Invertibility in the misère multiverse

Alfie Davies Vishal Yadav

Submitted: Oct 19, 2024; Accepted: Jun 4, 2025; Published: Oct 3, 2025

Abstract

Understanding invertibility in restricted misère play has been challenging; in particular, the possibility of non-conjugate inverses posed difficulties. Advances have been made in a few specific universes, but a general theorem was elusive. We prove that every universe has the conjugate property, and also give a characterisation of the invertible elements of each universe. We then explore when a universe can have non-trivial invertible elements, leaving a slew of open problems to be further investigated.

Mathematics Subject Classifications: 91A46, 06F05, 20M10

1 Introduction

The current direction of misère research is moving fiercely towards studying restrictions of the full misère structure $\mathcal{M}$. Larsson, Nowakowski, and Santos, in their work on absolute combinatorial game theory, introduced the notion of an (absolute) *universe*[1] [6, Definitions 12 and 13 on pp. 113–114], which is a set of games that satisfies additive, conjugate, hereditary, and dicotic closure. These closure properties, aside from the dicotic closure, are naturally occurring in many rulesets that are typically studied, and so these universes are not entirely arbitrary sets of games to study.

A terrific tool was built in [6, Theorem 4 on p. 103] that yields a comparison test when working modulo a universe, which is analogous to the test one enjoys when working in normal play—this cements the concept of the universe as warranting further study. It was then shown in [7, Theorem 26 on p. 239], again by Larsson, Nowakowski, and Santos, that there are infinitely many (misère) universes. In fact, Siegel strengthened this and

Department of Mathematics and Statistics, Memorial University of Newfoundland, Canada (research@alfied.xyz,vkyadav@mun.ca).

[1]We will follow Siegel [15, Footnote 1 on p. 192] in dropping the adjective 'absolute' from 'absolute universe', and note that a universe (in the original, weaker sense) is absolute if and only if it is parental (see [6, Definition 23, Theorem 24, and Corollary 27 on pp. 117–118]).

 https://doi.org/10.37236/13504

showed that there are uncountably many universes lying between the dicot universe $\mathcal{D}$ and the dead-ending universe $\mathcal{E}$ [15, pp. 204–205].[2]

It is immediate that every universe is a monoid, and hence it is natural to ask about invertibility. Indeed, Milley writes in [10, p. 13] that "[a] better understanding of misère invertibility is a significant open problem in the growing theory of restricted misère play." In the near-decade since [10] was published, there have been numerous improvements. For instance, it was shown by Larsson, Milley, Nowakowski, Renault, and Santos in [5, Theorems 25 and 27 on pp. 262–263] that both $\mathcal{D}$ and $\mathcal{E}$ have the conjugate property [5, Definition 2 on p. 248]; that is, if $G \in \mathcal{D}$ and there exists some $H \in \mathcal{D}$ such that $G + H \equiv_{\mathcal{D}} 0$, then $H \equiv_{\mathcal{D}} \overline{G}$, and similarly for $\mathcal{E}$. Furthermore, characterisations were then found for the $\mathcal{D}$-invertible elements of $\mathcal{D}$ by Fisher, Nowakowski, and Santos [4, Theorem 12 on p. 7], and also for the $\mathcal{E}$-invertible elements of $\mathcal{E}$ by Milley and Renault [12, Theorems 19 and 22 on pp. 11–12].

In their survey on partizan misère theory, Milley and Renault posed two open problems [11, §7 on p. 122]. The first concerned the possibility of non-conjugate inverses. They asked "[in] what universes does $G + H \equiv_{\mathcal{U}} 0$ imply $\overline{G} \equiv_{\mathcal{U}} H$?" In particular, they wonder whether the implication can be proved for parental, dense universes (they are using "universe" in the weaker sense). This is equivalent to asking if it is true for every (absolute) universe (going back to our terminology). Such questions have been asked before. Indeed, Milley conjectured in [9, Conjecture 2.1.7 on p. 10] that being closed under conjugation would suffice to yield the conjugate property. This turned out to be false, as discussed again by Milley in [10, p. 13] where they then asked "is there some condition on the universe $\mathcal{U}$ so that G being invertible implies $G + \overline{G} \equiv_{\mathcal{U}} 0$?" The answer is yes, and this is what we now prove a decade or so later!

Notably, our results here were not formed from a steady trickle of small improvements, but rather were hammered into existence by a recent discovery: Siegel's simplest forms [15, §5]. These forms allow us to do away with the complications that can arise from reversing through ends, and indeed they effectively solve the second open problem posed in [11, §7 on p. 122] as it pertains to (absolute) universes. We will give the necessary background for this theory, as well as other important ideas, in Section 2.

In Section 3, we make the strides towards the better understanding of invertibility that we have just discussed: we show that every universe has the conjugate property (Theorem 16). We also give a characterisation of the invertible elements of each universe (Theorem 17), which we then contrast with the known characterisations in $\mathcal{D}$ and $\mathcal{E}$.

In Section 4, we explore when a universe has no (non-trivial) invertible elements; we call such a universe *reduced*[3]. We give a characterisation to test for when a universe is reduced, although the result is far from practical.

[2] There is some subtlety here, however, about when two universes should be called 'distinct', and this is not yet a well-trodden idea in the literature: should they be distinct if they are different sets of games, or should they be distinct if they have different equivalence classes, et cetera. See [7, Definition 12 and Observation 13 on p. 234].

[3] A monoid whose invertible subgroup is trivial is often called reduced, and we use that same language for our universes here (since every universe is a monoid).

In Section 5, we introduce the notion of a *weak* universe (and also a weak set of games), which we will see is equivalent to meaning one that induces the same partial order relation as full misère. (The actual definition requires more material than we have in this introduction; see Definition 34.) When a universe is weak, it can have no (non-zero) invertible elements; it is reduced. But it is unclear whether the reverse implication also holds. We discover elements, such as $\{ \mid 2\}$, whose mere presence in a universe is enough to render it weak. Roughly speaking, this is because, for each game G, Right can win $G + n \cdot \{ \mid 2\}$ going first (if n is sufficiently large) by playing on the copies of $\{ \mid 2\}$ to leave Left with too many moves to make (see Corollary 43).

Finally, in Section 6, we briefly discuss the landscape and set some future directions.

2 Preliminaries

The main object of our study here is the universe, and so we had better define it! We will not, however, recall all of the basic ideas of the field of Combinatorial Game Theory, and the reader is directed to Siegel's wonderful book for such information [14].

A non-empty set of games $\mathcal{U}$ is called a *universe* if the following closure properties are satisfied for all $G, H \in \mathcal{U}$:

- (additive closure) $G + H \in \mathcal{U}$;
- (conjugate closure) $\overline{G} \in \mathcal{U}$;
- (hereditary closure) $G' \in \mathcal{U}$ for all options G' of G;
- (dicotic closure) if $\mathscr{G}, \mathscr{H} \subseteq \mathcal{U}$ are finite and non-empty, then

$$\{\mathscr{G} \mid \mathscr{H}\} \in \mathcal{U}.$$

Given a universe $\mathcal{U}$, Siegel defines a game G to be Left $\mathcal{U}$-strong if $\mathrm{o}(G + X) \geqslant \mathscr{N}$ for all Left ends $X \in \mathcal{U}$ [15, Definition 2.3 on p. 195].

Now, as we alluded to in the introduction, our results make heavy use of Siegel's simplest forms. These are new ideas in misère theory, and the reader is most strongly encouraged to read Siegel's original presentation of the material [15, §5] that includes excellent motivations and additional information to what we will provide here; we include only enough to get by.

Say we have a universe $\mathcal{U}$, and some game $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$. A Left option G^L of G is *$\mathcal{U}$-reversible* if there exists some G^{LR} such that $G \geqslant_{\mathcal{U}} G^{LR}$. In normal play, when such an option is reversible, the reduction that occurs is replacing the G^L with all of the Left options of the G^{LR}. When G^{LR} is not a Left end, we may do the same thing in (restricted) misère. The problem occurs precisely when G^L is a Left end. This is because, if G has a $\mathcal{U}$-reversible option G^L that is a Left end, then G is Left $\mathcal{U}$-strong, and removing that G^L (i.e. replacing it with nothing) may not preserve enough information to keep the form Left $\mathcal{U}$-strong. But, as Siegel shows, we can get rid of almost all of the extra information encoded in G^L.

The key idea is to remove the end-reversible option and replace it with an abstract symbol " ∎ " called a *tombstone*.[4] This tombstone just acts as a marker, or a flag, to the fact that the game is Left / Right $\mathcal{U}$-strong, and this is what the reader should keep in their head. Siegel calls the tombstones (if they are present) the *tombstone options* of a game, and the remaining options are the *ordinary options*. But, as a reminder, the tombstones are *not* games, and they cannot be moved to. When we write G^L (or G^R), it will never refer to tombstone options, but instead only to ordinary options.

Borrowing Siegel's simple example [15, p. 208], consider the form $G = \{0, * \mid *\}$ modulo the dicot universe $\mathcal{D}$. Note that $\mathrm{o_L}(G) = \mathscr{L}$, so G is Left $\mathcal{D}$-strong. The Left option $*$ of G is reversible since $G \geqslant_{\mathcal{D}} 0$, which could be checked with Theorem 4 below. And so, if one were to proceed according to how reversibility works in normal play, one would obtain the form $G' = \{0 \mid *\}$. But observe that $\mathrm{o_L}(G') = \mathscr{R}$; that is, G' is *not* Left $\mathcal{D}$-strong. This means that G cannot be equal to G'; we need to retain a witness to the fact that G is Left $\mathcal{D}$-strong, which Siegel's theory does by placing a tombstone: the form $\{\blacksquare, 0 \mid *\}$ is indeed equal to G modulo $\mathcal{D}$.

We write $\mathcal{M}_{\mathrm{aug}}$ for the set of all game forms where each subposition might additionally have a Left or Right tombstone, or both; these are Siegel's *augmented forms* (cf. [15, Definition 5.1 on p. 212]). Siegel then illustrates a beautiful theory, naturally extending the definitions of addition and outcomes and other useful things. The extension for addition is slightly nuanced, so we repeat it here for clarity. Note that, if an augmented form is a Left end (i.e. there exists no G^L) or contains a Left tombstone, then Siegel calls it *Left end-like* (and similarly for *Right end-like*).

Definition 1 ([15, Definition 5.4 on p. 213]). If $G, H \in \mathcal{M}_{\mathrm{aug}}$, then the sum $G + H$ is defined as follows:

- the Left ordinary options of $G + H$ are given by $\{G^L + H, G + H^L\}$ as usual, and similarly for the Right options;
- $G + H$ has a Left tombstone if and only if G and H are both Left end-like, and at least one of them has a Left tombstone; and similarly for a Right tombstone.

