The $\boldsymbol{P}$-associahedron $\boldsymbol{f}$-vector is a comparability invariant

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Abstract

For any finite, connected poset P, we show that the f-vector of Galashin's P-associahedron $\mathscr{A}(P)$ only depends on the comparability graph of P. In particular, this allows us to produce a family of polytopes with the same f-vectors as permutohedra, but that are not combinatorially equivalent to permutohedra.

Mathematics Subject Classifications: 05A19, 52B05

1 Introduction

Recall that the *comparability graph* of a poset P is a graph $C(P)$ whose vertices are the elements of P and where i and j are connected by an edge if i and j are comparable. A property of P is said to be *comparability invariant* if it only depends on $C(P)$. Properties of finite posets known to be comparability invariant include the order polynomial and number of linear extensions [7], the fixed point property [1], and the Dushnik-Miller dimension [10].

For a finite connected poset P, Galashin introduced the *P-associahedron* $\mathscr{A}(P)$ [2]. P-associahedra generalize Stasheff's associahedron [8] to the setting of properadic composition. That is, instead of having a sequence of operations with one input and one output, one may view a poset as a collection of operations with multiple inputs and multiple outputs "wired together" by covering relations. A vertex of a P-associahedron disambiguates the order of composition. For more details, see [3, 6, 9].

The f-polynomial of a d-dimensional polytope Q is

$$f_Q(z) := \sum_{i=0}^{d} f_i z^i$$

where f_i is the number of faces of Q in dimension i. We call $(f_0, \ldots, f_d)$ the f-vector of Q. The following is our main result:

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Theorem 1. *The f-vector of $\mathscr{A}(P)$ is a comparability invariant.*

Theorem 1 may lead one to ask if $C(P) \simeq C(P')$ implies that $\mathscr{A}(P)$ and $\mathscr{A}(P')$ are necessarily combinatorially equivalent.

Definition 2. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $a_i \geqslant 1$ for each i. Define the *complete graded poset* of type $\mathbf{a}$ to be the poset

$$P_{\mathbf{a}} := \{x_{11}, \ldots, x_{1a_1}, x_{21}, \ldots, x_{2a_2}, \ldots\},$$

where $x_{ij} \prec x_{i'j'}$ if and only if $i < i'$. That is, $P_{\mathbf{a}}$ is the ordinal sum of antichains.

Observe that $C(P_{\mathbf{a}})$ is invariant under permutation of $\mathbf{a}$. This observation, together with Theorem 1, yields an immediate corollary.

Corollary 3. *For any $\mathbf{a}$, $f_{\mathscr{A}(P_{\mathbf{a}})}(z)$ is invariant under permutation of $\mathbf{a}$.*

This class of examples is sufficiently rich to answer our question in the negative.

Theorem 4. *Let $m, n \geqslant 2$. Then $\mathscr{A}(P_{(m,1,n)})$ is combinatorially equivalent to the permutohedron, but $\mathscr{A}(P_{(1,m,n)})$ is not.*

Corollary 5. *There exists a family of polytopes with the same f-vector as the permutohedron, but which are not combinatorially equivalent.*

2 Background

2.1 Flips of autonomous subsets

Definition 6. Let P and S be posets and let $a \in P$. The *substitution* of a for S is the poset $P(a \to S)$ on the set $(P - \{a\}) \sqcup S$ formed by replacing a with S.

More formally, $x \preceq_{P(a\to S)} y$ if and only if one of the following holds:

