambda' := \lambda_{|G|}$. Note that $\lambda \geqslant \sqrt{m}$. By Lemmas 10 and 11,

$$f(G) \geqslant \frac{1}{8}(\lambda^4 + \lambda'^4) - \frac{1}{4}M(G) \geqslant \frac{1}{8}(\lambda^4 + \lambda'^4) - \frac{|G|}{4}\lambda^2 \geqslant \frac{1}{8}\lambda'^4 - \frac{b}{4}\lambda^2. \tag{11}$$

If $b < -\frac{m}{125}$, then

$$f(G) \geqslant -\frac{b}{4}\lambda^2 \geqslant \frac{m}{500}\lambda^2 \geqslant \frac{m^2}{500},$$

a contradiction. Thus, $b \geqslant -\frac{m}{125}$.

On the other hand, recall that $e(B) = 0$ and $\delta(G) \geqslant 2$. Then

$$m \geqslant e(B, A) \geqslant 2|B| = 2(|G| - |A|) = 2(\frac{m}{2} + b - |A|).$$

Thus, $b \leqslant |A|$, as desired. □

Claim 18. *$\Delta(G) \leqslant \frac{2}{15}m$, where $\Delta(G)$ is the maximum degree of G.*

Proof. We know that $\sum_{i=1}^{|G|} \lambda_i^2 = 2m$. Thus, $\lambda^2 = \lambda_1^2 \leqslant 2m$. Combining with (11) and $b \leqslant |A| \leqslant 9\sqrt{m}$, we have

$$f(G) \geqslant \frac{1}{8}\lambda'^4 - \frac{b}{4}\lambda^2 \geqslant \frac{1}{8}\lambda'^4 - 9m^{\frac{3}{2}}. \tag{12}$$

Now if there exists some $u \in V(G)$ with $d_G(u) > \frac{2}{15}m$, then

$$|N_B(u)| \geqslant d_G(u) - |A| > \frac{2}{15}m - 9\sqrt{m}.$$

Since $e(B) = 0$, G contains $K_{1,|N_B(u)|}$ as an induced subgraph. By Lemma 9,

$$\lambda' \leqslant -\sqrt{|N_B(u)|} < -\sqrt{\frac{2}{15}m - 9\sqrt{m}},$$

and by (12) we have

$$f(G) \geqslant \frac{1}{8}\lambda'^4 - 9m^{\frac{3}{2}} \geqslant \frac{1}{8}\Big(\frac{2}{15}m - 9\sqrt{m}\Big)^2 - 9m^{\frac{3}{2}} > \frac{m^2}{500},$$

for $m \geqslant 1.8 \times 10^9$. We have a contradiction. Therefore, $\Delta(G) \leqslant \frac{2}{15}m$. □

Claim 19. *Let $B^* = \{u \in V(G) : d_G(u) = 2\}$. Then $B^* \subseteq B$ and*

$$\frac{m}{2} + 3(b - |A|) \leqslant |B^*| \leqslant \frac{m}{2}. \tag{13}$$

Proof. Let $u \in B^*$ and $N_G(u) = \{u_1, u_2\}$. Then $\lambda x_u = x_{u_1} + x_{u_2} \leqslant 2$. Since $\lambda \geqslant \sqrt{m}$, we have $x_u \leqslant \frac{2}{\sqrt{m}} < \frac{1}{3\sqrt[4]{m}}$, and so $u \in B$.

Recall that $e(B) = 0$. Thus, $e(B^*) = 0$, and $m \geqslant e(B^*, A) \geqslant 2|B^*|$. This gives $|B^*| \leqslant \frac{m}{2}$. On the other hand, note that $|B| = |G| - |A| = \frac{m}{2} + b - |A|$, then

$$m \geqslant e(B, A) \geqslant 2|B^*| + 3(|B| - |B^*|) = \frac{3}{2}m + 3(b - |A|) - |B^*|.$$

It follows that $|B^*| \geqslant \frac{m}{2} + 3(b - |A|)$. □

Claim 20. *Let $A^* = \{v \in N_G(u) : u \in B^*\}$. Then $A^* \subseteq A$ and $|A^*| \leqslant 24$.*

Proof. Since $e(B) = 0$, we have $N_G(u) \subseteq A$ for any $u \in B^*$. Thus, $A^* \subseteq A$. Furthermore, we will see that $\frac{1}{25} < x_v^2 \leqslant \frac{2}{17}$ for each $v \in A^*$.