We will extend Siegel's definition of being Left $\mathcal{U}$-strong to work for arbitrary sets of games, rather than just universes, since we will have use for it later. This is a critical notion not just in the established theory (as we will see in Definition 3 and Theorem 4), but also for our later characterisation of $\mathcal{U}$-invertible elements when working modulo a universe $\mathcal{U}$ (see Theorem 17).

Definition 2 (cf. [15, p. 213 and Definition 2.3 on p. 195]). If $\mathcal{A}$ is a set of games, then we say $G \in \mathcal{M}_{\mathrm{aug}}$ is *Left $\mathcal{A}$-strong* if $\mathrm{o_L}(G + X) = \mathscr{L}$ for all Left ends $X \in \mathcal{A}$. (We define *Right $\mathcal{A}$-strong* analogously.)

[4] Siegel actually writes Σ^L and Σ^R for a Left and Right tombstone respectively, but this has no consequence for the theory, and we have chosen to simplify the notation here and emphasise that this is an abstract symbol acting as a flag, rather than possibly some game Σ with Left or Right options.

Most importantly, Siegel extends the terrific comparison test of [6, Theorem 4 on p. 103] to work for all pairs of augmented forms. Before we state it below in Theorem 4, we extract some items that will allow us to state things more concisely.

Definition 3. If $\mathcal{A}$ is a set of games and $G, H \in \mathcal{M}_{\text{aug}}$, then we say the ordered pair (G, H) satisfies the $\mathcal{A}$-*maintenance* if

1. for every G^R, either there exists some H^R with $G^R \geqslant_{\mathcal{A}} H^R$, or else there exists some G^{RL} with $G^{RL} \geqslant_{\mathcal{A}} H$; and

2. for every H^L, either there exists some G^L with $G^L \geqslant_{\mathcal{A}} H^L$, or else there exists some H^{LR} with $G \geqslant_{\mathcal{A}} H^{LR}$.

Furthermore, we say (G, H) satisfies the $\mathcal{A}$-*proviso* if

1. if H is Left end-like, then G is Left $\mathcal{A}$-strong;

2. if G is Right end-like, then H is Right $\mathcal{A}$-strong.

Theorem 4 ([15, Theorem 5.5 on p. 213])**.** *If $\mathcal{U}$ is a universe and $G, H \in \mathcal{M}_{aug}$, then $G \geqslant_{\mathcal{U}} H$ if and only if (G, H) satisfies the $\mathcal{U}$-maintenance and the $\mathcal{U}$-proviso.*

If we weaken the requirement of Theorem 4 that $\mathcal{U}$ be a universe, then we obtain the following simple results. These results are not significant developments; we are including them here because writing them down in an organised way will be useful to us later!

Proposition 5. *If $\mathcal{A}$ is a set of games, and $G, H \in \mathcal{M}_{aug}$ with $G \geqslant_{\mathcal{A}} H$, then (G, H) satisfies the $\mathcal{A}$-proviso.*

Proof. Since $G \geqslant_{\mathcal{A}} H$, we must have $\mathrm{o}(G + X) \geqslant \mathrm{o}(H + X)$ for all $X \in \mathcal{A}$ by definition. If H is Left end-like, then $\mathrm{o}(H + X) \geqslant \mathscr{N}$ for all Left ends $X \in \mathcal{A}$, and hence G must be Left $\mathcal{A}$-strong. The rest follows by symmetry. □

In the following proposition, we can follow (almost verbatim) one of the directions of Siegel's proof [15, Proof of Theorem 5.5 on pp. 215–216].

Proposition 6. *If $\mathcal{A}$ is a hereditary set of games, and $G, H \in \mathcal{M}_{aug}$ whereby (G, H) satisfies the $\mathcal{A}$-maintenance and the $\mathcal{A}$-proviso, then $G \geqslant_{\mathcal{A}} H$.*

Proof. We need to show that $\mathrm{o}(G + X) \geqslant \mathrm{o}(H + X)$ for all $X \in \mathcal{A}$. It suffices to show that $\mathrm{o_L}(G + X) \geqslant \mathrm{o_L}(H + X)$ for all $X \in \mathcal{A}$, since an identical argument will prove $\mathrm{o_R}(G + X) \geqslant \mathrm{o_R}(H + X)$. We proceed by induction on the formal birthday of $G + H + X$.

We need only consider when $\mathrm{o_L}(H + X) = \mathscr{L}$. We must be in one of three cases:

1. $\mathrm{o_R}(H + X^L) = \mathscr{L}$ for some X^L.

   By induction, since $\mathcal{A}$ is hereditary, we have that $\mathrm{o_R}(G + X^L) = \mathscr{L}$, and hence $\mathrm{o_L}(G + X) = \mathscr{L}$.

2. $o_R(H^L + X) = \mathscr{L}$ for some H^L.

   By hypothesis, either there exists some G^L with $G^L \geqslant_{\mathcal{A}} H^L$, or else there exists some H^{LR} with $G \geqslant_{\mathcal{A}} H^{LR}$. In the first case, we have immediately that $o_R(G^L + X) \geqslant o_R(H^L + X)$, and hence $o_L(G + X) = \mathscr{L}$. In the latter, we observe that $o_L(H^{LR}+X) = \mathscr{L}$ (since $o_R(H^L+X) = \mathscr{L}$), and so $o_L(G+X) \geqslant o_L(H^{LR}+X) = \mathscr{L}$.

3. $H + X$ is Left end-like.

   Since $X \in \mathcal{A}$, it must follow that X is a Left end and H is Left end-like. By hypothesis, G is Left $\mathcal{A}$-strong, and hence $o_L(G + X) = \mathscr{L}$. □

The other major piece of the theory we will require is the notion of the $\mathcal{U}$-simplest form (see [15, Definition 5.19 on p. 219]). When we have a universe $\mathcal{U}$, and some augmented form G, we can repeatedly perform the following reductions on all subpositions of G:

1. bypass $\mathcal{U}$-reversible options, replacing end-reversible ones with a tombstone;

2. remove $\mathcal{U}$-dominated options; and

3. remove unnecessary tombstones.

Siegel proves that the form we arrive at (i.e. when we can no longer perform any more reductions), which is called the *$\mathcal{U}$-simplest form*, is unique. That is, if $G \equiv_{\mathcal{U}} H$, then the $\mathcal{U}$-simplest form of G is isomorphic to the $\mathcal{U}$-simplest form of H.

So, when the reader sees a game in $\mathcal{U}$-simplest form in our arguments, they should have in their mind that there are no $\mathcal{U}$-dominated options, no $\mathcal{U}$-reversible options, and no unnecessary tombstones in any of its subpositions.

For each universe $\mathcal{U}$, Siegel then defines a special set of augmented forms called $\widehat{\mathcal{U}}$; simply put, this set contains every augmented form whose tombstones can be replaced with end-reversible options in $\mathcal{U}$ such that we can recover a game form in $\mathcal{U}$ (Siegel says that such a game has a *$\mathcal{U}$-expansion*). So, each game in $\widehat{\mathcal{U}}$ is equal (modulo $\mathcal{U}$) to some game in $\mathcal{U}$. We will not need the formal definitions and theory here. Instead, we need just two results.

Lemma 7 ([15, Lemma 5.21 on p. 220])**.** *If $\mathcal{U}$ is a universe and $G \in \mathcal{U}$, then the $\mathcal{U}$-simplest form of G is in $\widehat{\mathcal{U}}$.*

Theorem 8 ([15, Theorem 5.23 on p. 222])**.** *If $\mathcal{U}$ is a universe, and $G, H, J \in \widehat{\mathcal{U}}$ satisfy $G \equiv_{\mathcal{U}} H$, then $G + J \equiv_{\mathcal{U}} H + J$.*

So, Siegel shows that $(\widehat{\mathcal{U}}, \equiv_{\mathcal{U}})$ is a monoid (in particular, note that $\widehat{\mathcal{U}}$ is closed under addition). We give a straightforward extension of this to show that $(\widehat{\mathcal{U}}, \geqslant_{\mathcal{U}})$ is a pomonoid, which is more apt for our purposes.

Theorem 9. *If $\mathcal{U}$ is a universe, then $(\widehat{\mathcal{U}}, \geqslant_{\mathcal{U}})$ is a pomonoid.*

Proof. We need to prove that, for all $G, H, J \in \widehat{\mathcal{U}}$ with $G \geqslant_{\mathcal{U}} H$, it follows that $G + J \geqslant_{\mathcal{U}} H + J$.

Since $G, H, J \in \widehat{\mathcal{U}}$, we know that we can find $G', H', J' \in \mathcal{U}$ such that

$$\begin{aligned} G' &\equiv_{\mathcal{U}} G, \\ H' &\equiv_{\mathcal{U}} H, \text{ and} \\ J' &\equiv_{\mathcal{U}} J. \end{aligned}$$

Observe then that $G' \equiv_{\mathcal{U}} G \geqslant_{\mathcal{U}} H \equiv_{\mathcal{U}} H'$. Since $(\mathcal{U}, \geqslant_{\mathcal{U}})$ is a pomonoid, it then follows from Theorem 8 that

$$G + J \equiv_{\mathcal{U}} G' + J \equiv_{\mathcal{U}} G' + J' \geqslant_{\mathcal{U}} H' + J' \equiv_{\mathcal{U}} H' + J \equiv_{\mathcal{U}} H + J. \qquad \square$$

3 Invertibility

Armed with the powerful tools referenced in our preliminaries, we now seek to further understand how invertibility works within a universe. Of course, we must define what we mean by *invertibility*! We do not restrict ourselves to universes, since we will have something to say later about more general monoids.

Definition 10. If $\mathcal{A}$ is a set of games, $G \in \mathcal{M}_{\text{aug}}$, and there exists some $H \in \mathcal{A}$ such that $G + H \equiv_{\mathcal{A}} 0$, then we say G is *$\mathcal{A}$-invertible*; we call H an *$\mathcal{A}$-inverse* of G.

For a general set of games $\mathcal{A}$, there does not necessarily exist a *unique* $\mathcal{A}$-inverse for an $\mathcal{A}$-invertible element, which is why we called H *an* $\mathcal{A}$-inverse of G in Definition 10, as opposed to *the* $\mathcal{A}$-inverse of G. But when $\mathcal{A}$ is a monoid, then the $\mathcal{A}$-inverse of a game *must* be unique (up to equivalence modulo $\mathcal{A}$, of course).

It is conceivable that one would like to consider when $G + H \equiv_{\mathcal{A}} 0$ for completely arbitrary forms; i.e. without restricting H to be an element of $\mathcal{A}$ like in Definition 10. Doing so here would add difficulties to our exposition, and our results do not currently apply to such a general case. (But it *is* simpler for us to allow G to be outside of $\mathcal{A}$, which is why we do so.) In the future, such things may be further explored.

Readers familiar with normal play will recall that the inverse of a game is always its conjugate. But, in misère, we have already discussed that this is not so obvious a claim.[5] As such, we give a separate definition for this case. As we mentioned in the introduction, the idea of conjugate invertibility is not new, but we state this definition for clarity.