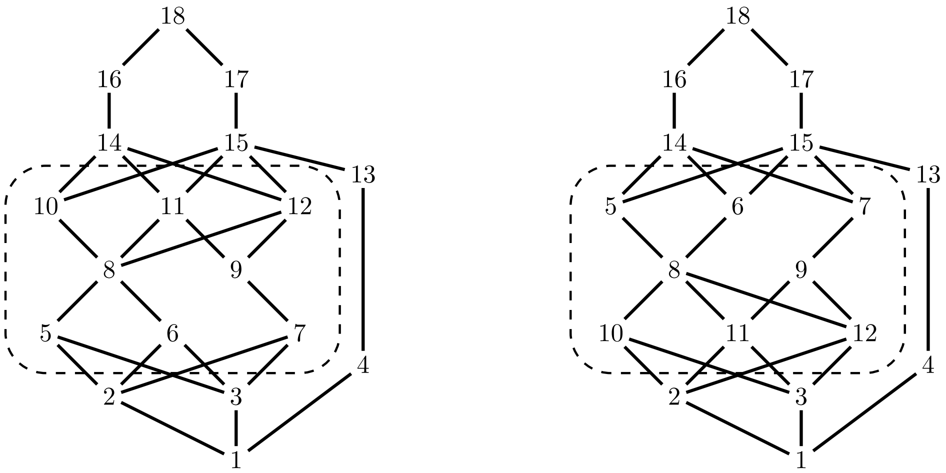
- $x, y \in P - \{a\}$ and $x \preceq_P y$
- $x, y \in S$ and $x \preceq_S y$
- $x \in S, y \in P - \{a\}$ and $a \preceq_P y$
- $y \in S, x \in P - \{a\}$ and $y \preceq_P a$.

Definition 7. Let P be a poset and let $S \subseteq P$. The subset S is called *autonomous* if there exists a poset Q and $a \in Q$ such that $P = Q(a \to S)$.

Equivalently, S is autonomous if for all $x, y \in S$ and $z \in P - S$, we have

$$(x \preceq z \Leftrightarrow y \preceq z) \text{ and } (z \preceq x \Leftrightarrow z \preceq y).$$

Definition 8. For a poset S, the *dual poset* S^{op} is defined on the same ground set where $x \preceq_S y$ if and only if $y \preceq_{S^{\mathrm{op}}} x$. A *flip* of S in $P = Q(a \to S)$ is the replacement of P by $Q(a \to S^{\mathrm{op}})$.



(a) An autonomous subset S of a poset P.

(b) A flip of S.

Figure 1

See Figure 1a for an example of an autonomous subset and Figure 1b for an example of a flip. One can see that if two posets are related by the flip of an autonomous subset, then they clearly have the same comparability graph. The following theorem shows that these flips suffice to connect any posets with the same comparability graph.

Theorem 9 ([1, Theorem 1])**.** *If P and P' are finite posets such that $C(P) = C(P')$ then P and P' are connected by a sequence of flips of autonomous subsets.*

In particular, a property is comparability invariant if and only if it is preserved under flips.

2.2 *P*-Associahedra

We recall the definition of P-associahedra.

Definition 10. Let P be a finite connected poset. A subset $\tau \subsetneq P$ is called a *proper pipe* if

- $2 \leqslant |\tau|$.
- τ is *convex*, i.e. for all $x, z \in \tau, y \in S$, we have

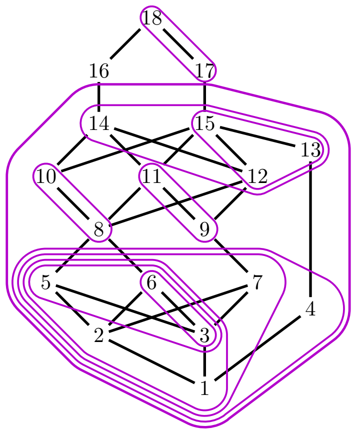
$$(x \preceq y \preceq z) \Rightarrow (y \in \tau).$$

- τ is *connected* as a subgraph of the Hasse diagram of P.

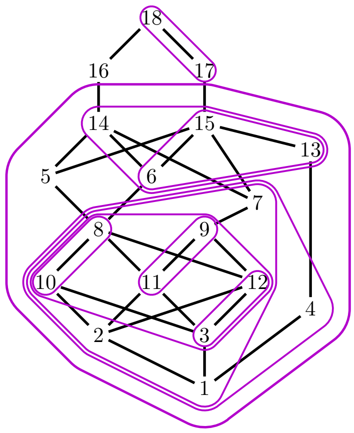
A collection T of proper pipes is called a *proper piping* if the following two conditions hold:

- The pipes in T are pairwise either nested or disjoint. That is, for all $\sigma, \tau \in T$, we have either $\sigma \subseteq \tau$, $\tau \subseteq \sigma$, or $\tau \cap \sigma = \emptyset$.
- The directed graph D_T is acyclic, where T is the vertex set of D_T and where (σ, τ) is an edge if $\sigma \cap \tau = \emptyset$ and there exist $x \in \sigma, y \in \tau$ such that $x \prec y$.