Let v be an arbitrary vertex in A^*. By Cauchy-Schwarz inequality,

$$(\lambda x_v)^2 = \Big(\sum_{u \in N_G(v)} x_u\Big)^2 \leqslant d_G(v) \sum_{u \in N_G(v)} x_u^2 \leqslant d_G(v)(1 - x_v^2) \leqslant \frac{2}{15}m(1 - x_v^2),$$

as $\Delta(G) \leqslant \frac{2}{15}m$. Since $\lambda \geqslant \sqrt{m}$, we have $x_v^2 \leqslant \frac{2}{17}$.

If there exists a vertex $v \in A^*$ with $x_v^2 \leqslant \frac{1}{25}$, then by the definition of A^*, we can find a vertex $u \in N_{B^*}(v)$. Clearly,

$$\lambda x_u \leqslant x_v + \sqrt{\frac{2}{17}} \leqslant \frac{1}{5} + \sqrt{\frac{2}{17}} < \frac{5}{9}.$$

Consequently,

$$x_u x_v < \frac{1}{\lambda} \cdot \frac{5}{9} \cdot \frac{1}{5} \leqslant \frac{1}{9\sqrt{m}},$$

which contradicts the condition of Lemma 14. Therefore, $x_v^2 > \frac{1}{25}$ for any $v \in A^*$, and so $|A^*| \leqslant 24$. □

Claim 21. *Let $V'' := (A \setminus A^*) \cup (B \setminus B^*)$. Then $|V''| \leqslant \frac{m}{60}$ and $e(V'') \leqslant \frac{m}{20}$.*

Proof. Recall that $|A \cup B| = |G| = \frac{m}{2} + b$. Combining with (13), we obtain that $|V''| \leqslant |G| - |B^*| \leqslant 3|A| - 2b$. Moreover, by Claims 16 and 17, we have $|A| \leqslant 9\sqrt{m}$ and $b \geqslant -\frac{m}{125}$. Thus, $|V''| \leqslant 27\sqrt{m} + \frac{2}{125}m \leqslant \frac{m}{60}$.

Now we estimate $e(V'')$. Again by $|A| \leqslant 9\sqrt{m}$, $b \geqslant -\frac{m}{125}$ and (13), we have

$$e(A^*, B^*) = 2|B^*| \geqslant m + 6(b - |A|) \geqslant m - \frac{6}{125}m - 54\sqrt{m}.$$

It follows that $e(V'') \leqslant m - e(A^*, B^*) \leqslant \frac{6}{125}m + 54\sqrt{m} \leqslant \frac{m}{20}$. □

Now we give the final part of the proof. For convenience, let $d'(u) := |N_{V''}(u)|$ for each $u \in V''$. Note that $e(V'', B^*) = 0$. Thus by Claim 20,

$$d_G(u) \leqslant d'(u) + |A^*| \leqslant d'(u) + 24$$

for each vertex $u \in V''$. Consequently,

$$\sum_{u \in V''} d_G^2(u) \leqslant \sum_{u \in V''} (d'(u) + 24)^2 = 96e(V'') + 24^2|V''| + \sum_{u \in V''} d'^2(u). \tag{14}$$

Since $e(V'') \leqslant \frac{m}{20}$, by Lemma 13 we have $\sum_{u \in V''} d'^2(u) \leqslant \frac{m^2}{400} + \frac{m}{20}$. Combining this with Claim 21 and (14), we have

$$\sum_{u \in V''} d_G^2(u) \leqslant 96 \cdot \frac{m}{20} + 24^2 \cdot \frac{m}{60} + \frac{m^2}{400} + \frac{m}{20} < \frac{m^2}{225}. \tag{15}$$