Definition 11. If $\mathcal{A}$ is a set of games, and $G \in \mathcal{M}_{\text{aug}}$, then we say G is *conjugate $\mathcal{A}$-invertible* if $\overline{G}$ is an $\mathcal{A}$-inverse of G.

One could rephrase this as: we say G is conjugate $\mathcal{A}$-invertible if $\overline{G} \in \mathcal{A}$ and $G + \overline{G} \equiv_{\mathcal{A}} 0$.

We will write $\mathcal{A}^{\times}$ and $\mathcal{A}^{\overline{\times}}$ for the set of $\mathcal{A}$-invertible and conjugate $\mathcal{A}$-invertible elements of $\mathcal{A}$ respectively; note that we are *not* considering those forms outside of $\mathcal{A}$ here. Of

[5] It is so non-obvious that it is *false* in some cases! This was mentioned in [10, p. 13], where the offending example comes from an element of period 6 in a monoid in [13, A.6 on p. 617].

course, every conjugate $\mathcal{A}$-invertible game is $\mathcal{A}$-invertible, and so $\mathcal{A}^{\overline{\times}} \subseteq \mathcal{A}^{\times}$. We can also make a trivial remark about 0 being conjugate $\mathcal{A}$-invertible.

Proposition 12. *If $\mathcal{A}$ is a set of games containing 0, then 0 is conjugate $\mathcal{A}$-invertible.*

Proof. Observe that $\overline{0} \cong 0 \in \mathcal{A}$, and so also $0 + \overline{0} \cong 0 \equiv_{\mathcal{A}} 0$. ☐

Given two groups $\mathcal{G}$ and $\mathcal{H}$, recall the notation '$\mathcal{G} \leqslant \mathcal{H}$' used to mean that $\mathcal{G}$ is a subgroup of $\mathcal{H}$. Also recall that the set of invertible elements of a monoid always forms a group. Now, if $\mathcal{A}$ is a monoid of games, is it true that $\mathcal{A}^{\overline{\times}} \leqslant \mathcal{A}^{\times}$? In particular, if $G, H \in \mathcal{A}^{\overline{\times}}$, is it true that $G + H \in \mathcal{A}^{\overline{\times}}$? Indeed, the answer is yes.

Proposition 13. *If $\mathcal{A}$ is a monoid of games, then $\mathcal{A}^{\overline{\times}} \leqslant \mathcal{A}^{\times}$.*

Proof. Let $G, H \in \mathcal{A}^{\overline{\times}}$. Since $\mathcal{A}^{\overline{\times}}$ is clearly closed under conjugation, we have $\overline{G}, \overline{H} \in \mathcal{A}^{\overline{\times}}$, and in particular that $\overline{G + H} \cong \overline{G} + \overline{H} \in \mathcal{A}$. Now observe

$$\begin{aligned} G + H + \overline{G + H} &\cong G + H + \overline{G} + \overline{H} \\ &\cong G + \overline{G} + H + \overline{H} \\ &\equiv_{\mathcal{A}} 0. \end{aligned}$$ ☐

Given Proposition 13, it is natural to wonder whether $\mathcal{A}^{\times} = \mathcal{A}^{\overline{\times}}$. In general, such a statement is not true. We give a definition here for when the $\mathcal{A}$-invertible elements (of a set of games $\mathcal{A}$) do coincide precisely with the conjugate $\mathcal{A}$-invertible elements; this is essentially the *conjugate property* that other authors mention, but here we have a different notation and a slightly more general context.

Definition 14. If $\mathcal{A}$ is a set of games such that $\mathcal{A}^{\overline{\times}} = \mathcal{A}^{\times}$, then we say $\mathcal{A}$ has the *conjugate property.*

We will begin to show now that every universe has the conjugate property. In order to do so, we will first prove a technical lemma that will also be helpful to us soon after in our characterisation of the invertible elements of each universe. The proof of this lemma uses a technique from the beginning of a proof of Ettinger's [3, Proof of Theorem 14 on pp. 48–51], and this technique has been mentioned and used several times in these kinds of arguments (for example, see [5, Proofs of Theorems 25 and 27 on pp. 262–263]).

Lemma 15. *If $\mathcal{U}$ is a universe, and $G, H \in \widehat{\mathcal{U}}$ are in $\mathcal{U}$-simplest form with $G + H \equiv_{\mathcal{U}} 0$, then: for all G^L, there exists H^R such that $G^L + H^R \equiv_{\mathcal{U}} 0$.*

Proof. If G is a Left end, then the conclusion is vacuously true. So, assume that G is not a Left end.

Since $G + H \equiv_{\mathcal{U}} 0$, we know in particular that $G + H \leqslant_{\mathcal{U}} 0$. Pick an arbitrary Left option G^{L_1}. By Theorem 4, it follows that there must exist some Right option $(G^{L_1} + H)^{R_1}$ of $G^{L_1} + H$ such that $(G^{L_1} + H)^{R_1} \leqslant_{\mathcal{U}} 0$. Such a Right option must be of one of the following forms:

1. $G^{L_1R_1} + H$; or

2. $G^{L_1} + H^{R_1}$.

In the first case, where we have $G^{L_1R_1} + H \leqslant_{\mathcal{U}} 0$, we may add G to both sides to obtain $G^{L_1R_1} \leqslant_{\mathcal{U}} G$. (Note that we are allowed to add G to both sides since $(\widehat{\mathcal{U}}, \geqslant_{\mathcal{U}})$ is a pomonoid by Theorem 9, and $G^{L_1R_1}, G, H \in \widehat{\mathcal{U}}$.) This tells us that G^{L_1} is a reversible Left option of G. But this contradicts G being in simplest form. Thus, we must be in the second case: the Right option must be of the form $G^{L_1} + H^{R_1}$.

So, recall that we have $G^{L_1} + H^{R_1} \leqslant_{\mathcal{U}} 0$. If we have equality, then we have the result. As such, we will suppose, for a contradiction, that we have the strict inequality $G^{L_1} + H^{R_1} <_{\mathcal{U}} 0$. Now, given H^{R_1}, we can apply a symmetric argument to the one before to obtain some G^{L_2} such that $G^{L_2} + H^{R_1} \geqslant_{\mathcal{U}} 0$. We can clearly continue in this way so that, given G^{L_i}, we may find some H^{R_i} such that

$$G^{L_i} + H^{R_i} \leqslant_{\mathcal{U}} 0; \tag{1}$$

and given H^{R_i}, we may find some $G^{L_{i+1}}$ such that

$$G^{L_{i+1}} + H^{R_i} \geqslant_{\mathcal{U}} 0. \tag{2}$$

We will show by induction that each inequality of type (2) is strict. If we have equality for $i = 1$, then adding G^{L_1} to both sides yields $G^{L_2} + G^{L_1} + H^{R_1} \equiv_{\mathcal{U}} G^{L_1}$. But $G^{L_1} + H^{R_1} <_{\mathcal{U}} 0$ by supposition, and so we obtain $G^{L_1} \leqslant_{\mathcal{U}} G^{L_2}$. Since G is in $\mathcal{U}$-simplest form, this implies $G^{L_1} \cong G^{L_2}$. But then we have

$$0 \leqslant_{\mathcal{U}} G^{L_2} + H^{R_1} \cong G^{L_1} + H^{R_1} <_{\mathcal{U}} 0,$$

which is a contradiction. Hence, $G^{L_2} + H^{R_1} >_{\mathcal{U}} 0$.

Now suppose $G^{L_{i+1}} + H^{R_i} >_{\mathcal{U}} 0$ for all $i \leqslant k$. Suppose further, for a contradiction, that $G^{L_{k+2}} + H^{R_{k+1}} \equiv_{\mathcal{U}} 0$. We may add $G^{L_{k+2}}$ to both sides of $G^{L_{k+1}} + H^{R_{k+1}} \leqslant_{\mathcal{U}} 0$ to obtain $G^{L_{k+1}} \leqslant_{\mathcal{U}} G^{L_{k+2}}$. Since G is in $\mathcal{U}$-simplest form, we must have $G^{L_{k+1}} \cong_{\mathcal{U}} G^{L_{k+2}}$.

We have $G^{L_{k+1}} + H^{R_k} \geqslant_{\mathcal{U}} 0$ from (2). Adding $H^{R_{k+1}}$ to both sides yields $G^{L_{k+1}} + H^{R_{k+1}} + H^{R_k} \geqslant H^{R_{k+1}}$. Since $G^{L_{k+2}} + H^{R_{k+1}} \equiv_{\mathcal{U}} 0$ by supposition, and $G^{L_{k+1}} \cong G^{L_{k+2}}$ from earlier, we obtain $H^{R_k} \geqslant_{\mathcal{U}} H^{R_{k+1}}$. Because H is in $\mathcal{U}$-simplest form, we must have $H^{R_k} \cong H^{R_{k+1}}$. But then observe

$$0 \geqslant_{\mathcal{U}} G^{L_{k+1}} + H^{R_{k+1}} \cong G^{L_{k+1}} + H^{R_k} >_{\mathcal{U}} 0,$$

which is a contradiction.

Recall that G and H are short games—in particular, they have only a finite number of options. Thus, there must exist integers $a < b$ such that $G^{L_a} \cong G^{L_b}$. Summing the inequalities of type (1) over this range, we obtain

$$S_1 \coloneqq \sum_{i=a}^{b-1} \left(G^{L_i} + H^{R_i} \right) \leqslant_{\mathcal{U}} 0.$$

Similarly, summing the inequalities of type (2), we obtain

$$S_2 \coloneqq \sum_{i=a}^{b-1} \left(G^{L_{i+1}} + H^{R_i} \right) >_{\mathcal{U}} 0.$$

But $G^{L_a} \cong G^{L_b}$, and so $S_1 \cong S_2$. And this implies $0 <_{\mathcal{U}} S_1 \leqslant_{\mathcal{U}} 0$, which is a contradiction. □

Recall again that $(\widehat{\mathcal{U}}, \geqslant_{\mathcal{U}})$ is a pomonoid (Theorem 9) and, importantly, that every game in $\widehat{\mathcal{U}}$ is equal to some game in $\mathcal{U}$. Thus, we may pass freely between the two in the following way: if we have some $G \in \widehat{\mathcal{U}}$ and find another $H \in \widehat{\mathcal{U}}$ such that $G + H \equiv_{\mathcal{U}} 0$, then we know G is $\mathcal{U}$-invertible, even though H might not be in $\mathcal{U}$—and this is because we know that there must exist some $H' \in \mathcal{U}$ with $H' \equiv_{\mathcal{U}} H$, and so $G + H' \equiv_{\mathcal{U}} G + H \equiv_{\mathcal{U}} 0$.