See Figure 2a for an example of a proper piping.



(a) A proper piping T on P.



(b) $\Phi_{P,S}(T)$

Figure 2: A piping and its image under $\Phi_{P,S}(T)$ from the flip in Figure 1

Theorem 11 ([2, Theorem 1.2])**.** *Let P be a finite, connected poset on at least 2 elements. Then the collection of proper pipings of P ordered by reverse inclusion is isomorphic to the face lattice of a simple $(|P|-2)$-dimensional polytope $\mathscr{A}(P)$. We call this polytope a P-associahedron.*

Lemma 12 ([2, Corollary 2.7])**.** *The codimension of $T \in \mathscr{A}(P)$ is equal to $|T|$.*

By an abuse of notation, we also use $\mathscr{A}(P)$ to refer to the set of proper pipings of P. Our strategy for proving Theorem 1 is to give a bijection between the pipings of $Q(a \to S)$ and of $Q(a \to S^{\text{op}})$ that preserves the number of pipes in a piping. See Figure 2 for an example of the map.

3 Proof of Theorem 1

3.1 Proof Sketch

Let $P = Q(a \to S)$ and $P' = Q(a \to S^{\text{op}})$. Our goal is to build a bijection

$$\Phi_{P,S} : \mathscr{A}(P) \to \mathscr{A}(P')$$

such that for any $T \in \mathscr{A}(P)$, we have $|T| = |\Phi_{P,S}(T)|$. Let $T \in \mathscr{A}(P)$. We will describe how to construct $T' := \Phi_{P,S}(T)$.

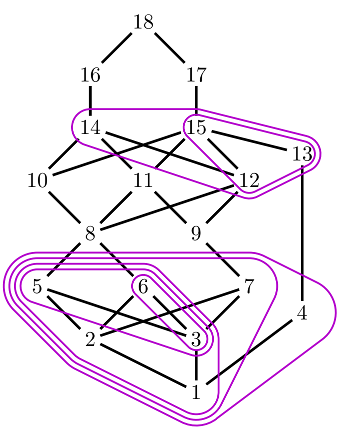


Figure 3: T_{bad} (left), T_{good} (middle), and T_{good} on P' (right).

Definition 13. A pipe $\tau \in T$ is *good* if $\tau \subseteq P - S$, $\tau \subseteq S$, or $S \subseteq \tau$ and is *bad* otherwise. We denote the set of good pipes by T_{good} and the set of bad pipes by T_{bad}.

All good pipes are also good pipes in P', and we add all good pipes to T'. See Figure 3 for an example of T_{good} and T_{bad}. It remains to handle the bad pipes.

Definition 14. A sequence of sets $(A_1, \ldots, A_r)$ is called *nested* if $A_i \subseteq A_j$ for all $i \leqslant j$. A *decorated* nested sequence is a nested sequence $(A_1, \ldots, A_r)$ paired with a function

$$f : \{1, \ldots, r\} \to \{0, 1\}.$$

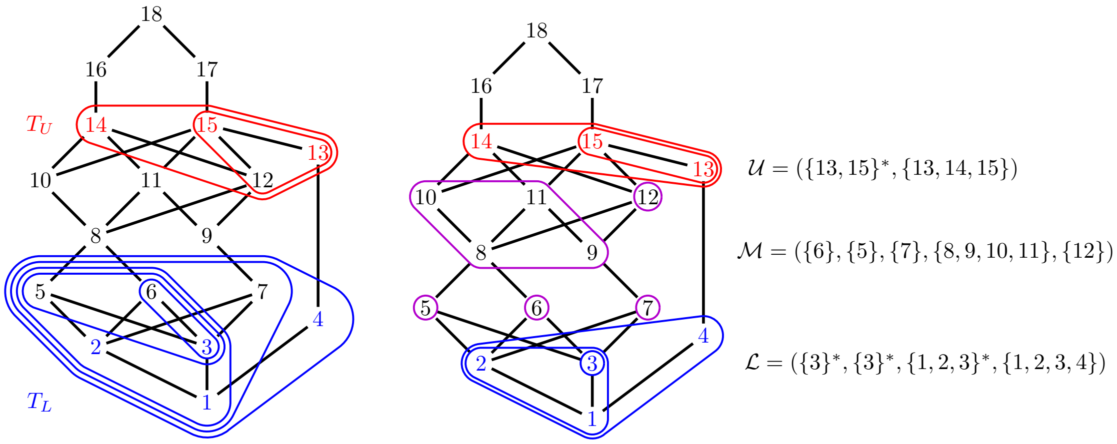
For brevity, instead of specifying f, we will instead mark A_i with a star if and only if $f(i) = 1$.