On the other hand, by Claim 18, $\Delta(G) \leqslant \frac{2}{15}m$, and so

$$\sum_{u \in A^*} d_G^2(u) \leqslant |A^*|(\Delta(G))^2 \leqslant \frac{96}{225}m^2$$

(as $|A^*| \leqslant 24$). Moreover, by Claim 19 $|B^*| \leqslant \frac{m}{2}$, and thus

$$\sum_{u\in B^*} d_G^2(u) = 4|B^*| \leqslant 2m.$$

Combining with (15), we get

$$M(G) = \sum_{u\in V''\cup A^*\cup B^*} d_G^2(u) \leqslant \frac{1}{225}m^2 + \frac{96}{225}m^2 + 2m < \frac{100}{225}m^2 = \frac{4}{9}m^2.$$

Now by Lemma 11, we have

$$f(G) \geqslant \frac{1}{8}\lambda^4 - \frac{1}{4}M(G) \geqslant \frac{1}{8}m^2 - \frac{1}{9}m^2 = \frac{1}{72}m^2 > \frac{1}{500}m^2,$$

a contradiction. This completes the proof. □

3.2 Nikiforov's deleting small eigenvalue edge method

Over the past decades, Nikiforov developed some novel tools and techniques for solving problems in spectral graph theory (see [24]). One is the method we called "deleting small eigenvalue edge method", or "Nikiforov's DSEE Method". Generally speaking, an edge $xy \in E(G)$ is called a *small eigenvalue edge*, if $x_u x_v$ is small where x_u, x_v are Perron components.

By using this method, Nikiforov [22] successfully proved the following results, of which some original ideas appeared in [24] earlier:

- Every graph on m edges contains a 4-cycle if $\lambda(G) \geqslant \sqrt{m}$ and $m \geqslant 10$, unless it is a star with possibly some isolated vertices (see Claim 4 in [22, pp. 2903]);
- Every graph on m edges satisfies that the booksize $bk(G) > \frac{\sqrt[4]{m}}{12}$ if $\lambda(G) \geqslant \sqrt{m}$, unless it is a complete bipartite graph with possibly some isolated vertices (see [25], this confirmed a conjecture in [29]).

One main ingredient in the proof of Theorem 4 is to use this method.

3.3 Proof of Theorem 4

Now we are ready to give the proof of Theorem 4.

Proof of Theorem 4. Let G be a graph with $|E(G)| = m$ and $\lambda(G) \geqslant \sqrt{m}$. By using the Nikiforov DESS Method [25], we first construct a sequence of graphs.

(i) Set $i := 0$ and $G_0 := G$.
(ii) If $i = \lfloor \frac{m}{2} \rfloor$, stop.
(iii) Let $X = (x_1, x_2 \ldots, x_{|G_i|})^T$ be the Perron vector of G_i.
(iv) If there exists $uv \in E(G_i)$ with $x_u x_v \leqslant \frac{1}{9\sqrt{|E(G_i)|}}$, set $G_{i+1} := G_i - uv$ and $i := i + 1$.
(v) If there is no such edge, stop.

Assume that G_k is the resulting graph of the graph sequence constructed by the above algorithm. Then $k \leqslant \lfloor \frac{m}{2} \rfloor$. We can obtain the following two claims.

Claim 22. $\lambda(G_{i+1}) \geqslant \sqrt{m-i-1}$ *for each* $i \in \{0, 1, \ldots, k-1\}$.