Theorem 16. *Every universe $\mathcal{U}$ has the conjugate property.*

Proof. We know that 0 is conjugate $\mathcal{U}$-invertible by Proposition 12, so let $G \in \widehat{\mathcal{U}}$ be a non-zero, $\mathcal{U}$-invertible game; write H for the $\mathcal{U}$-inverse of G (and note that it must also be non-zero). We may assume that G and H are in $\mathcal{U}$-simplest form. We want to show that $G + \overline{G} \equiv_{\mathcal{U}} 0$. Since $G + \overline{G}$ is a symmetric form, it suffices to show that either $G + \overline{G} \geqslant_{\mathcal{U}} 0$ or $G + \overline{G} \leqslant_{\mathcal{U}} 0$. It also suffices to show that $\overline{G} \equiv_{\mathcal{U}} H$. (We will make use of each sufficient condition.)

We show first that H and $\overline{G}$ have identical ordinary options; we induct on $\tilde{\mathrm{b}}(G + H)$. By Lemma 15, for every Left option G^L, there exists a Right option H^R such that $G^L + H^R \equiv_{\mathcal{U}} 0$. By induction, for every G^L, we have $G^L + \overline{G^L} \equiv_{\mathcal{U}} 0$, and hence $H^R \cong \overline{G^L}$ (since G and H are both in $\mathcal{U}$-simplest form). By a symmetric argument: for every Right option G^R, there exists a Left option H^L such that $G^R + H^L \equiv_{\mathcal{U}} 0$. By induction, we have $G^R + \overline{G^R} \equiv_{\mathcal{U}} 0$, and hence $H^L \cong \overline{G^R}$. We have now shown that the ordinary options of $\overline{G}$ form a subset of the ordinary options of H. By an identical argument, it follows that the ordinary options of H form a subset of the ordinary options of $\overline{G}$.

Note that, if $\overline{G}$ is comparable with H (modulo $\mathcal{U}$), then we are done: say, without loss of generality, that $\overline{G} \geqslant_{\mathcal{U}} H$, then we obtain $G + \overline{G} \geqslant_{\mathcal{U}} 0$ by adding G to both sides.

It is clear that adding a Left tombstone, or removing a Right tombstone, to a form can only result in a form at least as good for Left, et similia. As such, since $\overline{G}$ and H have the same ordinary options, it is clear that the only cases where it is not immediately obvious that $\overline{G}$ and H are comparable are where G has both tombstones and H has none, and the symmetric case where G has no tombstones and H has both. By symmetry, we need only consider the former case.

We will show that $G + \overline{G} \geqslant_{\mathcal{U}} 0$ via Theorem 4. The argument will be clearer if we write down the options of $G + \overline{G}$ explicitly:

$$\begin{aligned} G + \overline{G} &\cong \left\{ \blacksquare, G^L + \overline{G}, G + \overline{G}^L \,\middle|\, G^R + \overline{G}, G + \overline{G}^R, \blacksquare \right\} \\ &\cong \left\{ \blacksquare, G^L + \overline{G}, G + \overline{G^R} \,\middle|\, G^R + \overline{G}, G + \overline{G^L}, \blacksquare \right\} \\ &\cong \left\{ \blacksquare, G^L + \overline{G}, G + H^L \,\middle|\, G^R + \overline{G}, G + H^R, \blacksquare \right\}. \end{aligned}$$

First, we show that for every $(G+\overline{G})^L$ there exists some $(G+\overline{G})^{LR} \geqslant_{\mathcal{U}} 0$. Since $G+\overline{G}$ is a symmetric form, we may assume that an arbitrary Right option is of the form $G^R+\overline{G}$. Since $G+H \equiv_{\mathcal{U}} 0$ by supposition, we know by Lemma 15 that there exists some H^L with $G^R+H^L \equiv_{\mathcal{U}} 0$. Since $\overline{G}$ has a Left option to H^L, we are done. Furthermore, since 0 has no Left options, the rest of the maintenance property is vacuously satisfied.

It remains to show the proviso. We know that 0 is Right $\mathcal{U}$-strong, which completes one part. For the other, $G+\overline{G}$ must be Left $\mathcal{U}$-strong since it has a Left tombstone. Thus, we have the result. □

Unsurprisingly, a misère monoid can indeed have the conjugate property without being a universe. The impartial misère monoid is one such example [5, Corollary 26 on p. 263]. For another, consider the monoid of (game) integers $\mathcal{S} = \{n : n \in \mathbb{Z}\}$.[6] Amusingly, $(\mathcal{S}, \equiv_{\mathcal{S}})$ is isomorphic (under the natural mapping) to the group of integers $\mathbb{Z}$ (under addition). Thus, $\mathcal{S}$ has the conjugate property, but is clearly not a universe—it does not satisfy the dicotic closure property (and this is the only property of a universe that it does not satisfy, just like the impartial monoid).

With our conjugate property in hand, we can already give a characterisation of $\mathcal{U}^{\times}$ for each universe $\mathcal{U}$.

Theorem 17. *If $\mathcal{U}$ is a universe and $G \in \widehat{\mathcal{U}}$, then: G is $\mathcal{U}$-invertible if and only if $G+\overline{G}$ is Left $\mathcal{U}$-strong and every option of the $\mathcal{U}$-simplest form of G is $\mathcal{U}$-invertible.*[7]

Proof. First, suppose that $G+\overline{G}$ is Left $\mathcal{U}$-strong and every option of the $\mathcal{U}$-simplest form of G is $\mathcal{U}$-invertible. We may assume that G is in simplest form. We want to show that $G+\overline{G} \equiv_{\mathcal{U}} 0$. Since $G+\overline{G}$ is a symmetric form, we need only show that $G+\overline{G} \geqslant_{\mathcal{U}} 0$. We know that $G+\overline{G}$ is Left $\mathcal{U}$-strong, hence also Right $\mathcal{U}$-strong, and so it suffices to show, without loss of generality, that, for every Right option G^R of G, there exists a Left option $(G^R+\overline{G})^L \geqslant_{\mathcal{U}} 0$. But G^R is an option of G, and hence $\mathcal{U}$-invertible by supposition, and hence also conjugate $\mathcal{U}$-invertible by Theorem 16. Thus, $G^R+\overline{G^R} \equiv_{\mathcal{U}} 0$, and we know that $\overline{G^R}$ is a Left option of $\overline{G}$, which now yields that G is $\mathcal{U}$-invertible.

Now suppose instead that $G \in \widehat{\mathcal{U}}$ is $\mathcal{U}$-invertible. We know that G is then also conjugate $\mathcal{U}$-invertible by Theorem 16. We may assume that G is in simplest form.

Since $G+\overline{G} \equiv_{\mathcal{U}} 0$, we know that $G+\overline{G}$ is Left $\mathcal{U}$-strong. Let G^L be a Left option of G. By symmetry, it remains only to show that G^L is $\mathcal{U}$-invertible: by Lemma 15, there exists a Right option $\overline{G}^R$ of $\overline{G}$ such that $G^L+\overline{G}^R \equiv_{\mathcal{U}} 0$, yielding the result. □

It is most interesting to compare Theorem 17 with the characterisation of invertibility in the dicot universe. We restate the theorem of [4] (with minor modifications to ease comparison).

Theorem 18 (cf. [4, Theorem 12 on p. 7])**.** *If $G \in \mathcal{D}$, then G is $\mathcal{D}$-invertible if and only if $\mathrm{o}(G'+\overline{G'}) = \mathscr{N}$ for every subposition G' of the $\mathcal{D}$-simplest form of G.*

[6] The reader must excuse our abuse of terminology here, but this is just a humorous incident.

[7] For those interested, we used the working term *strong mirror* to describe a game G where $G'+\overline{G'}$ is Left $\mathcal{U}$-strong for every subposition G' of G. It can be readily observed that a game $G \in \widehat{\mathcal{U}}$ is $\mathcal{U}$-invertible if and only if its $\mathcal{U}$-simplest form is a strong mirror.

Since 0 is the only Left end that is also a dicot, a game G is Left $\mathcal{D}$-strong if and only if $\mathrm{o}(G) = \mathscr{N}$. As such, it is not hard to see that our theorem (Theorem 17) is stated in essentially the same way, and that it is a straightforward generalisation. The reader is likely thinking "of course this was the case, what other form could it have taken?" And therein lies some intrigue; let us restate the theorem of [12] for invertibility in the dead-ending universe (with minor modifications again).

Theorem 19 (cf. [12, Theorems 19 and 22 and pp. 11–12])**.** *If $G \in \mathcal{E}$, then G is $\mathcal{E}$-invertible if and only if no subposition of its $\mathcal{E}$-simplest form has outcome $\mathscr{P}$.*

This is peculiar. The similarity between our Theorem 17 and this result is not as obvious it was for the dicot characterisation. They must yield the same invertible forms, of course, but the characterisation in terms of the simplest form being free of subpositions with outcome $\mathscr{P}$ is fascinating; to eschew references to being Left $\mathcal{E}$-strong, and instead make reference only to the outcomes of the subpositions, results in what we consider to be a cleaner statement than ours.[8]

If it were the case that a game in $\widehat{\mathcal{E}}$ did not have outcome $\mathscr{P}$ if and only if $G + \overline{G}$ were Left $\mathcal{E}$-strong, then the two results would indeed be trivial translations of each other. But this is not the case: for example, 1 does not have outcome $\mathscr{P}$, but $1 + \overline{1}$ is certainly Left $\mathcal{E}$-strong. What *is* true, however, is that $G + \overline{G}$ being Left $\mathcal{E}$-strong implies G cannot have outcome $\mathscr{P}$. This follows from a result in [2], but is not too hard to see: the proof is essentially a local response strategy, where Right can leverage a waiting game W_n of sufficiently large formal birthday to always follow the local $\mathscr{P}$-strategy and win $G + \overline{G} + W_n$ going second if G has outcome $\mathscr{P}$ (thus, $G + \overline{G}$ could not be Left $\mathcal{E}$-strong). Recall that the waiting game W_n is defined recursively as $W_n \coloneqq \{\ \mid 0, W_{n-1}\}$ for $n > 0$, where $W_0 \coloneqq 0$.

We leave it as an open problem to investigate when alternative characterisations of invertibility like this exist.

Open Problem 20. What alternative characterisations exist for the invertible elements of misère universes? What about for the dicot universe, specifically?

4 Reduced universes

Even with the results of Section 3, the investigations into misère invertibility are not over! There are still many questions to answer. In this section, we will explore when a universe has no invertible elements at all.

Recall that a monoid is called reduced if the identity is the only invertible element. Since a universe of games is a monoid (where 0 is the identity), we will use the same terminology and call a universe *reduced* if it is reduced as a monoid. Equivalently, a universe is called reduced if $\mathcal{U}^\times$ is the trivial group.

The full misère universe $\mathcal{M}$ is an example of a reduced universe, thanks to the well-known theorem of Mesdal and Ottaway [8, Theorem 7 on p. 5]. There exist many more

[8]Theorem aesthetics are subjective, of course, but we find it undeniable that there is something special about the invertibility characterisation of [12].