The key idea for defining $\Phi_{P,S}$ is to decompose T_{bad} into a triple $(\mathcal{L}, \mathcal{M}, \mathcal{U})$ where $\mathcal{L}$ and $\mathcal{U}$ are decorated nested sequences of sets contained in $P - S$ and $\mathcal{M}$ is an ordered set partition of S. In particular, we split T_{bad} into two disjoint subsets T_L (called *lower pipes*) and T_U (called *upper pipes*), for details see Definition 15. Each of T_L and T_U form a nested sequence of pipes, so by taking the intersection of the pipes in T_L and T_U with $P - S$, we get the nested sequences $\mathcal{L}$ and $\mathcal{U}$. Furthermore, this decomposition allows us to recover T_{bad} (uniquely) from $(\mathcal{L}, \mathcal{M}, \mathcal{U})$. See Figure 4 for an example of this decomposition.

We then build T'_{bad} by applying the recovery algorithm to the triple $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$ where $\overline{\mathcal{M}}$ is the reverse of $\mathcal{M}$. We then add T'_{bad} to T'. See Figure 5 for an example of the recovery algorithm applied to $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$. See Figure 2b for the image of T under $\Phi_{P,S}$ (including T_{good}).

3.2 Proof details

Definition 15. A pipe $\tau \in T_{\text{bad}}$ is called *lower* (resp. *upper*) if there exist $x \in \tau - S$ and $y \in \tau \cap S$ such that $x \preceq y$ (resp. $y \preceq x$). We denote the set of lower pipes by T_L and the set of upper pipes by T_U.



(a) T_L is blue and T_U is red. (b) $\mathcal{L}$ is blue, $\mathcal{M}$ is purple, and $\mathcal{U}$ is red.

Figure 4: The decomposition of T_{bad}.

Lemma 16 (Structure Lemma). *T_{bad} is the disjoint union of T_L and T_U. Furthermore, T_L and T_U each form a nested sequence.*

Proof. We first show that T_{bad} is the disjoint union of T_L and T_U. Suppose that τ is a pipe such that $\tau \in T_L \cap T_U$, i.e. there exist $x_1, x_2 \in \tau - S$ and $y_1, y_2 \in \tau \cap S$ such that

$$x_1 \preceq y_1 \text{ and } y_2 \preceq x_2.$$

Then as S is autonomous, for all $y \in S$, we have $x_1 \preceq y \preceq x_2$. As τ is convex, this implies $S \subseteq \tau$ and hence that τ is good. Therefore T_L and T_U are disjoint. Next observe that if $\tau \in T_{\text{bad}}$, then by connectivity there exist $x \in \tau \cap S$ and $y \in \tau - S$ such that x and y are comparable. Hence $\tau \in T_L \cup T_U$, so $T_{\text{bad}} = T_L \sqcup T_U$.

Finally, we show that T_L is nested. The result on T_U follows analogously. It suffices to show that T_L is pairwise nested. Let $\sigma, \tau \in T_L$. As T is a piping, if σ and τ are not nested, then they are disjoint. Suppose, for the sake of contradiction, that $\sigma \cap \tau = \emptyset$, and let $x_1 \in \tau - S$, $x_2 \in \sigma - S$, $y_1 \in \tau \cap S$, and $y_2 \in \sigma \cap S$ such that $x_1 \preceq y_1$ and $x_2 \preceq y_2$. Then as S is autonomous, we have $x_1, x_2 \preceq y_1, y_2$. Thus (σ, τ) and (τ, σ) are both edges in D_T, so D_T is not acyclic, a contradiction. □

We decompose T_L (resp. T_U) into a sequence of nested sets contained in $P - S$ and a sequence of disjoint sets contained in S as follows.