Proof. Let X be the Perron vector of G_i with component x_u corresponding to $u \in V(G_i)$. Then, there exists $uv \in E(G_i)$ with $x_u x_v \leqslant \frac{1}{9\sqrt{|E(G_i)|}}$. Thus,

$$\lambda(G_{i+1}) \geqslant X^T A(G_{i+1}) X = X^T A(G_i) X - 2x_u x_v \geqslant \lambda(G_i) - \frac{2}{9\sqrt{|E(G_i)|}}.$$

Hence,

$$\lambda(G_0) \leqslant \lambda(G_1) + \frac{2}{9\sqrt{m}} \leqslant \cdots \leqslant \lambda(G_{i+1}) + \sum_{j=0}^{i} \frac{2}{9\sqrt{m-j}}.$$

It follows that

$$\lambda(G_{i+1}) \geqslant \lambda(G_0) - \frac{2(i+1)}{9\sqrt{m-i}} \geqslant \sqrt{m} - \frac{2(i+1)}{9\sqrt{m-i-1}}. \tag{16}$$

This implies that $\lambda(G_{i+1}) \geqslant \sqrt{m-i-1}$, as $i+1 \leqslant k \leqslant \lfloor \frac{m}{2} \rfloor$. □

Now we may assume that all isolated vertices are removed from each G_i, where $i \in \{0, 1, \ldots, k\}$.

Claim 23. G_k *cannot be a star unless* $G_k = G_0 \cong K_{1,m}$.

Proof. Suppose to the contrary that $k \geqslant 1$ while G_k is a star. Since $|E(G_k)| = m-k$, we have $G_k \cong K_{1,m-k}$. Let u_0 be the central vertex of G_k and $u_1, \ldots, u_{m-k}$ be the leaves. We now let $G_k = G_{k-1} - uv$ and X be the Perron vector of G_{k-1}.

If uv is a pendent edge incident to u_0, say $uv = u_0 u_{m-k+1}$, then

$$\lambda(G_{k-1}) = \sqrt{|E(G_{k-1})|} = \sqrt{m-k+1}$$

and $\lambda(G_{k-1}) x_{u_i} = x_{u_0}$ for $i \in \{1, 2, \ldots, m-k+1\}$. Hence, $\|X\|_2 = \sum_{i=0}^{m-k+1} x_{u_i}^2 = 2x_{u_0}^2$, which gives $x_{u_0}^2 = \frac{1}{2}$. It follows that

$$x_{u_0} x_{u_{m-k+1}} = \frac{x_{u_0}^2}{\sqrt{|E(G_{k-1})|}} > \frac{1}{9\sqrt{|E(G_{k-1})|}},$$

which contradicts the definition of G_k.

If uv is an isolated edge or a pendent edge not incident to u_0, then G_{k-1} is bipartite but not complete bipartite. By Lemma 12, $\lambda(G_{k-1}) < \sqrt{|E(G_{k-1})|}$, which contradicts Claim 22.

Now we conclude that uv is an edge within $V(G_k) \backslash \{u_0\}$, say $uv = u_1 u_2$, then $x_{u_1} = x_{u_2}$ and $\lambda(G_{k-1}) x_{u_1} = x_{u_0} + x_{u_2}$. Hence, $x_{u_1} = \frac{x_{u_0}}{\lambda(G_{k-1}) - 1} < \frac{1}{2} x_{u_0}$, as $\lambda(G_{k-1}) \geqslant \sqrt{m-k+1}$ by Claim 22. Consequently,

$$\lambda^2(G_{k-1}) x_{u_0} = \sum_{i=1}^{m-k} \lambda(G_{k-1}) x_{u_i} = (m-k) x_{u_0} + (x_{u_1} + x_{u_2}) < (m-k+1) x_{u_0}.$$

It follows that $\lambda(G_{k-1}) < \sqrt{m-k+1}$, which also contradicts Claim 22. □

Now we finish the proof of Theorem 4. Assume that G is not a star. Then G_k is not a star by Claim 23; moreover, $\lambda(G_k) \geqslant \sqrt{m-k} = \sqrt{|E(G_k)|}$ by Claim 22. If $k < \lfloor \frac{m}{2} \rfloor$, then $x_u x_v > \frac{1}{9\sqrt{|E(G_k)|}}$ for any edge $uv \in E(G_k)$. Since $|E(G_k)| = m-k > \frac{m}{2}$, by Lemma 14 $f(G_k) \geqslant \frac{|E(G_k)|^2}{500} > \frac{m^2}{2000}$, and so $f(G) > \frac{m^2}{2000}$.