(uncountably many, in fact), but we will not prove such a statement until the next section where we prove something stronger still (see the discussion after Corollary 43).

Our first result here concerns the possible formal birthdays of $\mathcal{U}$-invertible games (for a universe $\mathcal{U}$). If there exists an invertible element of formal birthday 1, then it is clear that there exists an invertible element of each formal birthday (just by taking sums of that element). It is unclear whether a universe that is not reduced necessarily contains an invertible element of formal birthday 1, but we can prove a weaker conclusion: there exists an invertible game in $\widehat{\mathcal{U}}$ of formal birthday 1.

Proposition 21. *If $\mathcal{U}$ is a universe that is not reduced, then there exists a $\mathcal{U}$-invertible game in $\widehat{\mathcal{U}}$ of formal birthday n for all $n \in \mathbb{N}$.*

Proof. By hypothesis, there exists some non-zero $G \in \mathcal{U}$ such that G is $\mathcal{U}$-invertible. Since it is non-zero, its $\mathcal{U}$-simplest form must have formal birthday at least 1, and this form must be $\mathcal{U}$-invertible (because G is). Thus, by the characterisation of $\mathcal{U}$-invertibility (Theorem 17), it follows that there must exist a $\mathcal{U}$-invertible subposition $G' \in \widehat{\mathcal{U}}$ of the $\mathcal{U}$-simplest form of G of formal birthday 1. We then observe that $n \cdot G \in \widehat{\mathcal{U}}$ is a $\mathcal{U}$-invertible game of birthday n, yielding the result. □

So, for a universe $\mathcal{U}$ that is not reduced, what we have proved is that $\mathcal{U}$ contains a $\mathcal{U}$-invertible element whose $\mathcal{U}$*-simplest form* (which lies in $\widehat{\mathcal{U}}$, not necessarily in $\mathcal{U}$) has formal birthday 1. The authors know of no counter-example to the stronger assertion (that there is always a $\mathcal{U}$-invertible element in $\mathcal{U}$ with formal birthday 1), which we pose as an open problem.

Open Problem 22. Does there exist a universe $\mathcal{U}$ that is not reduced such that there exists no $\mathcal{U}$-invertible $G \in \mathcal{U}$ of formal birthday 1? If not, then what about for other formal birthdays?

As a result of Proposition 21, to characterise those universes that are reduced monoids, we can look at all of the augmented forms born by day 1 and try to characterise which universes they are invertible in. The 16 augmented forms born by day 1 are as follows:

- $0 \coloneqq \{\ \mid\ \}$;
- $\{\ \mid 0\}$;
- $\{\ \mid \blacksquare\}$;
- $\{\ \mid 0, \blacksquare\}$;
- $1 \coloneqq \{0 \mid\ \}$;
- $* \coloneqq \{0 \mid 0\}$;
- $\{0 \mid \blacksquare\}$;
- $\{0 \mid 0, \blacksquare\}$;
- $\{\blacksquare \mid\ \}$;
- $\{\blacksquare \mid 0\}$;
- $\{\blacksquare \mid \blacksquare\}$;
- $\{\blacksquare \mid 0, \blacksquare\}$;
- $\{\blacksquare, 0 \mid\ \}$;
- $\{\blacksquare, 0 \mid 0\}$;
- $\{\blacksquare, 0 \mid \blacksquare\}$;
- $\{\blacksquare, 0 \mid 0, \blacksquare\}$.

By considering symmetry and reductions (see [15, Definitions 5.15 & 5.17 and Lemma 5.16 on p. 218]), we need only consider the following 6 forms.

- 0;
- 1;
- $*$;
- $\{\blacksquare, 0 \mid \ \}$;
- $\{\blacksquare, 0 \mid 0\}$;
- $\{\blacksquare, 0 \mid 0, \blacksquare\}$.

It is trivial that 0 is $\mathcal{U}$-invertible for all universes $\mathcal{U}$ (Proposition 12). It is useful to note that a game G of formal birthday 1 is $\mathcal{U}$-invertible if and only if $G + \overline{G}$ is Left $\mathcal{U}$-strong, which is just a corollary of Theorem 17.

Corollary 23. *If $\mathcal{U}$ is a universe and $G \in \widehat{\mathcal{U}}$ has formal birthday 1, then G is $\mathcal{U}$-invertible if and only if $G + \overline{G}$ is Left $\mathcal{U}$-strong.*

Proof. By Theorem 17, it suffices to show that every option of the $\mathcal{U}$-simplest form of G is $\mathcal{U}$-invertible. But G has formal birthday 1, and hence 0 is the only possible option of the $\mathcal{U}$-simplest form of G, which is trivially $\mathcal{U}$-invertible (Proposition 12). □

It is tempting to think that we should be able to weaken the hypothesis of Corollary 23 to let $\mathcal{U}$ be a monoid that might *not* be a universe, but this is unclear. Compare Corollary 23 with the following result that we can indeed prove; in particular, if the monoid has the conjugate property, then we would obtain essentially the same conclusion.

Proposition 24. *If $\mathcal{A}$ is a hereditary set of games and $G \in \mathcal{M}_{aug}$ has formal birthday 1, then: $G + \overline{G} \equiv_{\mathcal{A}} 0$ if and only if $G + \overline{G}$ is Left $\mathcal{A}$-strong.*

Proof. This follows immediately from Propositions 5 and 6. □

We now begin to investigate each of the five augmented forms born on day 1 (up to conjugation and reduction), attempting to determine which universes they are invertible in.

4.1 Disintegrators

We begin with the game 1. By Corollary 23, we know that, if $1 \in \widehat{\mathcal{U}}$, then 1 is $\mathcal{U}$-invertible if and only if $1 + \overline{1}$ is Left $\mathcal{U}$-strong. Being Left $\mathcal{U}$-strong means that Left wins going first on $1 + \overline{1} + X$ for all Left ends X in $\mathcal{U}$. This must be equivalent to Left winning going second on $\overline{1} + X$. We give a definition to help characterise those X such that Right will win going first on $\overline{1} + X$.

Definition 25. If $G \in \mathcal{M}$, then we say that G is a *disintegrator*[9] if there exists a Right option G^R of G that is not a Left end such that, for all Left options G^{RL} of G^R, either

1. $\mathrm{o_L}\left(G^{RL}\right) = \mathscr{R}$; or
2. G^{RL} is a disintegrator.

[9]The words *disintegrator* and *integer* both share the Proto-Indo-European root **tag-*.

For example, any game with a Right option of the form $\{* \mid G^R\}$ must be a disintegrator (where G^R can be arbitrary).

The following proposition is the reason disintegrators were defined in this way, and the reader can likely easily convince themselves of its correctness. The proof is tedious and just a straightforward application of the definition, so we have relegated it to Appendix A.1.

Proposition 26. *If $\mathcal{U}$ is a universe and $1 \in \widehat{\mathcal{U}}$, then 1 is $\mathcal{U}$-invertible if and only if $\mathcal{U}$ contains no Left end that is a disintegrator.*

A dream result would be one which says adding two Left ends that are not disintegrators cannot result in a disintegrator. This is because universes are characterised by their Left ends (see [1, Proposition 2.3 on p. 6]), and we would thus be able to look at a generating set of the Left ends in a universe and determine whether 1 is invertible. But we must wake ourselves from this dream, for it is not true. As an example, consider the Left end $G = \{\ \mid \{1 \mid 0, \overline{1}\} + \{\overline{1}, 1 \mid\ \}\}$, where G is not a disintegrator, but $G + G$ is (which the reader may readily check via the definition). We will see this example again soon.

4.2 Starkillers

We now turn our attention to $*$, which the reader might have guessed from the title of this subsection. In the same way that we just investigated 1, we give here a definition to help characterise those Left ends X such that Right wins going first on $* + X$.

Definition 27. If $G \in \mathcal{M}$, then we say that G is a *starkiller* if there exists a Right option G^R of G such that $\mathrm{o_R}\left(G^R\right) = \mathscr{R}$ and, for all Left options G^{RL} of G^R, either

1. $\mathrm{o_L}\left(G^{RL}\right) = \mathscr{R}$, or
2. G^{RL} is a starkiller.

Much like Proposition 26, we defined a starkiller precisely to get the following proposition. Its proof is similarly tedious and straightforward, so we have relegated it to Appendix A.2.

Proposition 28. *If $\mathcal{U}$ is a universe, then $*$ is $\mathcal{U}$-invertible if and only if $\mathcal{U}$ contains no Left end that is a starkiller.*

Just as for disintegrators, our dream of having Left ends that are not starkillers being closed under addition is quickly shattered. Take, for example, the Left end $G = \{\ \mid \{1 \mid 0, \overline{1}\} + \{\overline{1}, 1 \mid\ \}\}$, where G is not a starkiller, but $G + G$ is. This is, in fact, the same example we gave in the previous subsection.

As an example application of Proposition 28, it is a trivial observation that a terminable Left end must be a starkiller; i.e. a Left end where Right has an option to 0. Since $\overline{1}$ is a terminable Left end, it is then a simple corollary that, if $\mathcal{U}$ is a dead-ending universe, then $*$ is $\mathcal{U}$-invertible if and only if $\mathcal{U} = \mathcal{D}$.

Corollary 29. *If $\mathcal{U}$ is a dead-ending universe, then, $*$ is $\mathcal{U}$-invertible if and only if $\mathcal{U} = \mathcal{D}$. That is, the dicot universe is the only dead-ending universe where $*$ is invertible.*

Proof. If $\mathcal{U} = \mathcal{D}$, then we know already that $*$ is $\mathcal{U}$-invertible. So, assume now that $\mathcal{U}$ is a dead-ending universe not equal to $\mathcal{D}$. It is then clear that $\overline{1}$ *must* be in $\mathcal{U}$. But $\overline{1}$ is a starkiller (we have $\mathrm{o_L}(* + * + \overline{1}) = \mathscr{R}$), and hence $*$ is not $\mathcal{U}$-invertible. □

Immediately from Propositions 26 and 28, we have a characterisation of those universes that contain a $\mathcal{U}$-invertible element of formal birthday 1, which we state now.

Theorem 30. *If $\mathcal{U}$ is a universe, then $\mathcal{U}$ admits a $\mathcal{U}$-invertible element (in $\mathcal{U}$) of formal birthday 1 if and only if at least one of the following is true:*

- *$\mathcal{U}$ admits no Left ends that are starkillers; or*
- *$1 \in \mathcal{U}$ and $\mathcal{U}$ admits no Left ends that are disintegrators.*

4.3 Super starkillers

We now consider the game $\{\blacksquare, 0 \mid 0\}$. Writing $G = \{\blacksquare, 0 \mid 0\}$, it is useful to note that

$$\begin{aligned} G + \overline{G} &\cong \left\{G, \overline{G} \,\middle|\, \overline{G}, G\right\} \\ &\equiv_{\mathcal{M}} \left\{G \,\middle|\, \overline{G}\right\}, \end{aligned}$$

which follows after recalling that adding a Left tombstone to a form can only make the form better for Left (and similarly for Right). In the same way as the previous two subsections, we give here a definition to help characterise those Left ends X such that Right wins going first on $\{0 \mid 0, \blacksquare\} + X$.