Definition 17 (Piping decomposition). Let $T_L = \{\tau_1, \dots\}$ where $\tau_i \subset \tau_{i+1}$ for all i. For convenience, we define $\tau_0 = \emptyset$. We define a decorated nested sequence $\mathcal{L} = (L_1, \dots)$ and a sequence of disjoint sets $\mathcal{M}_L = (M_L^1, \dots)$ as follows.

- For each $i \geqslant 1$, let $L_i = \tau_i - S$, and mark L_i with a star if $(\tau_i - \tau_{i-1}) \cap S \neq \emptyset$.
- If L_i is the j-th starred set, let $M_L^j = (\tau_i - \tau_{i-1}) \cap S$.

We define the sequences $\mathcal{U}$ and $\mathcal{M}_U$ analogously. We make the following definitions.

- Let $\hat{M} := S - \bigcup_{\tau \in T_{\text{bad}}} \tau$.
- For sequences $\mathbf{a}$ and $\mathbf{b}$, let the sequence $\mathbf{a} \cdot \mathbf{b}$ be $\mathbf{b}$ appended to $\mathbf{a}$.
- For a sequence $\mathbf{a}$, let $\overline{\mathbf{a}}$ be the reverse of $\mathbf{a}$.
- We define

$$\mathcal{M} := \begin{cases} \mathcal{M}_L \cdot \overline{\mathcal{M}_U} & \text{if } \hat{M} = \emptyset \\ \mathcal{M}_L \cdot (\hat{M}) \cdot \overline{\mathcal{M}_U} & \text{if } \hat{M} \neq \emptyset \end{cases}$$

  where $(\hat{M})$ is the sequence containing $\hat{M}$.
- The *decomposition* of T_{bad} is the triple $(\mathcal{L}, \mathcal{M}, \mathcal{U})$.

See Figure 4 for an example of a decomposition.

Lemma 18 (Reconstruction algorithm)**.** *T_{bad} can be reconstructed from its decomposition.*

Proof. Let $\mathcal{M} = (M_1, \ldots, M_n)$. To reconstruct T_L, we set $\tau_1 = L_1 \cup M_1$ and take

$$\tau_i = \begin{cases} \tau_{i-1} \cup L_i & \text{if } L_i \text{ is not starred} \\ \tau_{i-1} \cup L_i \cup M_j & \text{if } L_i \text{ is marked with the } j\text{-th star.} \end{cases}$$

For T_U, we set $\tau_1 = U_1 \cup M_n$ and

$$\tau_i = \begin{cases} \tau_{i-1} \cup U_i & \text{if } U_i \text{ is not starred} \\ \tau_{i-1} \cup U_i \cup M_{n-j+1} & \text{if } U_i \text{ is marked with the } j\text{-th star.} \end{cases}$$

In each case, the efficacy of the algorithm follows easily from induction on i. □

Definition 19 (Flip map for pipings)**.** Let $T = T_{\text{good}} \sqcup T_{\text{bad}}$. The *flip map*

$$\Phi_{P,S} : \mathscr{A}(P) \to \mathscr{A}(P')$$

sends T to a piping $T' = T'_{\text{good}} \sqcup T'_{\text{bad}}$ on P' where $T_{\text{good}} = T'_{\text{good}}$ and T'_{bad} has the decomposition $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$.

In Lemma 23, we show that applying the reconstruction algorithm to $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$ indeed yields a proper piping T'_{bad} of P'. In Lemma 24, we show that $T_{\text{good}} \sqcup T'_{\text{bad}}$ is a proper piping on P' and hence that $\Phi_{P,S}$ is well-defined.