If $k = \lfloor \frac{m}{2} \rfloor$, then by (16) we have

$$\lambda(G_k) \geqslant \sqrt{m} - \frac{2k}{9\sqrt{m-k}} \geqslant \sqrt{m} - \frac{m}{9\sqrt{\frac{m}{2}}} = \Big(1 - \frac{\sqrt{2}}{9}\Big)\sqrt{m},$$

and so

$$\lambda^4(G_k) \geqslant (1 - \frac{\sqrt{2}}{9})^4 m^2 > 0.5047m^2 > 0.504m^2 + 4m.$$

On the other hand, by Lemma 13,

$$M(G_k) \leqslant |E(G_k)|^2 + |E(G_k)| = \left\lceil \frac{m}{2} \right\rceil^2 + \left\lceil \frac{m}{2} \right\rceil \leqslant 0.25m^2 + 2m.$$

Thus by Lemma 11,

$$f(G_k) \geqslant \frac{1}{8}\lambda^4(G_k) - \frac{1}{4}M(G_k) > \frac{1}{8}(0.504 - 0.5)m^2 = \frac{1}{2000}m^2,$$

and so $f(G) > \frac{m^2}{2000}$. This completes the proof. ■

4 Concluding remarks

We do not try our best to optimize the constant "$\frac{1}{2000}$" in Theorem 4. So it is natural to pose the following problem:

Problem 24. Determine $\lim\limits_{m\to\infty} \frac{f(m)}{m^2}$. (We think that the upper bound in Proposition 5 is close to the truth.)

By Theorem 4 and an inequality $\lambda(G) \geqslant \frac{2m}{n}$ due to Collatz and Sinogowitz [7], we deduce the following immediately.

Theorem 25. *Let G be a graph on n vertices and m edges. If $m > \max\{\frac{n^2}{4}, 3.6 \times 10^9\}$, then G contains $\frac{n^4}{32000}$ copies of 4-cycles.*

An anonymous referee suggested the following improvement of the above theorem: The number of C_4 is equal to $\frac{1}{2}\sum_{S\in\binom{V}{2}} \binom{|N(S)|}{2}$, where S takes over all 2-sets of $V(G)$, and $N(S)$ denotes the set of common neighbors of vertices of S. By Jensen's inequality, we have $\#C_4 \geqslant \frac{1}{2}\binom{n}{2}\binom{N}{2}$, where

$$N = \frac{1}{\binom{n}{2}} \sum_{S\in\binom{V}{2}} |N(S)| = \frac{1}{\binom{n}{2}} \sum_{v\in V} \binom{d(v)}{2} \geqslant \frac{1}{\binom{n}{2}} \cdot n \binom{\frac{1}{n}\sum_{v\in V} d(v)}{2}.$$

Since $m > \frac{n^2}{4}$, we get $\frac{2m}{n} > \frac{n}{2}$ and so $N \geqslant \frac{n}{4} - \frac{1}{2}$. Therefore, we can get

$$\#C_4 \geqslant \frac{1}{2}\binom{n}{2}\binom{\frac{n}{4}-\frac{1}{2}}{2} = \frac{1}{128}n^4 - \frac{9}{128}n^3 + \frac{5}{32}n^2 - \frac{3}{32}n > \left(\frac{1}{128} - o(1)\right)n^4.$$

Let $B_{r,k}$ be the join of an r-clique with an independent set of size k. If $k = 1$, then $B_{r,k}$ is the complete graph K_{r+1}. We conclude this article with a new conjecture appeared in [13] which extends Theorem 1 and Theorem 7.

Conjecture 26 (Conjecture 1.20 in[13])**.** Let m be large enough and G be a $B_{r,k}$-free graph on m edges without isolated vertices. Then $\lambda(G) \leqslant \sqrt{(1-\frac{1}{r})2m}$, with equality if and only if G is a complete bipartite graph for $r = 2$, and G is a regular complete r-partite graph for $r \geqslant 3$.

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