Definition 31. If $G \in \mathcal{M}$, then we say that G is a *super starkiller* if there exists a Right option G^R of G that is not a Left end such that $\mathrm{o_R}\left(G^R\right) = \mathscr{R}$ and, for all Left options G^{RL} of G^R, either

1. $\mathrm{o_L}\left(G^{RL}\right) = \mathscr{R}$, or
2. G^{RL} is a super starkiller.

It should be noted that, from a simple comparison of Definitions 25, 27 and 31, a super starkiller is necessarily both a disintegrator and a starkiller. To give explicit examples that illustrate the differences, consider the following table:

	disintegrator	starkiller	super starkiller
$\{\ \mid \{* \mid 0\}\}$	✓	✗	✗
$\overline{1}$	✗	✓	✗
$\{\ \mid \{* \mid \overline{1}\}\}$	✓	✓	✓

Just as for disintegrators and starkillers, the proof of the following proposition is simply a tedious application of the definition, so we relegate it to Appendix A.3.

Proposition 32. *If $\mathcal{U}$ is a universe and $\{\blacksquare, 0 \mid 0\} \in \widehat{\mathcal{U}}$, then $\{\blacksquare, 0 \mid 0\}$ is $\mathcal{U}$-invertible if and only if $\mathcal{U}$ contains no Left end that is a super starkiller.*

To complete the triad of disappointments, consider the Left end $G = \{\ \mid \{1 \mid 0, \overline{1}\} + \{\overline{1}, 1 \mid\ \}\}$, where G is not a super starkiller, but $G + G$ is. Once again, this is the same example we gave before (a powerful counter-example, indeed, but there exist others, which the reader is more than welcome to find).

4.4 The remaining forms

The augmented forms left to consider are $\{\blacksquare, 0 \mid\ \}$ and $\{\blacksquare, 0 \mid 0, \blacksquare\}$. Consider first $G = \{\blacksquare, 0 \mid\ \}$. We calculate $G + \overline{G} \cong \{\blacksquare, \overline{G} \mid G, \blacksquare\}$. Since this is Left $\mathcal{U}$-strong for all universes $\mathcal{U}$, it is clear from Proposition 24 that $G + \overline{G} \equiv_{\mathcal{U}} 0$ for all universes $\mathcal{U}$. That is, if $G \in \widehat{\mathcal{U}}$, then $\mathcal{U}$ admits an invertible element.

Now consider $G = \{\blacksquare, 0 \mid 0, \blacksquare\}$. We calculate $G + \overline{G} \cong \{\blacksquare, G \mid G, \blacksquare\}$. Since this is Left $\mathcal{U}$-strong for all universes $\mathcal{U}$, we have again by Proposition 24 that $G + \overline{G} \equiv_{\mathcal{U}} 0$ for all universes $\mathcal{U}$. That is, if $G \in \widehat{\mathcal{U}}$, then $\mathcal{U}$ admits an invertible element.

It is here that we must interrupt the narrative to assuage the concerns of our alarmed readers, for what we have just said may appear strange. We have found that, for all universes $\mathcal{U}$,

$$\{\blacksquare, 0 \mid\ \} + \{\ \mid 0, \blacksquare\} \equiv_{\mathcal{U}} 0, \text{ and}$$
$$2 \cdot \{\blacksquare, 0 \mid 0, \blacksquare\} \equiv_{\mathcal{U}} 0.$$

The informed reader will recall that the full misère universe $\mathcal{M}$ is reduced, meaning that it has no invertible elements. That is, there exists no $\mathcal{M}$-invertible $G \in \mathcal{M}$. And therein lies the antidote: the $\mathcal{M}$-equivalence classes of the games $\{\blacksquare, 0 \mid\ \}$ and $\{\blacksquare, 0 \mid 0, \blacksquare\}$ do not contain any games in $\mathcal{M}$! They do not have $\mathcal{M}$-expansions, or, equivalently here, they are not the simplest forms of any games in $\mathcal{M}$. So, we are *not* saying that these two games $\{\blacksquare, 0 \mid\ \}$ and $\{\blacksquare, 0 \mid 0, \blacksquare\}$ are $\mathcal{U}$-invertible for every universe $\mathcal{U}$, but just for those universes where these games have a $\mathcal{U}$-expansion (equivalently here, for those universes where these games are the $\mathcal{U}$-simplest forms of some games in $\mathcal{U}$).

Collecting together Propositions 26, 28 and 32, we have the following characterisation of reduced universes. But it is important to note that this result is not immediately practical. For example, how does one check whether $\mathcal{U}$ contains a Left end that is a starkiller? How does one check whether $\{\blacksquare, 0 \mid\ \} \in \widehat{\mathcal{U}}$? These do not appear to be easy problems to solve.

Theorem 33. *If $\mathcal{U}$ is a universe, then it is reduced if and only if the following statements are all true:*

1. *if $1 \in \widehat{\mathcal{U}}$, then there exists a Left end in $\mathcal{U}$ that is a disintegrator;*
2. *there exists a Left end in $\mathcal{U}$ that is a starkiller;*
3. *if $\{\blacksquare, 0 \mid 0\} \in \widehat{\mathcal{U}}$, then there exists a Left end in $\mathcal{U}$ that is a super starkiller;*

4. $\{\blacksquare, 0 \mid \ \} \notin \widehat{\mathcal{U}}$*; and*

5. $\{\blacksquare, 0 \mid 0, \blacksquare\} \notin \widehat{\mathcal{U}}$.

Proof. This follows immediately from Propositions 26, 28 and 32 and the previous discussion in Section 4.4. □

Earlier, in Open Problem 22, we asked whether there exists a universe $\mathcal{U}$ that is not reduced such that there exists no $\mathcal{U}$-invertible $G \in \mathcal{U}$ of formal birthday 1 (and also other birthdays). Theorems 30 and 33 might present a path to try and tackle such a question.

Recalling that the full misère universe $\mathcal{M}$ is reduced, it follows clearly that it must be the unique universe that is maximal with respect to being reduced. But what about universes that are *minimal* with respect to being reduced? We will see in the next section (in the discussion after Corollary 43) that $\mathcal{D}(\{\ \mid 2\})$ is one such universe (since its only subuniverses are $\mathcal{D}$ and $\mathcal{D}(\overline{1})$, neither of which is reduced). We will also see that there are uncountably many reduced universes.

5 Weak universes

In any given universe, it is always trivially the case that a Left end-like form is Left $\mathcal{U}$-strong. But the converse is not always true: in our favourite universes $\mathcal{D}$ and $\mathcal{E}$, it is trivial to find games that are Left $\mathcal{D}$- and $\mathcal{E}$-strong (respectively) but which are not Left ends. For the full misère universe $\mathcal{M}$, however, being Left $\mathcal{M}$-strong is exactly equivalent to being Left end-like. We will call a universe exhibiting this behaviour (like $\mathcal{M}$) a *weak* universe, since it has no more strong elements than absolutely required.

Definition 34. If $\mathcal{A}$ is a set of games such that $G \in \mathcal{M}_{\text{aug}}$ is Left $\mathcal{A}$-strong if and only if G is Left end-like, then we say $\mathcal{A}$ is *Left weak*. We define *Right weak* analogously, and furthermore say that $\mathcal{A}$ is *weak* if it is both Left and Right weak.

Of course, a set of games that is conjugate closed is Left weak if and only if it is Right weak; and so it is simply either weak or not.

As we have discussed, the universe $\mathcal{M}$ is weak—and we will give more examples soon. But we first give a short proof for completeness and to highlight this fact.

Proposition 35. *The universe* $\mathcal{M}$ *is weak.*

Proof. Let $G \in \mathcal{M}_{\text{aug}}$ be an augmented form that is not Left end-like. Let $X = \{\ \mid \tilde{\text{b}}(G)\}$, and consider Left playing first on $G + X$. Since G is not Left end-like, Left does not win immediately and so she must play to some $G^L + X$. Right can respond to $G^L + \tilde{\text{b}}(G)$, and by construction will clearly run out of moves before Left, hence winning the game. Thus, G is not Left $\mathcal{M}$-strong. □

It is not shocking that all weak universes induce the same relation; in particular, all weak universes induce the same partial order as $\mathcal{M}$ does (with $\geqslant_{\mathcal{M}}$).

Proposition 36. *If $\mathcal{U}$ and $\mathcal{W}$ are weak universes, then the relations $\geqslant_{\mathcal{U}}$ and $\geqslant_{\mathcal{W}}$ agree on $\mathcal{M}_{aug}$.*

Proof. From Theorem 4, we observe that two universes induce the same relation on $\mathcal{M}_{\text{aug}}$ if they have the same proviso. Since a game in $\mathcal{M}_{\text{aug}}$ is Left $\mathcal{U}$-strong if and only if it is Left $\mathcal{W}$-strong, it is clear that $\geqslant_{\mathcal{U}}$ and $\geqslant_{\mathcal{W}}$ have the same proviso, and hence they agree on $\mathcal{M}_{\text{aug}}$; i.e. they are the same relation. □

It is also easy to see that every weak monoid of games is also a reduced monoid. More precisely: every invertible element of a Left (Right) weak monoid is a Left (Right) end.

Proposition 37. *If $\mathcal{A}$ is a Left (Right) weak monoid of games, then every element of the invertible subgroup of $\mathcal{A}$ is a Left (Right) end. In particular, if $\mathcal{A}$ is weak, then $\mathcal{A}$ is reduced.*

Proof. Let $G, H \in \mathcal{A}$ with $G + H \equiv_{\mathcal{A}} 0$ and $\mathcal{A}$ Left weak. By Proposition 5, it follows that $G + H$ is Left $\mathcal{A}$-strong. Since $\mathcal{A}$ is Left weak, we must have that $G + H$ is Left end-like. But G and H have no tombstone options, so $G + H$ is a Left end, which implies G and H are both Left ends. The result for $\mathcal{A}$ being Right weak follows by symmetry. Finally, if $\mathcal{A}$ is weak, then our G and H from above would necessarily be both Left ends and Right ends, which implies they are both isomorphic to 0, yielding the result. □

We could also have realised that every weak universe is reduced from Proposition 36: every weak universe induces the same partial order as $\mathcal{M}$, which is reduced, and hence every weak universe is reduced. What is unclear, however, is whether the reverse implication holds. That is, if we have a reduced universe, must it also be weak? The authors know of no such examples, and leave it as an open problem.

Open Problem 38. Must a reduced universe be weak?

The reader may or may not be surprised to learn that there exist Left ends (Right ends) whose mere presence in a semigroup of games (i.e. an additively closed set of games) forces the set to be Left weak (Right weak). We call such games *weakening*.