Observation 20. *By construction, the decomposition of T'_{bad} is $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$, so applying $\Phi_{P',S}$ returns T. In particular, $\Phi_{P,S}$ is a bijection.*

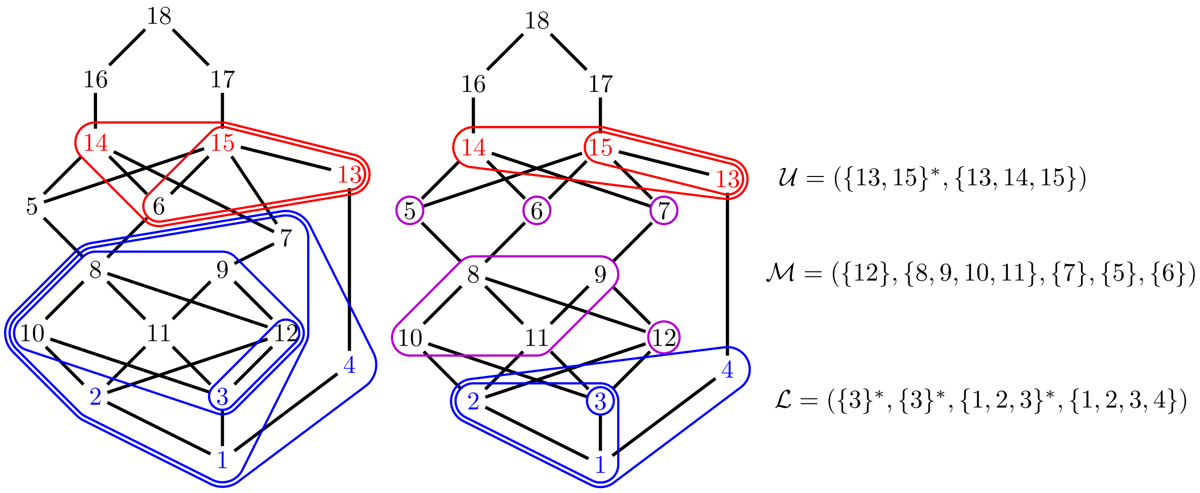


Figure 5: T'_{bad} and its decomposition.

Definition 21. Let $\mathbf{A} = (A_1, \dots, A_n)$ be a sequence of disjoint subsets of P. We say $\mathbf{A}$ is *weakly increasing* if for all $i < j$ we have $(x \in A_i \text{ and } y \in A_j) \Rightarrow y \not\prec x$.

Lemma 22. *$\mathcal{M}$ is weakly increasing.*

Proof. First we show that $\mathcal{M}_L$ is weakly increasing. Indeed, suppose to the contrary that $1 \leqslant i < j \leqslant |\mathcal{M}_L|$ but that there exist $x \in M_i$ and $y \in M_j$ such that $y \prec x$. As $i < j$, there exists a pipe $\tau \in T_L$ such that $x \in \tau$ but $y \notin \tau$. Furthermore, as τ is a lower pipe, there exists $z \in \tau - S$ such that $z \preceq y$. Then since τ is convex, $y \in \tau$, a contradiction.

Next, we show that $\mathcal{M}_L \cdot \hat{M}$ is weakly increasing. Let $x \in M_i$ and $y \in \hat{M}$ such that $1 \leqslant i \leqslant |\mathcal{M}_L|$. Then there exists a pipe $\tau \in T_L$ such that $x \in T_L$. Again, there exists $z \in \tau - S$ such that $z \preceq y$. Then by the same convexity argument, if $y \prec x$ we have $y \in \tau$, contradicting the definition of $\hat{M}$. Hence $\mathcal{M}_L \cdot \hat{M}$ is weakly increasing.

By symmetry, we have that $\hat{M} \cdot \overline{\mathcal{M}_U}$ is weakly increasing. It remains to show that for all $x \in \bigcup_{A \in \mathcal{M}_L} A$ and $y \in \bigcup_{A \in \mathcal{M}_U} A$ we have $y \not\prec x$.

Suppose to the contrary that there are such x and y. Then there exist $\sigma \in T_L$ and $\tau \in T_U$ with $x \in \sigma$ and $y \in \tau$. Furthermore, there exist $a \in \sigma$ and $b \in \tau$ such that

$$a \preceq x \text{ and } y \preceq b.$$

But then we have a cycle in D_T, a contradiction.

□

Lemma 23. *T'_{bad} is a proper piping on P' such that $|T'_{bad}| = |T_{bad}|$.*

Proof. By construction, for all $\sigma, \tau \in T'_{\text{bad}}$, σ and τ are nested or disjoint. Furthermore, observe that in the construction of $T'_L = (\tau'_1, \dots)$, if L_i is empty then it is necessarily starred. Thus for all i, we have $\tau'_i \subsetneq \tau'_{i+1}$. Then $|T'_L| = |\mathcal{L}| = |T_L|$. Similarly, $|T'_U| = |T_U|$. Hence

$$|T'_{\text{bad}}| = |T_L| + |T_U| = |T_{\text{bad}}|.$$

It remains to show that $D_{T'_{\mathrm{bad}}}$ is acyclic. It suffices to show that $A := \bigcup_{\tau' \in T'_L} \tau'$ and that $B := \bigcup_{\tau' \in T'_U} \tau'$ do not form a directed cycle. Observe that as $\mathcal{M}$ is weakly increasing in P, $\overline{\mathcal{M}}$ is weakly increasing in P'. Hence (A, B) is weakly increasing, so A and B do not form a directed cycle. □

Lemma 24. *$T_{good} \sqcup T'_{bad}$ is a proper piping on P'.*

Proof. This is most easily seen by observing how $\Phi_{P,S}$ interacts with quotients of good pipes. Galashin [2, Corollary 2.7] observes that faces of P-associahedra are products of P-associahedra. In particular, given $T \in \mathscr{A}(P)$ and $\tau \in T \cup \{P\}$, we define an equivalence relation $\sim_\tau$ on τ by $i \sim_\tau j$ if there exists $\sigma \in T$ such that $i, j \in \sigma$ and $\sigma \subsetneq \tau$. Then the facet corresponding to T is combinatorially equivalent to the product $\prod_{\tau \in T \cup \{P\}} \mathscr{A}(\tau/\sim_\tau)$.

Let $\tau \in T_{\mathrm{good}} \cup \{P\}$ be minimal such that $S \subseteq \tau$. One may verify that $\Phi_{P,S}$ on any piping containing T_{good} is equivalent to applying $\Phi_{T/\sim_\tau, S/\sim_\tau}$ on the factor of $T/\sim_\tau$ in the product decomposition. Then either $T_{\mathrm{good}} = \emptyset$ and $\Phi_{P,S}$ is well-defined by Lemma 23 or $\Phi_{P,S}$ is well-defined by induction on the size of P. □

We can finally prove Theorem 1.

Proof of Theorem 1. By Observation 20, $\Phi_{P,S} : \mathscr{A}(P) \to \mathscr{A}(P')$ is a bijection. Furthermore, for any piping $T \in \mathscr{A}(P)$, we have

$$|\Phi_{P,S}(T)| = |T_{\mathrm{bad}}| + |T_{\mathrm{good}}| = |T|.$$

Hence the f-vectors of $\mathscr{A}(P)$ and $\mathscr{A}(P')$ are equal. By Lemma 9, the f-vector of $\mathscr{A}(P)$ is a comparability invariant.

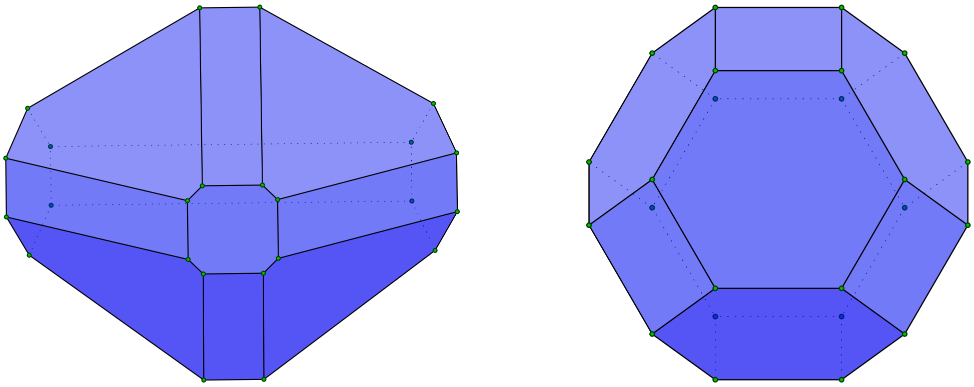
□

4 Proof of Theorem 4

Observation 25 ([3, 4])**.** *If the Hasse diagram of P is a tree, then $\mathscr{A}(P)$ is combinatorially equivalent to the graph associahedron [5] of the line graph of the Hasse diagram of P.*