Definition 39. If $X \in \mathcal{M}$ is a Left end (Right end) such that every semigroup of games containing X is Left weak (Right weak), then we say X is *weakening*.

A simple example of a weakening end is the form $\{\ \mid 2\}$. A key property of this game is that Right can effectively *give back* more moves to Left than Left can give back to him; in particular here, Right has to play only 1 move in order to give back 2 to Left.

Theorem 40. *If $\mathcal{A}$ is a semigroup of games containing $\{\ \mid 2\}$, then $\mathcal{A}$ is Left weak. That is, $\{\ \mid 2\}$ is weakening.*

Proof. Let G be an augmented form that is not Left end-like. Now consider Left playing first on $G + \tilde{\mathrm{b}}(G) \cdot \{\ \mid 2\}$. We will show that Right wins.

Left must play to some $G^L + \tilde{\mathrm{b}}(G) \cdot \{\ \mid 2\}$. Right can then respond with $G^L + 2 + (\tilde{\mathrm{b}}(G) - 1) \cdot \{\ \mid 2\}$. Regardless of what Left does, Right will continue to play on the $\{\ \mid 2\}$

components until they are exhausted. Clearly Left cannot run out of moves before Right accomplishes this task. At the moment that Right plays on the last $\{\ \mid 2\}$ component, we have a game of the form $G' + n$, where G' is a subposition of G^L, and $n \geqslant \tilde{\mathrm{b}}(G) + 1$. Thus, Right will win. □

An almost identical proof clearly yields a countably infinite number of weakening ends. For example, *every* Left end with an option to 2 must be weakening. And it need not have 2 as an option either; it could instead have an option to some $n > 2$. But even this is not exhaustive, and the reader may amuse themselves finding other more complex examples. We leave it as an open problem to try and characterise such games.

Open Problem 41. Find a characterisation of the weakening Left ends.

What is also unclear is whether a weak universe necessarily contains a weakening Left end. (Or, more generally, whether a Left weak semigroup necessarily contains a weakening Left end.) We leave this as an open problem.

Open Problem 42. Is it true that a universe is weak if and only if it contains a weakening Left end?

For universes in particular, we have the following obvious corollary of Theorem 40, which states that a universe containing $\{\ \mid 2\}$ must be weak (and reduced).

Corollary 43. *If $\mathcal{U}$ is a universe containing $\{\ \mid 2\}$, then $\mathcal{U}$ is weak (and hence also reduced).*

Proof. It follows from Theorem 40 that $\mathcal{U}$ is Left weak. Now, every universe is conjugate closed, and hence $\mathcal{U}$ must also be Right weak. That is, $\mathcal{U}$ is weak, and hence also reduced (Proposition 37). □

It then follows swiftly that $\mathcal{D}(\{\ \mid 2\})$ must be a universe that is minimal with respect to being reduced, since its only subuniverses are $\mathcal{D}$ and $\mathcal{D}(\overline{1})$, neither of which is reduced. We can also give a very simple construction for uncountably many weak universes here (and hence also uncountably many reduced universes). Take any subset $\mathcal{S}$ of $\mathbb{N}_{\geqslant 3}$, and then construct the universal closure of the Left ends $\{\{\ \mid 2, n\} : n \in \mathcal{S}\}$. This defines an injective map from the subsets of $\mathbb{N}_{\geqslant 3}$ to the set of universes, and hence there must be uncountably many weak universes.

Going to back to monoids of games (that are not necessarily universes), we have another corollary of Theorem 40.

Corollary 44. *If $\mathcal{A}$ is a monoid of games containing $\{\ \mid 2\}$, then every $\mathcal{A}$-invertible game is a Left end. In particular, $\mathcal{A}^{\overline{\times}}$ is the trivial group.*

Proof. By Theorem 40, we know that $\mathcal{A}$ is Left weak. So, by Proposition 37, every invertible element in $\mathcal{A}$ is a Left end.

If $G \in \mathcal{A}$ is non-zero, then $G + \overline{G}$ is not a Left end. If $\overline{G} \in \mathcal{A}$, then clearly $G + \overline{G}$ cannot be Left $\mathcal{A}$-strong since $\mathcal{A}$ is Left weak. Thus, there cannot exist a non-zero element in $\mathcal{A}$ that is conjugate $\mathcal{A}$-invertible: $\mathcal{A}^{\overline{\times}}$ is the trivial group. □

Note that Corollary 44 does not mean that the invertible subgroup of $\mathcal{A}$ is trivial—although, the conjugate invertible subgroup of $\mathcal{A}$ certainly is. Indeed, in Milley's monoid of PARTIZAN KAYLES $\mathcal{K}$, it is the case that $\overline{1} + W_2 \equiv_{\mathcal{K}} 0$ [10, Corollary 4 on p. 11], and it is obvious that $\mathcal{K}^{\overline{\times}}$ is the trivial group. And notice that both $\overline{1}$ and W_2 are Left (dead) ends. But, if we want to give an example of a Left weak monoid that has a non-trivial invertible subgroup, then $\mathcal{K}$ will not do it for us, since it is not Left weak: recall that $\mathcal{K} \subseteq \mathcal{E}$, and also that $1 + \overline{1} \equiv_{\mathcal{E}} 0$; in particular, $1 + \overline{1}$ is Left $\mathcal{E}$-strong, and hence also Left $\mathcal{K}$-strong; but $1 + \overline{1}$ is not Left end-like, so $\mathcal{K}$ cannot be Left weak. We will now give a true example, demonstrating further how strange misère monoids can be.

Let $\mathcal{J} = \langle \overline{1}, \{\ \mid 2\} \rangle$; that is, the semigroup generated by the (additive) closure of $\overline{1}$ and $\{\ \mid 2\}$ (we will show soon that it is a monoid). Observe that

$$\mathrm{o}(\overline{n} + m \cdot \{\ \mid 2\}) = \begin{cases} \mathscr{L} & n > m \\ \mathscr{N} & n \leqslant m. \end{cases}$$

It then clearly follows that $\overline{n} + m \cdot \{\ \mid 2\} \equiv_{\mathcal{J}} 0$ if and only if $n = m$. Hence $\mathcal{J}$ must be a monoid: $\overline{1} + \{\ \mid 2\}$ is the identity. Since $\{\ \mid 2\} \in \mathcal{J}$, we know that $\mathcal{J}$ is Left weak by Theorem 40. Now we have

$$\begin{aligned} \mathcal{J} &\cong \langle x, y \mid xy \rangle \\ &\cong \mathbb{Z}. \end{aligned}$$

So, our Left weak monoid $\mathcal{J}$ is not only *not* reduced, but it is a group! We will stress again that such a thing cannot occur in a universe (or indeed any misère monoid with the conjugate property); if a universe is weak, then it is reduced.

Regarding Milley's monoid of PARTIZAN KAYLES $\mathcal{K}$, they write in [10, p. 13] that it is the only known partizan example where a game may have an inverse that is not its conjugate. We have demonstrated, with our strange monoid $\mathcal{J}$, another example. Indeed, using the ideas we have discussed, it is not difficult to find many more. But we make a remark on the invertibility here. Milley writes $\overline{1} + W_2 \equiv_{\mathcal{K}} 0$, and observes that $1 \not\equiv_{\mathcal{K}} W_2$. Now, even though $\overline{1}$ and W_2 are not conjugate $\mathcal{K}$-invertible, they do satisfy the following relations:

$$\begin{aligned} 1 + \overline{1} &\equiv_{\mathcal{K}} 0, \\ W_2 + \overline{W_2} &\equiv_{\mathcal{K}} 0. \end{aligned}$$

But, in our monoid $\mathcal{J}$, we have a stronger behaviour:

$$\begin{aligned} 1 + \overline{1} &\not\equiv_{\mathcal{J}} 0, \\ \{\ \mid 2\} + \overline{\{\ \mid 2\}} &\not\equiv_{\mathcal{J}} 0. \end{aligned}$$

This is because $\mathcal{J}$ is Left weak, and hence only Left ends might be equal to 0 ($\mathcal{K}$ is *not* Left weak). It would be useful to collate a number of examples and their behaviours so that we may get a better lay of the land with regards to the strangeness of partizan misère monoids.

6 Final remarks

In his brilliant paper, Siegel laments the possible lack of utility of his simplest forms [15, §7 on p. 226]. Indeed, in the subsection "Is any of this useful for anything?", he gives some computational evidence for the small amounts of information we are able to discard from a game tree, and he writes that the theory of simplest forms is "unlikely to be applicable to specific case studies in a way that provides much real insight." We have seen in this paper that the simplest forms were instrumental in coordinating our proofs. We now know far more about invertibility in universes, which will indeed have practical benefit to analysing specific rulesets. As such, we push back: both in theory and in practice, Siegel's simplest forms are wonderful.

It is curious how the development of the simplest forms led so swiftly to our results on the conjugate property and the characterisation of the invertible elements. It would be interesting to construct an alternative proof that makes no use of these forms. This has, of course, been tried before, but the challenge of overcoming end-reversibility is a great one. It is left as a challenge to the reader.

Open Problem 45. Can we find a proof of each universe having the conjugate property without using simplest forms? And the characterisation of the invertible elements?

Given that we do now have a characterisation of the invertible elements of each universe, an almost entirely open area that is ripe for future work is understanding what their group structure is (the invertible elements of a monoid always form a group). These groups must be countable and abelian, of course, but what else can we say? It might be wise to look at the subgroups generated by those invertible games born by days 2 and 3, and see what patterns emerge.

Open Problem 46. What can we say about the group structure of the invertible subgroup of each universe?

Finally, for the reader's convenience, we briefly summarise the other open problems that were scattered throughout the paper:

> Open Problem 20 on Page 12: what alternative characterisations of invertible elements in misère universes are there?
>
> Open Problem 22 on Page 13: is there a universe that is not reduced but contains no invertible element of formal birthday 1?
>
> Open Problem 38 on Page 19: are all reduced universes weak?
>
> Open Problem 41 on Page 20: can we characterise the weakening Left ends?
>
> Open Problem 42 on Page 20: do all weak universes contain a weakening Left end?

Many of these seem tantalising. Perhaps the likeliest to bear fruit quickly is Open Problem 41. We hope to see many advancements soon!

Acknowledgements

The authors thank the reviewer for their comments that undoubtedly improved the quality of the paper, and also Aaron Siegel for fruitful discussions.

References

[1] Alfie Davies. On factorisations of Left dead ends, 2024. arXiv:2409.17871v1.

[2] Alfie M. Davies, Neil A. McKay, Rebecca Milley, Richard J. Nowakowski, and Carlos P. Santos. Pocancellation theorems in Combinatorial Game Theory. Submitted, 2024.

[3] John Mark Ettinger. *Topics in combinatorial games*. ProQuest LLC, Ann Arbor, MI, 1996. Thesis (Ph.D.)–The University of Wisconsin - Madison.

[4] Michael Fisher, Richard J. Nowakowski, and Carlos Pereira dos Santos. Invertible elements of the dicot misère universe. *Integers*, 22:Paper No. G6, 11, 2022.

[5] Urban Larsson, Rebecca Milley, Richard J. Nowakowski, and Carlos P. Santos. Reversibility, canonical form, and invertibility in dead-ending misère play. In *Games of no chance 6*, volume 71 of *Math. Sci. Res. Inst. Publ.*, pages 245–266. Cambridge Univ. Press, New York, 2025.

[6] Urban Larsson, Richard J. Nowakowski, and Carlos P. Santos. Absolute combinatorial game theory. In *Games of no chance 6*, volume 71 of *Math. Sci. Res. Inst. Publ.*, pages 99–134. Cambridge Univ. Press, New York, 2025.

[7] Urban Larsson, Richard J. Nowakowski, and Carlos P. Santos. Infinitely many absolute universes. In *Games of no chance 6*, volume 71 of *Math. Sci. Res. Inst. Publ.*, pages 229–244. Cambridge Univ. Press, New York, 2025.

[8] G. A. Mesdal and P. Ottaway. Simplification of partizan games in misère play. *Integers*, 7:G06, 12, 2007.

[9] Rebecca Milley. *Restricted Universes of Partizan Misère Games*. Phd thesis, Dalhousie University, Halifax, Canada, March 2013.

[10] Rebecca Milley. Partizan Kayles and misère invertibility. *Integers*, 15:Paper No. G3, 14, 2015.

[11] Rebecca Milley and Gabriel Renault. Restricted developments in partizan misère game theory. In *Games of no chance 5*, volume 70 of *Math. Sci. Res. Inst. Publ.*, pages 113–123. Cambridge Univ. Press, Cambridge, 2019.

[12] Rebecca Milley and Gabriel Renault. The invertible elements of the monoid of dead-ending misère games. *Discrete Math.*, 345(12):Paper No. 113084, 13, 2022. doi:10.1016/j.disc.2022.113084.

[13] Thane E. Plambeck and Aaron N. Siegel. Misère quotients for impartial games. *J. Combin. Theory Ser. A*, 115(4):593–622, 2008. doi:10.1016/j.jcta.2007.07.008.

[14] Aaron N. Siegel. *Combinatorial game theory*, volume 146 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. doi:10.1090/gsm/146.

[15] Aaron N. Siegel. On the general dead-ending universe of partizan games. In *Games of no chance 6*, volume 71 of *Math. Sci. Res. Inst. Publ.*, pages 191–228. Cambridge Univ. Press, New York, 2025.

A Miscellaneous proofs

A.1 Disintegrators

Lemma 47. *If X is a disintegrator, then $\mathrm{o_R}(\overline{1}+X)=\mathscr{R}$.*

Proof. Since X is a disintegrator, there exists some X^R that is not a Left end such that, for all Left options X^{RL}, either

1. $\mathrm{o_L}(X^{RL})=\mathscr{R}$; or
2. X^{RL} is a disintegrator.

Right should play to this X^R, leaving $\overline{1}+X^R$. Left must then respond to some $\overline{1}+X^{RL}$ (recall that X^R is not a Left end). If $\mathrm{o}(X^{RL})=\mathscr{R}$, then Right can play on $\overline{1}$ to 0, leaving himself with a winning position. Instead, if X^{RL} is a disintegrator, then Right wins by induction. □

Lemma 48. *If X satisfies $\mathrm{o_L}(X)=\mathscr{L}$ and is not a disintegrator, then $\mathrm{o_R}(\overline{1}+X)=\mathscr{L}$.*

Proof. Right cannot play on $\overline{1}$ to 0, leaving X, since this is winning for Left. So, Right must play on X to some X^R, leaving $\overline{1}+X^R$. Since X is not a disintegrator, we must be in one of two cases:

1. X^R is a Left end.

   In this case, Left wins immediately since she has no moves.

2. There exists a Left option X^{RL} such that $\mathrm{o_L}(X^{RL})=\mathscr{L}$ and X^{RL} is not a disintegrator.

   In this case, Left can play to this X^{RL}, which must be winning by induction. □

Proposition 26. *If $\mathcal{U}$ is a universe and $1\in\widehat{\mathcal{U}}$, then 1 is $\mathcal{U}$-invertible if and only if $\mathcal{U}$ contains no Left end that is a disintegrator.*

Proof. By Corollary 23, we know that 1 is $\mathcal{U}$-invertible if and only if $1+\overline{1}$ is Left $\mathcal{U}$-strong, which means that $\mathrm{o_L}(1+\overline{1}+X)=\mathscr{L}$ for all Left ends $X\in\mathcal{U}$. Since X is a Left end, this is equivalent to saying that $\mathrm{o_R}(\overline{1}+X)=\mathscr{L}$ for all Left ends $X\in\mathcal{U}$.

The result then follows immediately from Lemmas 47 and 48. □

A.2 Starkillers

Lemma 49. *If X is a starkiller, then* $\mathrm{o_R}(* + X) = \mathscr{R}$*.*

Proof. By Definition 27, we know that there exists some Right option X^R such that $\mathrm{o_R}(X^R) = \mathscr{R}$ and, for all Left options X^{RL}, either

1. $\mathrm{o_L}(X^{RL}) = \mathscr{R}$; or
2. X^{RL} is a starkiller.

So, let us consider Right moving to $* + X^R$. If Left plays on $*$ to leave simply X^R, then Right wins since $\mathrm{o}(X^R) = \mathscr{R}$. Otherwise, Left moves to some $* + X^{RL}$. If $\mathrm{o_L}(X^{RL}) = \mathscr{R}$, then Right can play on $*$ to leave X^{RL}, which is winning. Otherwise, X^{RL} must be a starkiller, and so Right wins by induction. □

Lemma 50. *If X satisfies $\mathrm{o_L}(X) = \mathscr{L}$ and is not a starkiller, then $\mathrm{o_R}(* + X) = \mathscr{L}$.*

Proof. Right cannot play on $*$ to 0, leaving X, otherwise Left would win immediately. Thus, Right must play on X to leave some $* + X^R$. Since X is *not* a starkiller, it follows (straight from Definition 27) that either $\mathrm{o_R}(X^R) = \mathscr{L}$, or else there exists some X^{RL} such that $\mathrm{o_L}(X^{RL}) = \mathscr{L}$ and X^{RL} is not a starkiller.

If we are in the first case, with $\mathrm{o_R}(X^R) = \mathscr{L}$, then Left can play on $*$ to leave X^R, which must be winning. In the second case, Left should move on X^R to the X^{RL} with properties just discussed, leaving $* + X^{RL}$, which must be winning by induction. □

Proposition 28. *If $\mathcal{U}$ is a universe, then $*$ is $\mathcal{U}$-invertible if and only if $\mathcal{U}$ contains no Left end that is a starkiller.*

Proof. By Corollary 23, we know that $*$ is $\mathcal{U}$-invertible if and only if $* + *$ is Left $\mathcal{U}$-strong, which means that $\mathrm{o_L}(* + * + X) = \mathscr{L}$ for all Left ends $X \in \mathcal{U}$. Since X is a Left end, this is equivalent to saying that $\mathrm{o_R}(* + X) = \mathscr{L}$ for all Left ends $X \in \mathcal{U}$.

The result then follows immediately from Lemmas 49 and 50. □

A.3 Super starkillers

Lemma 51. *If X is a super starkiller, then* $\mathrm{o_R}(\{\blacksquare, 0 \mid 0\} + X) = \mathscr{R}$*.*

Proof. Since X is a super starkiller, Right can play on X to some X^R that is not a Left end such that $\mathrm{o_R}(X^R) = \mathscr{R}$ and, for all Left options X^{RL} of X^R, either

1. $\mathrm{o_L}(X^{RL}) = \mathscr{R}$; or
2. X^{RL} is a super starkiller.

Since X^R is not a Left end, the tombstone of $\{\blacksquare, 0 \mid 0\}$ does not yield that Left wins going first here. If Left plays on $\{\blacksquare, 0 \mid 0\}$ to 0, leaving X^R, then Right wins since $\mathrm{o_R}(X^R) = \mathscr{R}$. So, Left must instead play on X^R to some X^{RL}. If $\mathrm{o_L}(X^{RL}) = \mathscr{R}$, then Right can play on $\{\blacksquare, 0 \mid 0\}$ to 0 and win the game. Otherwise, if X^{RL} is a super starkiller, then Right wins by induction. □

Lemma 52. *If X satisfies $\mathrm{o_L}(X) = \mathscr{L}$ and is not a super starkiller, then $\mathrm{o_R}(\{\blacksquare, 0 \mid 0\} + X) = \mathscr{L}$.*

Proof. Right cannot play on $\{\blacksquare, 0 \mid 0\}$ to 0, otherwise Left would win. So, Right must play to some $\{\blacksquare, 0 \mid 0\} + X^R$. Since X is not a super starkiller, we must be in one of the following cases:

1. X^R is a Left end.

   In this case, since $\{\blacksquare, 0 \mid 0\}$ is Left $\mathcal{U}$-strong (it has a Left tombstone), we observe that Left wins going first.

2. $\mathrm{o_R}(X^R) = \mathscr{L}$.

   In this case, Left can play on $\{\blacksquare, 0 \mid 0\}$ to 0, leaving herself a winning position.

3. there exists a Left option X^{RL} such that $\mathrm{o_L}(X^{RL}) = \mathscr{L}$ and X^{RL} is not a super starkiller.

   In this case, Left can play to this X^{RL}, which must be winning by induction. □

Proposition 32. *If $\mathcal{U}$ is a universe and $\{\blacksquare, 0 \mid 0\} \in \widehat{\mathcal{U}}$, then $\{\blacksquare, 0 \mid 0\}$ is $\mathcal{U}$-invertible if and only if $\mathcal{U}$ contains no Left end that is a super starkiller.*

Proof. By Corollary 23, we know that $\{\blacksquare, 0 \mid 0\}$ is $\mathcal{U}$-invertible if and only if $\{\blacksquare, 0 \mid 0\} + \{0 \mid 0, \blacksquare\}$ is Left $\mathcal{U}$-strong. We calculate (or recall from Section 4.3) that

$$\{\blacksquare, 0 \mid 0\} + \{0 \mid 0, \blacksquare\} \equiv_{\mathcal{M}} \{\{\blacksquare, 0 \mid 0\} \mid \{0 \mid 0, \blacksquare\}\}.$$

So, $\{\blacksquare, 0 \mid 0\}$ is $\mathcal{U}$-invertible if and only if $\mathrm{o_R}(\{\blacksquare, 0 \mid 0\} + X) = \mathscr{L}$ for all Left ends $X \in \mathcal{U}$.

The result then follows immediately from Lemmas 51 and 52. □