Proof of Theorem 4. By Observation 25, for any $m, n \geqslant 1$, $\mathscr{A}(P_{m,1,n})$ is combinatorially equivalent to the permutohedron Π_{m+n}.

However, for $m, n \geqslant 2$, $\mathscr{A}(P_{1,m,n})$ has an octagon for a 2-dimensional face which permutohedra never do. In particular, an octagon is a factor of the facet given by any pipe isomorphic to $P_{2,2}$. □



$\mathscr{A}(P_{(1,2,2)})$ $\mathscr{A}(P_{(2,1,2)})$

Figure 6: $\mathscr{A}(P_{(1,2,2)})$ has an octagonal face, but $\mathscr{A}(P_{(2,1,2)})$ does not.

5 Open questions

Question 26. In [7], Stanley defines the *order polytope* and the *chain polytope*, with the latter defined purely in terms of the comparability graph. He constructs a piecewise linear volume preserving map between the two polytopes which sends vertices to vertices.

In particular, this shows that the number of vertices of the order polytope is a comparability invariant. Can a similar geometric map be defined on the realization of P-associahedra in [6]?

Question 27. More generally, can we define $f_{\mathscr{A}(P)}(z)$ purely in terms of $C(P)$? It would also be interesting to answer this question even for f_0.

Question 28. It remains open to find an interpretation of

$$h_{\mathscr{A}(P)}(z) := f_{\mathscr{A}(P)}(z-1)$$

in terms of the combinatorics of P. Can $h(z)$ be defined purely in terms of $C(P)$?

Question 29. The flip map can be analogously defined for *affine poset cyclohedra* [2], where an autonomous subset S has at most one representative from each residue class. Again, it preserves the f-vector of the affine poset cyclohedron. Does Lemma 9 (and hence Theorem 1) hold for affine posets?

Acknowledgements

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References

[1] Dreesen, B., Poguntke, W., & Winkler, P. (1985). Comparability invariance of the fixed point property. *Order*, 2, 269–274. doi:10.1007/BF00333133

[2] Galashin, P. (2024). P-associahedra. *Selecta Mathematica*, 30(1), 6. doi:10.1007/s00029-023-00896-1

[3] Laplante-Anfossi, G. (2022). The diagonal of the operahedra. *Advances in Mathematics*, 405, 108494. doi:10.1016/j.aim.2022.108494

[4] Mantovani, C., Padrol, A., & Pilaud, V. (2024). Acyclonestohedra. *Sém. Lothar. Combin.*, 91B, Article 28.

[5] Postnikov, A., Reiner, V., & Williams, L. (2008). Faces of generalized permutohedra. *Documenta Mathematica*, 13, 207–273. doi:10.4171/DM/248

[6] Sack, A. (2025). A realization of poset associahedra. *Combinatorial Theory*, to appear.

[7] Stanley, R. P. (1986). Two poset polytopes. *Discrete & Computational Geometry*, 1(1), 9–23. doi:10.1007/BF02187680

[8] Stasheff, J. (1997). From operads to "physically" inspired theories. In *Operads: Proceedings of Renaissance Conferences (Hartford, CT / Luminy, 1995)*, Contemp. Math., 202, 53–81. Amer. Math. Soc., Providence, RI. doi:10.1090/conm/202/02593

[9] Stoeckl, K. (2024). Koszul operads governing props and wheeled props. *Advances in Mathematics*, 454, 109869. doi:10.1016/j.aim.2024.109869

[10] Trotter, W. T., Moore, J. I., & Sumner, D. P. (1976). The dimension of a comparability graph. *Proceedings of the American Mathematical Society*, 60(1), 35-38. doi:10.2307/2041106